Question 4:
BOOTSTRAPPING (Using Nelson-Siegel model)

Multiple bonds can be priced together by creating a discounted cashflow matrix (DCFM), which gives a discounted coupon/terminal value at each payment time for a bond and zero otherwise.

E.g. for 3 bonds with coupons = {3, 4, 5} and time to maturity = {1.03, 2.07, 3.03}. If the rates r (i) are given in increasing order of payment dates, then the DCFM will be:

	B1	B2	B3
t = 0.03	3 * exp(-0.03*r(1))	0	5* exp(-0.03*r(1))
t = 0.07	0	4*exp(-0.07*r(2))	0
t = 1.03	103*exp(-1.03*r(3))	0	5*exp(-1.03*r(3))
t = 1.07	0	4*exp(-1.07*r(4))	0
t = 2.03	0	0	5*exp(-2.03*r(5))
t = 2.07	0	104*exp(-2.07*r(6))	0
t = 3.03	0	0	105* exp(-3.03*r(7))

The no arbitrage value (price) of the 3 bonds will simply be sum across the rows for each of the three columns. Or price array can also be calculated as $\boldsymbol{Z^T C}$ where **C** is the cashflow matrix of size length (rates) * length (coupons) and Z is the vector of discount factors of size length (rates) * 1.
As a matter of fact, any array of securities (e.g. a mix of deposits, fixed-float swaps and FRAs) with deterministic cashflow has to follow the no-arbitrage equation: $\boldsymbol{V = Z^T C}$

Where, V is the vector of securities' values, Z is the discount vector and C is the cashflow matrix arising from all the cashflows of these securities. Given these values and cashflows, we can re-engineer the discount factors and hence the interest rates to create an interest rate curve with a process known as **Bootstrapping**. A very simple example of bootstrapping is what we did in the previous question. A method to bootstrap zero rates curve with deposits, FRAs/Futures and swaps was very common before the financial crisis but not anymore. That's a story for another time.

The rates as a function of time are something known as a yield curve but it is not possible to directly observe interest rates at arbitrary times for example time = 1.0707 exactly. There must be some bootstrapping methodology with interpolation scheme defined to calculate such values. In this question, we will discuss a parametric method to bootstrap which is commonly used for modelling treasury yield curves, known as the **Nelson-Siegel model (NS).**

The NS model assume that the interest rates follow the below deterministic functional form w.r.t time to maturity: -

$$y(m) = \beta_0 + \beta_1 \frac{[1-\exp(-m/\tau)]}{m/\tau} + \beta_2 \left(\frac{[1-\exp(-m/\tau)]}{m/\tau} - \exp(-m/\tau) \right)$$

Here, y (m) is the interest rate at time to maturity m. $[\beta_0, \beta_1, \beta_2, \tau]$ are parameters that need to be calibrated using real-world data. β_0 $r$epresents the level of yield curve i.e., value of long-term interest rates, β_1 represents the negative of slope of the yield curve and $\beta_0 + \beta_1$ would approximate the short-term interest rates. A positive β_1 aka negative slope means an inverted yield curve which some believe to be a leading indicator for recession. β_2 is known as the curvature of yield curve and is used to model the hump observed. Γ controls the location of the hump (maxima/minima) on the time to maturity axis.

Additional references (not necessarily required for the question): -
https://cepr.org/sites/default/files/events/1854_NS_1987.pdf ,
https://www.cnbc.com/2019/08/14/the-inverted-yield-curve-explained-and-what-it-means-for-your-money.html

As mentioned, before we have to calibrate the Nelson-Siegel parameters according to real world data. This is done by minimizing the squared error between NS model predicted bond prices (using the rates) and actual market observed bond prices. i.e.

$$\beta_0, \beta_1, \beta_2, \tau = argmin\, \Sigma(PriceArray_{NS} - PriceArray_{market})^2$$

For this question, you are given the Nelson Siegel parameters as input in the form of an array of size 4 and in the order: $[\beta_0, \beta_1, \beta_2, \tau]$ and a set of bonds with an array of time to maturity (in increasing order), array of coupons for the bonds and the actual market traded prices for those bonds. You need to return the sum of squared errors b/w the bond prices according to NS model and market traded prices i.e.

$$return\ \Sigma\ (PriceArray_{NS} - PriceArray_{market})^2$$

Note: *You don't need to do any minimization, you only need to write a function as described above that takes in NS parameters and outputs the price error. This function can be used to optimize for the NS parameters but is out of scope for this question.*

STEPS:

1. Calculate the interest rates using nelson Siegel equation and payment times determined with the cashflow matrix
2. Input these interest rates to discount the cashflow matrix
3. Return squared sum of errors b/w the calculated bond prices and input market traded prices.

Input Specification:

input1: An array of Nelson Siegel parameters in the order $[\beta_0, \beta_1, \beta_2, \tau]$
input2: An array of market traded bond prices
input3: An array of time to maturity bonds in increasing order. Can have duplicate values, can take floating decimal values.
input4: An array of coupons for the bonds. Bonds are in increasing order of their time to maturity.
input5: An integer n representing the size of bond price.

Output Specification:

Return sum of squared error between market bond Prices and calculated prices rounded to 2 decimal places.