UNIT-3 ELEMENTARY COMBINOTORICS

Sum Rule: If an event occur in m ways and another event occurs in n ways and two events cannot occur simultaneously, then one of two events can occur in $m+n$ ways.

Ex: In a class there are 9 boys and 10 girls, in how many ways we can select a CR?

From 9 boys we can select a CR in 9 ways

From 10 girls we can select a CR in 10 ways

The no. of ways of selecting a CR from boys or girls is $9+10$ ways

Product Rule: If an event E_1 occurs in m_1 ways, event E_2 occurs in m_2 ways ------ event E_n occurs in m_n ways, then the sequence of events $E_1, E_2, ---- E_n$ can occur in $m_1 \times m_2 \times ---- \times m_n$ ways.

Ex: A computer password consists of an alphabet, followed by 4 digits. Find the total number of passwords that can be created a] with repetition

b] without repetition

a] with repetion:

0 - 9 $\rightarrow$ 10 digits

A - Z $\rightarrow$ 26 Alphabets

□□□□

$= 26 \times 10 \times 10 \times 10$

$= 26 \times 10^4$ ways

b) without repetion:

$= 26 \times 10 \times 9 \times 8 \times 7$ ways

Permutation: A r- permutation of n objects is an ordered arrangement of n-objects taken r at a time

It is given by $^nP_r = \frac{n!}{(n-r)!}$

Ex: What is 2-permutation of the word KMIT-A

Given word KMITA

$n = 5$

$r = 2$

2-permutation of 5 letter word $^5P_2 = \frac{5!}{(5-2)!} = 20$

KM	MK	IK	AK	TK
KI	MI	IM	AM	TM
KT	MT	IT	AI	TI
KA	MA	IA	AT	TA

Combination: A r-combination of n objects is an unordered arrangement of n objects taken r at a time

It is given by $^nC_r = \frac{n!}{r!(n-r)!} = {}^nC_r = \binom{n}{r}$

Ex: what is 2- combination of the word KMITA.

Given word KMITA

$n = 5$

$r = 2$

2- Combination of 5 letter word $^5C_2 = \frac{5!}{2!(5-2)!} = 10$

KM	MI	IA
KI	MT	TA
KT	MA	
KA	IT	

Q] A committee of 5 students is formed; selecting from 6 boys and 5 girls such that atleast one boy and one girl is included. How many different committees can be formed.

B G

$\cancel{6 \times 5 \times 9 \times 8 \times 7 = 15120}$

$= \cancel{15120}$ ~~committees can be formed.~~

$\cancel{{}^6C_1 + {}^5C_4 \times}$

$${}^6C_1 \times {}^5C_4 + {}^6C_2 \times {}^5C_3 + {}^6C_3 \times {}^5C_2 + {}^6C_4 \times {}^5C_1$$

$$(6 \times 5) + (15 \times 10) + (20 \times 10) + (15 \times 5)$$

$$30 + 150 + 200 + 75$$

$$= 455$$

Q] How many ways a committee of 4 teachers and 5 students can be chosen from 9 teachers and 15 students

Sol: No. of ways of selecting 4 teachers from 9 is ${}^9C_4$ ways

No. of ways of selecting 5 students out of 15 is ${}^{15}C_5$ ways

No. of ways of selecting a comittee with 4 teachers and 5 students is ${}^9C_4 \times {}^{15}C_5$

$$= 126 \times 3003$$

$$= 378378$$

Q] In a cricket team of 11 to be selected out of 14 players of whom 5 are bowlers, find the no. of ways in which this can be done so as to include atleast 3 bowlers

~~$^5C_3 + {}^5C_4 + {}^5C_5$~~

No. of ways of selecting 11 players with 3 bowlers and 8 rest is $^5C_3 \times {}^9C_8$

or

No. of ways of selecting 11 players with 4 bowlers and 7 rest is $^5C_4 \times {}^9C_7$

or

No. of ways of selecting 11 players with 5 bowlers and ~~7~~ 6 rest is $^5C_5 \times {}^9C_6$

Total No. of ways of selecting 11 players with atleast 3 bowlers is $^5C_3 \times {}^9C_8 + {}^5C_4 \times {}^9C_7 + {}^5C_5 \times {}^9C_6$

$= 10 \times 9 + 5 \times 36 + 1 \times 84$

$= 90 + 180 + 84$

$= 354$ ways

Q] English alphabets contain 21 consonents and 5 vowels consider [i] 7 letter word with 3 different vowels and 4 different consonants.

No. of ways of selecting 3 vowels out of 5 is $^5C_3$ ways

No. of ways of selecting 5 consonents out of 21 is $^{21}C_4$

No. of ways of selecting 3 vowels and ~~5~~ 4 consonents is $^5C_3 \times {}^{21}C_4$

7 Letters can be internally arranged in 7! ways

∴ Total no. of ways of forming a 7 letter word with 3 vowels and 4 consonents is $^5C_3 \times {}^{21}C_4 \times 7!$ ways

(ii) How many such words contain the letter a

$^4C_2 \times {}^{21}C_4 \times 7!$

$\because$ one vowel (a) is already taken

Permutations with repetitions:

The no. of permutations of n objects taking r at a time with unlimited repetitions is n^r

Combination with repetitions:

The no. of combinations of n objects taking r at a time with unlimited repetitions is $c(n-1+r, r)$

$$= c(n-1+r, n-1)$$
$$= \frac{(n-1+r)!}{(n-1)!\, r!}$$

Observations:

We use combination with repetition under the following cases.

1. To find the no. of non negative integral solution to the equation $x_1 + x_2 + \ldots + x_n = r$
2. To find the no. of ways of distributing all r similar balls into n numbered boxes.
3. To find the no. of binary numbers with $(n-1)$ one's and r zeroes

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Q) Find the no. of 3-combinations of 5 objects with unlimited repetitions.

Sol An r-combination of n objects with unlimited repetitions is given by $\frac{(n-1+r)!}{(n-1)!\,r!}$

$$= \frac{(5-1+3)!}{(5-1)!\,3!}$$

$$= \frac{7!}{4! \times 3!} = 35 \text{ 3-combinations}$$

Q) Find the no. of non negative integral solution to

$x_1+x_2+x_3+x_4+x_5 = 50$

Sol To find no. of non-negative integral solution to

$x_1+x_2+ \dots +x_n = r$ we use r combination of n objects with unlimited repetitions formula

$x_1+x_2+ \dots +x_5 = 50$ generally $n < r$ (repetitions)

Here $n=5$ $r=50$

$$\frac{(5-1+50)!}{(5-1)!\,50!} = 316251 \text{ solutions}$$

8] Find the no. of binary numbers with 10 1's and 5 0's

To find the no. of binary numbers with $(n-1)$ one's and r zeroes we use combination with repetition formula.

A r combination of n objects with unlimited objects is given by $c(n-1+r, r) = \frac{(n-1+r)!}{(n-1)!\,r!}$

$n-1 = 10$ $r=5$

$n = 11$

$$\frac{(11-1+5)!}{(11-1)!\,5!} = 3003$$

Q) How many integral solutions are there to $x_1+x_2+\dots+x_5=20$ where $x_1 \geq 3$, $x_2 \geq 2$, $x_3 \geq 4$, $x_4 \geq 6$, $x_5 \geq 0$

$$x_1+x_2+x_3+x_4+x_5 = 20 \rightarrow ①$$

$x_1 \geq 3 \Rightarrow x_1 - 3 \geq 0$

Let $y_1 = x_1 - 3$

$x_1 = y_1 + 3$ with $y_1 \geq 0$

$x_2 \geq 2$

$x_2 = y_2 + 2$

$x_3 \geq 4$

$x_3 = y_3 + 4$

$x_4 \geq 6$

$x_4 = y_4 + 6$

$x_5 \geq 0$

$x_5 = y_5$

$\therefore$ sub $x_1, x_2, x_3, \dots x_5$ in ①

$$y_1+3+y_2+2+y_3+4+y_4+6+y_5 = 20$$

$$y_1+y_2+y_3+y_4+y_5 = 5$$

The no. of non -ve integral solutions of eq ② is obtained by applying combination with repetition.

A r-combination of n objects with unlimited repetitions is given by $c(n-1+r, r)$

$$= \frac{(5-1+5)}{(5-1)!\,5!} = \frac{9!}{4!\,5!} = \frac{9 \times \cancel{8}^{2} \times 7 \times \cancel{6}\,2}{\cancel{4} \times \cancel{3} \times \cancel{2} \times 1} = 126$$

Q] How many integral solutions are there to

$x_1 + x_2 + \cdots + x_5 = 20$ where $x_1 \geq 2$, $x_2 \geq 3$, $x_3 \geq 4$, $x_4 \geq 2$, $x_5 \geq 0$

$$x_1 + x_2 + x_3 + x_4 + x_5 = 20 \rightarrow (1)$$

$x_1 \geq 2 \Rightarrow$	$x_1 = y_1 + 2$	$y_1 \geq 0$
$x_2 \geq 3$	$x_2 = y_2 + 3$	$y_2 \geq 0$
$x_3 \geq 4$	$x_3 = y_3 + 4$	$y_3 \geq 0$
$x_4 \geq 2$	$x_4 = y_4 + 2$	$y_4 \geq 0$
$x_5 \geq 0$	$x_5 = y_5$	

$\therefore$ $y_1 + 2 + y_2 + 3 + y_3 + 4 + y_4 + 2 + y_5 = 20$

$$y_1 + y_2 + y_3 + y_4 + y_5 = 9 \rightarrow (2)$$

The no. of non -ve integral solutions of eq (2) is obtained by applying combination with repetition.
r-combination of n objects with unlimited repetitions is given by $C(n-1+r, r)$

$$= \frac{(5-1+9)!}{(5-1)!\,9!}$$

$$= 715$$

Q] Find the no. of distinct ordered tiple (x_1, x_2, x_3) of non -ve integers satisfying the inequality $x_1 + x_2 + x_3 < 6$

Given eq $x_1 + x_2 + x_3 < 6$ can be written as

$$x_1 + x_2 + x_3 \leq 5$$

$x_1+x_2+x_3=r$

$r=1,2,3,4,5$

$x_1+x_2+x_3=1$

$n=3 \quad r=1$

$= \dfrac{(3-1+1)!}{(3-1)!\,1!}$

$= \dfrac{3!}{2!} = 3$

$x_1+x_2+x_3=2$

$n=3 \quad r=2$

$= \dfrac{(3-1+2)!}{(3-1)!\,2!}$

$= \dfrac{4!}{2!.2!} = 6$

$x_1+x_2+x_3=3$

$n=3 \quad r=3$

$\dfrac{(3-1+3)!}{(3-1)!\,3!}$

$= \dfrac{5!}{2!\,3!} = 10$

$x_1+x_2+x_3=4$

$n=3 \quad r=4$

$= \dfrac{(3-1+4)!}{(3-1)!\,4!}$

$= \dfrac{6!}{2!\,4!} = 15$

$x_1+x_2+x_3=5$

$n=3 \quad r=5$

$= \dfrac{(3-1+5)!}{(3-1)!\,5!}$

$= 21$

$x_1+x_2+x_3=0$

$n=3, \ r=0$

$=$ ~~undefined~~ $^3C_0=1$

$\therefore$ Total = 56 solutions

Q] Find the no. of non -ve integral solutions of the equation $x_1+x_2+x_3+x_4=50$ where $x_1 \geq -4$, $x_2 \geq 7$, $x_3 \geq -14$, $x_4 \geq 10$

$x_1+x_2+x_3+x_4 = 50 \rightarrow (1)$

$x_1 \geq -4$

$x_1+4 \geq 0$

$y_1 = x_1+4$

$x_1 = y_1-4, \ y_1 \geq 0$

$\frac{||}{ly}$ $x_2 = y_2+7$

$x_3 = y_3-14$

$x_4 = y_4+10$

$y_1 - 4 + y_2 + 7 + y_3 - 14 + y_4 + 10 = 50$

$y_1 + y_2 + y_3 + y_4 = 51$

The no. of non -ve integral solutions of eq ② is obtained by applying combination with repetitions

r- combination with n objects with unlimited repetitions is $c(n-1+r, r)$

$= {}^{54}C_{51}$

$= 24804$

Permutation with constrained restictions

The number of permutations of a n letter word of which r_1 are of one type

r_2 are of one type

⋮

r_k are of one types is $\dfrac{n!}{r_1! \, r_2! \cdots r_k!}$

Q] Find the no. of arrangements of the word

(i) ENGINEERING

No. of letters $n = 11$

No. of E's $= 3$

N's $= 3$

G's $= 2$

I's $= 2$

R's $= 1$

No. of ways we can arrange is $\dfrac{n!}{r_1! \, r_2! \cdots r_k!}$

$= \dfrac{11!}{3! \, 3! \, 2! \, 2! \, 1!}$

$= \dfrac{39916800}{144} = 277200$ ways

(ii) MISSISSIPPI

No. of letters $n = 11$ No. of ways $= \dfrac{11!}{1!\,4!\,4!\,2!}$

No. of M's $= 1$

I's $= 4$ $= \dfrac{39916800}{1152}$

S's $= 4$

P's $= 2$ $= 34650$ ways

Binomial Theorem

If n is a +ve integer then

$$(x+y)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r} y^r$$

Observations:

1. The expansion of $(x+y)^n$ contains $(n+1)$ terms
2. The $(r+1)^{th}$ term of the expansion of $(x+y)^n$ is

$${}^nC_r\, x^{n-r} y^r$$

3. The expansions of $(x+y)^n$ and $(y+x)^n$ are equal

Q] Find the middle term of $\left(2x - \frac{1}{3x}\right)^{10}$

Sol The no. of terms of the expansion of $\left(2x - \frac{1}{3x}\right)^{10}$ contain 11 terms.

∴ The middle term of the expansion is 6th term

We know that $(r+1)^{th}$ term of the expansion of $(x+y)^n$ is ${}^nC_r\, x^{n-r} y^r$

Here $r+1 = 6$

$r = 5$

$n = 10$

$^{10}C_5 \, x^5 y^5$

$= 252 \, x^5 y^5$

$= 252 \, (2x)^5 \left(\frac{-1}{3x}\right)^5$

$= -\frac{252 \times 32}{243} \, x^5 \left(\frac{+1}{x^5}\right)$

$= \frac{896}{27}$

[ii] $\left(x - \frac{3}{y}\right)^9$

No. of terms of the expansion of $\left(x - \frac{3}{y}\right)^9$ is 10 terms

$\therefore$ The middle term of the expansion is 5, 6 terms

we know that $(r+1)^{th}$ term of the expansion of $(x+y)^n$ is $^nC_r \, x^{n-r} y^r$

a] $r + 1 = 5$

$r = 4$

$n = 9$.

$^9C_4 \, (x)^5 \left(\frac{-3}{y}\right)^4$

$\frac{126 \, x^5 \times 81}{y^4} = 10206 \frac{x^5}{y^4}$

b] $r + 1 = 6$

$r = 5$

$n = 9$

$^9C_5 \, (x)^4 \left(\frac{-3}{y}\right)^5$

$126 \cdot x^4 \cdot \frac{-243}{y^5} = -30618 \frac{x^4}{y^5}$

Q] Find the coefficient of $x^9 y^3$ in the expansion of $(2x - 3y)^{12}$

Sol We have by binomial expansion

$$(2x - 3y)^{12} = \sum_{r=0}^{12} {}^{12}C_r \, (2x)^{n-r} \, (-3y)^r$$

$$(2x - 3y)^{12} = \sum_{x=0}^{12} {}^{12}C_r \, 2^{n-r} \, x^{n-r} \, (-3)^r \, y^r$$

To find coefficient of x^9y^3

$r=3$

$n=12$

$\therefore$ The coeff of x^9y^3 in $(2x-3y)^{12}$ is

$$^{12}C_3 \cdot 2^{12-3}(-3)^3$$

$$= -3041280$$

Q] Find the term independent of x in the expansion of $\left(x^2+\frac{1}{x}\right)^{12}$

Sol The $(r+1)^{th}$ term in the expansion of $(x+y)^n$ is

$$^nC_r\, x^{n-r} y^r$$

for $\left(x^2+\frac{1}{x}\right)^{12}$, $(r+1)^{th}$ term will be

$$^{12}C_r\, (x^2)^{12-r}\left(\frac{1}{x}\right)^r$$

$$^{12}C_r\, x^{24-2r-r}$$

$$^{12}C_r\, x^{24-3r}$$

To get a term independent of x we must have

$24-3r=0$

$r=8$

$\therefore$ Independent term of x is $^{12}C_8 = 495$

Q] Find the two successive terms in the expansion of $(1+x)^{24}$ whose coefficients are in the ratio 4:1

Sol The two successive terms are r and $r+1$ in the expansion of $(x+y)^n$ is

r^{th} term $\rightarrow$ $^nC_{r-1}\, x^{n-(r-1)} y^{r-1}$

$(r+1)^{th}$ term $\rightarrow {}^nC_r\, x^{n-r} y^r$

For given expression $(1+x)^{24}$

r^{th} term $= {}^{24}C_{r-1}\,(1)^{24-(r-1)}\, x^{r-1}$

$(r+1)^{th}$ term $= {}^{24}C_r\,(1)^{24-r}\, x^r$

$\because$ Ratio of coefficients of two successive terms $r, r+1$
$= 4:1$

$$\frac{{}^{24}C_{r-1}}{{}^{24}C_r} = \frac{4}{1} \Rightarrow 4\,{}^{24}C_r = {}^{24}C_{r-1}$$

$$4 \times \frac{24!}{(24-r)!\,r!} = \frac{24!}{(24-r+1)!\,(r-1)!}$$

$$4\,(24-r)!\,r! = 4\,(24-r+1)!\,(r-1)!$$

$$(24-r)\cancel{(24-r-1)!}\; r\,\cancel{(r-1)!} = 4\,\cancel{(24-r-1)!}\,\cancel{(r-1)!}$$

$$(24-r)\,r = 4$$

$$4(24-r+1)\,\cancel{(24-r)!}\,\cancel{(r-1)!} = \cancel{(24-r)!}\,(r)\,\cancel{(r-1)!}$$

$$4(24-r+1) = r$$

$$96 - 4r + 4 = r$$

$$100 = 5r$$

$$r = 20$$

r^{th} term $= {}^{24}C_{19}\, x^{19}$

$(r+1)^{th}$ term $= {}^{24}C_{20}\, x^{20}$

Properties of Binomial Coefficients:

1. Prove that $c(n,0) + c(n,1) + \ldots + c(n,n) = 2^n$

Sol We have by Binomial Expansion

$$(1+x)^n = \sum_{r=0}^{n} c(n,r)\, 1^{n-r} x^r$$

$= c(n,0) + c(n,1)x + c(n,2)x^2 + \dots + c(n,n)x^n$

put $x = 1$

$2^n = c(n,0) + c(n,1) + c(n,2) + \dots + c(n,n)$

$\therefore \sum_{r=0}^{n} c(n,r) = 2^n$

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2. Prove that $c(n,0) + c(n,2) + c(n,4) + \dots =$
$c(n,1) + c(n,3) + \dots = 2^{n-1}$

By binomial expansion we have

$(1+x)^n = \sum_{r=0}^{n} c(n,r)\, 1^{n-r} x^r$

~~put x = -1~~

$(1+x)^n = c(n,0)x^0 + c(n,1)x + c(n,2)x^2 + \dots$

put $x = -1$

$0^n = c(n,0) - c(n,1) + c(n,2) - c(n,3) + \dots$

$c(n,0) + c(n,2) + \dots = c(n,1) + c(n,3) + \dots \rightarrow (1)$

We know that $c(n,0) + c(n,1) + \dots c(n,n) = 2^n \rightarrow (2)$

from (2) $\Rightarrow 2(c(n,1) + c(n,3) + c(n,5) + \dots) = 2^n$

$\therefore c(n,1) + c(n,3) + c(n,5) + \dots = 2^{n-1}$

proved

from (1) $c(n,0) + c(n,2) + c(n,4) + \dots = 2^{n-1}$

Hence proved.

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3. Prove that $c(n,1) + 2c(n,2) + 3c(n,3) + \dots$

$+ nc(n,n) = n2^{n-1}$

By binomial expansion we have

$$(1+x)^n = \sum_{r=0}^{n} c(n,r) 1^{n-r} x^r$$

$$(1+x)^n = c(n,0)x^0 + c(n,1)x + c(n,2)x^2 + \dots c(n,n)x^n$$

diff w.r.t x

$$n(1+x)^{n-1} = 0 + c(n-1) + 2c(n,2)x + \dots nc(n,n)x^{n-}$$

put $x = 1$

$$n2^{n-1} = c(n,1) + 2c(n,2) + 3c(n,3) + \dots + nc(n,n)$$

Hence proved

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4. Prove that $c(n,1) - 2c(n,2) + 3c(n,3) - 4c(n,4) +$

$\dots = 0$

By binomial theorem we have

$$(1+x)^n = \sum_{r=0}^{n} c(n,r) 1^{n-r} x^r$$

$$(1+x)^n = c(n,0)x^0 + c(n,1)x + c(n,2)x^2 + \dots c(n,n)x^n$$

diff w.r.t x

$$n(1+x)^{n-1} = 0 + c(n,1) + 2c(n,2)x + \dots nc(n,n)x^{n-1}$$

put $x = -1$

$$0 = c(n,1) - 2c(n,2) + 3c(n,3) - \dots$$

Hence proved

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5. Show that $C_0 + 3C_1 + 5C_2 + \cdots + (2n+1)C_n = (n+1)2^n$

$C_0 + 3C_1 + 5C_2 + \cdots$

we can write it as

$S = (2n+1)C_n + (2n-1)C_{n-1} + (2n-3)C_{n-2} + \cdots + C_0 \rightarrow (1)$

since $C_{n-r} = C_r$

$S = (2n+1)C_0 + (2n-1)C_1 + (2n-3)C_2 + \cdots + C_n \rightarrow (2)$

Adding (1) and (2)

$2S = (2n+2)C_0 + (2n+2)C_1 + (2n+2)C_2 + \cdots + (2n+2)C_n$

$2S = (2n+2)[C_0 + C_1 + C_2 + \cdots + C_n]$

$2S = (2n+2)(2^n)$

$S = \dfrac{\cancel{2}(n+1)(2^n)}{\cancel{2}}$

$S = (n+1)2^n$

Hence proved

Multinomial theorem:

Let n be a +ve integer then $\forall x_1, x_2, \ldots x_r$

$(x_1 + x_2 + x_3 + \cdots + x_r)^n = \sum_{r=0}^{n} \left(\dfrac{n!}{q_1! q_2! \cdots q_r!}\right) x_1^{q_1} x_2^{q_2} \cdots x_r^{q_r}$

It is denoted by $P(n; q_1, q_2, q_3, \ldots q_r)$

Q] Evaluate $P(18; 4, 5, 6, 3)$

Sol We know that $P(n; q_1, q_2, q_3, q_4) = \dfrac{n!}{q_1! q_2! q_3! q_4!}$

$= \dfrac{18!}{4!5!6!3!} = 514594080$

8] Find the coefficient of $x^5y^{10}z^5w^5$ in the expansion of $(x+7y+3z+w)^{25}$

By multinomial theorem the general term of the expansion of $(x_1+x_2+\cdots+x_r)^n$ is $P(n;q_1,q_2,\ldots).x_1^{q_1}x_2^{q_2}\ldots x_r^{q_r}$

Given expression $(x+7y+3z+w)^{25}$

It's general term is $P(n;q_1,q_2,q_3,q_4)\,x_1^{q_1}x_2^{q_2}x_3^{q_3}x_4^{q_4}$

$= P(25;q_1,q_2,q_3,q_4)\,x^{q_1}.(7y)^{q_2}.(3z)^{q_3}.(w)^{q_4}$

$= \left[\dfrac{25!}{q_1!q_2!q_3!q_4!}\right] 7^{q_2}.3^{q_3}.x^{q_1}y^{q_2}z^{q_3}w^{q_4}$

To find coefficient of $x^5y^{10}z^5w^5$

Here $q_1=5$ $q_2=10$ $q_3=5$, $q_4=5$

$\therefore$ The required coefficient of $x^5y^{10}z^5w^5$ is

$\dfrac{25!}{5!\,10!\,5!\,5!}\,7^{10}.3^5$

9] Find the coefficient of $x^{11}y^4z$ in the expansion of $(2x^3+3xy^2+z^2)^6$

Given expression $(2x^3+3xy^2+z^2)^6$

It's general term is $P(n;q_1,q_2,q_3)\,x_1^{q_1}x_2^{q_2}x_3^{q_3}$

$= P(6;q_1,q_2,q_3)(2x^3)^{q_1}(3xy^2)^{q_2}(z^2)^{q_3}$

$= \dfrac{6!}{q_1!\,q_2!\,q_3!}\,2^{q_1}3^{q_2}.x^{3q_1}x^{q_2}y^{2q_2}z^{2q_3}$

$= \dfrac{6!}{q_1!q_2!q_3!}\,2^{q_1}3^{q_2}.x^{3q_1+q_2}y^{2q_2}z^{2q_3}$

Here $3q_1+q_2=11$ $\quad 2q_2=4$ $\quad q_3=1$

$3q_1=9$ $\quad q_2=2$

$q_1=3$

$\therefore$ Required coefficient of $x^{11}y^4z$ is

$$\frac{6!}{3!\,2!\,1!}(2^3 \cdot 3^2) = 60 \times 8 \times 9 = 4320$$

Q) Determine the no. of terms of the expansion $(2-7y+3z-w)^{25}$

Sol The no. of terms in the expansion of $(x_1+x_2+\cdots x_r)^n$

is $c(n-1+r, r)$

$n=25 \quad r=4$

$$^{25-1+4}C_4 = {}^{28}C_4$$

$= 20475$ no. of terms

Principle of Inclusion and Exclusion

Theorem: Let A and B be any two subsets of the universal set U then $|A\cup B| = |A| + |B| - |A\cap B|$

$|\,|\rightarrow$ no. of elements.

2. If A, B and C are any three sets and U is the universal set then ~~$|A\cup B|$~~.

$|A\cup B\cup C| = |A| + |B| + |C| - |A\cap B| - |B\cap C| - |C\cap A| + |A\cap B\cap C|$

Note:

$|\bar{A}\cap\bar{B}\cap\bar{C}| = |\overline{A\cup B\cup C}|$ (De' Morgan's Law)

$= U - |A\cup B\cup C|$

3. Let A_1, A_2, A_3, A_4 be any 4 subsets of the universal set U then

$$|A_1 \cup A_2 \cup A_3 \cup A_4| = |A_1| + |A_2| + |A_3| + |A_4| - |A_1 \cap A_2|$$

$$- |A_2 \cap A_3| - |A_3 \cap A_4| - |A_4 \cap A_1| - |A_1 \cap A_3| - |A_2 \cap A_4|$$

$$+ |A_1 \cap A_2 \cap A_3| + |A_2 \cap A_3 \cap A_4| + |A_3 \cap A_4 \cap A_1| +$$

$$|A_4 \cap A_1 \cap A_2| - |A_1 \cap A_2 \cap A_3 \cap A_4|$$

Q] A sample of 80 people revealed that 25 like movies and 60 like TV programs find the no. of people who like both.

Let A_1 be the set of people who like movies

Let A_2 be the set of people who like TV programs

$|A_1| = 25$, $|A_2| = 60$

$|A_1 \cup A_2| = 80$

we know that

Using principle of Inclusion and Exclusion

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$80 = 25 + 60 - |A \cap B|$$

$$|A \cap B| = 5$$

$\therefore$ No. of people who like both is 5

Q] From a group of 10 professors how many ways can a committee of 5 members be formed so that atleast one professor A and one professor B were included

Let A_1 be the set of committees with professor A included.

$$|A_1| = {}^9C_4$$

Let A_2 be the set of committees with professor B included

$$|A_2| = {}^9C_4$$

Set of committees with Professor A and professor B both included is

$$|A_1 \cap A_2| = {}^8C_3$$

Using the principle of Inclusion and Exclusion

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

$$= {}^9C_4 + {}^9C_4 - {}^8C_3$$

$$= 252 - 56$$

$$= 196$$

Q] In a language survey of students it is found that 80 students know English, 60 know French, 50 know German, 30 know english & french, 20 know french and German, 15 know english and German and 10 know all the 3 languages. How many students know

(i) atleast one language

(ii) english only

(iii) students who know french and one but not both out of English and German

$|A_1| = 80$

$|A_2| = 60$

$|A_3| = 50$

$|A_1 \cap A_2| = 30$ $\quad$ $|A_1 \cap A_2 \cap A_3| = 10$

$|A_2 \cap A_3| = 20$

$|A_3 \cap A_1| = 15$

(i) $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A \cap B| - |A \cap C| - |B \cap C)$

$+ |A_1 \cap A_2 \cap A_3|$

$= 80 + 60 + 50 - 30 - 20 - 15 + 10$

$= 135$

(ii) $= |A_1| - |A_1 \cap A_2| - |A_1 \cap A_3| + |A_1 \cap A_2 \cap A_3|$

$= 80 - 30 - 15 + 10$

$= 35 + 10$

~~Let~~ $= 45$

(iii) Let $|S_1| = |E \cap F| = \cancel{80} + \cancel{B/B} = 45$

$|S_2| = |F \cap G| = 30$

$|S_1 \cap S_2| = |F \cap G \cap E| - 10$

$|S_1 \cup S_2| = 30 + 20 - 10$

$= 40$

$\therefore$ Ans $= ||S_1 \cup S_2| - |S_1 \cap S_2||$

$= |40 - 10|$

$= 30$ (No. of students who know French and one but not both) ~~to~~ languages ~~(English~~

$\therefore$ No of students who know French and one but not both languages (English and German) = 30

Note :

$[x]$ = floor of x

= greatest integer $\leq x$

Ex : ~~[4.9~~ $\lfloor 5.9 \rfloor = 5$

$\lfloor 5.1 \rfloor = 5$

$\lfloor 7.3 \rfloor = 7$

Q] Find the no. of integers between 1 and 250 which are divisible by any of the integers 2, 3, 5, 7 and hence find the no. of integers between 1 and 250 which are not divisibly by 2, 3, 5 or 7

Sol Let U be the set of integers between 1 and 250

Let A_1, A_2, A_3, A_4 be the set of integers of U divisible by 2, 3, 5, 7 respectively

Let $A_1 \cap A_2$ be the set of all integers which are divisibleby 2 and 3 i.e LCM of 2 and 3 $\Rightarrow$ no. divisible by 6

Let $A_1 \cap A_3$ be the set of all integers which are divisible by 2 and 5 i.e LCM of 2 and 5 $\Rightarrow$ no divisible by 10

Let $A_1 \cap A_4$ be the set of all integers which are divisible by 2 and 7 i.e LCM of 2 and 7 $\Rightarrow$ no. divisibly by 14

Let $A_2 \cap A_3$ be the set of all integers which are divisible by 3 and 5 i.e LCM of 3 and 5 $\Rightarrow$ no. divisible by 15

Let $A_2 \cap A_4$ be the set of ~~b~~ all integers divible by both 3 and 7 $\Rightarrow$ by 21

Let $A_3 \cap A_4$ be the set of all integers divisible by both 5 and 7 $\Rightarrow$ by 35

Let $A_1 \cap A_2 \cap A_3$ be the set of no.s from U which are divisible by 2, 3, 5 $\Rightarrow$ by 30

Let $A_1 \cap A_2 \cap A_4$ be the set of no.s from U which are divisible by 2, 3, 7 $\Rightarrow$ by 42

Let $A_1 \cap A_3 \cap A_4$ be the set of no.s from U which are divisible by 2, 5, 7 $\Rightarrow$ by 70

Let $A_2 \cap A_3 \cap A_4$ be the set of no.s from U which are divisible by 3, 5, 7 $\Rightarrow$ by 105

Let $A_1 \cap A_2 \cap A_3 \cap A_4$ be the set of no.'s from U which are divisible by 2, 3, 5, 7 $\Rightarrow$ by 210

$|A_1| = \left\lfloor \frac{250}{2} \right\rfloor = 125$

$|A_2| = \left\lfloor \frac{250}{3} \right\rfloor = 83$

$|A_3| = \left\lfloor \frac{250}{5} \right\rfloor = 50$

$|A_4| = \left\lfloor \frac{250}{7} \right\rfloor = 35$

$|A_1 \cap A_2| = \left\lfloor \frac{250}{6} \right\rfloor = 41$

$|A_1 \cap A_3| = \left\lfloor \frac{250}{10} \right\rfloor = 25$

$|A_1 \cap A_4| = \left\lfloor \frac{250}{14} \right\rfloor = 17$

$|A_2 \cap A_3| = \left\lfloor \frac{250}{15} \right\rfloor = 16$

$|A_2 \cap A_4| = \left\lfloor \frac{250}{21} \right\rfloor = 11$

$|A_3 \cap A_4| = \left\lfloor \frac{250}{35} \right\rfloor = 7$

$|A_1 \cap A_2 \cap A_3| = \left\lfloor \frac{250}{30} \right\rfloor = 8$

$|A_1 \cap A_2 \cap A_4| = \left\lfloor \frac{250}{42} \right\rfloor = 5$

$|A_1 \cap A_3 \cap A_4| = \left\lfloor \frac{250}{70} \right\rfloor = 3$

$|A_2 \cap A_3 \cap A_4| = 2$

$|A_1 \cap A_2 \cap A_3 \cap A_4| = 1$

The no. of elements between 1 and 250 divisible by 2, 3, 5 or 7 is $|A_1 \cup A_2 \cup A_3 \cup A_4|$

$|A_1 \cup A_2 \cup A_3 \cup A_4| = |A_1| + |A_2| + |A_3| + |A_4| - |A_1 \cap A_2| - |A_1 \cap A_3|$

$|A_1 \cap A_4| - |A_2 \cap A_3| - |A_2 \cap A_4| - |A_3 \cap A_4| + |A_1 \cap A_2 \cap A_3|$

$+ |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4| -$

$|A_1 \cap A_2 \cap A_3 \cap A_4|$

$= 125 + 83 + 50 + 35 - 41 - 25 - 17 - 16 - 11 - 7$

$+ 8 + 5 + 3 + 2 - 1$

$= 311 - 118$

$= 193$

∴ The no. of elements which are not divisible by 2, 3, 5, 7 is $250 - 193$

$= 57$

Pigeon Hole Principle:

If k pigeons are assigned to n pigeon holes then one of the pigeon hole must contain atleast $\left\lfloor \frac{k-1}{n} \right\rfloor + 1$ pigeons

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Q] Prove that if 30 people are selected then we may chose a subset of 5 so that all five were born on the same day of the week

Sol We assign each day of the week to a person in which he/she were born.

We have 30 people and 7 days in a week.

Using the principle of Pigeon Hole principle i.e

if k pigeons are assigned to n pigeon holes then one of the pigeon hole must contain atleast $\left\lfloor \frac{k-1}{n} \right\rfloor + 1$ pigeons.

Here $k = 30$
$n = 7$

$\therefore \left\lfloor \frac{30-1}{7} \right\rfloor + 1 = \left\lfloor \frac{29}{7} \right\rfloor + 1 = \lfloor 4.14 \rfloor + 1 = 4+1 = 5$

$\therefore$ 5 people ~~will~~ be ~~sharing~~ were born on the same day of the week.