30/8/24

Unit - 2 : Probability Distributions

There are two types of probability distributions

I. Discreate Probability Distributions

1) Binomial Distribution
2) Poisson Distribution

$\}$ only 2^{nd} unit

3) Rectangular Distribution
4) Negative Binomial Distribution
5) Geometric Distribution

II. Continous Probability Distributions

1) Normal Distribution
2) Student - t distribution
3) F - Distribution
4) χ^2 (chi-square) Distribution

Bernoulli's Theorem :-

Here we take '1' as success and '0' as failure and we take 'p' probability of success and 'q' as probability of failure.

X	0	1
P(x)	q	p

$p+q=1$

The ~~property~~ Probability of getting exactly 'r' times success out of 'n' independent trails is

$$\boxed{{}^{n}C_{r}\, p^{r}\, q^{n-r}}$$

Ex : $n=10$, $r=4$

$${}^{10}C_{4}\left(\tfrac{1}{2}\right)^{4}\left(\tfrac{1}{2}\right)^{6}$$

Binomial Distribution :-

A random variable x has the Binomial Distribution if it assumes only non-negative values and its Probability Distribution function is given by

$$P(X=r) = P(r) = \begin{cases} {}^{n}C_{r}\, p^{r} q^{n-r}, & \text{if } r = 0,1,2,3,\ldots n \\ 0, & \text{otherwise} \end{cases}$$

Measures of Binomial Distribution :-

1) Mean :

$\mu = np$

2) Varience :

$\sigma^2 = npq$

3) mode :

$$\text{mode} = \begin{cases} \text{Integral part of } (n+1)p, & \text{if } (n+1)p \text{ is not an Integer (odd)} \\ (n+1)p,\ (n+1)p-1, & \text{if } (n+1)p \text{ is integer (even)} \end{cases}$$

31/08/24

4) ~~Recur~~ Reccu~~rr~~arance Relation :

Reccurance Relation of ~~bion~~ binomial Distribution is $p(r+1) = \frac{n-r}{r+1} \cdot \frac{p}{q}\, p(r)$

Important points in binomial Distribution :-

n : total number of Probabilities.

p : Probability of success

1) A fair coin tossed 6 times find the Probability of getting four heads.

$n=6$, $r=4$

p = Probability of getting ~~tail~~ head $=\frac{1}{2}$

q = Probability of not getting head $=\frac{1}{2}$

$P(r) = {}^nC_r\, p^r\, q^{n-r}$

$P(r) = {}^6C_4 (1/2)^4 . (1/2)^{6-4} = \frac{15}{2^6} = \frac{15}{64}$

2) A dice is thrown 6 times if getting an even number is success find the Probabilities of

i) atleast 1 success ii) atmost 3 success

iii) exactly 4 success

$P(r) = {}^nC_r\, p^r q^{n-r} = {}^6C_r (1/2)^6$

$n=6$ p = Probability of getting even $= 1/2$

q = Probability of not getting even $= 1/2$

i) $P(r \geq 1) = 1 - P(r<1) = 1 - P(0) = 1 - {}^6C_0 (1/2)^6$

$= 1 - \frac{1}{64} = \frac{63}{64}$

ii) $P(r \leq 3) = 1 - P(r>3) = 1 - [P(4)+P(5)+P(6)] = 1 - [{}^6C_4 (1/2)^6 + {}^6C_5 (1/2)^6 + {}^6C_6 (1/2)^6]$

$= 1 - [\frac{1}{64}[15+6+1]] = 1 - [\frac{1}{64}(22)]$

$= 1 - \frac{22}{64} = \frac{42}{64}$

$= \frac{21}{32}$

iii) $P(4) = \frac{15}{64}$

3) Two dice are thrown 5 times. Find the Probability of getting ~~sum~~ 7 as the sum
i) atleast 1 ii) 2 times iii) $P(1<r<5)$

$n = 5$

p = Probability of getting 7 as sum $= \frac{6}{36} = \frac{1}{6}$

q = Probability of not getting 7 as sum $= \frac{5}{6}$

$P(r) = {}^nC_r\, p^r q^{n-r} = {}^5C_r \left(\frac{1}{6}\right)^r \left(\frac{5}{6}\right)^{5-r}$

i) $P(r \geq 1) = 1 - P(0) = 1 - {}^5C_0 \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^5 = \frac{4651}{7776}$

ii) $P(2) = {}^5C_2 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^3 = 0.1607$

iii) $P(1<r<5) = P(2) + P(3) + P(4)$

$= {}^5C_2 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^3 + {}^5C_3 \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^2 + {}^5C_4 \left(\frac{1}{6}\right)^4 \left(\frac{5}{6}\right)^1$

$= 0.196$

4) If the probability of getting defective bolt is 1/8, find the mean and varience of the distribution of defective bolts out of 640.

$n = 640$, $p = 1/8$ $q = 7/8$

mean $\mu = np = 640 \times 1/8 = 80$

Varience $\sigma^2 = npq = 640 \times 1/8 \times 7/8 = 70$

5) 20% of items produced from a factory are defective find the probability that in a sample of 5 choosen at random i) None is defective

ii) 1 is defective

iii) $P(1<r<4)$

$n=5 \quad p=1/5 \quad q=4/5 \quad P(r)={}^5C_r (1/5)^r (4/5)^{5-r}$

i) $P(0) = {}^5C_0 (1/5)^0 (4/5)^5 = \frac{1024}{3125} = 0.327$

ii) $P(1) = {}^5C_1 (1/5)^1 (4/5)^4 = \frac{256}{625} = 0.4096$

iii) $P(1<r<4) = P(2)+P(3) = {}^5C_2 (1/5)^2 (4/5)^3 + {}^5C_3 (1/5)^3 (4/5)^2$

$= \frac{32}{125} = 0.256$

03/09/24

6) Fit a Binomial Distribution to the following frequency Distribution.

x	0	1	2	3	4	5	6
f	13	25	52	58	32	16	4

observed frequency

$N=200$, $n=6$,

$\mu = \Sigma x_i f_i / N = \frac{0+25+104+174+128+80+24}{200}$

$\mu = 2.675$

$\mu = np \Rightarrow 2.675 = 6 \times p \Rightarrow p = \frac{2.675}{6} = 0.4458$

$q = 1-p = 0.5542$

$P(r) = {}^nC_r p^r q^{n-r} = {}^6C_r (0.4458)^r (0.5542)^{6-r}$

$N \cdot P(0) = 200 \times {}^6C_0 (0.4458)^0 (0.5542)^{6-0} = 5.79 \simeq 6$

$N \cdot P(1) = 200 \times {}^6C_1 (0.4458)^1 (0.5542)^{6-1} = 27.96 \simeq 28$

$N \cdot P(2) = 200 \times {}^6C_2 (0.4458)^2 (0.5542)^{6-2} = 56.18 \simeq 56$

$N \cdot P(3) = 200 \times {}^6C_3 (0.4458)^3 (0.5542)^{6-3} = 60.33 \simeq 60$

$N \cdot P(4) = 200 \times {}^6C_4 (0.4458)^4 (0.5542)^{6-4} = 36.431 \simeq 36$

$N \cdot P(5) = 200 \times {}^6C_5 (0.4458)^5 (0.5542)^{6-5} = 11.68 \simeq 12$

$N \cdot P(6) = 200 \times {}^6C_6 (0.4458)^6 (0.5542)^{6-6} = 1.574 \simeq 2$

f	6	28	56	60	36	12	2

expected ~~estimated~~ frequency

Poisson Distribution:-

If X is a Discreate R.V that can assume value 0, 1, 2..... such that its probability distribution function is given by

$$P(x) = \frac{e^{-\lambda} \cdot \lambda^{x}}{x!} \; ; \; x = 0, 1, 2, ...$$

Then X is said to follow a poisson Distribution with parameter λ.

Poisson Distribution as limiting form of binomial Distribution:-

Poisson Distribution is a limiting case of binomial Distribution under following conditions:-

1) n, no of trails is indefinitly Large
i.e $n \rightarrow \infty$

2) P, Probability of success is very small
i.e $p \rightarrow 0$

3) $\lambda = np$ is finite

10/09/24

measures of Poission Distribution:-

1) mean: $\mu = \lambda$

2) varience : $\sigma^2 = \lambda$

3) mode :

$$\text{mode} = \begin{cases} \text{integral part of } \lambda, & \text{if } \lambda \text{ is not integer} \\ \lambda, \lambda - 1, & \text{if } \lambda \text{ is an integer.} \end{cases}$$

4) Reccurance Relation:

$$P(r+1) = \frac{\lambda}{r+1} P(r)$$

1) 2% of items of a factory are defective the items are packed in boxs of 100 items. what is the probability that there will be
i) 2 defective items ii) atleast 3 defective items in a box.

$n = 100$, $P = 2\% = 0.02$

$\lambda = np = 100 \times 0.02 = 2$ $\quad P(x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-2}2^x}{x!}$

i) $P(2) = \frac{e^{-2}2^2}{2!} = 0.2706$

ii) $P(x \geq 3) = 1 - \{P(x < 3)$

$= 1 - [P(0) + P(1) + P(2)]$

$= 1 - e^{-2}\left[\frac{2^0}{0!} + \frac{2^1}{1!} + \frac{2^2}{2!}\right] = 1 - 5e^{-2} = 0.3233$

2) using Reccurance formula find the Probabilities when $x = 0, 1, 2, 3, 4, 5$ if the mean of Poisson distribution is 3.

$\lambda = 3$ $\quad P(r) = \frac{e^{-3}3^r}{r!}$

$P(0) = \frac{e^{-3}3^0}{0!} = 0.04978$

$P(r+1) = \frac{\lambda}{r+1} P(r)$

if $r = 0 \Rightarrow P(1) = \frac{\lambda}{0+1} P(0) = 3(0.04978) = 0.14934$

if $r = 1 \Rightarrow P(2) = \frac{\lambda}{1+1} P(1) = \frac{3}{2}(0.14934) = 0.2241$

if $r = 2 \Rightarrow P(3) = \frac{\lambda}{2+1} P(2) = \frac{3}{3} P(2) = 0.2241$

if $r = 3 \Rightarrow P(4) = \frac{\lambda}{3+1} P(3) = \frac{3}{4} P(3) = 0.1681$

if $r = 4 \Rightarrow P(5) = \frac{\lambda}{4+1} P(4) = \frac{3}{5} P(4) = 0.1008$

3) If the varience of Poisson distribution is 3 then find the probabilities that

i) 0 ii) $0 < x \leq 3$ iii) $1 \leq x < 4$

$\lambda = 3$ $P(x) = \frac{e^{-3} 3^x}{x!}$

i) $P(0) = \frac{e^{-3} 3^0}{0!} = 0.04978$

ii) $P(0 < x \leq 3) = P(1) + P(2) + P(3)$

$= 0.1494 + 0.2241 + 0.2241$

$= 0.5974$

iii) $P(1 \leq x < 4) = P(1) + P(2) + P(3)$

$= 3e^{-3} + \frac{9e^{-3}}{2} \times 2$

$= 12 e^{-3} = 0.5974$

4) If X is a poisson varient such that $3P(x=4) = \frac{1}{2}P(x=2) + P(x=0)$. find

i) mean ii) $P(x \leq 2)$ $P(x) = \frac{e^{-\lambda} \lambda^x}{x!}$

$3P(4) = \frac{1}{2} P(2) + P(0)$

$3 \frac{e^{-\lambda} \lambda^4}{4!} = \frac{1}{2} \frac{e^{-\lambda} \lambda^2}{2!} + \frac{e^{-\lambda} \lambda^0}{0!}$

$\frac{3}{24} e^{-\lambda} \lambda^4 = \frac{1}{4} \lambda^2 e^{-\lambda} + e^{-\lambda}$

$\frac{1}{8} \lambda^4 e^{-\lambda} = e^{-\lambda}(\frac{1}{4}\lambda^2 + 1) \Rightarrow \lambda^4 = 8(\frac{1}{4}\lambda^2 + 1)$

$\Rightarrow \lambda^4 = 2\lambda^2 + 8$

$\Rightarrow \lambda^4 - 2\lambda^2 - 8 = 0$

$\Rightarrow \lambda^2 = 4, -2$

$\Rightarrow \lambda = \pm 2, \pm\sqrt{2}i$

$\boxed{\lambda = 2}$ since it can't be negative or complex

i) mean $\Rightarrow \mu = 2$

ii) $P(x \le 2) = P(0) + P(1) + P(2)$

$$= \frac{e^{-2} 2^0}{0!} + \frac{e^{-2} 2^1}{1!} + \frac{e^{-2} 2^2}{2!}$$

$$= e^{-2}\left[\frac{2^0}{0!} + \frac{2^1}{1!} + \frac{2^2}{2!}\right] = e^{-2}[5] = 0.6766$$

5) $P(X=2) = 9$, $P(x=4) + 90P(x=6)$. Find the mean and variance of X

$$P(x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

$$P(2) = 9P(4) + 90P(6)$$

$$\frac{e^{-\lambda}\lambda^2}{2!} = 9 \cdot \frac{e^{-\lambda}\lambda^4}{4!} + 90\frac{e^{-\lambda}\lambda^6}{6!}$$

$$\frac{\lambda^2}{2} = \frac{9\lambda^4}{24} + 90\frac{\lambda^6}{720}$$

$$\frac{\lambda^2}{2} = \frac{3\lambda^4}{8} + \frac{1}{8}\lambda^6$$

$$\frac{3}{4}\lambda^2 + \frac{1}{4}\lambda^4 = 1$$

$$3\lambda^2 + \lambda^4 - 4 = 0$$

$\lambda^2 = 1, -4$ $\quad \lambda = \pm 1, \pm 2i$ $\quad \therefore \lambda = 1$

$\therefore \mu = \lambda = 1 \quad \sigma^2 = \lambda = 1$

6) If X is a poisson variant such that $P(X=0) = P(X=1) = k$ determine k.

Given, $P(X=0) = P(X=1) = k$

$$P(0) = P(1)$$

$$\frac{e^{-\lambda}\lambda^0}{0!} = \frac{e^{-\lambda}\lambda^1}{1!}$$

$$1 = \lambda$$

$$P(0) = k$$

$$k = \frac{e^{-1}(1)}{1!} = \frac{1}{e} = 0.367$$

7) fit a poisson distribution to the following frequency distribution.

x	0	1	2	3	4
f	109	65	22	3	1

$N = \Sigma f_i = 200$, $n = 4$

$$\lambda = \mu = \frac{\Sigma x_i f_i}{N} = \frac{0+65+44+9+4}{200} = 0.61$$

$$P(x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-(0.61)}\cdot(0.61)^x}{x!}$$

$$N\cdot P(0) = 200\cdot\frac{e^{-(0.61)}(0.61)^x}{0!} = 108.6 \simeq 109$$

$$N\cdot P(1) = 200\cdot\frac{e^{-(0.61)}(0.61)^1}{1!} = 66.28 \simeq 66$$

$$N\cdot P(2) = 200\cdot\frac{e^{-(0.61)}(0.61)^2}{2!} = 20.21 \simeq 20$$

$$N\cdot P(3) = 200\cdot\frac{e^{-(0.61)}(0.61)^3}{3!} = 4.11 \simeq 4$$

$$N\cdot P(4) = 200\cdot\frac{e^{-(0.61)}(0.61)^4}{4!} = 0.62 \simeq 1$$

x	0	1	2	3	4
obtained freq	109	66	22	3	1
Expected freq	109	66	20	4	1

24/09/24

Co-Variance :-

If X & Y are two random variables then the covariance b/w X & Y is defined as covariance of (x, y). denoted as cov(x, y)

$$cov(x,y) = E[(x - E(x))(y - E(y))]$$

$$= E(xy) - E(x) \cdot E(y)$$

Properties:-

1) If x, y are independent then ~~cov~~ $cov(x,y) = 0$
2) $cov(ax, by) = ab\,cov(x,y)$ where a, b are constants
3) $cov(ax+b, cy+d) = ac\,cov(x,y)$
4) $cov(x+y, z) = cov(x,z) + cov(y,z)$

Linear combination:-

Let $x_1, x_2, \ldots x_n$ be n random variable and $a_1, a_2 \ldots a_n$ be orbitary constants then $y = a_1x_1 + a_2x_2 + \ldots + a_nx_n$ is called the linear combination of $x_1, x_2 \ldots x_n$

Mean of linear combination :-

$$E(Y) = E(a_1x_1 + a_2x_2 + \ldots + a_nx_n)$$

$$= a_1E(x_1) + a_2E(x_2) + \ldots a_nE(x_n$$

Varience of linear combination:-

$$\sigma^2 = var(Y) = a_1^2\,var(x_1^2) + a_2^2\,var(x_2^2) + \ldots + a_n^2\,var(x_n^2)$$

$$+ a_1a_2\,cov(x_1, x_2) + a_1a_3\,cov(x_1, x_3) + \ldots + a_{n-1}a_n\,cov(x_{n-1}, x_n)$$

$$= \sum_{i=1}^{n} a_i^2\,Var(x_i^2) + 2\sum_{\substack{i=1 \\ i<j}}^{n}\sum_{j=1}^{n} a_ia_j\,cov(x_i, x_j)$$

Chebyshev's inequality

If x is a random variable with mean μ and varience σ^2 then probability of

$$P(|X-\mu| \geq K) \leq \frac{\sigma^2}{K^2} \text{ (or) } P(|X-\mu| \leq K) \geq 1-\frac{\sigma^2}{K^2}$$

$$\Rightarrow P(\mu-K \leq X \leq \mu+K) \geq 1-\frac{\sigma^2}{K^2}$$

27/09/24

1) If the chebyshev's inequality for the R.V X is given by $P(-2<X<8) \geq \frac{21}{25}$ Find mean & σ^2.

$$P(\mu-K < X < \mu+K) \geq 1-\frac{\sigma^2}{K^2}$$

$\mu - K = -2$ — ①

$\mu + K = 8$ — ②

$2\mu = 6 \Rightarrow \mu = 3$

Put $\mu=3$ in eq ① . $3-K=-2 \Rightarrow K=5$

$$1-\frac{\sigma^2}{K^2} = \frac{21}{25} \Rightarrow 1-\frac{21}{25} = \frac{\sigma^2}{K^2}$$

$$\sigma^2 = 25\left(\frac{9}{25}\right) = 4$$

2) The number of components manufactured in the factory during a one month period is a R.V with mean 600 and varience 100. What is the probability that the production will be between 500 and 700 over a month.

$\mu = 600$, $\sigma^2 = 100$ $\quad P(500<X<700)$

$\mu-K=500 \Rightarrow K=600-500=100$

$\mu+K=700$

$$1-\frac{\sigma^2}{K^2} = 1-\frac{100}{(100)^2} = 1-\frac{1}{100} = \frac{99}{100}$$

$$\therefore P(500<X<700) \geq \frac{99}{100} = 0.99$$

3) For the following Joint probability distribution find covar(x, y).

$X \mid Y$	1	2	3	P_i
0	2/12	1/12	1/12	4/12
1	1/12	1/12	2/12	4/12
2	3/12	1/12	0	4/12
P_j	6/12	3/12	3/12	1

$$\mathrm{Cov}(x, y) = E(xy) - E(x) \cdot E(y)$$

$$E(x) = \sum x_i p_i = 0 \times \frac{4}{12} + 1 \times \frac{4}{12} + 2 \times \frac{4}{12} = 1$$

$$E(y) = \sum y_i P_j = 1 \times \frac{6}{12} + 2 \times \frac{3}{12} + 3 \times \frac{3}{12} = \frac{21}{12} = \frac{7}{4}$$

$$E(xy) = \sum \sum x_i y_j P_{ij}$$

$$= 0 \times 1 \times \frac{2}{12} + 0 \times 2 \times \frac{1}{12} + 0 \times 3 \times \frac{1}{12} + 1 \times 1 \times \frac{1}{12} + 1 \times 2 \times \frac{1}{12}$$

$$+ 1 \times 3 \times \frac{2}{12} + 2 \times 1 \times \frac{3}{12} + 2 \times 2 \times \frac{1}{12} + 2 \times 3 \times 0$$

$$= 19/12$$

$$\mathrm{cov}(x, y) = E(xy) - E(x) \cdot E(y)$$

$$= \frac{19}{12} - 1 \times \frac{7}{4}$$

$$= \frac{19}{12} - \frac{7}{4} = -\frac{1}{6}$$