CSE 544

PROBABILITY AND STATISTICS FOR DATA SCIENCE

ASSIGNMENT 1

Team members:

Name	SBU ID
Venkata Satwik Chilukuri	113263262
Akshay Somayaji	113322316
Shardul Churi	114403283

1. Nerdy NBA

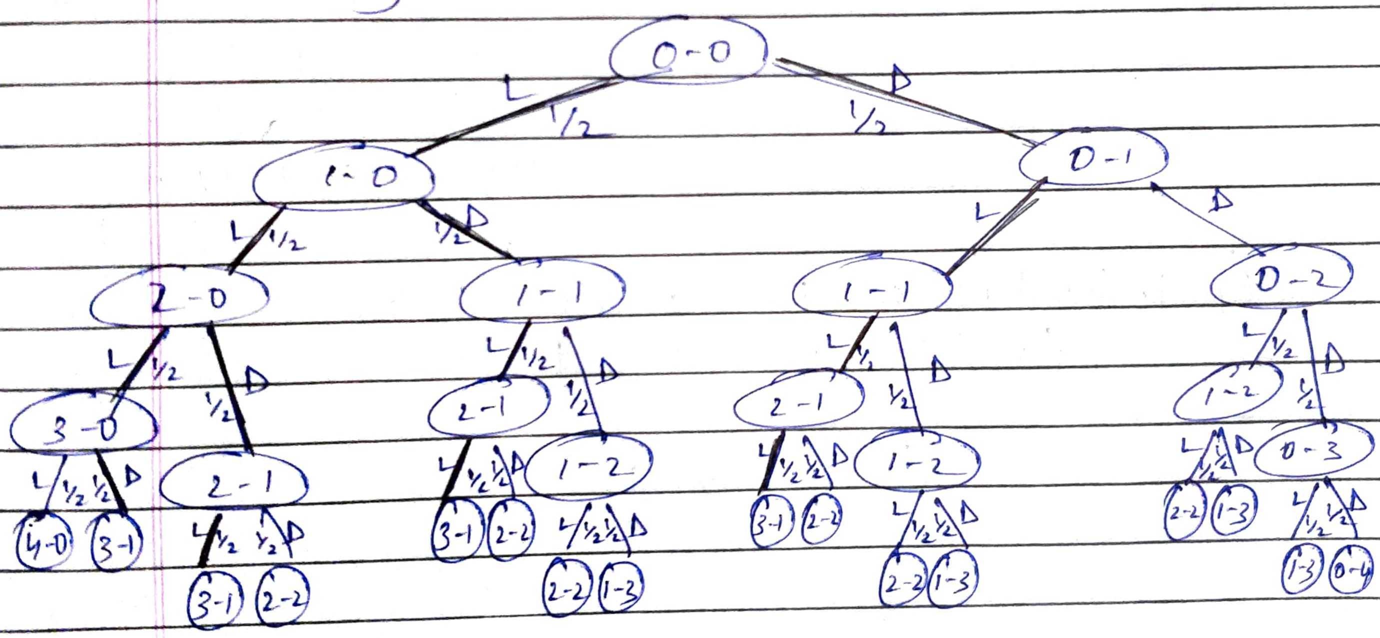
Los Angeles Clippers (LAC) v/s Denver Nuggets (DEN)

(a) Probability of either team winning = 0.5

$\therefore P(LAC) = P(DEN) = 0.5$

To find : Probability of LAC being 3-1 up

Probability Tree :

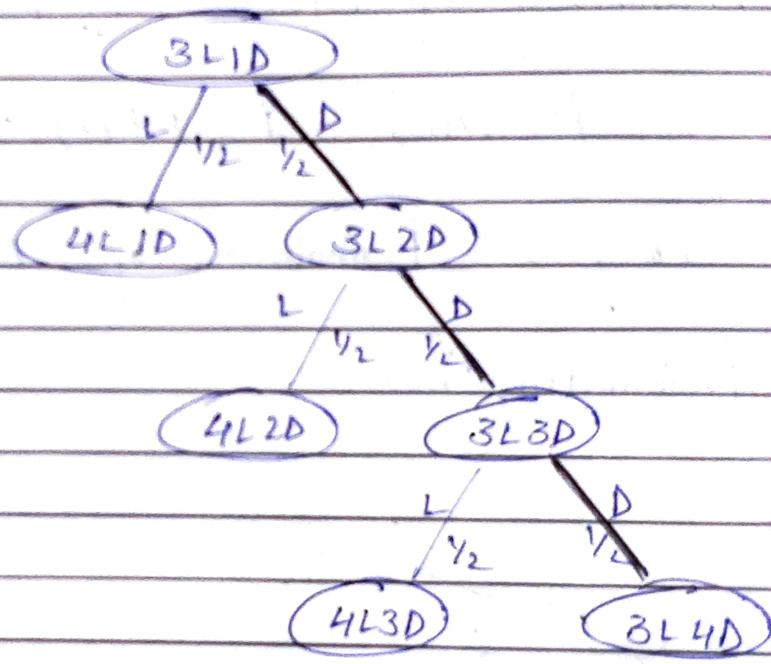


4 Paths - LLLD, LLDL, LDLL, DLLL give us the 3-1 scoreline.

$$P(LAC\ 3\text{-}1\ DEN) = P(LLLD) + P(LLDL) + P(LDLL) + P(DLLL)$$
$$= (1/2)^4 + (1/2)^4 + (1/2)^4 + (1/2)^4$$
$$= 1/16 + 1/16 + 1/16 + 1/16$$
$$= 4/16 = 1/4$$

b) P(LAL = P(DEN) = 0.5

Probability Tree starting from LAC 3-1 DEN



c) Probability of DEN winning 4-3,
from the probability tree,

$$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$$

$$= \frac{1}{8}$$

d) Home team now has a higher chance of winning.

P (home win) = 0.75

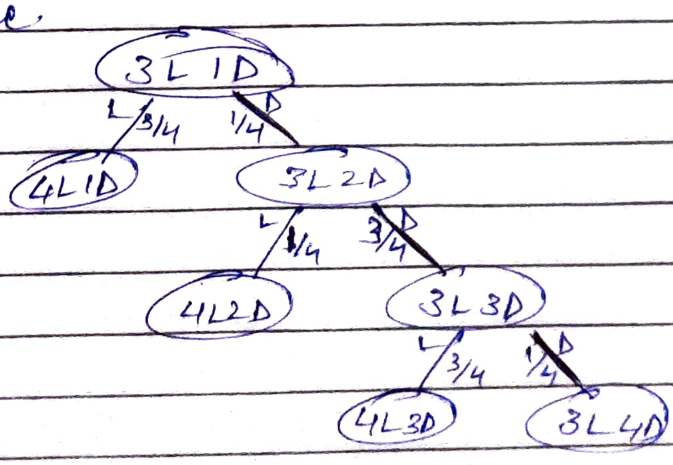
∴ P (away win) = 1 − 0.75 = 0.25

Games 5 & 7 were held in LAC.
Game 6 was held in DEN.

Probabilities of winning.

Team / Game	LAC	DEN
5	0.75	0.25
6	0.25	0.75
7	0.75	0.25

Probability Tree.



e) Probability of DEN winning 4-3, from probability tree $= \frac{1}{4} \times \frac{3}{4} \times \frac{1}{4}$

$= \frac{3}{64}$

1f) Simulation Results (for seed = 1)

n	(a)	(c)	(e)
3	0.257	0.1361	0.03139
4	0.2499	0.13085	0.05125
5	0.25052	0.12079	0.04671
6	0.25002	0.12441	0.0476
7	0.2501954	0.12502	0.04668

2) We need to find the probability of atleast one undiscarded phone, i.e, of KEEPING atleast one phone.

→ Let E_i be event of picking iPhone i at step i, i.e, E_i is the event of keeping iPhone i.

→ The probability of keeping atleast 1 iPhone translates to the union of events of keeping a minimum of atleast 1 iPhone upto a maximum of keeping all n iPhones.

The above statement can be represented as

$$P(\text{keep atleast 1 iPhone}) = P(E_1) \cup P(E_2) \cup P(E_3) \dots P(E_n)$$

$$= P\left(\bigcup_{i=1}^{n} E_i\right) = \text{LHS}$$

Now, we are asked to find LHS based on PIE, i.e,

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_i P(E_i) - \sum_{i<j} P(E_i \cap E_j) + \sum_{i<j<k} P(E_i \cap E_j \cap E_k) - \dots +$$

$$(-1)^{n+1} P(E_i \cap \dots \cap E E_j) \qquad (i)$$

Now, let us look at some of the terms in RHS of (i).

→ $P(E_i)$ = Probability of picking iPhone i at step i.

$$= \frac{\text{number of ways of keeping iPhone } i}{\text{all possible arrangements}}$$

$$= \frac{1.(\text{number of ways of ordering remaining } n-1)}{\text{all possible arrangements}}$$

$$= \frac{1.(n-1)!}{n!} = \frac{(n-1)!}{n!}$$

Now, for $i = 1$ to n, there are ${}^nC_1$ ways of choosing 1 iphone i from n correctly.

$$\therefore \sum_i P(E_i) = {}^nC_1 \cdot \frac{(n-1)!}{n!} \qquad (ii)$$

→ $P(E_i \cap E_j)$ → Probability of keeping 2 iPhones at step i & j, given $i < j$

$$= \frac{1.1.(\text{ways of ordering remaining } n-2 \text{ iphones})}{\text{all possible arrangements}}$$

$$= \frac{(n-2)!}{n!}$$

$\Rightarrow \sum_{i<j} P(E_i \cap E_j)$ = all possible probabilities of keeping 2 iphones out of n

$$= \frac{{}^nC_2 \cdot (n-2)!}{n!} \quad \text{(iii)}$$

→ From (ii) and iii), we can extrapolate that to keep k iPhones out of n, then

$$\sum_{i<j<\dots k}^{n} P(E_i \cap E_j \cap \dots E_k) = {}^nC_k \frac{(n-k)!}{n!} \quad \text{(iv)}$$

Using result (iv) in (ii), we reduce it to

→ $$P\left(\bigcup_{i=1}^{n} E_i\right) = {}^nC_1 \frac{(n-1)!}{n!} - {}^nC_2 \frac{(n-2)!}{n!} + {}^nC_3 \frac{(n-3)!}{n!} + \dots + (-1)^{n+1}\, {}^nC_n \frac{(n-n)!}{n!}$$

$$= \frac{\cancel{n!}}{\cancel{(n-1)!}\,1!} \cdot \frac{\cancel{(n-1)!}}{\cancel{n!}} + \frac{\cancel{n!}}{\cancel{(n-2)!}\,2!} \cdot \frac{\cancel{(n-2)!}}{\cancel{n!}} + \frac{\cancel{n!}}{\cancel{(n-3)!}\,3!} \cdot \frac{\cancel{(n-3)!}}{\cancel{n!}} + \dots$$

$$= \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \dots (-1)^{n+1} \frac{1}{n!}$$

$$\Rightarrow P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{j=1}^{n} \frac{(-1)^{j+1}}{j!} \quad \text{(v)}$$

Date ______
Page ______

Let us assume for large value of n, such that $n \to \infty$

Now, we know by Taylor series of infinite expansion,

$$e^{\lambda} = 1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \dots \frac{\lambda^n}{n!} + \dots$$

$$e^{\lambda} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = 1 + \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} \quad \text{(vi)}$$

Now, let us solve for $\lambda = -1$, then (vi) becomes

$$\Rightarrow e^{-1} = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!}$$

multiplying both sides by -1, we get

$$-e^{-1} = -1\left(1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!}\right) = -1 + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!}$$

when $n \to \infty$, we can replace (v) with above as $\Rightarrow$

$$-e^{-1} = -1 + P\left(\bigcup_{i=1}^{\infty} E_i\right)$$

$$\therefore P\left(\bigcup_{i=1}^{\infty} E_i\right) = 1 - e^{-1} = 1 - \frac{1}{e}$$

$$= \frac{e-1}{e} \simeq \underline{\underline{0.632}} \quad (\text{taking } e = 2.72)$$

3a)

Assume R represents the event that the Ring is the One-Ring i.e, $\bar{R}$ represents the event where the Ring is not the one Ring.

Assume S represents the event where the owner will have an Above-Average life span i.e, $\bar{S}$ represents the event where the ower will not have an Above Average life span.

Now Given,

① There are 10,000 Rings in the Middle east

$P(R) = \frac{1}{10000}$ which Means $P(\bar{R}) = \frac{9999}{10000}$

$[P(\bar{R}) = 1 - P(R)]$

② if the ring is One Ring, there is 95% chance of owner having an Above-average life space

i.e, $P(S/R) = \frac{95}{100}$

③ if the ring is not the one Ring, there is 75% chance of owner not having Above-average life space

i.e, $P(\bar{S}/\bar{R}) = \frac{75}{100}$ which Means

$P(S/\bar{R}) = 1 - \frac{75}{100} = \frac{25}{100}$

To find,

probability of Ring is, infact, the One Ringsea, given the owner is having above Average life span

i.e $P(R/S)$

By Bayes theorem,

$$P(E_i/A) = \frac{P(A/E_i) \cdot P(E_i)}{\sum_{j=1}^{n} P(A/E_j) \cdot P(E_j)}$$

$$\Rightarrow P(R/S) = \frac{P(S/R) \cdot P(R)}{P(S/R) \cdot P(R) + P(S/\bar{R}) \cdot P(\bar{R})}$$

By using all the given values.

$$\Rightarrow P(R/S) = \frac{\frac{95}{100} \times \frac{1}{10000}}{\frac{95}{100} \times \frac{1}{10000} + \frac{25}{100} \times \frac{9999}{10000}}$$

$$\Rightarrow P(R/S) = \frac{95}{250070}$$

$$\boxed{\Rightarrow P(R/S) = 0.00037989}$$

3 b)

Assume w represents the event that writing appears on it.

Now Given,

(i) if it is the one Ring, probability of writing appearing on it is 0.9

$$P(W/R) = 0.9$$

(ii) if it is not the one Ring, writing may still apper on with with a probability of 0.05

$$P(W/\bar{R}) = 0.05$$

(iii) Both the tests are conditionally independent on the ring being the one Ring and also on the the Ring not being the one Ring.

i.e $W \cap S \perp R$, $W \cap S \perp \bar{R}$

$$\Rightarrow P(W \cap S / R) = P(W/R) \cdot P(S/R) \quad - (1)$$

$$\Rightarrow P(W \cap S / \bar{R}) = P(W/\bar{R}) \cdot P(S/\bar{R}) \quad - (2)$$

(iv) from 3a question

$$P(S/R) = \frac{95}{100} \quad , \quad P(S/\bar{R}) = \frac{25}{100}$$

To find,

Given the writing appears on the ring and the owner has an above Average life span, what is the probability that the ring is the One Ring.

$$P(R/wns)$$

By Bayes theorem,

$$P(E_i/A) = \frac{P(A/E_i)\cdot P(E_i)}{\sum_{j=1}^{n} P(A/E_j)\cdot P(E_j)}$$

$$\Rightarrow P(R/wns) = \frac{P(wns/R)\cdot P(R)}{P(wns/R)\cdot P(R) + P(wns/\bar{R})\cdot P(\bar{R})}$$

Using ①, ② and Given values.

$$\Rightarrow P(R/wns) = \frac{P(w/R)\cdot P(s/R)\cdot P(R)}{P(w/R)\cdot P(s/R)\cdot P(R) + P(w/\bar{R})\cdot P(s/\bar{R})\cdot P(\bar{R})}$$

$$= \frac{0.9 \times 0.95 \times \frac{1}{10000}}{0.9\times 0.95 \times \frac{1}{10000} + 0.05 \times 0.25 \times \frac{9999}{10000}}$$

$$= \frac{0.855}{0.855 + 124.9875} = \frac{0.855}{125.8425}$$

$$\boxed{\therefore P(R/wns) = 0.00679421}$$

4) Given, X is a +ve, integer valued RV, i.e, $X \in \mathbb{Z}^+$

To Prove - $E[X] = \sum_{x=0}^{\infty} P(X > x)$

Now, $\sum_{x=0}^{\infty} P(X > x) = P(x>0) + P(x>1) + P(X>2) + \ldots$ (i)

→ Let us see what each term in RHS of (i) means.
Since X is an integer, we can say that
$P(X > x) = P(X = x+1) + P(X = x+2) + P(X = x+3) + \ldots$

→ Using above relation, we get
$P(X>0) = P(X=1) + P(X=2) + P(X=3) + \ldots$ (ii)
$P(X>1) = P(X=2) + P(X=3) + P(X=4) + \ldots$ (iii)
$P(X>2) = P(X=3) + P(X=4) + P(X=5) + \ldots$ (iv)

Substituting (ii), (iii) & (iv) in (i) and entrapolating for rest of the series, we get

$$\sum_{x=0}^{\infty} P(X > x) = 1.P(X=1) + 2.P(X=2) + 3.P(X=3) + \ldots\ x.P(X=x) + \ldots$$

$$= \sum_{x=1}^{\infty} x \cdot p_X(x)$$

$$\Rightarrow \sum_{x=0}^{\infty} P(X > x) = E[X]$$

5)

5a)

Indicator Rv introduced in class is

for a Event E

$X \sim$ Indicator $(E) = I(E)$ or I_E

$$X = \begin{cases} 1 & \text{if event } E \text{ occurs} \\ 0 & \text{if event } E \text{ does not occur} \end{cases}$$

This Random variable ~~X~~ depends on another

Random variable E

<u>pmf</u>

$$P_X(x) = \begin{cases} Pr(E) & \text{if } x = 1 \\ 1 - Pr(E) & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

Now,

$$E(X) = \sum_x x \cdot P_X(x)$$

$$= 0 \cdot P_X(0) + 1 \cdot P_X(1)$$

$$= P_X(1)$$

when $x = 1$ we know that $P_X(x) = Pr(E)$

$\Rightarrow E(x) = Pr(E)$

As $x \sim I_E$

$$\boxed{E(I_E) = Pr(E)}$$

5b)

To find,

$Var(I_E)$ in terms of $Pr(E)$

we know that,

$$Var(x) = E[x^2] - (E[x])^2 \quad — ①$$

Now

$$E[x^2] = \sum_x x^2 \cdot P_x(x) \quad \left[\text{As } E[x^i] = \sum_x x^i P_x(x) \right]$$

i^{th} Moment of x

$$\Rightarrow E[x^2] = 0 \cdot P_x(0) + (1^2) \cdot P_x(1)$$

$$= 0 + P_x(1)$$

$$= Pr(E)$$

putting $E[x^2]$ in ①

$$var(x) = Pr(E) - (E[x])^2$$

we know that $E(x] = Pr(E)$ from 5(a)

$$\therefore var(x) = Pr(E) - (Pr(E))^2 = Pr(E)[1 - Pr(E)]$$

as $x \sim I_E$

$$\boxed{var(I_E) = Pr(E)\left[1 - Pr(E)\right]}$$

5c)

Given,

$$x \sim Geometric(p), \quad p < 1$$

<u>pmf</u>

$$P_x(i) = Pr(x = i) = (1-p)^{i-1} \cdot p \quad — ①$$

To find,

$$E[x].$$

Now we know that,

$$E[x] = \sum_{i=1}^{\infty} i \cdot P_x(i)$$

using ① in the above equation

$$\Rightarrow E[x] = \sum_{i=1}^{\infty} i \cdot (1-p)^{i-1} \cdot p \quad — ②$$

we can write i as $(i-1)+1$

$$\Rightarrow E[x] = \sum_{i=1}^{\infty} \left[(i-1)+1\right] \cdot (1-p)^{i-1} \cdot p$$

$$\Rightarrow E[x] = \sum_{i=1}^{\infty} \left[(i-1)+1\right] \cdot (1-p)^{i-1} \cdot p$$

$$\Rightarrow E[x] = \underbrace{\sum_{i=1}^{\infty} (i-1)(1-p)^{i-1} \cdot p}_{\text{PART A}} + \underbrace{\sum_{i=1}^{\infty} (1-p)^{i-1} \cdot p}_{\text{PART B}} \quad \text{— (3)}$$

Lets solve part A

$$= \sum_{i=1}^{\infty} (i-1)(1-p)^{i-1} \cdot p$$

let $i-1 = j$

i.e, $i = j+1$

$$= \sum_{j+1=1}^{\infty} j \cdot (1-p)^{j} \cdot p$$

$$= \sum_{j=0}^{\infty} j \cdot (1-p)^{j} \cdot p$$

This is basically $\sum_{j=1}^{\infty} j \cdot (1-p)^{j} \cdot p$ as $j=0$ is zero

anyway

$$= \sum_{j=1}^{\infty} j \cdot (1-p)^{j} \cdot p$$

This is basically $(1-p) \cdot E[x]$ from (2)

$$= (1-p) \times E[x]$$

lets solve part B

$$= \sum_{i=1}^{\infty} (1-p)^{i-1} \cdot p$$

as p is constant get it out of Σ

$$= p \sum_{i=1}^{\infty} (1-p)^{i-1}$$

Given the infinite series,

$$1 + x + x^2 + \dots\dots = \frac{1}{1-x} \quad \text{for } x < 1$$

Also Given $p < 1$ so,

$$= p \sum_{i=1}^{\infty} (1-p)^{i-1} = p \left[(1-p)^0 + (1-p) + (1-p)^2 + \dots \right]$$

$$= p \left[1 + (1-p) + (1-p)^2 + (1-p)^3 + \dots \right]$$

$$= p \left[\frac{1}{1-(1-p)} \right]$$

$$= p \left[\frac{1}{1-1+p} \right] = p \cdot \frac{1}{p}$$

$$= 1$$

putting back part A part B in equation (3)

$$\Rightarrow E[x] = (1-p)\,E[x] + 1$$

$\Rightarrow E[x] - [(1-P) E[x]] = 1$

$\Rightarrow E[x] [1 - (1-P)] = 1$

$\Rightarrow E[x] [1-1+P] = 1$

$$\boxed{\therefore E[x] = \frac{1}{P}}$$

5d)

we know,

$Var(x) = E(x^2) - (E(x))^2$, $P_x(i) = (1-P)^{i-1} \cdot P$

we know that for i^{th} moment of x

$$E(x^i) = \sum_x x^i P_x(x)$$

Now,

$$Var(x) = E(x^2) - (E(x))^2$$

$$= \sum_{i=1}^{\infty} i^2 \cdot P_x(i) - \left(\sum_{i=1}^{\infty} i \cdot P_x(i) \right)^2$$

from 5(c) we know $E(x) = \frac{1}{P}$

$$= \sum_{i=1}^{\infty} i^2 \cdot P_x(i) - \frac{1}{P^2} \quad \text{--- } (1)$$

Now,

$$E(x^2) = \sum_{i=1}^{\infty} i^2 \cdot p \cdot (1-p)^{i-1} \quad — \quad 2$$

write i as $(i-1)+1$

$$\Rightarrow E(x^2) = \sum_{i=1}^{\infty} (i-1+1)^2 \cdot p \cdot (1-p)^{i-1}$$

$$\Rightarrow E(x^2) = \sum_{i=1}^{\infty} ((i-1)+1)^2 \cdot p \cdot (1-p)^{i-1}$$

Now, $(a+b)^2 = a^2+2ab+b^2$ use this formulae

$$\Rightarrow E(x^2) = \sum_{i=1}^{\infty} \left[(i-1)^2 + 1 + 2(i-1)\right] \cdot p \cdot (1-p)^{i-1}$$

$$\Rightarrow E(x^2) = \underbrace{\sum_{i=1}^{\infty} (i-1)^2 \cdot p \cdot (1-p)^{i-1}}_{\text{Part-A}} + \left.\sum_{i=1}^{\infty} p \cdot (1-p)^{i-1}\right\} \text{Part-B}$$

$$+ \underbrace{2\sum_{i=1}^{\infty} p \cdot (i-1) \cdot (1-p)^{i-1}}_{\text{part-C}} \quad — \quad 3$$

lets solve part-A

$$= \sum_{i=1}^{\infty} (i-1)^2 \cdot p \cdot (1-p)^{i-1} \qquad \text{let } i-1 = j$$

$$\text{i.e., } i = j+1$$

$$= \sum_{j=0}^{\infty} j^2 \cdot p \cdot (1-p)^j$$

At $j=0$ whole value will be zero, so change limits

$$= \sum_{j=1}^{\infty} j^2 \cdot p \cdot (1-p)^j$$

This is basically $(1-p) \cdot E(x^2)$ from ②

$$= (1-p)\, E(x^2)$$

lets solve part - B :-

$$= \sum_{i=1}^{\infty} p \cdot (1-p)^{i-1}$$

As p is constant put it out of Σ

$$= p \sum_{i=1}^{\infty} (1-p)^{i-1}$$

$$= p \left[(1-p)^0 + (1-p)^1 + (1-p)^2 + (1-p)^3 + \cdots \right]$$

we know the infinite series,

$$1 + x + x^2 + \cdots = \frac{1}{1-x} \quad \text{when } x < 1$$

Here $p < 1$ so

$$= p \left[\frac{1}{1-(1-p)} \right]$$

$$= p \left[\frac{1}{1-1+p} \right] = p \times \frac{1}{p} = 1$$

lets solve part c:

$$= 2 \sum_{i=1}^{\infty} p \cdot (i-1) \cdot (1-p)^{i-1}$$

let $i-1 = j$ i.e $i = j+1$

$$= 2 \sum_{j+1=1}^{\infty} p \cdot (j) \cdot (1-p)^{j}$$

$$= 2 \sum_{j=0}^{\infty} p \cdot (j) \cdot (1-p)^{j}$$

At $j=0$ whole value will be 0 so change limits

$$= 2 \sum_{j=1}^{\infty} p \cdot (j) \cdot (1-p)^{j}$$

This is basically $(1-p)\, E(x)$ from $J(C)$

$$= 2(1-p)\, E(x)$$

Now, substituting part A, part B, part C solutions back in (3)

$$\Rightarrow E(x^2) = (1-p)\, E(x^2) + 1 + 2(1-p)\, E(x)$$

$$\Rightarrow E(x^2) - \left[(1-p)\, E(x^2)\right] = 2(1-p)\, E(x) + 1$$

$$\Rightarrow E(x^2)\left[1 - (1-p)\right] = 2(1-p)\, E(x) + 1$$

$\Rightarrow E(x^2)(p) = 2(1-p)E(x) + 1$

from J (c) we know that $E(x) = \frac{1}{p}$

$$\Rightarrow E(x^2)(p) = \frac{2(1-p)}{p} + 1$$

$$\Rightarrow E(x^2)(p) = \frac{2-2p+p}{p}$$

$$\Rightarrow E(x^2)(p) = \frac{2-p}{p}$$

$$\Rightarrow E(x^2) = \frac{2-p}{p^2}$$

putting this back in 1

$$var(x) = E(x^2) - (E(x))^2$$

$$= \frac{2-p}{p^2} - \left(\frac{1}{p}\right)^2$$

$$= \frac{2-p}{p^2} - \frac{1}{p^2}$$

$$\boxed{\therefore\ var(x) = \frac{1-p}{p^2}}$$

6) a) Given, pmf of Poisson distribution is denoted by

$$p_x(i) = \frac{e^{-\lambda}\lambda^i}{i!} \quad \forall\ i \geq 0$$

To find/prove, $\sum_{i=0}^{\infty} p_x(i) = 1$

$$\sum_{i=0}^{\infty} p_x(i) = \sum_{i=0}^{\infty} \frac{e^{-\lambda}\lambda^i}{i!} = e^{-\lambda} \cdot \sum_{i=0}^{\infty} \lambda^i / i!$$

$$= e^{-\lambda} \cdot \left(1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \dots \frac{\lambda^n}{n!} + \dots \right) \quad i)$$

Now, using infinite series expansion of exponentials, that is also known as Taylor Series, we know

$$1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \frac{x^n}{n!} + \dots = e^x \quad ii)$$

Using ii) in i)

$$\Rightarrow \sum_{i=0}^{\infty} p_x(i) = \cancel{e^{-\lambda}} \cdot \cancel{e^{\lambda}} = 1$$

Date / /

CISCO University Recruiting

b) $E[X] = \sum_{x=0}^{\infty} x \cdot p_X(x) = \sum_{x=0}^{\infty} x \cdot \frac{e^{-\lambda}\lambda^{x}}{x!}$

now, discarding $x=0$ term in the series

$E[X] = e^{-\lambda} \sum_{x=1}^{\infty} \frac{x}{x!} \cdot \lambda^{x} = e^{-\lambda} \cdot \sum_{x=1}^{\infty} \frac{\lambda^{x}}{(x-1)!}$

$= e^{-\lambda} \cdot \lambda \cdot \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$ (i)

Let $y = x-1$. when $x=1$, $y=0$

$\Rightarrow$ Replacing $x-1=y$ in (i)

$\Rightarrow E[X] = e^{-\lambda} \cdot \lambda \cdot \sum_{y=0}^{\infty} \lambda^{y}/y!$ (ii)

Using Taylor series of expansion of exponents, (ii) reduces to

$E[X] = \cancel{e^{-\lambda}} \cdot \lambda \cdot \cancel{e^{\lambda}}$

$E[X] = \lambda$

7)

7a)

Given,

$$x \sim \text{Pareto}(\alpha), \quad 1 < \alpha < 2$$

p.d.f,

$$f_X(x) = \alpha x^{-\alpha-1}, \quad x \geq 1 \quad -①$$

$x \geq 1$ so,

$$Pr(x \in (1,\infty)) = \int_1^\infty f_X(x) \cdot dx$$

from ①

$$= \int_1^\infty \alpha \cdot x^{-\alpha-1} \, dx$$

Here α is constant

$$= \alpha \int_1^\infty x^{-\alpha-1} \, dx$$

using the formulae $\int x^n dx = \frac{x^{n+1}}{n+1}$

$$= \alpha \left[\frac{x^{-\alpha-1+1}}{-\alpha-1+1} \right]_1^\infty$$

$$= \alpha \left[\frac{x^{-\alpha}}{-\alpha} \right]_1^\infty$$

$$= \alpha \left[\frac{+x^{-\alpha}}{-\alpha} \right]_1^\infty$$

$$= \left[-x^{-\alpha} \right]_1^\infty$$

changing the integral from $[\,]_1^\infty$ to $[\,]_\infty^1$ so

Add - at front

$$= -\left[-x^{-\alpha} \right]_\infty^1$$

$$= \left[x^{-\alpha} \right]_\infty^1$$

$$= \left[\frac{1}{x^\alpha} \right]_\infty^1$$

$$= \left[\frac{1}{1} - \frac{1}{\infty} \right]$$

$$= 1$$

$\therefore$ pdf integrates to 1.

7b)

Here $x \geq 1$

So,

$$E[x] = \int_1^\infty x \cdot f_x(x)\, dx$$

we know that $f_x(x) = \alpha x^{-\alpha-1}$

$$\Rightarrow E[x] = \int_1^\infty x \cdot \alpha \cdot x^{-\alpha-1} \, dx$$

α is constant get it out of integral

$$\Rightarrow E[x] = \alpha \int_1^\infty x \cdot x^{-\alpha-1} \, dx$$

using the formulae $x^m \cdot x^n = x^{m+n}$

$$\Rightarrow E[x] = \alpha \int_1^\infty x^{1-\alpha-1} \, dx$$

$$\Rightarrow E[x] = \alpha \int_1^\infty x^{-\alpha} \, dx$$

using the formulae $\int x^n dx = \frac{x^{n+1}}{n+1}$

$$\Rightarrow E[x] = \alpha \left[\frac{x^{-\alpha+1}}{-\alpha+1} \right]_1^\infty$$

changing the integral signs from $[\;]_1^\infty$ to $[\;]_\infty^1$

$$\Rightarrow E(x) = \frac{-\alpha}{-\alpha+1} \left[x^{-\alpha+1} \right]_\infty^1$$

$$\Rightarrow E[x] = \frac{\alpha}{\alpha-1} \left[x^{-(\alpha-1)} \right]_\infty^1$$ As $1 < \alpha < 2$

$$E[x] = \frac{\alpha}{\alpha-1}\left[\frac{1}{x^{\alpha-1}}\right]_{\infty}^{1}$$

$$= \frac{\alpha}{\alpha-1}\left[\frac{1}{1} - \frac{1}{\infty}\right]$$

$$= \frac{\alpha}{\alpha-1}\left[1-0\right]$$

$$\boxed{\therefore\ E[x] = \frac{\alpha}{\alpha-1}}$$

7c)

we know that,

$$\text{var}[x] = E[x^2] - (E[x])^2 \quad — ①$$

we know that for i^{th} moment of x

$$E[x^i] = \int_{-\infty}^{\infty} x^i . f(x)\, dx$$

as $x \geq 1$

$$\Rightarrow E[x^2] = \int_{1}^{\infty} x^2 . f(x)\, dx$$

$$\Rightarrow E[x^2] = \int_{1}^{\infty} x^2 . \alpha . x^{-\alpha-1}\, dx$$

As α is constant

$$\Rightarrow E[x^2] = \alpha \int_{1}^{\infty} x^2 . x^{-\alpha-1}\, dx$$

Using the formulae $x^m \times x^n = x^{m+n}$

$$\Rightarrow E[x^2] = \alpha \int_1^\infty x^{2-\alpha-1}\, dx$$

$$\Rightarrow E[x^2] = \alpha \int_1^\infty x^{1-\alpha}\, dx$$

Using the formulae $\int x^n\, dx = \frac{x^{n+1}}{n+1}$

$$\Rightarrow E[x^2] = \alpha \left[\frac{x^{1-\alpha+1}}{1-\alpha+1}\right]_1^\infty$$

$$\Rightarrow E[x^2] = \alpha \left[\frac{x^{2-\alpha}}{2-\alpha}\right]_1^\infty$$

Here, as $1 < \alpha < 2$ and variance cannot be negative no need to change the limits.

$$\Rightarrow E[x^2] = \frac{\alpha}{2-\alpha}\left[x^{2-\alpha}\right]_1^\infty$$

$$\Rightarrow E[x^2] = \frac{\alpha}{2-\alpha}\left[\infty - 1\right]$$

$$\Rightarrow E[x^2] = \infty$$

Substituting in ①

$$var[x] = E(x^2) - (E(x))^2$$

$$\Rightarrow var[x] = \infty - (E(x))^2$$

$$\boxed{\therefore var[x] = \infty}, \quad \text{for } 1 < \alpha < 2$$

Date / /

CISCO University Recruiting

8a) Given, F is a CDF that is continuous & strictly increasing, and hence also has an inverse F^{-1}.
We also have RV $U \sim \text{Uniform}(0,1)$, and another RV X such that $X = F^{-1}(U)$

→ Since U is Uniformly distributed, its CDF is represented by $F_U(x) = \begin{cases} 0 & , \text{when } x \leq 0 \\ \frac{x-0}{1-0} = x & , \text{when } 0 \leq x \leq 1 \\ 1 & , \text{when } x \geq 1 \end{cases}$

$\therefore F_U(x) = P(U \leq x) = x$, when $0 < x < 1$ (i)

→ Now, we need to find CDF of X, that is,

$CDF_X(x) = P(X \leq x)$

$= P(F^{-1}(U) \leq x)$ (Given)

$= P(U \leq F(x))$ (since F is a strictly increasing fn)

$= F(x)$ (from (i))

$\therefore$ CDF of $X = F(x)$ for all $0 < x < 1$

Date / /

8b) Given, Y is a RV with CDF F. To prove - new RV, let's say Z, which is defined as $Z = F(Y) \sim$ Uniform $(0,1)$

Now, CDF for Z can be represented as

$$F_Z(z) = P(Z \leq z)$$
$$= P(F(Y) \leq z) \quad (i)$$

Since F is strictly increasing, there exists an inverse such that $Y = F_Y^{-1}(z)$

$\Rightarrow$ i) becomes $F_Z(z) = P(Y \leq F_Y^{-1}(z))$

$$= \cancel{F_Y}(\cancel{F_Y}^{-1}(z)) = z$$

$\therefore$ CDF of Z, i.e., $F_Z(Z) = z \quad$ (ii)

Now we know that the CDF for a uniform distribution of RV $X = x$ for $0 < x < 1$

Hence, $Z = F(Y) \sim$ Uniform $(0,1)$