Question 3:
BOOTSTRAPPING

Bonds represent debt undertaken by the issuer and these debts are sliced up and sold to investors in smaller units, fixed-rate bonds are one of the most wide-spread and uncomplicated debt instruments. Each of these types of bonds have a pre-specified maturity date, when the issuer's debt obligation to the holder is finished.

A fixed-rate coupon bond pays to the investor a fixed cashflow (coupon) at regular intervals of time determined by the time to maturity, frequency and facevalue and the facevalue itself at the maturity date. For the purpose of this question, we will always assume that frequency is annual (i.e. coupons are paid once a year until maturity) and facevalue = $100.

The price of a coupon bond for an investor can be calculated today by summing up all the discounted future cashflows (payments) to that investor. The payments in the future need to be discounted by a fraction known as discount factor due to a time value of money. For the purpose of this question, we will assume that the discount factor (**Z**) is from continuously compounded interest rates i.e.:

$$\boldsymbol{Z(0,t) = \exp(-r(t) * t)}$$

Where, *Z* (0, t) is the discount factor used to bring payments in the future at time = t in today's money (t = 0) and r (t) is the interest rate today for time t.

With this information a value of a bond can be written as:

$$P_b = [\Sigma_{\mathrm{t}\,\epsilon\tau} c * \exp(-\mathrm{r_t} * \mathrm{t})] + [(FV + c) * \exp\,(-\mathrm{r}_T * \mathrm{T})]$$

Where, τ is the set of payments dates except the terminal payoff, r_t is the interest rate today for time t. *FV* is the facevalue (100), c is the coupon value and T is the time to maturity or the terminal payoff date.

E.g., a bond with a dollar coupon value of $5 and 3.25 years left to maturity will pay $5 at t = 0.25,1.25,2.25 years each and pay $105($5 + $100) at t = 3.25 years, the maturity date.
Assume the interest rates at time t = 0.25, 1.25.2.25 and 3.25 are 3%, 3.5%, 4% and 4.5% respectively.
Then the price can be calculated as P = 5*exp (-0.25*0.03) + 5*exp (-1.25*0.035) + 5*exp (-2.25*0.04) + 105*exp (-3.25*0.045) = $103.89

Another example can be found here: https://financetrain.com/how-to-price-a-bond-using-spot-rates-zero-curve

You are given an array of bond prices (prices) in increasing order of their corresponding time to maturities (tmat). All the maturities are integer valued. The first bond in the array will always have a time to maturity = 1Y. The coupons corresponding to the bonds are given in an array (coupons). All the bonds have FV = 100 and annual payment frequency of coupons. You will need to do linear extrapolation with two nearest maturities for any missing time to maturity as:

$$r(t_x) = \frac{[r(t_{i+1}) - r(t_i)]}{(t_{i+1} - t_i)} X\,(t_x - t_i) + r(t_i)$$

Where, in this case $$t_x > t_{i+1} > t_i$$

Assume discount factors with continuously compounded interest rates. You need to develop an iterative algorithm to calculate the interest rate at the maximum time to maturity (i.e., last element of the time to maturities array).

e.g. For an array of bond prices [98,97,99,98] with coupons of [3,4,5,6] and time to maturity (mat) = [1,2,3,5]. You can calculate r (1) simply as -log (98/103). Then you can use this r (1) to discount the first coupon for the 2nd bond and thus calculate r (2) as

$$-0.5 * log[\frac{97-4*exp(-r(1))}{104}]$$

And do it similarly for r(3). For the last bond you will need to calculate r (4) by extrapolation from r (2) and r (3) using the above equation. After r (4) is found, the last bond price can be used to calculate the answer: r (5)

NOTE: In case of more than one missing rates e.g. in the case of mat = [1, 2, 5]. You will need to extrapolate twice once for r(3) using r(1) and r(2) and then for r(4) using r(2) and r(3). You can assume r(0) = 0

Input Specification:

input1: an integer array of bond prices
input2: a double array of coupons for the bonds. Bonds are in increasing order of their time to maturity
input3: an integer array of unique time to maturity for bonds in increasing order. First element of the array mat [0] = 1. Can only take integer values.

Output Specification:

Return the interest rate at maximum time to maturity among the given bonds. i.e. interest rate at mat [-1] and round the value to 3 decimal places.