FAKULTÄT
FÜR !NFORMATIK
Faculty of Informatics

A Hybrid Algorithm for the Partition Coloring Problem

DIPLOMARBEIT

zur Erlangung des akademischen Grades

Diplom-Ingenieurin

im Rahmen des Studiums

Computational Intelligence

eingereicht von

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an der
Fakultät für Informatik der Technischen Universität Wien

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Wien, 21.Oct.2013

(Unterschrift Verfasserin) (Unterschrift Betreuung)

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Danksagung

Ich danke meinen Betreuern, ao. Univ.-Prof. Dipl.-Ing. Dr.techn. Günther Raidl und Univ.Ass. Dipl.-Ing. Dr.techn. Bin Hu für ihre Unterstützung bei der Erstellung dieser Arbeit durch ihr konstruktives Feedback und ihre Ideen, welche es mir ermöglicht haben, immer neue Aspekte der Problemstellung zu erkennen.

Besonderer Dank gilt meinen Eltern, Franz und Justine Fritz, sowie meiner Partnerin Odnoo und meinen Freunden für Ihre Unterstützung.

Abstract

The Partition Coloring Problem (PCP) is a generalization of the the classical Vertex Coloring Problem (VCP), partitioning the set of nodes into clusters and seeking a coloring for the subgraph induced by selecting exactly one node from each cluster.

Diese Arbeit beschäftigt sich mit dem Partition Coloring Problem (PCP). Es handelt sich dabei um eine Generalisierung des Knotenfärbungsproblems und ist ein Optimierungsproblem der Komplexitätsklasse $\mathcal{NP}$.
Gegeben ist ein Graph, dessen Knotenmenge in disjunkte Partitionen unterteilt ist. Aus jeder Partition muss ein Knoten gewählt werden. Der durch die gewählten Knoten induzierte Subgraph soll unter der Bedingung eingefärbt werden, dass kein zueinander adjazentes Knotenpaar die gleiche Farbe annimmt. Ziel ist es, die Gesamtanzahl der verwendeten Farben - die sogenannte chromatische Zahl - zu minimieren.
Zur Lösung dieses Problems sollen mittels heuristischer Verfahren initiale Lösungen erstellt und diese mittels Tabusuche und wiederholter, partieller Neueinfärbung verbessert werden. Das Problem der Neueinfärbung wird mit unterschiedlichen Ansätzen gelöst.

Kurzfassung

Todo

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CHAPTER 2

Problem Solving Approach

In this chapter, the algorithms and models of the new hybrid approach for the PCP will be described and analysed in detail. First in section 2.1, the main procedure is explained. Section 2.2 then analyses the two different construction heuristics used in this work, namely *OneStepCD* and *DANGER*. The improvement phase is split into two phases: algorithms, that assign a new, but not mandatorily feasible coloring to a chosen set of nodes, here consisting of one random-, one heuristic- and two exact approaches, are applied in the first phase. Below, this thesis will refer to these algorithms to as the "recoloring" algorithms 2.3. As for the second phase, section 2.4 presents the tabu search, which tries to find a coloring that makes the conflict-prone solution created by one of the recoloring algorithms feasible. Finally, two variants, one of them consisting in a modified ILP formulation and the other one in adding lately recolored areas to the tabulist, are considered in section 2.5.

2.1 Main Procedure

The idea of the approach in general is to start from a feasible solution, pick a color and eliminate it. For each color c, the set of nodes colored with c is reselected as well as recolored without considering color c. Using this strategy, it might not be possible to find a feasible solution. Therefore, infeasible solutions are accepted, i.e. solutions including at least one conflict, that is a pair of adjacent nodes $\{i, j\} \in E, i \in V, j \in V$ colored with the same color. One of the two nodes is chosen and in the following refered to as the "conflicting node"[1], its encasting cluster to as the "conflicting cluster". Further, a cluster is said to be of color c if and only if it contains a selected node colored with color c. The conflicting clusters form the starting points for the tabu-search, that tries to find an alternative color for each of them. This process eventually produces further conflicts, which then again have to be eliminated. If no feasible coloring can be found withing a specified number of iterations, the next color $c + 1$ is considered. If all conflicts can be eliminated, the algorithm has successfully decreased the chromatic number and repeats the

[1] How the node is chosen is shown in detail in 2.3

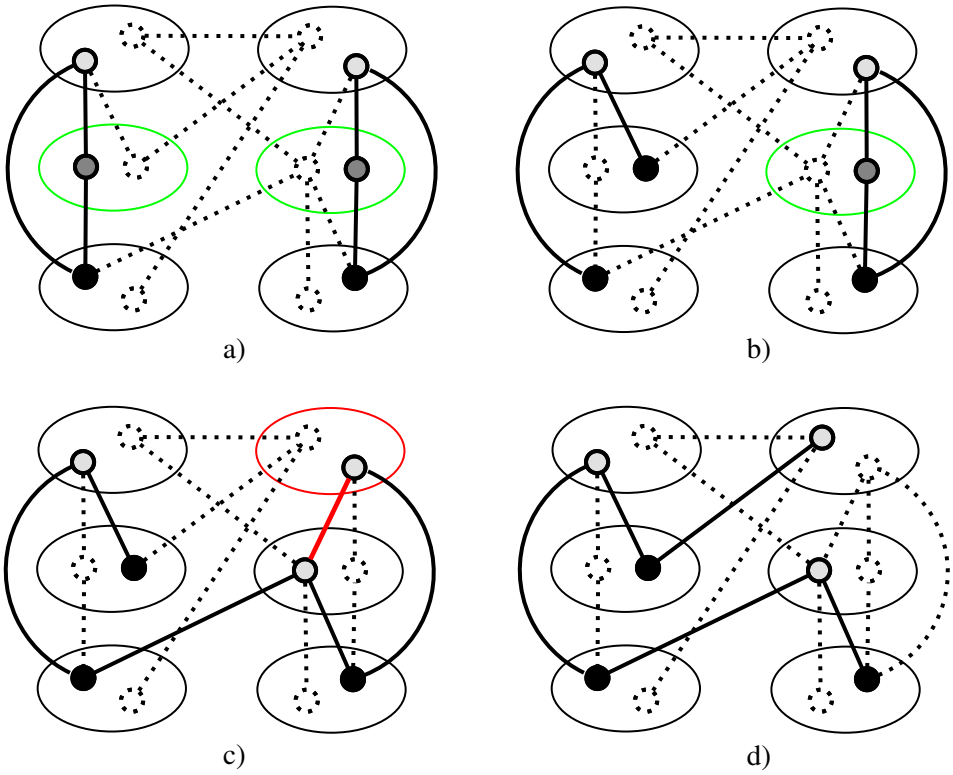

Figure 2.1: a) a feasible solution with 3 colors; b,c) recoloring phase: darkgrey is intended to be eliminated. An infeasible solution with one conflict results; d) elimintation of the conflict by tabu search

whole process.
This approach differs from the strategy presented in [24] mainly by the effort that is investigated in the recoloring phase. There, Noronha et.al. reassign colors in a random way and therefore produce a random number of conflicts. The main innovation presented in this work is the minimization of the number of conflicts produced by an advanced recoloring algorithm, in order to increase the chance of eliminating these conflicts by the tabu search.
An example of a possible sequence of steps performed by the algorithm is given in Figure 2.1. There, a feasible solution is shown in figure 2.1a). The color darkgrey is chosen to be eliminated. In the following two steps, for each darkgrey cluster, another node is chosen and colored with any color other than darkgrey. After recoloring, the resulting graph shown in 2.1c) is infeasible. Then, starting from the conflicting cluster outlined in red, the tabu search looks up the node-color pair inside that cluster, that causes the fewest conflicts. In 2.1d), a node-color pair is found, which does not produce any new conflicts. If this solution would have produced conflicts again, the local search would go on searching until a feasible solution was found or the maximum amount of iteration was reached.

Listing 1 provides an overview of the algorithm that have been implemented. The algorithm takes an instance P of PCP, an algorithm $INITIAL$ computing an initial solution, as well as a recoloring algorithm $RECOLOR$ as input. As described in 1, an instance of PCP consists of an uncolored graph $G = (V, E)$, where V is divided into k clusters. Parameter $INITIAL$ can be any algorithm that creates a feasible solution for PCP. Two of them have been taken into account in this work and are described in section 2.3, others are proposed e.g. in [21].
In line 1, the initial solution is calculated and assigned to S. The chromatic number of S is assigned to $cmax$ in line 2. Line 3 initializes an empty set X. Line 5 to line 9 are preformed for each color $c \in \{1, \ldots, cmax\}$. In line 5 all nodes in V are selected that are colored with color c and denoted by V_c. In line 6 a copy S' of S is created where all nodes in V_c are recolored by the algorithm $RECOLOR$, excluding color c. The set of conflicting nodes is denoted by C_c in line 7. The triple consisting of a tentative solution S_c, the set of nodes that have been recolored V_c and the set of conflicting nodes C_c are added to X in line 8. The elements are sorted, such that the triples with the fewest conflicting nodes are first, since it is assumed by the author that it is more likely to eliminate a lower amount of conflicts by tabu search. The loop in line 11 creates a tentative solution S'_c for each solution $S_c, c \in \{1, \ldots, k\}$, applies a tabu search and breaks in case of a feasible assignment could be found. If so, the chromatic number $cmax$ is reduced and the whole process is repeated with S'_c as new solution. If the tabu search could not find a feasible solution for all the solutions in X, the algorithm returns the latest feasible solution S.

Algorithm 1: PCP Hybrid

Input: A problem instance $\mathcal{P}$, an algorithm $INITIAL$ and an algorithm $RECOLOR$
Output: A feasible Solution S

1 $S \leftarrow INITIAL(\mathcal{P})$;
2 $cmax \leftarrow$ the chromatic number of S;
3 List $X \leftarrow \emptyset$;
4 **for** $c = 1, \ldots, cmax$ **do**
5 Let $\langle S_c, C_c \rangle$ be the solution and its conflicts created by applying $RECOLOR(S, c)$;
6 $X \leftarrow X \cup \langle S_c, C_c \rangle$
7 Sort elements in X ascendingly by $|C_i|$;
8 $reduction \leftarrow$ false;
9 **for** $\langle S_c, C_c \rangle \in X$ **do**
10 $S'_c \leftarrow TabuSearch(S_c, C_c)$;
11 **if** S'_c *is free of conflicts* **then**
12 $reduction \leftarrow$ true;
13 break;
14 **if** $reduction$ **then**
15 $S \leftarrow S'_c$;
16 $cmax = cmax - 1$;
17 **goto** line 3;
18 **return** S;

2.2 Constructional Heuristics

Construction heuristics create a solution from scratch. In this work this solution is required to be feasible, while a construction heuristic by definition may return infeasible solutions as well. This section presents the two construction heuristics implemented by the author.

OneStepCD

OneStepCD is the best performing out of six constructional greedy algorithms for PCP presented by Li and Simha in [21]. The algorithms have originally been constructed for VCP by [**?**] and have been adapted to PCP. Listing 2 shows the algorithm.
The main criterion considered for selecting and coloring the next node is the so called *color degree* of *saturation degree*, which is defined as the amount of different colors of nodes adjacent to the considered node. The idea behind OneStepCD is first for each (unselected) cluster to select the node with the lowest color degree. Out of the resulting set, select the node with the highest color degree and color it with the lowest possible color. This procedure is repeated until all clusters are colored, i.e. hold one selected and colored node.
Intuitivley, this approach leads to good results, since for *selecting* a node it would not be efficient to choose a node out of the cluster with a high color degree, while for *coloring* with each iteration it gets more and more dangerous to not color the node with highest degree.

Algorithm 2: OneStepCD

Input: A problem instance P
Output: A feasible Coloring S

```
1  Remove from G all edges (i,j) ∈ E : i,j ∈ V_k for some k = 1,...,q;
2  Set S ← ∅;
3  while |S| < q do
4      Set X ← ∅;
5      for k = 1,...,q : V_k ∩ S = ∅ do
6          X ← X ∪ argmin{CD(i) : i ∈ V_k};
7      x ← argmax{CD(i) : i ∈ X};
8      S ← S ∪ {x};
9      Assign the minimum possible colour to x;
10 return S;
```

DANGER

The DANGER heuristic is a method to color a graph introduced by Parker et.al. in [**?**]. It has been constructed for VCP and therefore does not concern about the peculiarities of PCP. In this work, inquiries have been made to adapt the DANGER heuristic to PCP by slotting an algorithm in ahead, which selects one node per cluster, in order to color the resulting subgraph as usual.
Listing 3 shows the preprocessing algorithm. In line 3, the node v is chosen which has the

minimum weighted sum of the already selected adjacent nodes $s(v)$ and the unselected but still selectable adjacent nodes $u(v)$. The weights c_s and c_u are constants. In line 4, the selected node is added to the set of selected nodes V' and in line 5, all the nodes contained in the cluster $p(v)$ enveloping v are removed from V.

Algorithm 3: Greedy Nodeselection

Input: A problem instance P
Output: A subproblem P' with $G' \subseteq G$

```
1 V' ← ∅;
2 while |V| > 0 do
3     v ← v ∈ V : min{c_s s(v) + c_u u(v)};
4     V' ← V' ∪ v;
5     V \ V_{p(v)};
```

Once one node per cluster is selected, the remaining problem is equivalent to VCP and DANGER can be applied in its original version. In contrast to OneStepCD, DANGER requires the allowed set of colors as parameter and is only successful if the graph can be colored within the given amount of colors. This makes the algorithm less flexibile and requires it to run several times in order to explore the lowest chromatic number. DANGER is based on two formulas *Node Danger* and *Color Danger*.
The former decides at each iteration which node shall be colored next. For every node, the algorithm evaluates how dangerous it is, NOT to color node i in this iteration. The node, that maximizes the following term is chosen:

$$NodeDanger(i) = \frac{C}{(max\_color - different\_colored(i))^k} + k_u \cdot uncolored(i) + k_a \frac{share(i)}{avail(i)}$$

$$ColorDanger(i) = \frac{k_1}{(max\_color - diff\_neighbours(c))^{k_2} + k_3 \cdot uncolored(n_c) - k_4 \cdot num(c)}$$

2.3 Recoloring

As explained above, the process of finding a new coloring for a set of clusters of same color, ommiting the actual color is of high relevance for this work. The recoloring algorithms intend to minimize the conflicts that arise from the new coloring, in order to increase the chance for the local search to eliminate them. An adaption of the already presented construction heuristic OneStepCD and two ILP models are shown in the following. For the purpose of comparison, a method assigning random colors has been implemented, too.

OneStepCD Adaption

The main idea behind the algorithm is to use the same strategy as the construction heuristic but restrict the colors to the domain $\{1, \ldots, cmax\} \setminus c$, where c is the color that is intended to be

eliminated. If no color in the domain can establish a feasible solution, the color that produces the minimum amount of conflicts is chosen.

Listing 4 shows the algorithm in detail

Algorithm 4: OneStepCD Recoloring

Input: A feasible solution S, a color c
Output: A possibly infeasible solution S' not using c

Let $V_{pcol(c)}$ be the set of nodes of all clusters colored with c in S;
Set $S' \leftarrow S \setminus V_c$;
Uncolor all nodes in V_c;
while *uncolored clusters exist* **do**
    Set $X \leftarrow \emptyset$;
    for $v \in V_c$ **do**
        $X \leftarrow X \cup argmin\{CD(i) : i \in V_{p(v)}\}$;
    $z \leftarrow argmax\{CD(x) : x \in X\}$;
    $cmin \leftarrow$ the minimum colour that can be assigned to z without producing a conflict;
    if $cmin > cmax$ **then**
        $cmin \leftarrow argmin\{conflicts(z, i) : i \in \{1, \ldots, cmax\} \setminus c\}$.
    Color z with $cmin$;
    $S' \leftarrow S' \cup \{z\}$;
return S';

ILP minimizing conflicts

Let $Q = Q_1, \ldots, Q_q$ be the set of Clusters. Every cluster Q_p consists of a set of nodes. Let $C = \{1, \ldots, cmax\}$ be the set of allowed colors. Let M be a 3-dimensional array of constants, storing for every cluster $p \in Q$, the number conflicts that would occur by selecting the pair $(v \in Q_p, c \in C)$. E denotes the set of edges and $P[v]$ the cluster of node v.

$$\underset{X}{\text{minimize}} \quad \sum_{p \in Q} \sum_{v \in Q_p} \sum_{c \in C} X_{pvc} * M_{pvc} \tag{1}$$

[h]

$$\text{subject to} \quad \sum_{v \in Q_p} \sum_{c \in C} X_{pvc} = 1, \qquad \forall p \in Q \tag{2}$$

$$X_{pvc} + X_{quc} \leq 1, \qquad \forall ((p,v),(q,u)) \in E, \forall c \in C \tag{3}$$

$$X_{pvc} \in \{0,1\}, \qquad \forall p \in Q, \forall v \in Q_p, \forall c \in C \tag{4}$$

ILP minimizing conflicting nodes

Let U be the set of uncolored nodes in uncolored clusters and $color[(p,v)]$ the color of the node v in partition p.

$$\underset{Z}{\text{minimize}} \quad \sum_{p \in Q} \sum_{v \in Q_p} \sum_{c \in C} Z_{pvc} \tag{1}$$

$$\text{subject to} \quad Z_{pvc} \geq X_{quc}, \qquad \forall ((p,v),(q,u)) \in E : (p,v) \notin U, (q,u) \in U, c = color[(p,v)] \tag{2}$$

[h]

$$\sum_{v \in Q_p} \sum_{c \in C} X_{pvc} = 1, \quad \forall p \in Q \tag{3}$$

$$X_{pvc} + X_{quc} \leq 1, \qquad \forall ((p,v),(q,u)) \in E, \forall c \in C \tag{4}$$

$$X_{pvc} \in \{0,1\}, \qquad \forall p \in Q, \forall v \in Q_p, \forall c \in C \tag{5}$$

2.4 Tabu Search

2.5 Variants

Algorithm 5: TabuSearch

Input: An infeasible solution S, the set of previously recolored nodes R, the set of conflicting nodes C

Output: A Solution $\bar{S}$

1 Set $C \leftarrow C \setminus R$;
2 Set $cmax \leftarrow$ the chromatic number of S;
3 Set $iter \leftarrow 0$;
4 Set $minConflicts \leftarrow \infty$;
5 Set $\bar{S} \leftarrow S$;
6 **while** $|C| > 0$ *and* $iter < maxiter$ **do**
7 **for** $V_{c(u)} : u \in C$ **do**
8 **for** $v \in V_{c(u)}$ *and* ***for*** $c = 1, \ldots, cmax$ **do**
9 Obtain a tentative solution S' by selecting and coloring node v with color c in $\bar{S}$;
10 **if** $conflicts(S') = 0$ **then**
11 $\bar{S} \leftarrow S', \bar{v} \leftarrow v, \bar{c} \leftarrow c$;
12 goto line 16;
13 **else if** *the pair* v, c *is not in the tabu list* **then**
14 **if** $conflicts(S') < minConflicts$ **then**
15 $minConflicts \leftarrow conflicts(S')$;
16 $\bar{S} \leftarrow S', \bar{v} \leftarrow v, \bar{c} \leftarrow c$;
17 insert pair $\bar{v}$, $\bar{c}$ in the tabu list for TabuTenure iterations;
18 $C \leftarrow C \setminus u$;
19 Let $C_{\bar{v}}$ be the set of nodes conflicting with $\bar{v}$;
20 $C \leftarrow C \cup C_{\bar{v}}$;
21 **return** $\bar{S}$;

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