SERVO MOTOR CONTROL DESIGN REPORT

Instructions: replace the yellow highlighted text with your own words and the requested plots (that is, delete the yellow text). Note the submitted must be saved from MATLAB (not screen captures).

Authorship details

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List others providing assistance	-

Text and plot answers

Question 1

Question ① : Initial Closed Loop

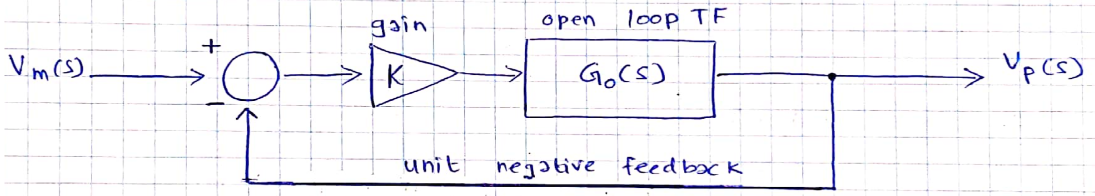
1) $K_m = 7.22$, $\alpha = 4.46$ } estimated values from Servo Motor Identification Task → Question 3

Open Loop System,

$$G_o(s) = \frac{V_p(s)}{V_m(s)} = \frac{K_m}{s(s+\alpha)}$$

$$G_o(s) = \frac{7.22}{s(s+4.46)}$$ (Open Loop Transfer Function)

Unit negative feedback closed-loop system with gain $K = 1$,



Cascade System $(C(s)) = K \cdot G_o(s)$; $K = 1$

$$C(s) = \frac{7.22}{s^2 + 4.46s}$$

Closed Loop System $(G_c(s)) = \frac{C(s)}{1 + H(s) \cdot C(s)}$

$$G_c(s) = \frac{K \cdot G_o(s)}{1 + H(s)\, K\, G_o(s)}$$

$H(s) = 1$ (Unit Negative Feedback)

$$\therefore \quad G_c(s) = \frac{G_o(s)}{1 + G_o(s)}$$

$$G_c(s) = \frac{\frac{7.22}{s(s+4.46)}}{1 + \frac{7.22}{s(s+4.46)}} \quad \begin{array}{l} \times \; s(s+4.46) \\ \times \; s(s+4.46) \end{array}$$

$$G_c(s) = \frac{7.22}{s(s+4.46) + 7.22}$$

$$G_c(s) = \frac{7.22}{s^2 + 4.46s + 7.22}$$ (Closed Loop Transfer Function)

$G_c(s)$ is in the form,

$$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\omega_n^2 = 7.22$$
$$\omega_n = \sqrt{7.22}$$

$$\therefore \; 2\zeta\omega_n = 4.46$$
$$\zeta\omega_n = \frac{4.46}{2}$$
$$\zeta\omega_n = 2.23$$

$$\therefore \; \zeta = \frac{2.23}{\sqrt{7.22}}$$
$$\zeta = 0.8299$$

% Overshoot,

$$\%OS = e^{-(\zeta\pi / \sqrt{1-\zeta^2})} \times 100\%$$

$$\%OS = e^{-(0.8299\pi / \sqrt{1-0.8299^2}} \times 100\%$$

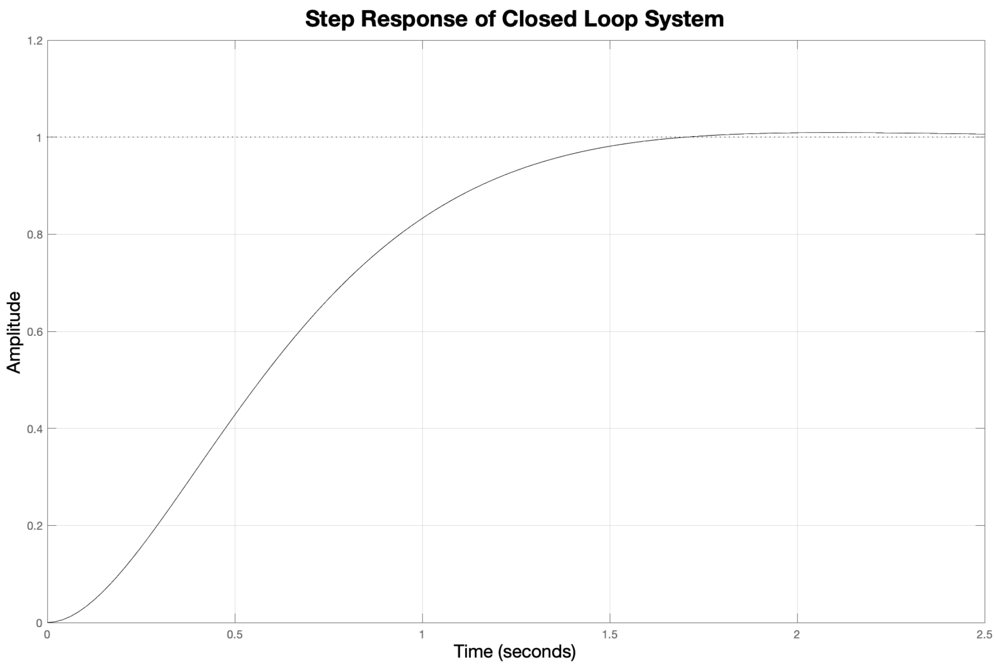
$$\%OS = 0.93\%$$

Settling Time,

$$T_s = \frac{4}{\zeta\omega_n}$$

$$T_s = \frac{4}{2.23}$$

$$T_s = 1.7937\,s$$



The step response of the closed loop system matches very closely with the analytical results as the response settles at around 1.8 seconds at a final value of 1 and has a very small % overshoot. The system is stable and also has zero steady state error for the given step input. The overshoot from the plot and the final value of the closed loop system is calculated below,

$$\%OS = \frac{(C_{max} - C_{final})}{C_{final}} * 100\% \qquad c(t) = \lim_{s \to 0} s \cdot G_C(s)$$

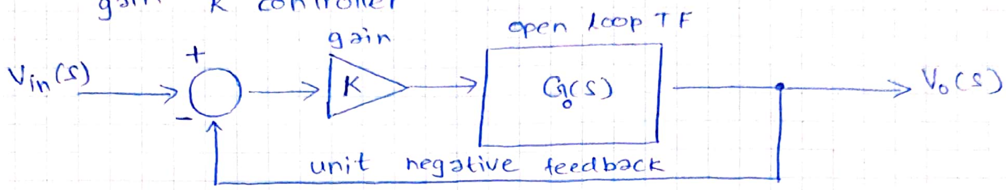
$$\%OS = \frac{1.01 - 1.00}{1.00} * 100\% \qquad c(t) = \lim_{s \to 0} \frac{1}{s} \cdot s \cdot \frac{7.22}{s^2 + 4.46s + 7.22}$$

$$\%OS = 1\% \approx 0.9339\% \qquad c(t) = 1 \;\; (Final\ Value)$$

Question 2

Question ② : Design Control For Requested Response

1) Unit negative feedback closed-loop system with gain K controller



$$\text{Cascade System } (C(s)) = K \cdot G_o(s)$$

$$C(s) = K \frac{7.22}{s(s+4.46)}$$

$$\text{Closed Loop System } (G_{os}(s)) = \frac{C(s)}{1 + H(s) \cdot C(s)}$$

$$G_{os}(s) = \frac{K \cdot G_o(s)}{1 + K H(s) G_o(s)} \quad ; H(s) = 1$$

$$G_{os}(s) = \frac{\frac{K \cdot 7.22}{s(s+4.46)}}{1 + \frac{K \cdot 7.22}{s(s+4.46)}}$$

$$G_{os}(s) = \frac{K \cdot 7.22}{s^2 + 4.46s + K \cdot 7.22}$$

(Closed Loop Transfer Function Expression)

2) Determing gain K for $\%OS = 5\%$

$\%OS = 5\%$

Damping ratio,

$$\zeta = \frac{\sqrt{\ln\left(\frac{OS}{100}\right)^2}}{\sqrt{\pi^2 + \ln\left(\frac{OS}{100}\right)^2}} \quad ; OS = 5$$

$$\zeta = \frac{\sqrt{\ln(5/100)^2}}{\sqrt{\pi^2 + \ln(5/100)^2}}$$

$$\zeta = 0.6901$$

$G_{os}(s)$ is in the form,

$$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$\therefore\ 2\zeta\omega_n = 4.46$

$$\omega_n = \frac{4.46}{2 \times \zeta}$$

$$\omega_n = \frac{4.46}{2 \times 0.6901}$$

$\omega_n = 3.2314$ (Natural Frequency)

$\therefore$ gain for $\%OS = 5\%$,

$$\omega_n^2 = K \cdot 7.22$$

$$K = \frac{\omega_n^2}{7.22}$$

$$K = \frac{3.2314^2}{7.22}$$

$$K = 1.4462$$

When $K = 1.4462$,

$$G_{os}(s) = \frac{10.4416}{s^2 + 4.46s + 10.4416}$$

Location of poles,

$$s^2 + 4.46s + 10.4416 = 0$$

$$s^2 + 4.46s + 4.9729 - 4.9729 + 10.4416 = 0$$

$$(s + 2.23)^2 + 5.4687 = 0$$

$$(s+2.23)^2 = -5.4687$$

$$s + 2.23 = \pm\sqrt{-5.4687}$$

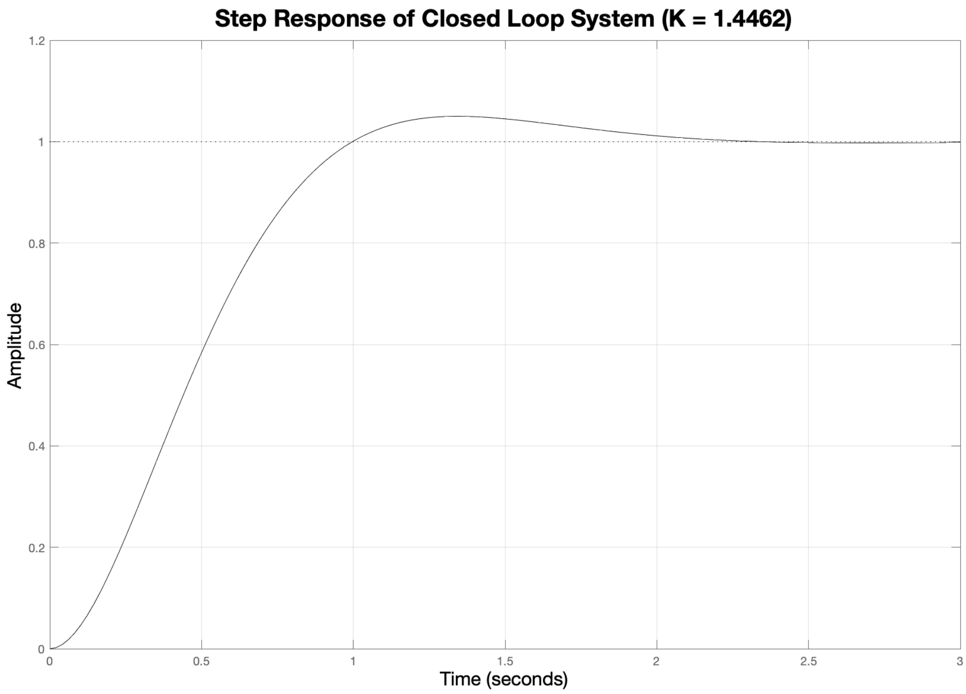
$$s + 2.23 = \pm j\,2.3385$$

$$s = -2.23 \pm j\,2.3385$$

$\therefore$ Closed Loop Poles,

$$s_1 = -2.23 + j\,2.3385$$

$$s_2 = -2.23 - j\,2.3385$$

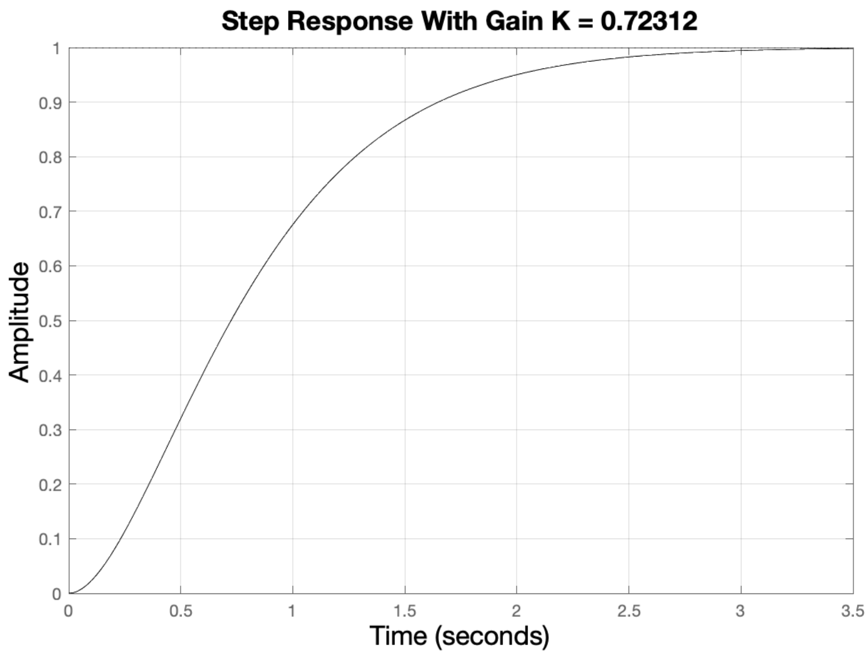
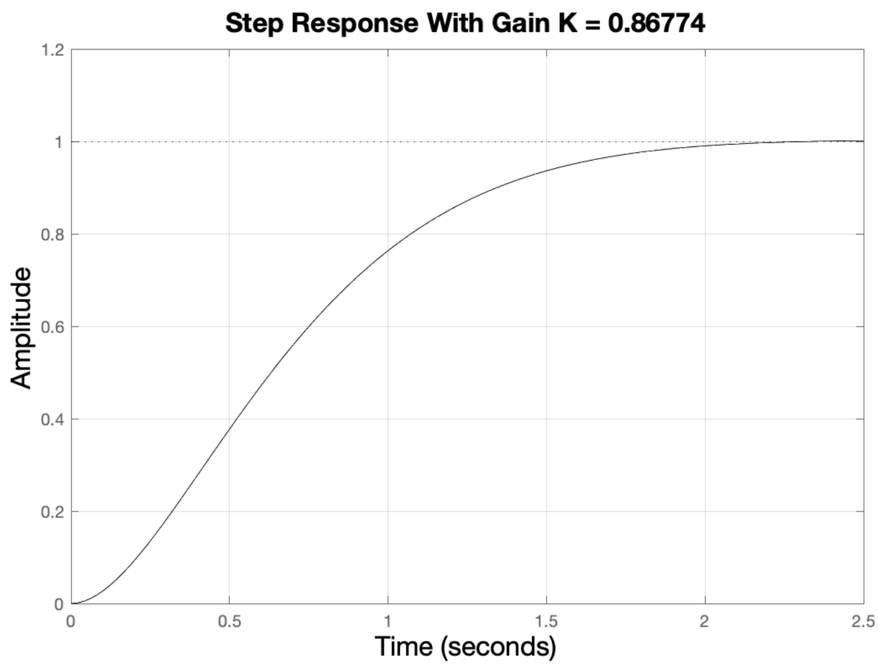


The step response of the closed loop system with 5% overshoot matches exactly with the analytical results as the response now has a higher but designed overshoot compared to the small % overshoot from the previous step response while maintaining the same final value of 1 as before. The system has maintained stability with zero steady state error for the given step input as well. The overshoot from the above plot is calculated below,

$$\%OS = \frac{(C_{max} - C_{final})}{C_{final}} * 100\%$$

$$\%OS = \frac{1.05 - 1.00}{1.00} * 100\%$$

$$\%OS = 5\%$$

<table>
<tr><td rowspan="4">Question 3</td><td>Gain Values Lower & Higher Than K = 1.4462</td><td>Closed Loop Pole Locations & Step Response Plot</td></tr>
<tr><td>K = 0.7231
(1.4462 * 0.5)</td><td>$s_1 = -2.23 + j0.498$
$s_2 = -2.23 - j0.498$

</td></tr>
<tr><td>K = 0.8677
(1.4462 * 0.6)</td><td>$s_1 = -2.23 + j1.137$
$s_2 = -2.23 - j1.137$

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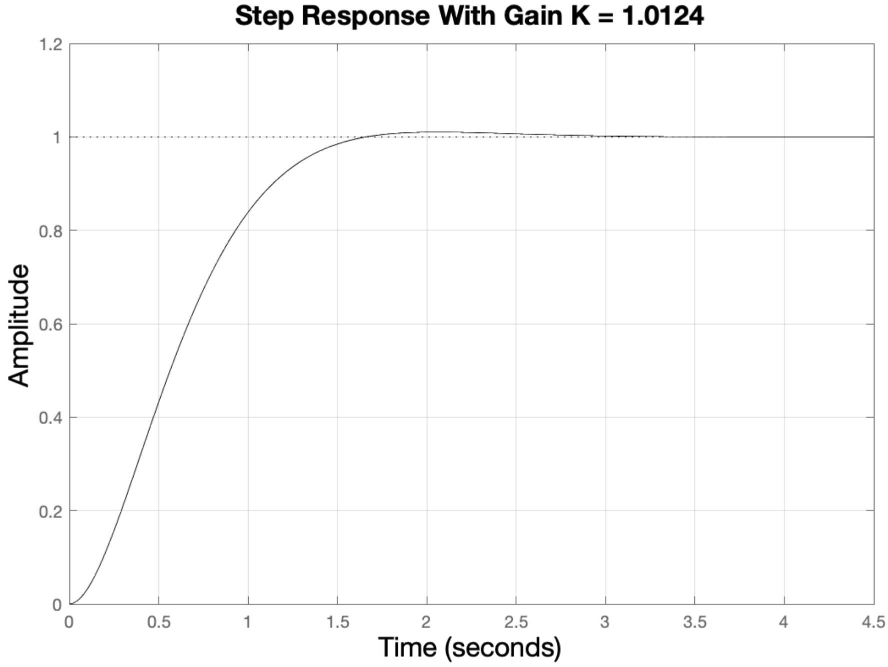
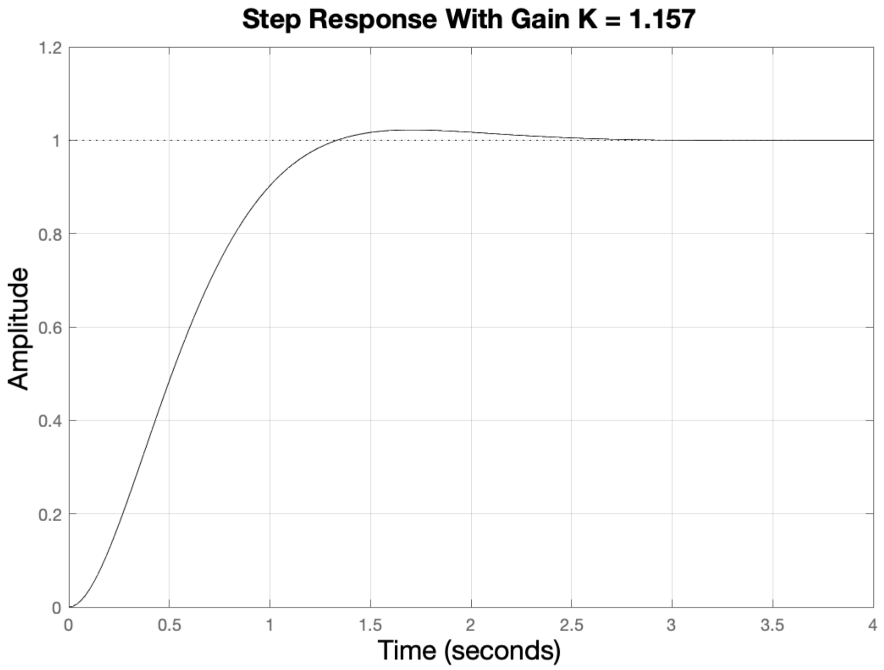
<table>
<tr><td></td><td>K = 1.0123
(1.4462 * 0.7)</td><td>$s_1 = -2.23 + j1.528$
$s_2 = -2.23 - j1.528$

</td></tr>
<tr><td></td><td>K = 1.157
(1.4462 * 0.8)</td><td>$s_1 = -2.23 + j1.839$
$s_2 = -2.23 - j1.839$

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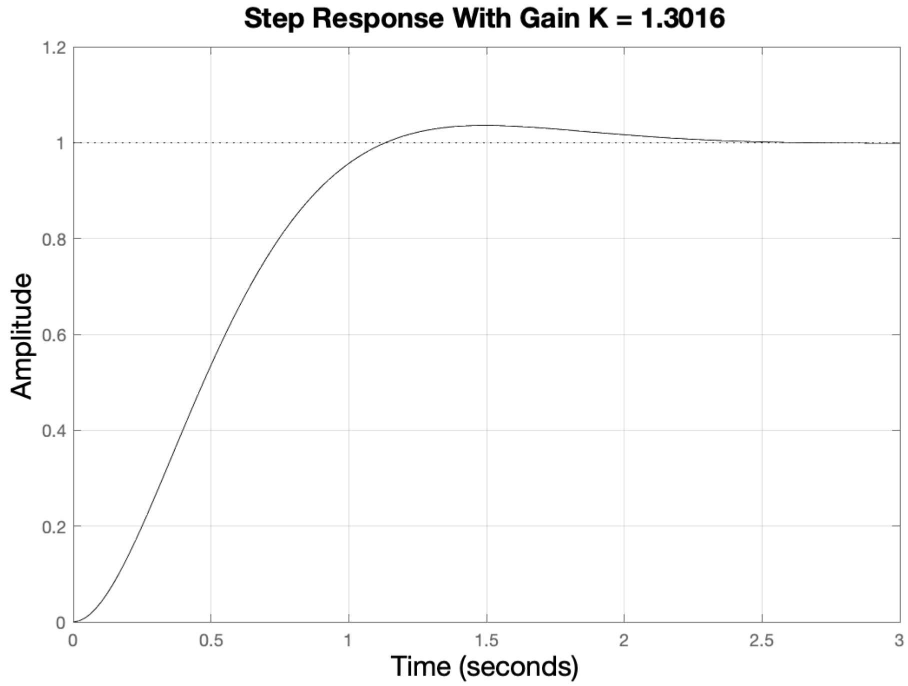
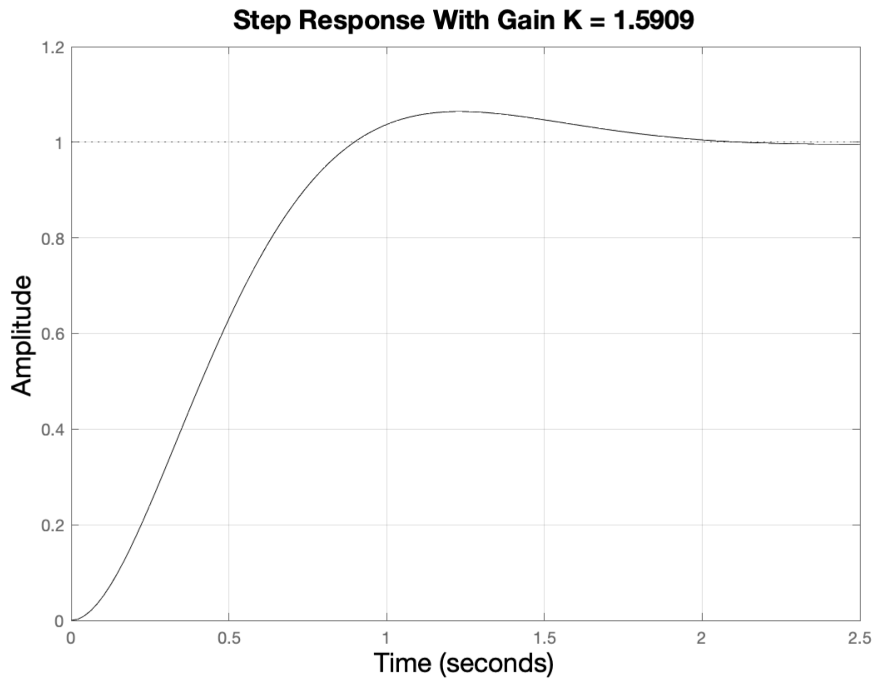
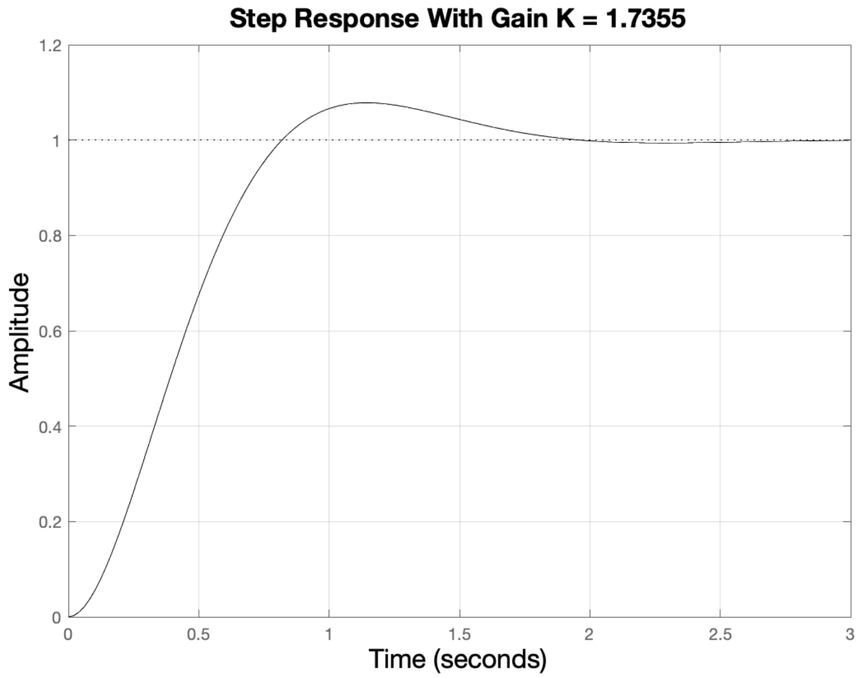
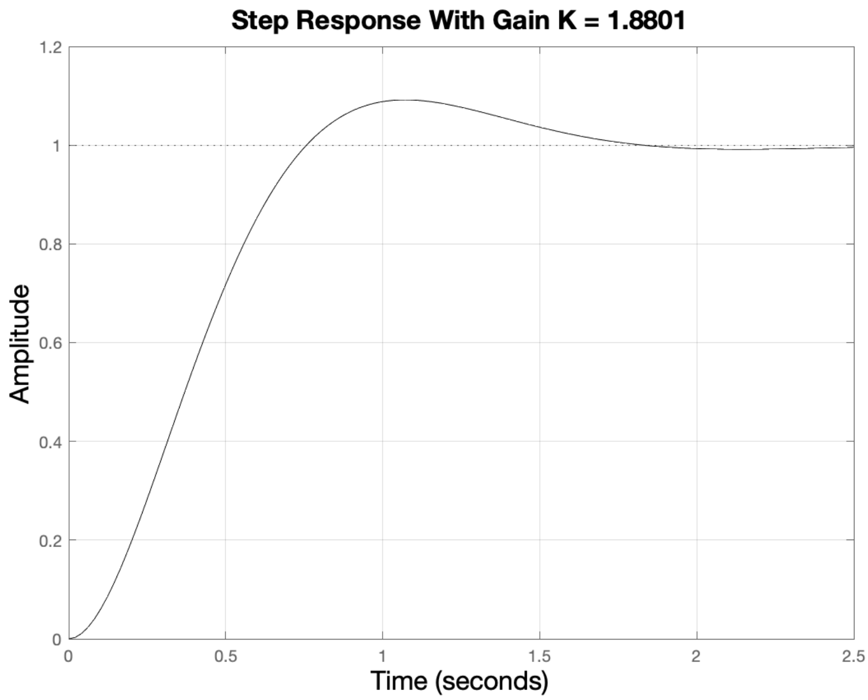
	K = 1.3016 (1.4462 * 0.9)	$s_1 = -2.23 + j2.103$ $s_2 = -2.23 - j2.103$ 
	K = 1.5908 (1.4462 * 1.1)	$s_1 = -2.23 + j2.552$ $s_2 = -2.23 - j2.552$ 

	K = 1.7354 (1.4462 * 1.2)	$s_1 = -2.23 + j2.7491$ $s_2 = -2.23 - j2.7491$ 
	K = 1.8801 (1.4462 * 1.3)	$s_1 = -2.23 + j2.933$ $s_2 = -2.23 - j2.933$ 

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<tr>
<td></td>
<td>K = 2.0247
(1.4462 * 1.4)</td>
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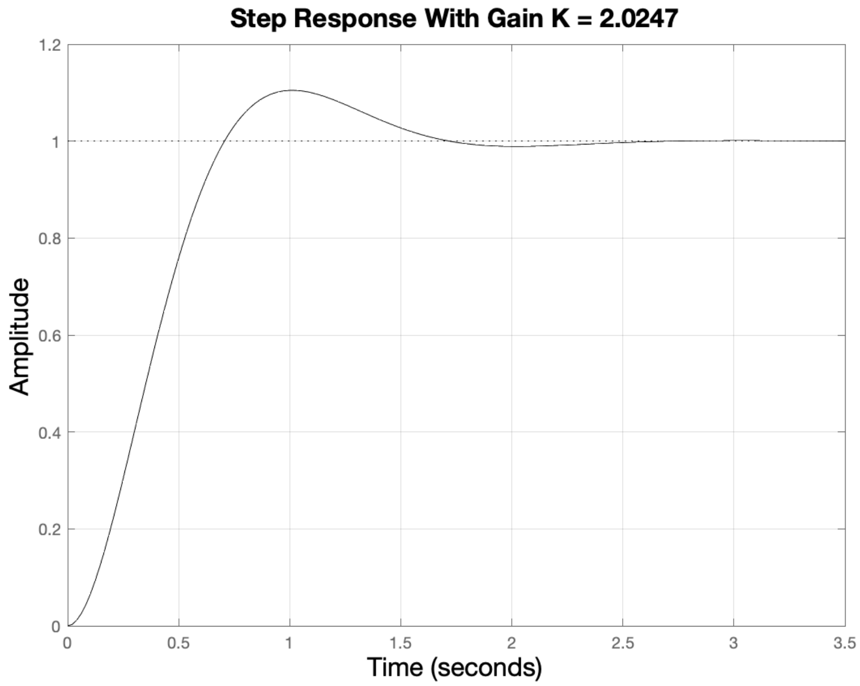
$$s_1 = -2.23 + j3.106$$
$$s_2 = -2.23 - j3.106$$



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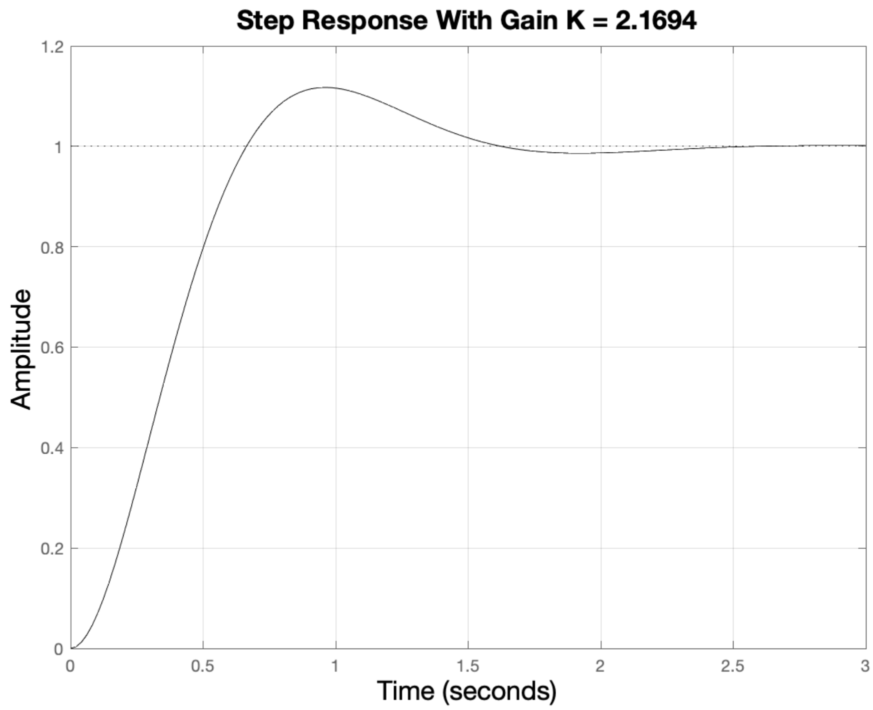
<tr>
<td></td>
<td>K = 2.1693
(1.4462 * 1.5)</td>
<td>

$$s_1 = -2.23 + j3.269$$
$$s_2 = -2.23 - j3.269$$



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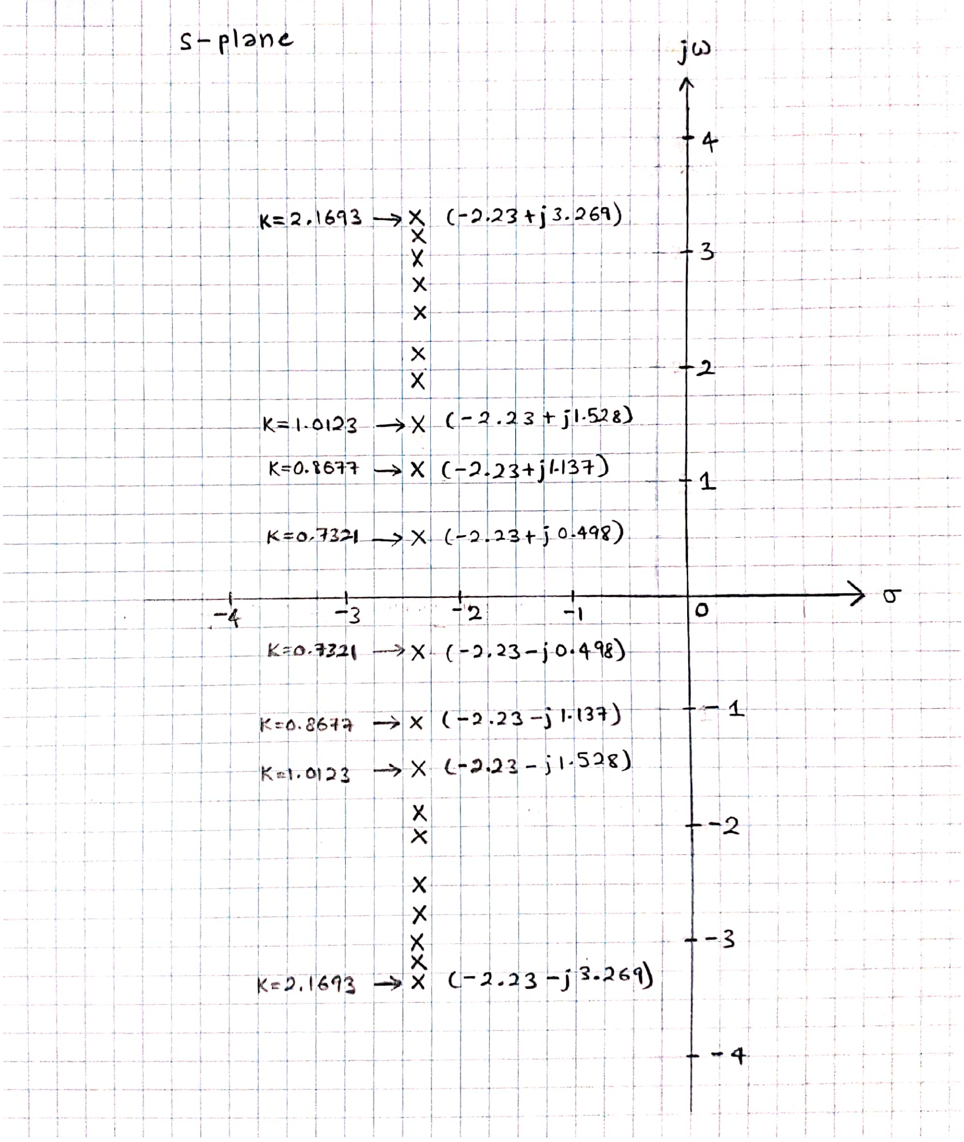
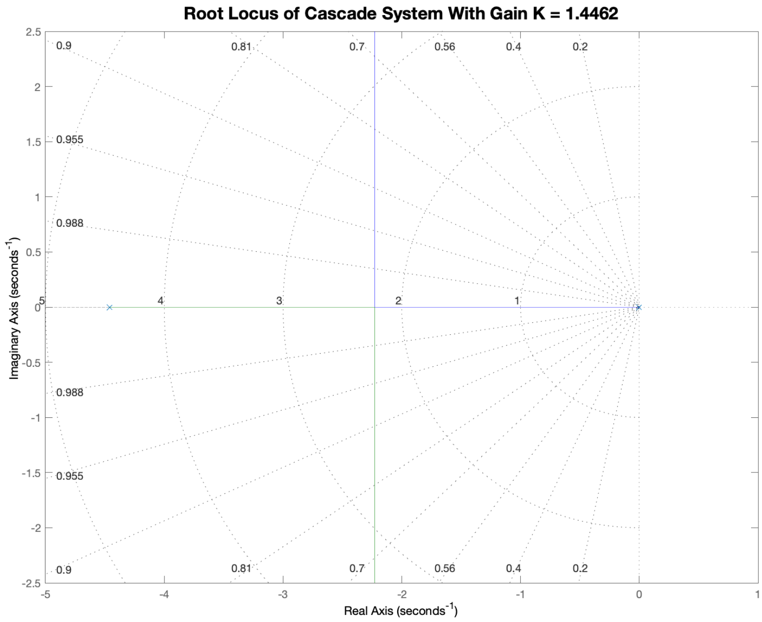
2) Graph in the s-plane that reports the 10 closed-loop pole locations



The above graph with the 10 closed-loop pole locations with lower and higher gain values to K = 1.4462 shows how the poles of the closed loop system will behave with respect to changes in gain value ‘K’ (gain controller) of the open loop system. If a root locus plot was to be drawn for the system’s open loop transfer function, the root loci branches would pass right through the middle of the poles as root locus in definition is the manner in which the closed loop system poles behave with respect to gain changes in its open loop system. This closed system has a fixed settling time at $T_s = 1.7937s$. That is, for any gain value possible, the system will exhibit this value of constant settling time. The overshoot of this system can be set from zero to its maximum amount through infinite gain adjustment (preferred gain values) which will then in turn increase the closed loop systems natural frequency (ω_n) as well making the poles travel higher and higher along the root loci branches.

Question 4	

To achieve half the settling time achieved previously, we first check if simple gain compensation is enough to fulfill the criteria through a root locus plot of the current system with gain $K = 1.4462$,



It can be observed from the above root locus plot that with the current gain compensation, the current settling time of $T_s = 1.7937s$ cannot be halved as both poles go to infinity (to their respective infinite zeros) at s = -2.23. Therefore, it is realized that dynamic compensation is needed to achieve half the settling time of T_s.

Current closed loop system,

$$G_{os}(s) = \frac{10.4416}{s^2 + 4.46s + 10.4416}$$

$$\text{Current } T_s = \frac{4}{\sigma}$$

$$\sigma = \zeta\omega_n$$

but, $4.46 = 2\zeta\omega_n$

$$\therefore \sigma = \frac{4.46}{2}$$

$$\sigma = 2.23$$

$$\therefore \text{Current } T_s = \frac{4}{2.23}$$

$$= 1.7937\,s$$

$$\text{New } T_s = \frac{1}{2}\text{ Current } T_s = \frac{1.7937}{2}$$

$$\text{New } T_s = 0.8967\,s$$

$$\text{New } T_s = \frac{4}{\sigma'}$$

$$\sigma' = \frac{4}{0.8967}$$

$$\sigma' = 4.46$$

$$\cos\theta = \zeta$$

$$\theta = \cos^{-1}(\zeta)$$

$$\theta = \cos^{-1}(0.6901)$$

$$\theta = 46.3619^\circ$$

$$\cos\theta = \frac{4.46}{\omega_n}$$

$$\omega_n = \frac{4.46}{\cos\theta}$$

$$\omega_n = 6.4628$$

$$\omega_d = \omega_n \cdot \sqrt{1-\zeta^2}$$

$$\omega_d = 6.4628 \cdot \sqrt{1-0.6901^2}$$

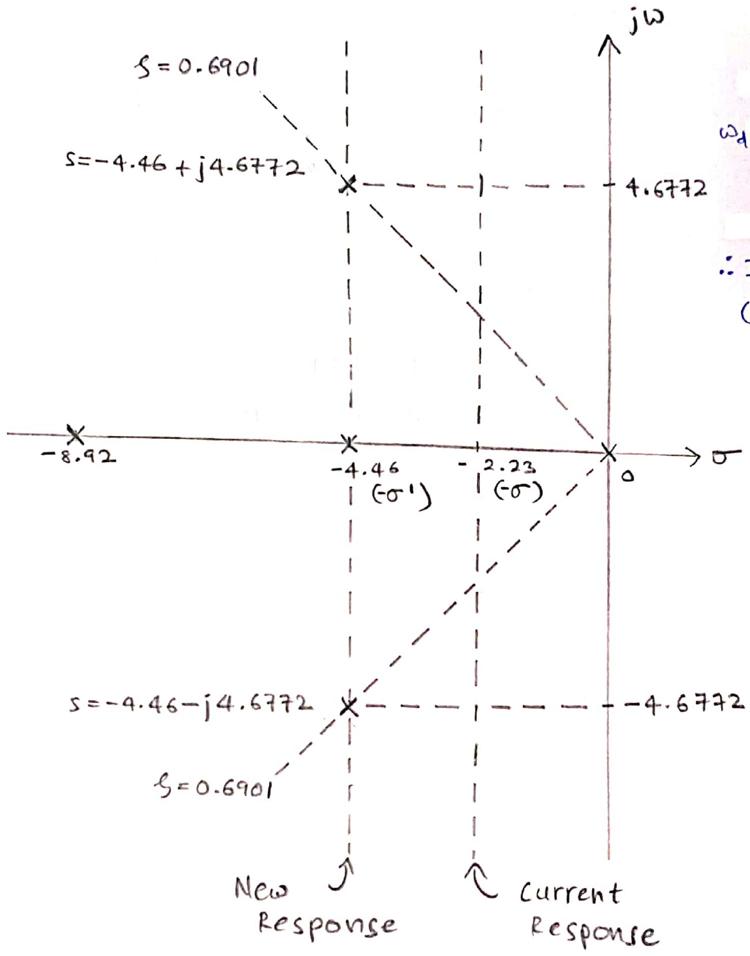
$$\omega_d = 4.6772$$

$$\downarrow$$

$$\therefore \pm j\omega_d = \pm j4.6772$$

(Imaginary axis intercept)

On an s-plane graph,



∴ Desired Closed Loop Pole Location To Achieve $T_s = 0.8967\,s$ (Half of current T_s) while maintaing the same steady state properties,

$$s_1 = -4.46 + j\,4.6772$$

$$s_2 = -4.46 - j\,4.6772$$

Using a lead compensator with transfer function,

$$G_{lead}(s) = K_c \cdot \frac{s+z_c}{s+P_c}$$; K_c = gain of compensator & system

with

$z_c < P_c$

Current closed loop system,

Root Locus Plot To Determine The Location of the compensator's poles & zeros.

$$\text{Cascade System} = \frac{10.4416}{s(s+4.46)}$$

Step 1:– poles $= -4.46, 0$

zeros $= -$

Step 2:– Combining real axis segments.

Step 3:– No. of infinite zeros $= 2$

Step 4:– Real axis intercept,

$$\sigma_0 = \frac{\Sigma \text{ finite poles} - \Sigma \text{ finite zeros}}{\# \text{ finite poles} - \# \text{ finite zeros}}$$

$$\sigma_0 = \frac{(-4.46 - 0) - 0}{2 - 0}$$

$$\sigma_0 = \frac{-4.46}{2}$$

$$\sigma_0 = -2.23$$

Angle,

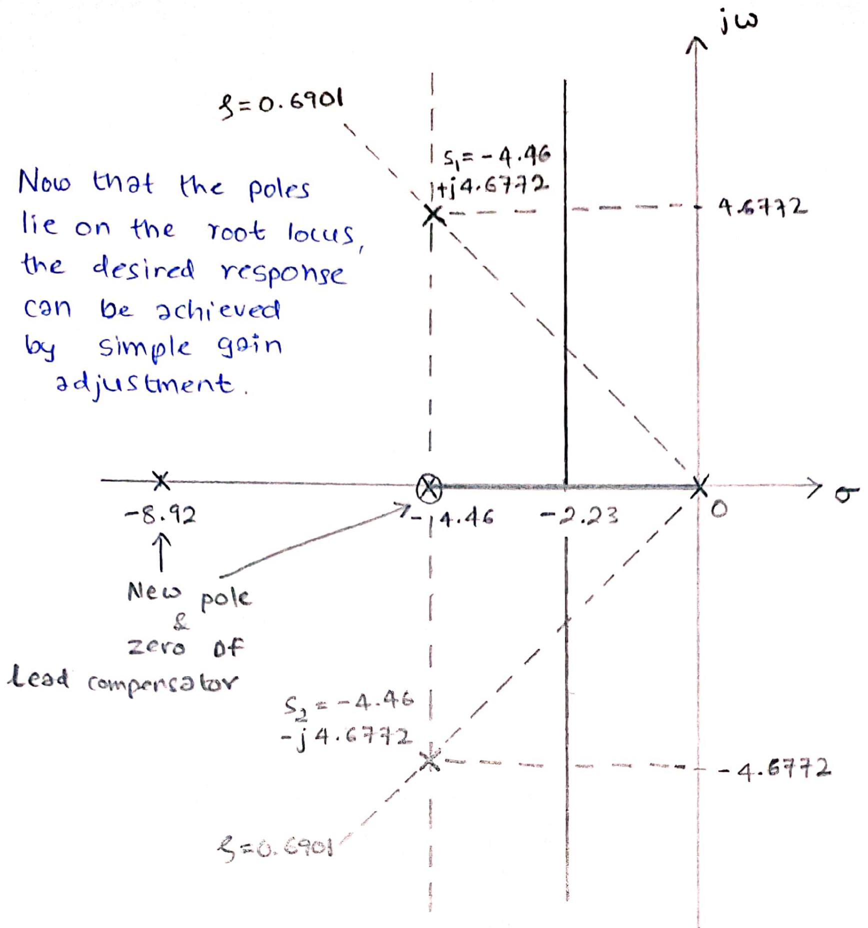
$$\theta_0 = \frac{(2K+1)\,180^\circ}{\# \text{ finite poles} - \# \text{ finite zeros}}$$

$$\theta_0 = \frac{(2K+1)\,180^\circ}{2}$$

$$\theta_{K=0} = \frac{180^\circ}{2} = 90^\circ$$

$$\theta_{K=1} = \frac{3 \times 180^\circ}{2} = 270^\circ = -90^\circ$$

only 2 required as No. of infinite zeros = 2



To achieve $T_s = 0.8967\,s \rightarrow \sigma' = 4.46$,

Therefore looking at the graph above, we can add a zero at $s = -4.46$ to cancel with the pole at $s = -4.46$ through pole-zero cancellation.

$$\therefore Z_c = 4.46$$

Then we can add a pole at $s = -8.92$ such that root locus (s-domain graph) poles go to their infinite zeros at $\sigma' = 4.46$ so that $T_s = 0.8967\,s$ is achieved.

$$\therefore P_c = 8.92$$

$$G_{lead}(s) = K_c \cdot \frac{(s + 4.46)}{(s + 8.92)}$$

New Cascade System with lead compensator,

$$G_c(s) = K_c \cdot \underbrace{\frac{(s+4.46)}{s(s+4.46)(s+8.92)}}_{G(s)}$$

Determining the gain that achieves the requested response of $\%OS = 5\%$ $(\zeta = 0.6901)$.

$$K_c = \frac{1}{|G(s)|}\Bigg|_{s=-4.46+j4.6772}$$

Desired closed loop pole locations,

$$s_{1,2} = -4.46 \pm j4.6772$$

$$K_c = \frac{1}{\left|\frac{(s+4.46)}{s(s+4.46)(s+8.92)}\right|}\Bigg|_{s=-4.46+j4.6772}$$

$$K_c = \frac{s\cancel{(s+4.46)}(s+8.92)}{\cancel{(s+4.46)}}\Bigg|_{s=-4.46+j4.6772}$$

$$K_c = |(-4.46+j4.6772)(-4.46+8.92+j4.6772)|$$

$$K_c = |-41.7678|$$

$$\therefore K_c = 41.7678$$ (required gain of the whole system)

but the system already had a gain of $K_m = 7.22$,

$$7.22\,K = K_c$$

$$K = \frac{41.7678}{7.22}$$

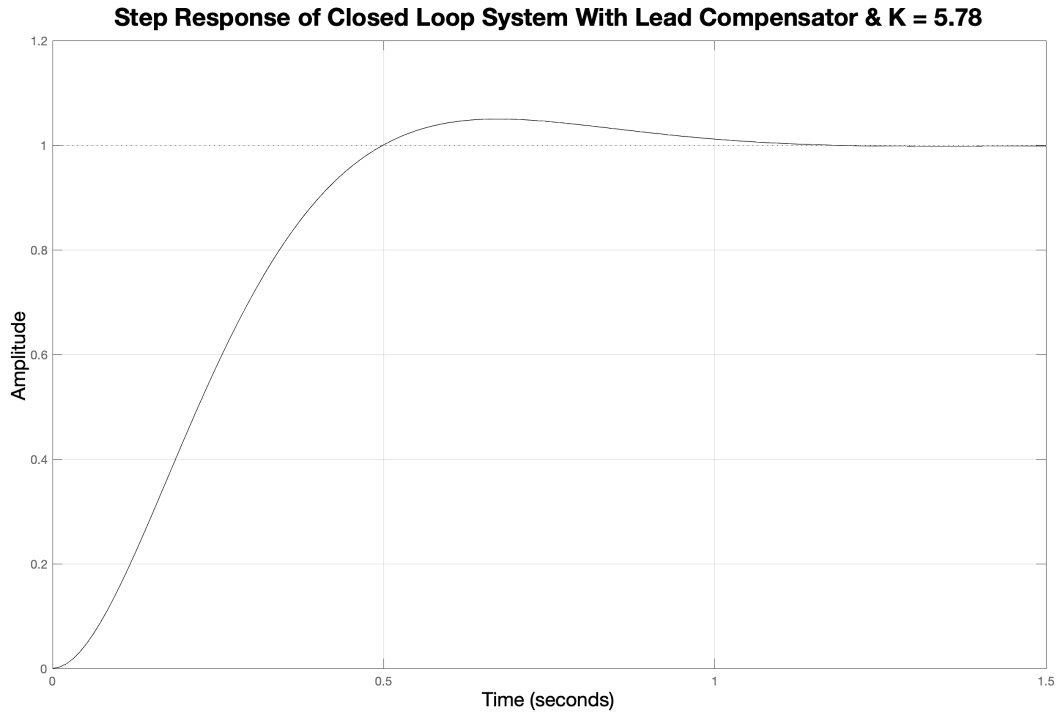
$$\therefore K = 5.78$$

$\therefore$ *Cascade System With Lead Compensator,*

$$G_C(s) = 5.78 \cdot \frac{7.22(s+4.46)}{s(s+4.46)(s+8.92)}$$

$\therefore$ *Closed Loop System With Lead Compensator,*

$$G_{OS,L} = \frac{41.77(s+4.46)}{s(s+4.46)(s+8.92) + 41.77(s+4.46)}$$



The settling time of the compensated system with the lead compensator is $T_s = 0.928s$ which is a tiny bit deviated (0.313s) from the expected value of $T_s = 0.8967s$ yet is very close enough to say that the lead compensator (noting that lead compensator requires active realization for implementation i.e. through an operational amplifier) has successfully achieved the requested response. The compensated system is stable and has also maintained the overshoot requirement of 5% having a value almost close with 4.9876% , settles at the expected final value of 1 at steady state and has zero steady state error for the given step input as well. The steady state error has been calculated as shown below,

$$Step\ input \rightarrow\ Position\ Error\ Constant\ (K_P),$$

$$K_P\ =\ \lim_{s\to 0} G_C(s)\ \ (Cascade\ System)$$

$$Cancelling\ the\ pole\ and\ zero\ at\ (s + 4.46),$$

$$K_P = \lim_{s\to 0} \frac{41.77}{s^2 + 8.92s}$$

$$K_P \rightarrow\ \infty$$

$$\therefore e_{step}(\infty) = \frac{1}{1 + K_P}$$

$$e_{step}(\infty) = \frac{1}{1 + \infty}$$

$$e_{step}(\infty) \rightarrow 0$$

Question 5	

Question (5) : Robustness Study - Model Parameter Error

K_m range $\rightarrow$ $0.9K_m$ to $1.1K_m$ = 6.498 to 7.942

α range $\rightarrow$ 0.9α to 1.1α = 4.014 to 4.906

When K_m and α are both at lower bound,

$$G_c(s) = \frac{K \cdot 6.498}{s^2 + 4.014s + K \cdot 6.498}$$

For 5% %OS $\rightarrow \zeta = 0.6901$,

$$2\zeta\omega_n = 4.014$$

$$\omega_n = \frac{4.014}{2 \times 0.6901}$$

$$\omega_n = 2.9083$$

$$\omega_n^2 = K \cdot 6.498$$

$$K = \frac{2.9083^2}{6.498}$$

$$K = 1.3016$$

When K_m and α are both at higher bound,

$$G_c(s) = \frac{K \cdot 7.942}{s^2 + 4.906s + K \cdot 7.942}$$

For 5% %OS $\rightarrow \zeta = 0.6901$,

$$2\zeta\omega_n = 4.906$$

$$\omega_n = \frac{4.906}{2 \times 0.6901}$$

$$\omega_n = 3.5546$$

$$\omega_n^2 = K \cdot 7.942$$

$$K = \frac{3.5546^2}{7.942} \rightarrow K = 1.5909$$

When α is low and K_m is high,

$$G_c(s) = \frac{K \cdot 7.942}{s^2 + 4.014 + K \cdot 7.942}$$

For 5% %OS $\rightarrow$ $\zeta = 0.6901$

$$2\zeta\omega_n = 4.014$$
$$\frac{4.014}{2 \times 0.6901}$$
$$\omega_n = 2.9083$$
$$\omega_n^2 = K \cdot 7.942$$
$$K = \frac{2.9083^2}{7.942}$$
$$K = 1.0650$$

When α is high and K_m is low,

$$G_c(s) = \frac{K \cdot 6.498}{s^2 + 4.906s + K \cdot 6.498}$$

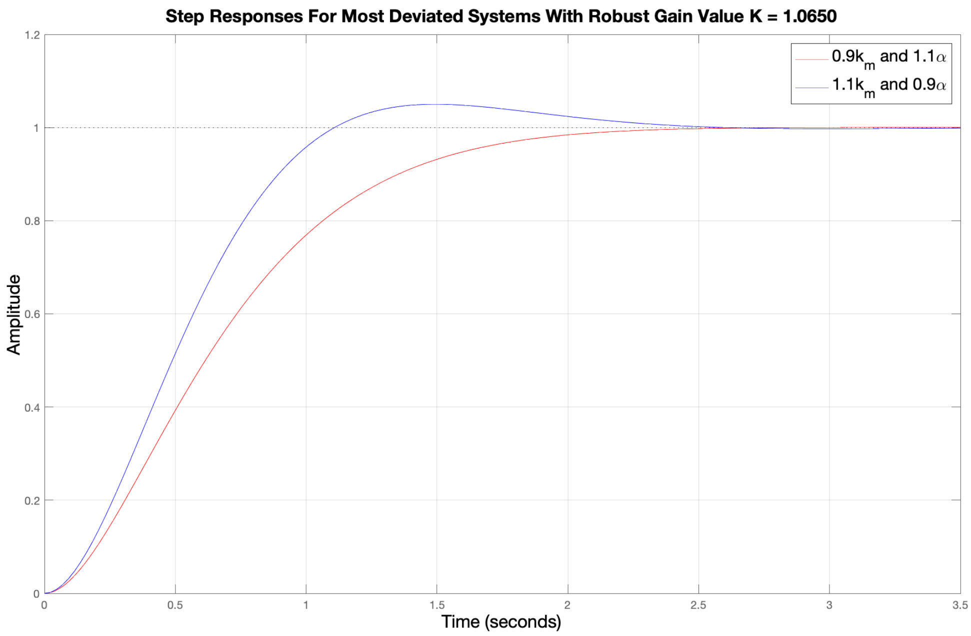
For 5% %OS $\rightarrow$ $\zeta = 0.6901$

$$2\zeta\omega_n = 4.906$$
$$\omega_n = \frac{4.906}{2 \times 0.6901}$$
$$\omega_n = 3.5546$$
$$\omega_n^2 = K \cdot 6.498$$
$$K = \frac{3.5546^2}{6.498}$$
$$K = 1.9444$$

Looking at the 4 systems above, it is seen that lowest 'K' value is 1.0650 indicating that at maximum that value is required for any closed loop system in the range of the unknown values from 0.9α to 1.1α & $0.9K_m$ to $1.1K_m$ in the system for it to produce a % overshoot no worse than the specified 5%.

$$\therefore K = 1.0650$$

The choice of $K = 1.0650$ for the gain compensator is robust for the system's k_m and α values as even in the event of the true k_m value being the highest possible of 1.1 k_m (7.942) and true alpha value being the lowest possible of 0.9 α (4.014) for the open loop system transfer function, the closed loop system will still have a % overshoot no worse than specified requirement of 5%. In the event of the opposite happening as true k_m value being the lowest possible of 0.9 k_m (6.498) and true alpha value being the highest possible of 1.1 α (4.906) for the open loop transfer function, the closed loop system will require a gain of $K = 1.9444$ for the system to produce an overshoot of 5% but having a lower gain value $'K'$ makes sure that the system will never go above the requirement overshoot and shows that the system is definitely stable and can handle uncertain conditions with changes in the range of k_m and α values. These two system cases were used for comparison as they showed the most deviated gain values 'K' making their parameters the most suitable for ranged calculations in the given model parameter error range. The gain value $'K'$ is higher for all systems other than the case of 0.9 α and 1.1 k_m to obtain the required overshoot of 5% thus having a robust lower $'K'$ value in the first place completely assures that at no point even due to an uncertainty will the system fail. Therefore, considering the above proven facts, the value of $K = 1.0650$ for the gain in the simple gain compensator for system is robust as it is able to handle uncertainty in model parameters efficiently while preserving the overall system requirements including fairly good settling times, stability and holding the final value of 1 with zero steady state error for step input. Shown below are step response plots of how the 2 above mentioned different cases of systems handled the determined robust gain value of $K = 1.0650$ with their respective % overshoots,



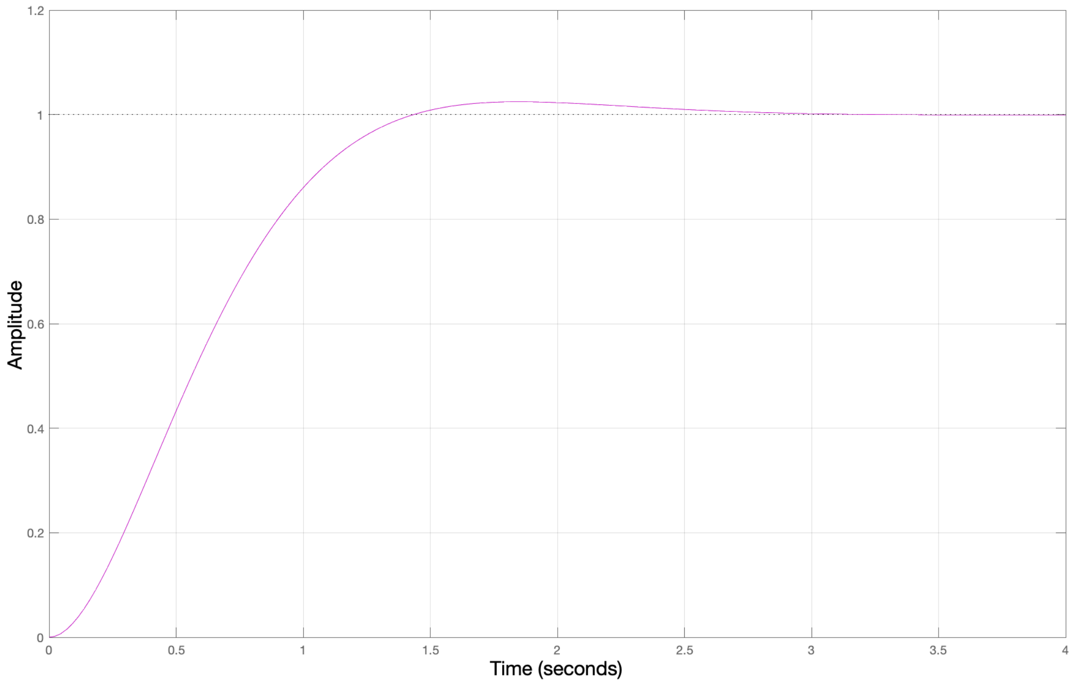
Closed Loop System with $k_m = 6.498$ & $\alpha = 4.906$,

```
    RiseTime: 1.1527
SettlingTime: 1.9241
 SettlingMin: 0.9045
 SettlingMax: 1.0003
   Overshoot: 0.0300
  Undershoot: 0
        Peak: 1.0003
    PeakTime: 3.3042
```

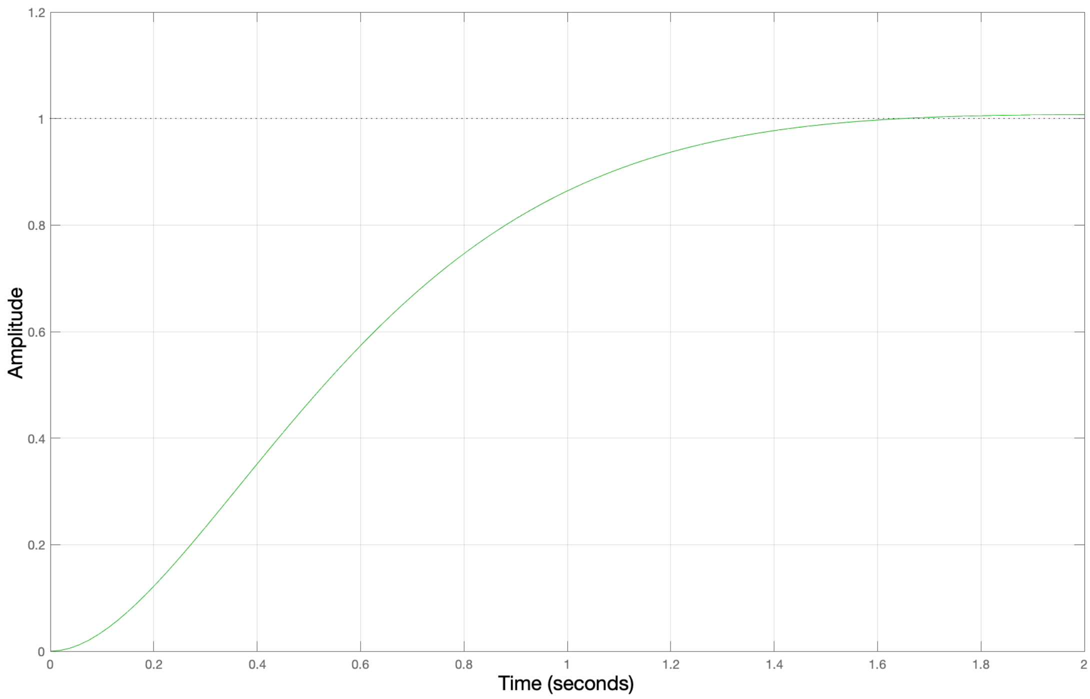
Closed Loop System with $k_m = 7.942$ & $\alpha = 4.014$,

```
    RiseTime: 0.7211
SettlingTime: 2.0615
 SettlingMin: 0.9005
 SettlingMax: 1.0500
   Overshoot: 5.0000
  Undershoot: 0
        Peak: 1.0500
    PeakTime: 1.4915
```

Step Response of 0.9k_m and 0.9α System With Robust Gain Value K = 1.0650



Step Response of 1.1k_m and 1.1α System With Robust Gain Value K = 1.0650



Robustness in control theory refers to controller design which explicitly handles problems in uncertainty. Robust control systems approaches are designed to function efficiently given that uncertain disturbances/parameters are found within the system. In the perspective of robustness, the system which exhibits the highest overshoot is the system with highest k_m value ($1.1\ k_m$) and lowest alpha value ($0.9\ \alpha$) with an overshoot of 5% (maximum possible according to specifications). The analysis below shows how the parameter values have affected the system in terms of robustness and overshoot,

$$\textit{From derived closed loop transfer function expression and}$$
$$2^{nd}\ \textit{order system general form,}$$
$$\omega_n^2 = K \cdot k_m \rightarrow\ \omega_n = \sqrt{1.0650 \cdot k_m}\ \ --(eq.1)$$
$$and\ \ 2 \cdot \zeta \cdot \omega_n = \alpha$$
$$\zeta = \frac{\alpha}{2 \cdot \omega_n}\ \ --(eq.2)$$
$$\therefore\ \zeta = \frac{\alpha}{2 \cdot \sqrt{1.0650 \cdot k_m}} --(eq.1)\ and\ (eq.2)$$
$$but\ \zeta \propto \frac{1}{\%\ OS}(higher\ damping\ ratio \rightarrow lower\ overshoot\ \&\ vice\ versa)$$
$$\therefore \frac{1}{\%\ OS} \propto \frac{\alpha}{2 \cdot \sqrt{1.0650 \cdot k_m}}$$
$$\therefore \%\ OS \propto \frac{2 \cdot \sqrt{1.0650 \cdot k_m}}{\alpha}$$

The above analytically proven expression explains the relationship between the model parameter values k_m and α with the % overshoot in the closed loop system which affects its robustness. It shows that the higher the k_m value and the lower the α value is in the system, the higher % overshoot the system will exhibit while the lower the k_m value and higher the α value is in the system, the lower % overshoot the system will exhibit. When comparing the range of overshoot values produced by all the systems in the model parameter range, it is calculated to be as follows by taking the difference between the 2 most deviated systems which were calculated earlier as follows,

$$\%\ OS_{max} - \%\ OS_{min} = 5\% - 0.03\% = 4.97\%\ (max.overshoot\ difference\ between\ 2\ systems)$$

Therefore, when comparing the above 2 systems, the settling time of the system with lowest overshoot is seen to settle at $T_s = 1.9241s$ while the system with the highest overshoot is seen to settle at $T_s = 2.0615s$. All systems in the different k_m and α value range possible are to settle at a settling time T_s in this range which is pretty decent when compared to the settling time of system with true k_m and α values at $T_s = 1.7937s$. All systems have also settled at a final value of 1 with zero steady state error making this system a robust system for the model parameter errors which are minimised with a robust gain value of $K = 1.0650$.

Question 6	

Question ⑥ : Robustness Study 2 - Measurement Error

Closed Loop System now,

$$G_c(s) = \frac{K \cdot 7.22}{s^2 + 4.46s + K \cdot H(s) \cdot 7.22}$$

feedback range (H) $\longrightarrow$ 0.9H to 1.1H = 0.9 to 1.1

When H is at its lowest possible value,

$$G_c(s) = \frac{K \cdot 7.22}{s^2 + 4.46s + K \cdot 6.498}$$

For 5% OS $\longrightarrow$ $\zeta = 0.6901$,

$$2\zeta\omega_n = 4.46$$

$$\omega_n = \frac{4.46}{2 \cdot \zeta}$$

$$\omega_n = \frac{4.46}{2 \times 0.6901}$$

$$\omega_n = 3.2314 \text{ rad/s}$$

and

$$\omega_n^2 = K \cdot 6.498$$

$$\therefore K = \frac{\omega_n^2}{6.498}$$

$$K = \frac{3.2314^2}{6.498}$$

$$K = 1.607$$

When H is at it's highest possible value,

$$G_c(S) = \frac{K \cdot 7.22}{S^2 + 4.46S + K \cdot 7.942}$$

For 5% OS $\rightarrow \zeta = 0.6901$,

$$2\zeta\omega_n = 4.46$$

$$\omega_n = \frac{4.46}{2 \cdot \zeta}$$

$$\omega_n = \frac{4.46}{2 \times 0.6901}$$

$$\omega_n = 3.2314 \text{ rad/s}$$

and

$$\omega_n^2 = K \cdot 7.942$$

$$\therefore K = \frac{\omega_n^2}{7.942}$$

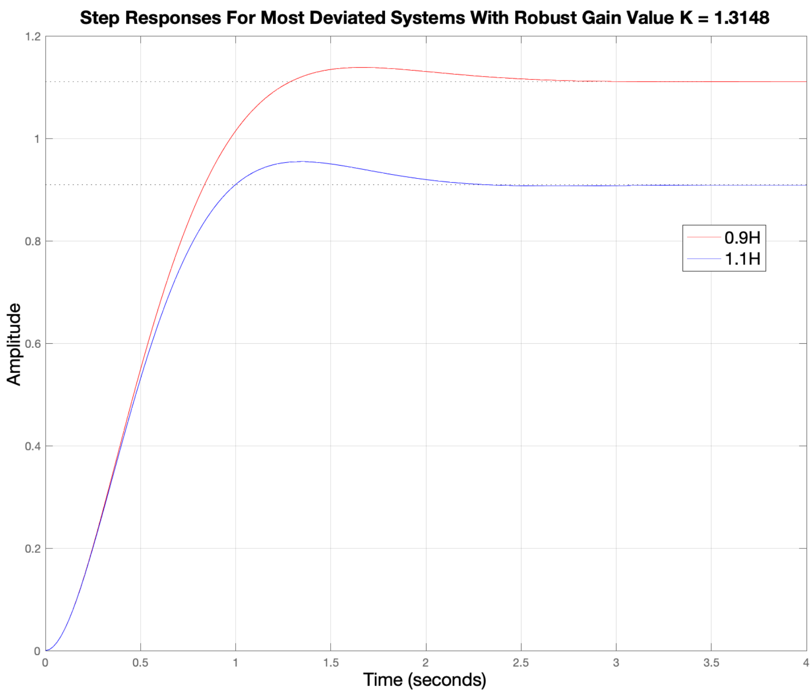
$$K = \frac{3.2314^2}{7.942}$$

$$K = 1.3148$$

Looking at 2 system cases above, it is seen that the lowest 'K' value is 1.3148 indicating that at maximum, that gain value is required for any closed loop system in the range of unknown feedback values from 0.9 H to 1.1 H in the system for it to produce a % overshoot no worse than the specified 5%.

$$\therefore K = 1.3148$$

The choice of $K = 1.3148$ for the gain value 'K' is robust for the system's varying feedback gain values 'H' as even in the event of the true feedback gain value 'H' being the highest possible of $1.1H$ (1.1), the closed loop system will still have a % overshoot no worse than specified requirement of 5%. In the event of the opposite happening as the true feedback gain value 'H' being the lowest possible of $0.9H$ (0.9), the closed loop system will require a gain of $K = 1.607$ for the system to produce an overshoot of 5% thus having a lower gain value $'K'$makes sure that the system will never go above the required % overshoot and shows that the system is definitely stable and can handle uncertain conditions with changes in the range of 'H'values. These two system cases were used for comparison as they showed the most deviated gain values 'K' making their parameters the most suitable for ranged calculations in the given measurement error range. The gain value $'K'$ is higher for systems with feedback gain lower than 1.1 H to obtain the required overshoot of 5% thus having a robust lower $'K'$ value in the first place completely assures that at no point even due to an uncertainty will the system fail. Therefore, considering the above proven facts, the value of $K = 1.3148$ for the gain value 'K' for system is robust as it is able to handle measurement error efficiently while preserving the overall system requirements including fairly good settling times and stability but has a finite(small) steady state error for the given step input as final value changes with different feedback gain values. Shown below are step response plots of how the 2 above mentioned different cases of systems handled the determined robust gain value of $K = 1.3148$ with their respective % overshoots,

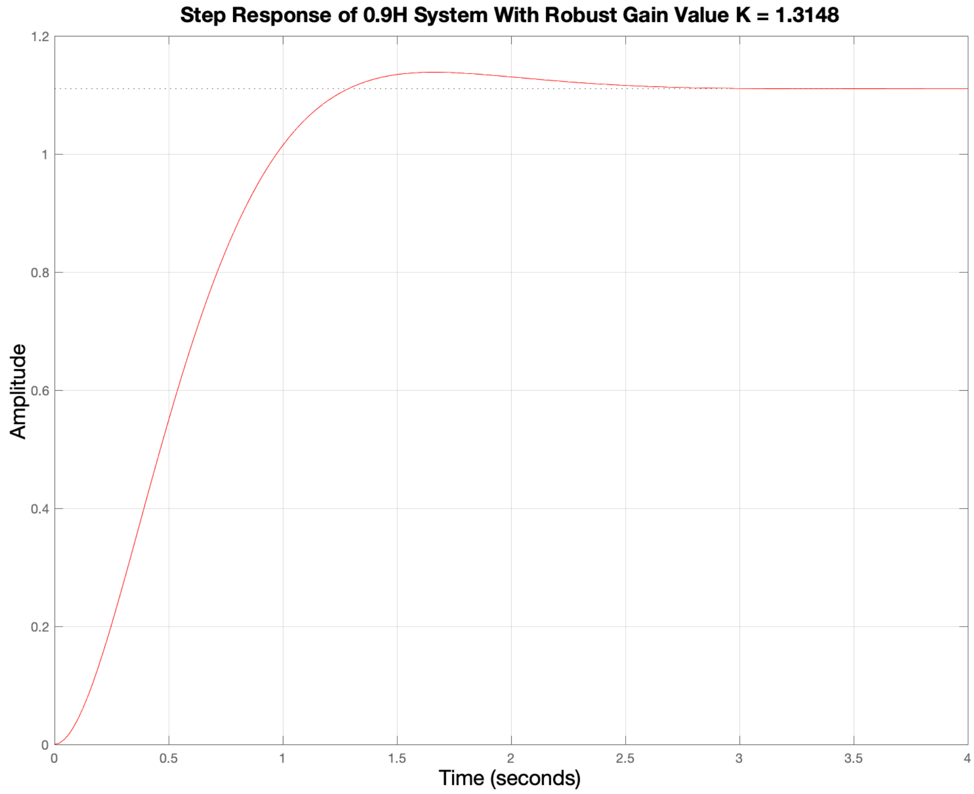


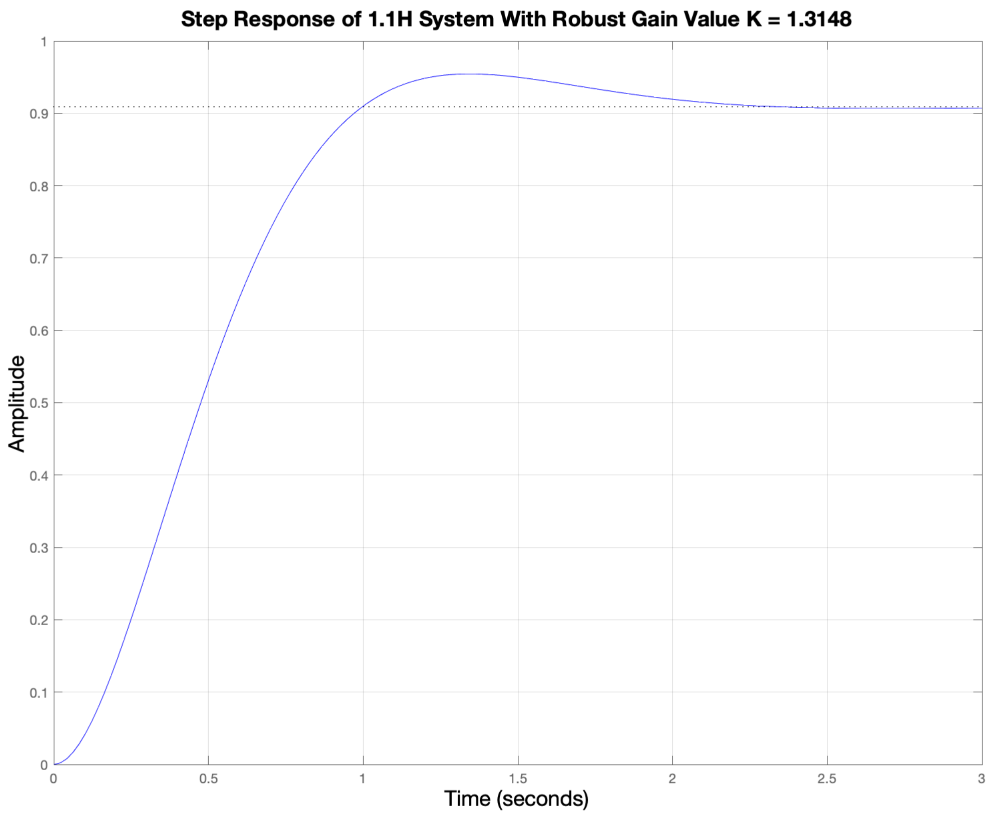
Closed Loop System with $H = 0.9$,

```
     RiseTime: 0.7982
 SettlingTime: 1.9145
  SettlingMin: 1.0101
  SettlingMax: 1.1384
    Overshoot: 2.4525
   Undershoot: 0
         Peak: 1.1384
     PeakTime: 1.6727
```

Closed Loop System with $H = 1.1$,

```
     RiseTime: 0.6490
 SettlingTime: 1.8554
  SettlingMin: 0.8186
  SettlingMax: 0.9545
    Overshoot: 5.0000
   Undershoot: 0
         Peak: 0.9545
     PeakTime: 1.3423
```


Step Response of 0.9H System With Robust Gain Value K = 1.3148
Amplitude
Time (seconds)
1.2
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2
2.5
3
3.5
4


Step Response of 1.1H System With Robust Gain Value K = 1.3148
Amplitude
Time (seconds)
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.5
1
1.5
2
2.5
3

In the perspective of robustness, the the system which exhibits the highest overshoot is the system with highest H feedback value ($1.1H$) with a % overshoot of 5% (maximum possible according to specifications) while the lowest overshoot was exhibited by the system with the lowest H feedback value ($0.9H$) with a % overshoot of 2.4525% . The analysis below shows how the measurement error has affected the system in terms of robustness and final value of the system,

$$From\ final\ value\ theorem,$$

$$c(t) = \lim_{s \to 0} s \cdot G_C(s)\ \ (theorem\ holds\ as\ all\ closed\ loop\ poles\ have\ a\ negative\ real\ part)$$

$$c(t) = \lim_{s \to 0} \frac{1}{s} \cdot s \cdot \frac{K \cdot k_m}{s^2 + \alpha \cdot s + H \cdot K \cdot k_m} \quad (for\ step\ input)$$

$$\therefore c(\infty) = \frac{K \cdot k_m}{H \cdot K \cdot k_m}$$

$$\therefore c(\infty) = \frac{1}{H} \ \ (final\ value\ of\ the\ system)$$

The above analytically proven expression explains the relationship between the feedback (measurement) value H with the final value $c(t)$ of the system which affects its steady state error. It shows that depending on the H value higher or lower than 1, the system is going to have different steady state error values as the system settles when time goes to infinity. When comparing the range of final values produced by all the systems in the measurement range, it is calculated to be as follows by taking the difference between the 2 most deviated systems which were found previously as follows,

$$c(\infty)_{max} - c(\infty)_{min} = \frac{1}{0.9H} - \frac{1}{1.1H} = \frac{1}{0.9} - \frac{1.1}{1.1}$$

$$c(\infty)_{max} - c(\infty)_{min} = 0.2020$$

Therefore, when comparing, we can expect a maximum steady state error of,

$$final\ value\ \pm \frac{steady\ state\ error\ range}{2} = 1\ \pm 0.1010\ (max.\, steady\ state\ error)$$

We get a wide range of systems' final values within that range from 0.9090 to 1.1111 with respect to the uncertainty present in the measurement values but at no point will any of the systems exceed the required specification value of 5% overshoot making this system a robust system for the measurement error which will have minimal impact with a robust gain value of $K = 1.3148$.

Question 7	

Open loop system with additional unmodelled fast pole β,

$$G_O(s) = \frac{V_P(s)}{V_m(s)} = \frac{k_m}{s \cdot (s+\alpha) \cdot (s+\beta)}$$

Cascade system,

$$C_\beta(s) = K \cdot G_O(s) = K \cdot \frac{k_m}{s \cdot (s+\alpha) \cdot (s+\beta)}$$

Closed loop system with unit negative feedback,

$$G_C(s) = \frac{C_\beta(s)}{1 + C_\beta(s)}$$

$$G_C(s) = \frac{\frac{K \cdot k_m}{s \cdot (s+\alpha) \cdot (s+\beta)}}{1 + \frac{K \cdot k_m}{s \cdot (s+\alpha) \cdot (s+\beta)}}$$

$$G_C(s) = \frac{1.4462 \cdot 7.22}{s \cdot (s+4.46) \cdot (s+\beta) + 1.4462 \cdot 7.22}$$

$$G_C(s) = \frac{10.4416}{s^3 + (4.46+\beta) \cdot s^2 + (4.46 \cdot \beta) \cdot s + 10.4416}$$

a) When $\beta = 10 \cdot \alpha$,

$$\beta = 10 \cdot 4.46 = 44.6$$

$$\therefore\ G_C(s) = \frac{10.4416}{s^3 + (4.46+44.6) \cdot s^2 + (4.46 \cdot 44.6) \cdot s + 10.4416}$$

$$G_C(s) = \frac{10.4416}{s^3 + 49.06 \cdot s^2 + 198.916 \cdot s + 10.4416}$$

Solving the 3^{rd} *order system poles through MATLAB using 'pole()' command,*

$$G_C(s) = \frac{10.4416}{(s+0.0532) \cdot (s+4.4010) \cdot (s+44.6058)}$$

The pole at $s = -0.0532$ *is a dominant pole as* $5 \cdot 0.0532 < 4.4010$ *and* 44.6058,

Therefore cancelling the poles at $s = -4.4010$ *and* $s = -44.6058$,

$$G_C(s) = \frac{10.4416}{(s+0.0532)}$$

$G_C(s)$ *is now in the form of a* 1^{st} *order system,*

$$\therefore T_s = \frac{4}{a} \text{ where } \; a = 0.0532$$

$$\therefore \underline{\underline{T_s = 75.188\,s}} \quad (1^{st} \text{ order approximation})$$

There is no overshoot for 1^{st} *order systems,*

$$\therefore \underline{\underline{\%\,OS = 0\%}} \quad (1^{st} \text{ order approximation})$$

$b)$ *When* $\beta = 2 \cdot \alpha$,

$$\beta = 2 \cdot 4.46 = 8.92$$

$$\therefore \; G_C(s) = \frac{10.4416}{s^3 + (4.46 + 8.92) \cdot s^2 + (4.46 \cdot 8.92) \cdot s + 10.4416}$$

$$G_C(s) = \frac{10.4416}{s^3 + 13.38 \cdot s^2 + 39.7832 \cdot s + 10.4416}$$

Solving the 3^{rd} *order system poles through MATLAB 'pole()' command,*

$$G_C(s) = \frac{10.4416}{(s + 0.2902) \cdot (s + 3.9275) \cdot (s + 9.1624)}$$

The pole at $s = -0.2902$ *is a dominant pole as* $5 \cdot 0.2902 < 3.9275$ *and* 9.1624,

Therefore cancelling the poles at $s = -3.9275$ *and* $s = -9.1624$,

$$G_C(s) = \frac{10.4416}{(s + 0.2902)}$$

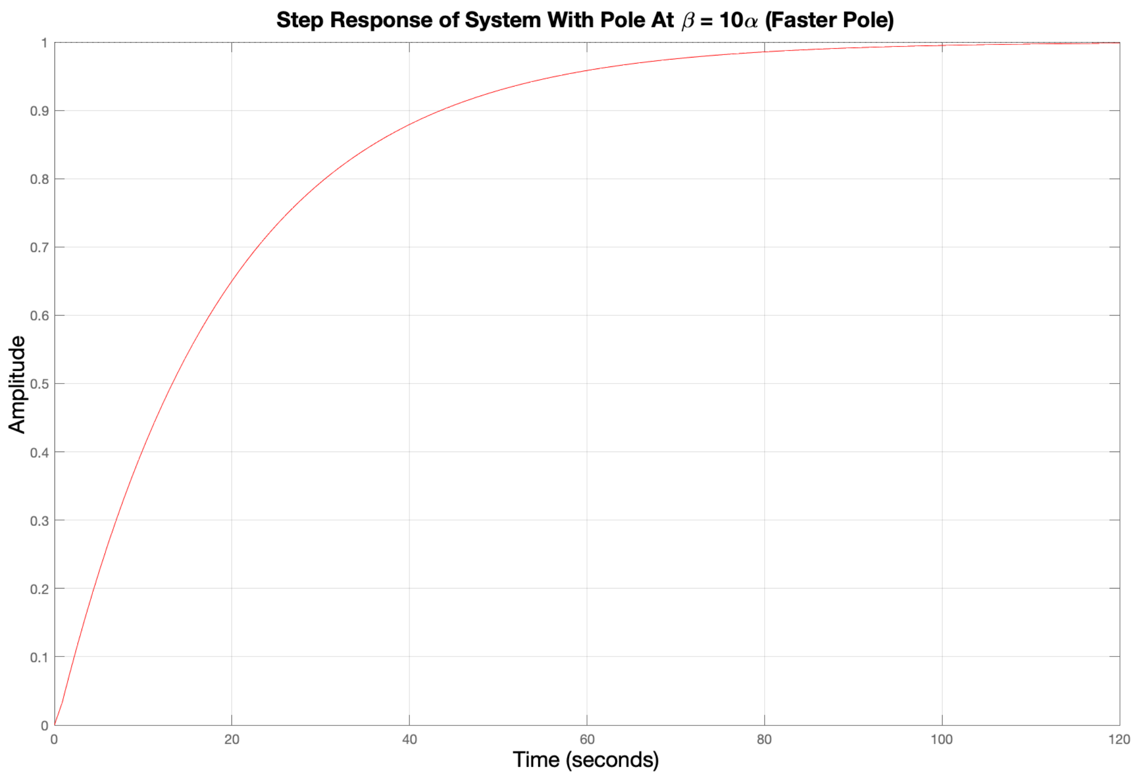
$G_C(s)$ *is now in the form of a* 1^{st} *order system,*

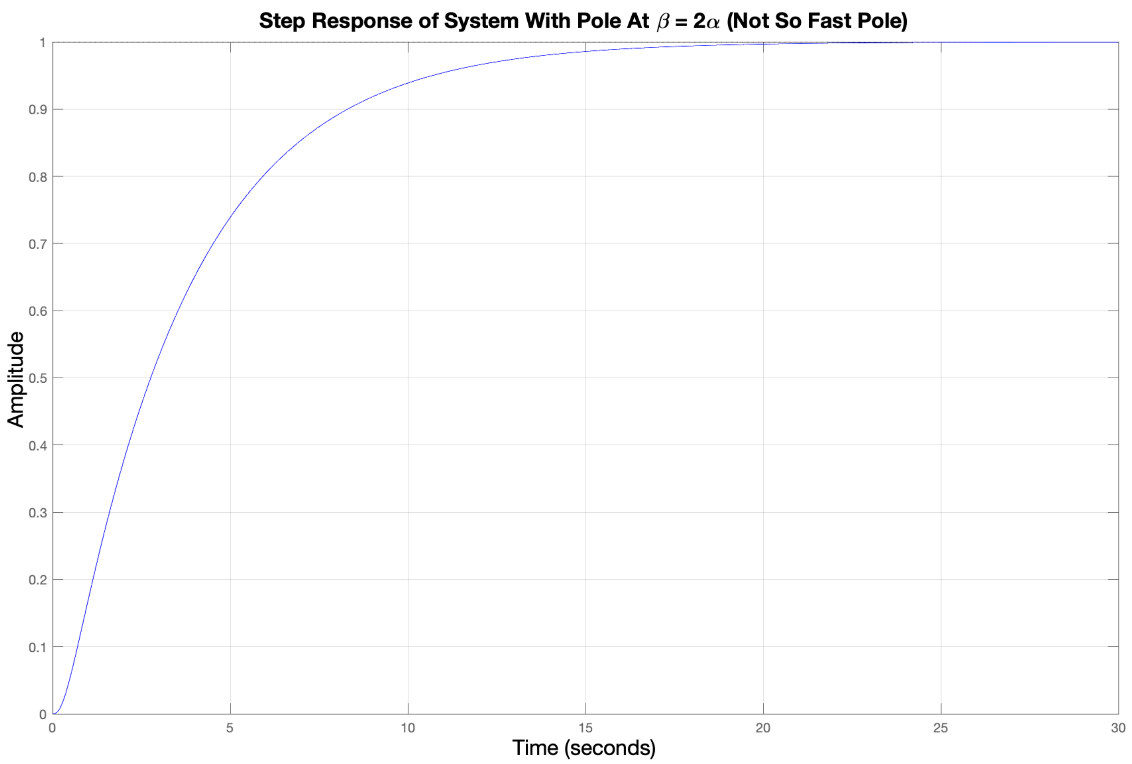
$$\therefore T_s = \frac{4}{a} \text{ where } \; a = 0.2902$$

$$\therefore \underline{\underline{T_s = 13.7836\,s}} \quad (1^{st} \text{ order approximation})$$

There is no overshoot for 1^{st} *order systems,*

$$\therefore \underline{\underline{\%\,OS = 0\%}} \quad (1^{st} \text{ order approximation})$$


Step Response of System With Pole At $\beta = 10\alpha$ (Faster Pole)
Amplitude
Time (seconds)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
20
40
60
80
100
120


Step Response of System With Pole At $\beta = 2\alpha$ (Not So Fast Pole)
Amplitude
Time (seconds)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
5
10
15
20
25
30

Given below are the step response data gathered from MATLAB for the above 2 different cases of systems using the '$stepinfo(\,)$' function,

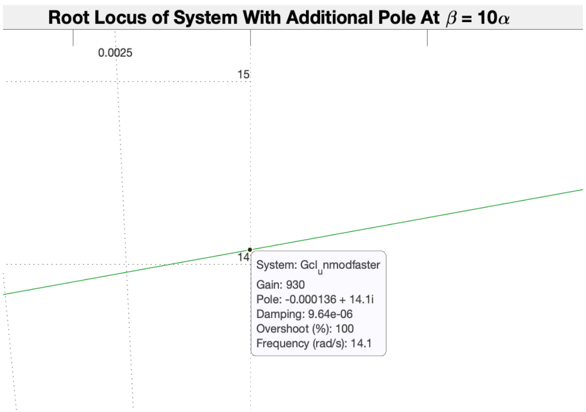
$Closed\ loop\ system\ with\ \beta = 10 \cdot \alpha,$

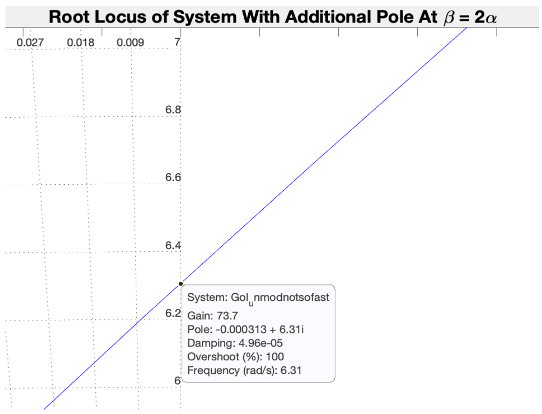
```
       RiseTime: 41.3088
   SettlingTime: 73.8037
    SettlingMin: 0.9032
    SettlingMax: 0.9993
      Overshoot: 0
     Undershoot: 0
           Peak: 0.9993
       PeakTime: 137.6633
```

$Closed\ loop\ system\ with\ \beta = 2 \cdot \alpha,$

```
       RiseTime: 7.6056
   SettlingTime: 13.8575
    SettlingMin: 0.9004
    SettlingMax: 0.9998
      Overshoot: 0
     Undershoot: 0
           Peak: 0.9998
       PeakTime: 29.7593
```

When comparing the above the systems, it can be clearly observed that the settling time T_s of the system with a pole at $\beta = 10 \cdot \alpha$ settles at an unreasonably long time with $T_s = 73.8037s$ compared to the case with $\beta = 2 \cdot \alpha$ with $T_s = 13.8575s$ which is quite a long time compared to the original (no β) system time of $T_s = 1.7937s$. The additional pole at the above β values have also gotten rid of the % overshoot (%OS = 0%) and this closely co-relates to the hand calculated 1^{st} order approximation although % overshoot was previously present in the system with no unmodelled pole where gain $K = 1.4462$. Both the systems however have maintained achieved the final value of 1 as seen from the above plots with zero steady state error for the given step input. The introduction of the new pole β also comes with an In-stability criteria such that at a certain gain value, the system is going to be unstable by having closed loop poles on the right half plane when represented on an s-domain graph. This is observed from the plots below with the identification of the value of gain at which point the root locus of the system crosses the imaginary axis (gain at which system is marginally stable – just before in-stability – poles on the imaginary axis).





Therefore, at gain K = 930 for the whole system with $\beta = 10 \cdot \alpha$ and K = 73.7 for the whole system with $\beta = 2 \cdot \alpha$ (gain including the k_m factor), the systems are going to be marginally stable. At gain values higher than that, the system poles are going to be in the right half plane of the s-domain graph. This makes the system to be unstable which then in turn makes the final value theorem not hold and thereby making the steady state error to be infinite. Therefore, it is seen that the additional unmodelled pole β has highly impacted the system in many ways including settling times, increased rise times, increased peak time, % overshoot, stability, final value and steady state error for step input.