

A Hybrid Algorithm for the Partition Coloring Problem

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Abstract

The Partition Coloring Problem (PCP) is a generalization of the the classical Vertex Coloring Problem (VCP), partitioning the set of nodes into clusters and seeking a coloring for the subgraph induced by selecting exactly one node from each cluster.

Diese Arbeit beschäftigt sich mit dem Partition Coloring Problem (PCP). Es handelt sich dabei um eine Generalisierung des Knotenfärbungsproblems und ist ein Optimierungsproblem der Komplexitätsklasse \mathcal{NP} .

Gegeben ist ein Graph, dessen Knotenmenge in disjunkte Partitionen unterteilt ist. Aus jeder Partition muss ein Knoten gewählt werden. Der durch die gewählten Knoten induzierte Subgraph soll unter der Bedingung eingefärbt werden, dass kein zueinander adjazentes Knotenpaar die gleiche Farbe annimmt. Ziel ist es, die Gesamtanzahl der verwendeten Farben - die sogenannte chromatische Zahl - zu minimieren.

Zur Lösung dieses Problems sollen mittels heuristischer Verfahren initiale Lösungen erstellt und diese mittels Tabusuche und wiederholter, partieller Neueinfärbung verbessert werden. Das Problem der Neueinfärbung wird mit unterschiedlichen Ansätzen gelöst.

Kurzfassung

Todo

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CHAPTER 1

Problem Definition

Problem Solving Approach

In this chapter, the algorithms and models of the new hybrid approach for the PCP will be described and analysed in detail. First in section 2.1, the main procedure is explained. Section 2.2 then analyses the two different construction heuristics used in this work, namely *OneStepCD* and *DANGER*. The improvement phase is split into two parts: algorithms, that assign a new, but not mandatorily feasible coloring to a chosen set of nodes, here consisting of one random-, one heuristic- and two exact approaches, are applied in the first part. Below, this thesis will refer to these algorithms to as the “recoloring” algorithms 2.3. As for the second part, section 2.4 presents the tabu search, which tries to find a coloring that makes the conflict-prone solution created by one of the recoloring algorithms feasible. Finally, two variants, one of them consisting in a modified ILP formulation and the other one in adding lately recolored areas to the tabulist, are considered in section 2.5.

2.1 Main Procedure

The idea of the approach in general is – starting from a feasible solution – to pick a color and eliminate it. For each color c , the set of nodes colored with c is reselected as well as recolored without using color c . Using this strategy, it might not be possible to find a feasible solution. Therefore, infeasible solutions are accepted, i.e. solutions including at least one conflict, that is, a pair of adjacent nodes $(i, j) \in E, i \in V, j \in V$ colored with the same color. One of the two nodes is chosen and in the following referred to as the “conflicting node”¹. These conflicting nodes form the starting points for the tabu-search, that tries to find an alternative color for each conflicting node. This process eliminates the conflicting node, but eventually produces further conflicts, which then again have to be eliminated. If no feasible coloring can be found withing a specified number of iterations, the next color $c + 1$ is considered. If all conflicts can be eliminated, the algorithm has successfully decreased the chromatic number and repeats the whole process.

¹How the node is chosen is shown in detail in 2.3

This approach differs from the strategy presented in [24] mainly by the effort that is investigated in the recoloring phase. There, Noronha et.al. reassign colors in a random way and therefore produce a random number of conflicts. The main innovation of the idea presented in this work is the minimization of the number of conflicts produced by an advanced recoloring algorithm, in order to increase the chance of eliminating these conflicts by the tabu search.

TODO: INSERT GRAPHIC!!!!

Listing 1 provides an overview of the single steps that have been implemented. The algorithm takes an instance P of PCP, an algorithm *INITIAL* computing an initial solution, and a recoloring algorithm *RECOLOR* as input. As described in 1, an instance of PCP consists of an uncolored graph $G = (V, E)$, where V is divided into k clusters. Parameter *INITIAL* can be any algorithm that creates a feasible solution for PCP. Two of them have been taken into account in this work and are described in section 2.3, others are proposed e.g. in [21].

In line 1, the initial solution is calculated and assigned to S . The chromatic number of S is assigned to $cmax$ in line 2. Line 3 initializes an empty set X . Line 5 to line 9 are preformed for each color $c \in \{1, \dots, cmax\}$. In line 5 all nodes in V are selected that are colored with color c and denoted by V_c . In line 6 a copy S' of S is created where all nodes in V_c are recolored by the algorithm *RECOLOR*, excluding color c . The set of conflicting nodes is denoted by C_c in line 7. The triple consisting of a tentative solution S_c , the set of nodes that have been recolored V_c and the set of conflicting nodes C_c are added to X in line 8. The elements are sorted, such that the triple with the fewest conflicting nodes is first, since it is assumed by the author that it is more likely to eliminate a lower amount of conflicts by tabu search. In line 10 the reduction phase starts by initializing variable *reduction* with **false**. The loop in line 11 creates a tentative solution S'_c for each solution S_c , $c \in \{1, \dots, k\}$, applies a tabu search and breaks in case of a feasible assignment could be found. If so, the chromatic number $cmax$ is reduced and the whole process is repeated with S'_c as new solution. If the tabu search could not find a feasible solution for all the solutions in X , the algorithm returns with the latest feasible solution S .

Algorithm 1: PCP Hybrid

Input: A problem instance \mathcal{P} , an algorithm *INITIAL* and an algorithm *RECOLOR*

Output: A feasible Solution S

```
1  $S \leftarrow INITIAL(\mathcal{P});$ 
2  $cmax \leftarrow$  the chromatic number of  $S$ ;
3 List  $X \leftarrow \emptyset$ ;
4 for  $c = 1, \dots, cmax$  do
5   Let  $V_c$  be the set of nodes coloured by the colour  $c$ ;
6   Let  $S_c$  be the solution created by applying RECOLOR( $V_c, \{1, \dots, cmax\} \setminus c$ ) on  $S$ ;
7   Let  $C_c$  be the set of all nodes involved in color conflicts of  $S_c$ ;
8    $X \leftarrow X \cup \langle S_c, V_c, C_c \rangle$ 
9 Sort elements in  $X$  ascendingly by  $|C_i|$ ;
10  $reduction \leftarrow$  false;
11 for  $\langle S_c, V_c, C_c \rangle \in X$  do
12    $S'_c \leftarrow TabuSearch(S_c, V_c, C_c)$ ;
13   if  $S'_c$  is free of conflicts then
14      $reduction \leftarrow$  true;
15     break;
16 if  $reduction$  then
17    $S \leftarrow S'_c$ ;
18    $cmax = cmax - 1$ ;
19   goto line 3;
20 return  $S$ ;
```

2.2 Constructional Heuristics

Algorithm 2: OneStepCD

Input: An uncolored Graph $G = (V, E)$

Output: A feasible Coloring V'

```
1 Remove from G all edges  $(i, j) \in E : i, j \in V_k$  for some  $k = 1, \dots, q$ ;  
2 Set  $V' \leftarrow \emptyset$ ;  
3 while  $|V'| < q$  do  
4   Set  $X \leftarrow \emptyset$ ;  
5   for  $k = 1, \dots, q : V_k \cap V' = \emptyset$  do  
6     Set  $X \leftarrow X \cup \operatorname{argmin}\{CD(i) : i \in V_k\}$ ;  
7   Set  $x \leftarrow \operatorname{argmax}\{CD(i) : i \in X\}$ ;  
8   Set  $V' \leftarrow V' \cup \{x\}$ ;  
9   Assign the minimum possible colour to  $x$ ;  
10  Remove from G all nodes in  $V_{c(x)} \setminus \{x\}$ ;  
11 return  $V'$ ;
```

2.3 Recoloring

Random

OneStepCD

Algorithm 3: OneStepCD Recoloring

Input: A partial Solution P , a number of maximum colours $cmax$

Output: A feasible Solution S

```
1 Let  $U$  be the set of uncolored nodes in  $P$ ;  
2 Set  $S \leftarrow \emptyset$ ;  
3 while  $|U| > 0$  do  
4   Set  $X \leftarrow \emptyset$ ;  
5   for  $u \in U$  do  
6      $X \leftarrow X \cup \operatorname{argmin}\{CD(i) : i \in V_{c(u)}\}$ ;  
7   Set  $x \leftarrow \operatorname{argmax}\{CD(i) : i \in X\}$ ;  
8   Set  $cmin \leftarrow$  the minimum possible colour that can be assigned to  $x$ ;  
9   if  $cmin \geq cmax$  then  
10     $cmin \leftarrow$  the color that produces the fewest conflicts.  
11  Assign  $cmin$  to  $x$ ;  
12   $S \leftarrow S \cup \{x\}$ ;  
13   $U \leftarrow U \setminus V_{c(x)}$ ;  
14 return  $V'$ ;
```

ILP minimizing conflicts

Let $Q = Q_1, \dots, Q_q$ be the set of Clusters. Every cluster Q_p consists of a set of nodes. Let $C = \{1, \dots, cmax\}$ be the set of allowed colors. Let M be a 3-dimensional array of constants, storing for every cluster $p \in Q$, the number conflicts that would occur by selecting the pair $(v \in Q_p, c \in C)$. E denotes the set of edges and $P[v]$ the cluster of node v .

$$\underset{X}{\text{minimize}} \quad \sum_{p \in Q} \sum_{v \in Q_p} \sum_{c \in C} X_{pvc} * M_{pvc} \quad (1)$$

$$[h] \text{ subject to } \sum_{v \in Q_p} \sum_{c \in C} X_{pvc} = 1, \quad \forall p \in Q \quad (2)$$

$$X_{pvc} + X_{quc} \leq 1, \quad \forall ((p, v), (q, u)) \in E, \forall c \in C \quad (3)$$

$$X_{pvc} \in \{0, 1\}, \quad \forall p \in Q, \forall v \in Q_p, \forall c \in C \quad (4)$$

ILP minimizing conflicting nodes

Let U be the set of uncolored nodes in uncolored clusters and $color[(p, v)]$ the color of the node v in partition p .

$$\underset{Z}{\text{minimize}} \quad \sum_{p \in Q} \sum_{v \in Q_p} \sum_{c \in C} Z_{pvc} \quad (1)$$

$$\text{subject to } Z_{pvc} \geq X_{quc}, \quad \forall ((p, v), (q, u)) \in E : (p, v) \notin U, (q, u) \in U, c = color[(p, v)] \quad (2)$$

$$[h] \quad \sum_{v \in Q_p} \sum_{c \in C} X_{pvc} = 1, \quad \forall p \in Q \quad (3)$$

$$X_{pvc} + X_{quc} \leq 1, \quad \forall ((p, v), (q, u)) \in E, \forall c \in C \quad (4)$$

$$X_{pvc} \in \{0, 1\}, \quad \forall p \in Q, \forall v \in Q_p, \forall c \in C \quad (5)$$

2.4 Tabu Search

2.5 Variants

Algorithm 4: TabuSearch

Input: An infeasible solution S , the set of previously recolored nodes R , the set of conflicting nodes C

Output: A Solution \bar{S}

```
1 Set  $C \leftarrow C \setminus R$ ;  
2 Set  $cmax \leftarrow$  the chromatic number of  $S$ ;  
3 Set  $iter \leftarrow 0$ ;  
4 Set  $minConflicts \leftarrow \infty$ ;  
5 Set  $\bar{S} \leftarrow S$ ;  
6 while  $|C| > 0$  and  $iter < maxiter$  do  
7   for  $V_{c(u)} : u \in C$  do  
8     for  $v \in V_{c(u)}$  and for  $c = 1, \dots, cmax$  do  
9       Obtain a tentative solution  $S'$  by selecting and coloring node  $v$  with color  $c$  in  
        $\bar{S}$ ;  
10      if  $conflicts(S') = 0$  then  
11         $\bar{S} \leftarrow S', \bar{v} \leftarrow v, \bar{c} \leftarrow c$ ;  
12        goto line 16;  
13      else if the pair  $v, c$  is not in the tabu list then  
14        if  $conflicts(S') < minConflicts$  then  
15           $minConflicts \leftarrow conflicts(S')$ ;  
16           $\bar{S} \leftarrow S', \bar{v} \leftarrow v, \bar{c} \leftarrow c$ ;  
17      insert pair  $\bar{v}, \bar{c}$  in the tabu list for TabuTenure iterations;  
18       $C \leftarrow C \setminus u$ ;  
19      Let  $C_{\bar{v}}$  be the set of nodes conflicting with  $\bar{v}$ ;  
20       $C \leftarrow C \cup C_{\bar{v}}$ ;  
21 return  $\bar{S}$ ;
```

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