Unit-3 Algebraic structures

→ **Binary operation :-**

→ Let G be a non empty set and a mapping $f: G \times G \to G$ is called a binary operation on G i.e. $G \times G$ is set of all ordered pairs such that $a, b \in G$

→ The symbols $+, \cdot, \circ, *$ etc are used to denote binary operations on a given set.

Algebraic structure :-

→ A non empty set 'G' together with the binary operation $*$ is called an algebraic structure and it is denoted by $(G, *)$.

→ An algebraic structure $(G, *)$ is said to be a group if it satisfies the following properties

i) Closure : $\forall\ a, b \in G \Rightarrow a \times b \in G$.

ii) Associative : $\forall a, b, c \in G \Rightarrow (a * b) * c = a * (b * c)$

iii) Existence of Identity : $\forall\ a \in G\ \exists\ e \in G$

$$a * e = e * a = a$$

Here 'e' is called identity element.

iv) Existence of Inverse : $\forall\ a, b \in G$ such that

$$a * b = b * a = e$$

Note :-

→ If a non empty set G equipped with a binary operation '$*$' i.e. $(G, *)$ is said to be groupoid if it satisfies only closure property.

→ If the algebraic structure $(G,*)$ satisfies closure, associative properties then it is said to be semi-group.

→ If the algebraic structure $(G,*)$ satisfies closure, associative, identity properties then it is said to be monoid.

Abelian Group :-

→ A group $(G,*)$ is said to be an abelian group if it satisfies commutative property i.e, $\forall a,b \in G$ $a*b = b*a$.

Remarks :-

i) "0" is called additive identity Eg:- $2+0 = 2$

ii) "1" is called multiplicative identity Eg:- $-2 \times 1 = -2$

iii) Under addition, the identity element is '0' and inverse of A is $-A$ i.e, $e=0$, $a+(-a) = 0e$

iv) Under multiplication, $e=1$ and inverse is $a \times \frac{1}{a} = 1e$

Examples :-

a) The algebraic structures $(Z,+)$ $(R,+)$, $(R,\cdot)$ a group.

→ $(N,+)$ is not a group since $0 \notin N$ but $(N,+)$ is a semi-group.

→ $(Z,\cdot)$ is also not a group since multiplicative inverse does not exist i.e $2 \times \frac{1}{2} = 1e \Rightarrow \frac{1}{2} \notin Z$. but $(N,\cdot)$ is having an identity element i.e. $e=1$

Hence $(N,\cdot)$ satisfies closure, associative, identity. $(N,\cdot)$ is a monoid.

Q) Show that set $G = \{1, w, w^2\}$ is an abelian group w.r.t to multiplication using composition table.

Sol Given $G = \{1, \omega, \omega^2\}$

C.T :-

$(\cdot)$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

W.k.t $\omega^3 = 1$, $\omega^4 = \omega$

ω is a complex $\omega = \frac{-1+i\sqrt{3}}{2}$

Closure; since all the entries in the table are in G $\therefore$ closure property is satisfied.

Associative : W.k.t multiplication of complex numbers is associative. Hence satisfied.

Existence of Identity :-

since each row contains identity element $e = 1$

Existence of Inverse :-

$1 \cdot 1 = 1 \Rightarrow (1)^{-1} = 1$

$\omega \cdot \omega^2 = 1 \Rightarrow (\omega)^{-1} = \omega^2$

$\omega^2 \cdot \omega = 1 \Rightarrow (\omega^2)^{-1} = \omega$

since every element of G has inverse $\therefore$ Inverse is satisfied. $\therefore (G, \cdot)$ is a group.

Abelian :-

Since $1^{st}, 2^{nd}, 3^{rd}$ rows coincides with $1^{st}, 2^{nd}, 3^{rd}$ columns $\therefore$ commutative property is satisfied.

Thus $(G, \cdot)$ is an abelian group under multiplication using C.T.

Q) Show that $G = \{1, -1, i, -i\}$ is an abelian group under multiplication using C.T.

Sol Given $G = \{1, -1, i, -i\}$.

C.T.

$(\cdot)$	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

Closure : since all entries in table satisfied.

Associative : Mul of complex numbers is possible. Hence associative.

Existence of Identity :- since each row contains $e = 1$. Hence satisfied.

Existence of Inverse : since every element of G has inverse. Hence satisfied

Abelian :- Since rows and columns coincide with each other. Hence satisfied.

Q) $G = \{0,1,2,3,4\}$ is an abelian group wrt addition and verify for multiplication, modulo,

Sol Given $G = \{0,1,2,3,4\}$.

Addition 5 :-

→ To find $(a+b) \in G$ it is enough to find the remainder of $(a+b)$ in G whether it is divisible by 5.

C.T :-

$(+_5)$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$1 +_5 4 = 0$

closure : All entries are in the table G.

Associative : Generally addition modulo is always associative.

Identity : since each row has additive identity $0 \Rightarrow e = 0$

Inverse : clearly $0^{-1} = 0$, $1^{-1} = 4$, $2^{-1} = 3$, $3^{-1} = 2$, $4^{-1} = 1$

$\therefore$ Each element in G has inverse.

Abelian : since rows and columns coincide. It satisfies.

Multiplication Modulo :-

$\cdot_5$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

i) closure :- It satisfies the closure property.

ii) Associative :- 5 satisfies associative property.

$$1 \times_5 (2 \times_5 4) = (1 \times_5 2) \times_5 4$$

iii) Existence of Identity :- We know that multiplicative identity is "1". But in the table 1st row doesn't '1'. It doesn't satisfy identity property.

Hence $(G, \times_5)$ is not a group but it satisfies closure and associative properties. Thus $(G, \times 5)$ is a semi-group.

8) A binary operation $*$ is defined on 'z' as $a*b = a+b+1 \ \forall \ a,b, \in Z$ then show that $(z,*)$ is a Abelian group.

Sol Given the binary operation $*$ defined as z as $a*b = a+b+1 \ \forall \ a,b \in Z$.

i) closure :- Let $a,b \in Z$ then $a*b = a+b+1 \in Z$

$$a*b \in Z$$

$\therefore$ satisfied.

ii) Associative :- Let $a,b,c \in Z$

Now $(a*b)*c = a*(b*c)$

$$(a+b+1)*c \cdot a*(b+c+1)$$

$$a+b+1+c+1 \quad a+b+c+1+1$$

$$a+b+c+2 \quad a+b+c+2 \quad \therefore \text{satisfied}$$

$$\therefore (a*b)*c = a*(b*c)$$

iii) Identity :- Let $a \in Z \ \exists \ a*e = a$ where 'e' is the identity element.

$$a*e = a+e+1 = a$$

$$e = -1 \in Z \quad \therefore e \text{ exists}$$

iv) Inverse :- Let $a,b \in Z \ \exists \ a*b = e$ where 'b' is the inverse of 'a'.

$a*b = -1$, $a+b+1 = -1 \Rightarrow a+b = -2 \Rightarrow b = -2-a$ (or) $b = -(2+$

$\therefore (G,*)$ is a group.

vi) Abelian group :- Let $a,b \in Z \ \exists \ a*b = a*b+1$

$$b+a+1 = b*a$$

$$\therefore a*b = b*a \quad \text{(commutative)}$$

$\therefore (G,*)$ is an abelian group.

Q) Show that the set $G = \{x / x = 2^a . 3^b$ for $a, b \in Z\}$

It is a group under multiplication.

Sol closure :- Let $x, y \in G$

Let $x = 2^a . 3^b$ and $y = 2^p . 3^q$ for $a, b, p, q \in Z$

$xy = 2^a . 3^b . 2^p . 3^q = 2^{a+p} . 3^{b+q}$

$\therefore$ for $(x, y) \in G \Rightarrow x \cdot y \in G$ $\therefore$ closure satisfied.

ii) Associative :- Let $x, y, z \in G \Rightarrow x(yz) = (xy)z$

Take $x = 2^a 3^b$, $y = 2^p . 3^q$, $z = 2^r . 3^s$

$(2^a . 3^b * 2^p . 3^q) \times 2^r . 3^s = 2^a . 3^b (2^p . 3^q * 2^r . 3^s)$

$2^{a+p+r} 3^{b+q+s} = 2^{a+p+r} . 3^{b+q+s}$

$(a * b) * c = a * (b * c)$ $\therefore$ Associative satisfied.

iii) Identity :- Let $x \in G \rightarrow xe = x$

Here $x = 2^a . 3^b$ and $e = 2^m . 3^n$ for $a, b, m, n \in Z$

$(2^a . 3^b)(2^m . 3^n) = 2^a . 3^b \Rightarrow xe = x$

$2^{a+m} . 3^{b+n} = 2^a . 3^b$

i.e, $a + m = a \Rightarrow m = 0$ $b + n = b \Rightarrow n = 0$ $\therefore e = 2^0 . 3^0 = 1$

since $0 \in Z$ $\therefore e = 2^0 . 3^0$ is identity element in G.

iv) Inverse : Let $x, y, \in G$ $xy = e$ where $y = x$

Let $x = 2^a 3^b$, $y = 2^p 3^q$, $a, b, p, q \in Z$

$2^a . 3^b . 2^p . 3^q = 2^0 3^0$ $p = -a$ $q = -b$

$2^{a+p} . 3^{b+q} = 2^0 3^0$ $\therefore y = 2^p . 3^q$

$a + p = 0$ $b + q = 0$ $y = 2^{-a} 3^{-b} \in G$ where $-a, -b \in Z$

Q) A binary operation $*$ is defined on 'Z' as $a * b = a + b - ab$ $\forall a, b \in Z$ then show that $(Z, *)$ is a group.

Sol $a * b = a + b - ab$ $\forall a, b \in Z$

i) closure :- Let $a, b \in Z$ then $a * b = a + b \in Z$

$a * b \in Z$

$\therefore$ closure satisfied.

ii) Associative :- Let $a,b,c \in Z$ then $(a*b)*c = a*(b*c)$

$(a+b-ab)*c = a*(b*c)$.

$(a+b-ab)*c = a*(b+c-bc)$

$(a+b-ab)+c-(a+b-ab)c = a+(b+c-bc)-a(b+c-bc)$

$a+b-ab+c-ac+bc+abc = a+b+c-ab-ac-bc+abc$.

$\therefore (a*b)*c = a*(b*c)$ $\therefore$ Associative satisfied.

iii) Identity :- Let $a \in Z \Rightarrow a*e = a$

$a*e = a+e-ae = a \Rightarrow e(1-a)=0 \Rightarrow e=0$

iv) Inverse :- $a,b \in Z \; \exists \; a*b = e = b*a$.

$a*b = a+b-ab = 0 \Rightarrow b-ab = -a \Rightarrow b(1-a) = -a$ $\quad b = -a/1-a$

$\therefore (Z,*)$ is a group.

v) Abelian group :- let $a,b \in Z \; \exists \; a*b = a+b-ab$

$b*a = b+a-ba$ $\quad \therefore a*b = b*a$ (commutative) $\therefore$ Satisfied.

Q) Let G be set of positive rational numbers and binary operation "o" is defined on "G" as $a_o b = ab/3$ $\forall a,b \in G$ then show (G,o) is an abelian group.

Sol i) closure :- Let $a,b \in G$, $a_o b = ab/3 \in G$

a and b are +ve rational numbers $a_o b = ab/3$ is also a positive rational number.

ii) Associative :- $\forall a,b,c \in G$ $(a_o b)oc = a_o(b_o c)$

$= ab/3 \, o \, c = a \, o \, bc/3 \Rightarrow \frac{abc}{a} = \frac{abc}{a}$ $\therefore$ satisfied

iii) Identity :- $a_o e = e_o a = a \Rightarrow \frac{\cancel{a}e}{3} = \cancel{a} \Rightarrow e = 3$

iv) Inverse :- $a_o b = b_o a = e$

$a_o b = e \Rightarrow ab/3 = 3 \Rightarrow b = 9/a$

v) Abelian group :- $a,b \in G$, $a_o b = b_o a$ (commutative)

$= a_o b = ab/3$

$= b_o a = ab/3$ $\quad \therefore$ Abelian group satisfied

$= a_o b = b_o a$

Q) verify that $G = \{-1, 0, 1\}$ is a group or not w.r.t to addition using C.T.

Sol C.T :-

+	-1	0	1
-1	-2	-1	0
0	-1	0	1
1	0	1	2

i) closure :- since the entries in the table are not in G. $\therefore$ closure is not satisfied. Hence (G,+) not a group.

Q) Every element in the group $(G,\cdot)$ has its own inverse then show that G is an abelian group. Is converse true?.

Sol Given $(G,\cdot)$ is a group.

Let us suppose that every element of $(G,\cdot)$ has its own inverse i.e, for $a, b \in G \Rightarrow a = a^{-1}, b = b^{-1}$.

Now we have to prove that G is abelian group which means we have to prove commutative property. i.e, $a,b \in G \Rightarrow$ ~~a, b~~ $ab \in G$ (by closure)

Now $ab = (ab)^{-1}$ ($\because$ Every element has its own inverse)

$\Rightarrow ab = b^{-1}a^{-1}$

$= ab = ba$

$\therefore$ G is abelian.

But the converse is not true, because if G is an abelian group it may not contain every element has its own inverse.

Let (R,+) is an abelian group but no element of R (except 0) has its own inverse.

Q) In a group G, for $a,b \in G$ such that $(ab)^2 = a^2 b^2$ if and only if G is abelian.

Sol Let $a,b \in G$

suppose that $(ab)^2 = a^2b^2$ to prove G is abelian

Now $(ab)^2 = a^2b^2$

$(ab)(ab) = (aa)(bb)$

$a(ba)b = a(ab)b$

using cancellation laws

$\therefore ba = ab \Rightarrow ab = ba \quad \therefore G$ is abelian.

conversely, suppose that 'G' is abelian

for $a,b \in G \Rightarrow ab = ba$ — ①

To prove $(ab)^2 = a^2b^2$

Now L.H.S $= (ab)^2$

$= (ab)(ab)$

$= a(ab)b = (aa)(bb) = a^2b^2$

Hence proved

(search for proof)

→ In a group (G,o) inverse of every element is unique

→ In a group (G,o) identity of element is unique.

→ Let G be a group then for $a,b,c \in G$ $ab = ac \Rightarrow b = c$ (Left cancellation law) $ba = ca \Rightarrow b = c$ (Right cancellation law).

→ Let $(G,\cdot)$ be a group, for any $a \in G$ prove that $(a^{-1})^{-1} = a$

→ Let $(G,\cdot)$ be a group, for any $a,b \in G$ prove that $(ab)^{-1} = b^{-1}a^{-1}$

→ If a,b any 2 elements of $(G,\cdot)$ which commute then prove i) a^{-1}, b ii) b^{-1}, a iii) a^{-1}, b^{-1}

Order of a group :-

→ Let G be a group then no of elements in G is called order of G denoted by $O(G)$.

Ex :- Let $G = \{1, \omega, \omega^2\}$ is a group under multiplication then $O(G) = 3$

Finite & Infinite group :-

→ If $O(G)$ is finite then G is a finite group else G is an infinite group.

Order of element in group :-

→ Let 'G' be a group and 'a' be any element in G then if there exist a least positive integer 'n' such that $a^n = e$. Here the order of element 'a' is 'n'.

Ex :- Let $G = (1, \omega, \omega^2)$ be a group under multiplication

Now, $(1)^1 = 1 \Rightarrow O(1) = 1$

$(\omega)^2 = 1 \Rightarrow O(\omega) = 3$

$(\omega^2)^3 = 1 \Rightarrow O(\omega^2) = 3$

Let $G = \{1, -1, i, -i\}$ be a group under multiplication

Now, $(1)^1 = 1 = e \Rightarrow O(1) = 1$

$(-1)^2 = 1 \Rightarrow O(-1) = 2$

$(i)^4 = 1 \Rightarrow O(i) = 4$

$(-i)^4 = 1 \Rightarrow O(-i) = -4$

Let $G = \{a, a^2, a^3, a^4, a^5, a^6 = e\}$

Now, $(a)^6 = e \Rightarrow O(a) = 6$

$(a^2)^3 = e \Rightarrow O(a^2) = 3$ $\quad$ $(a^5)^6 = e \Rightarrow O(a^5) = 6$

$(a^3)^2 = e \Rightarrow O(a^3) = 2$ $\quad$ $(a^6)^1 = e \Rightarrow O(a^6) = 1$

$(a^4)^3 = e \Rightarrow O(a^4) = 3$

Sub Group :-

→ Let $(G, *)$ be any group and 'H' is a nonempty subset of G. $(H, *)$ is said to be a subgroup of $(G, *)$ if $(H, *)$ is a group of its own right.

Ex :- Let $G = \{1, -1, i, -i\}$ is a group under multiplication then a subset $H_1 = \{i, -i\}$ is not a subgroup of G since H_1 is not a group under multiplication because closure property fails and $e = 1 \notin H_1$.

→ A subset $H_2 = \{1, -1\}$ is a subgroup of 'G' since H_2 satisfies closure, associative, identity and inverse group.

Criteria for a subgroup :- (Two-step theorem).

→ A non empty subset 'H' of a group 'G' is a subgroup of 'G' if and only if.

i) $a \in H, b \in H \Rightarrow a * b \in H$

ii) $a \in H \Rightarrow a^{-1} \in H$.

Proof :- suppose a subset 'H' of a group 'G' is a subgroup of 'G'.

Now we have to prove the two conditions i, ii.

→ since H is a subgroup implies H is also a group which means that H satisfies closure, associative, identity and inverse properties.

By closure :- $a \in H, b \in H \Rightarrow a * b \in H$

By inverse :- $a \in H \Rightarrow a^{-1} \in H$.

conversely suppose that, To prove 'H' is a subgroup of 'G'.

From condition i) closure property is satisfied in 'H'.

Since G is a subset of G where G is already a group ∴ The elements of H also satisfies associative property.

Identity :- From condition ii) i.e, Let $a \in H, a^{-1} \in H \Rightarrow$ $a * a^{-1} \in H$ (by condition (i))

$= e \in H$ ∴ Identity element exists in H

clearly by condition ii) a^{-1} exists in H. Hence H satisfies all the four properties of a group G.

Thus H is a subgroup of G.

One-step theorem :-

→ The necessary and sufficient condition for a non empty subset 'H' of a group $(G, *)$ is a subgroup of G is $a \in H, b \in H \Rightarrow a * b^{-1} \in H$

Group Homomorphism :-

→ A mapping $f: (G, *) \to (G', o)$ is said to be a homomorphism if $f(a*b) = f(a) o f(b)$ $\forall a, b \in G$.

Here $(G, *)$ and (G', o) be any two groups.

Monomorphism :-

→ A homomorphism $f: G \to G'$ is said to be monomorphism. If f is 1-1 i.e, A 1-1, homomorphism is called monomorphism.

Ep

→ A homomorphism $f: G \to G'$ is said to be epimorphism if f is onto function i.e, A onto homomorphism is called epi-morphosim.

· Endomorphism :-

→ A homomorphism $f: G \to G$ i.e, itself is called an endomorphism.

· Isomorphism :-

→ A homomorphism $f: G \to G'$ is called an isomorphi[sm] if f is both 1-1 and onto i.e, bijective.

· Automorphism :-

→ An isomorphism $f: G \to G$ is called an automorphism.

Q) If $f: (R,+) \to (R^+, \cdot)$ is defined by $f(x) = e^x$ then show that f is isomorphism.

Sol Given $f(x) = e^x$

To verify f is isomorphism we have to check homomorphism, 1-1, onto.

Homomorphism : consider, $f(x+y) = e^{x+y}$ (by def)

$$= e^x \cdot e^y$$

$$= f(x) \cdot f(y)$$

$\therefore$ f is homomorphism.

1-1 :- Let $f(x_1) = f(x_2)$

$$e^{x_1} = e^{x_2}$$

$$x_1 = x_2$$ i.e, f is 1-1

Onto :- Let c belongs to R^+ $\exists$ $\ln c \in R$ then take

$f(\ln c) = e^{\ln c} = c$

$\therefore$ $c = f(\ln c)$ for $\ln c \in R$

$\therefore$ f is onto.

$\therefore$ f is bijective homomorphism i.e, Isomorphism

Q) If $f : (R^+, \cdot) \to (R, +)$ defined by $f(x) = \log x$ then show that f is homomorphism.

Sol $f(x * y) = \log(x * y)$

$= \log x + \log y$

$= f(x) + f(y)$

$\therefore$ f is homomorphism

Q) verify $f : Z \to Z$ defined by $f(x) = -x \ \forall x \in Z$ then show that f is automorphism.

Unit-4 Elementary combinatorics

Basics of counting :-

→ There are two fundamental principles of all counting problems.

i) Sum Rule

ii) Product Rule

i) Sum Rule :-

→ If an event is occurring in 'm' ways and another event can occur in 'n' ways and if these two events cannot occur simultaneously. Then one of the two events can occur in 'm+n' ways.

Note :-

→ The sum rule can also be formulated in terms of choices.

Ex-1 :- If there are 5 boys and 4 girls in a class then there are 5+4 = '9' ways of selecting one student (either boy or a girl) as a CR.

Ex-2 :- The student can choose a computer project from one of the three lists contain 23,18,10 possible projects then the no of projects can be chosen are 23+18+10 = 51 ways.

Product Rule :-

→ If an event can occur in 'm' ways and a second event can occur in 'n' ways and if no of ways the second event occurs does not depend how the first event occurs. Then the two events can occur simultaneously is mn ways.

Ex :- If 2 dice are rolled then the first die can fall in 6 ways and the second die can fall in 6 ways. Hence there are 36 outcomes when two dice are rolled.

Ex :- The chairs in the auditorium are to be labeled with a letter and a +ve integer not exceeding 100 then the largest no of chairs that can be labeled differently in $26 \times 100 = 2600$

Ex :- 3 persons enter into a car where there are 5 seats. In how many ways can they take up their chair?

Sol No of persons = 3

No of seats = 5

1st person can sit in any one of the five seats. So the no of ways for 1st person = 5

∴ Remaining seats = 4

2nd person can sit in one of the four seats. so the no of ways for 2nd person = 4.

∴ Remaining seats = 3

3rd person can sit in any one of the three seats the no of ways for 3rd person = 3

∴ By product rule, the no of ways in which all the 3 persons can take up their seats =

$5 \times 4 \times 3 = 60$ ways.

→ In set theory the above two rules can be written as

Sum Rule - If a, b are disjoint sets then order of $|A+B| = |A|+|B|$

Product Rule :- If $a \times b$ is cartesian product of sets A, B then order of $|a \times b| = |a| \cdot |b|$

Factorial notation :-

→ The product of first n natural numbers is denoted by 'n!' i.e, $n! = n(n-1)(n-2) \cdots \times 3 \times 2 \times 1$

Ex :- $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1$

Note :- $n! = n(n-1)!$ & $6! = 6 \times 5!$

Thus the above definition n! is recursive.

Ex :- Find n if $(n+1)! = 12(n-1)!$

Sol $\frac{(n+1)!}{(n-1)!} = 12 \Rightarrow \frac{(n+1)\,n\,\cancel{(n-1)!}}{\cancel{(n-1)!}} = 12$

$n^2+n-12=0$

$n^2+4n-3n-12=0$

$n(n+4)-3(n+4)=0$

$n = 3, -4$

since $n \neq -4$

$\underline{n=3}$

Permutations & combinations :-

Permutation :-

→ A permutation of 'n' objects taken 'r' at a time is called an ~~un~~ordered selection of 'r' $(r \leq n)$ out of 'n' objects.

Combination :-

→ A combination of 'n' objects taken r at a time is called is called an unordered selection of 'r' $(r \geq n)$ out of 'n' objects.

Note :-

→ The order of the things is not considered in combinations whereas the order of the things is considered in permutations.

Formulae :-

→ The no of perm' of 'n' distinct objects taken r at a time is, $nP_r = \frac{n!}{(n-r)!}$

→ The no of comb' of 'n' distinct objects taken r at a time is $nC_r = \frac{n!}{(n-r)!\,r!}$

Note :-

→ The above two formulae is used for without repetitions.

Results :-

1) $nC_0 = nC_n = 1$

2) $nC_r = nC_{n-r}$

3) $nP_r = nC_r(r!)$

4) $nC_r = nC_s$ $r = s$ or $r+s = n$

→ nC_r is greatest if $r = \frac{n}{2}$ if n is even, $r = \frac{n-1}{2}$ or $\frac{n+1}{2}$ if n is odd.

Ex:- The permutations formed by two objects at a time from set of A, B, C are AB, BA, BC, CB, CA, AC $= (3P_2 = 6)$

Ex :- The combinations formed by two objects at a time from a set A, B, C are AB, BC, CA $(3C_2 = 3)$

Enumeration of permutation & combination (Without repetition):-

→ Let nP_r denotes the number of r permutation without repetition is $nP_r = \frac{n!}{(n-1)!}$

Note :-

→ There are 'n!' permutation of n distinct object

Ex :- compute $P(8,5) = ?$

Sol $P(8,5) = 8P_5$

$= \frac{8!}{(8-5)!} = \frac{8!}{3!} = \underline{6720}$

Ex :- The no of 3 permutations of {a,b,c,d,e} is

$5P_3 = 5P_3 = \frac{5!}{2!} = 60$

Ex :- The no of 5 letter words using set of {a,b,c,d,e} is $5P_5 = 5! = 120.$

Ex :- Compute n, r if $nP_r = 3024$

Sol $P(n,r) = nP_r = 3024$

$= 9 \times 8 \times 7 \times 6$

$= 9P_4$ $\therefore$ $\underline{n=9,\ r=4}$

Q) Find n if $P(n-1,3) : P(n+1,3) = 5:12$

Sol $\frac{P(n-1,3)}{P(n+1,3)} = \frac{5}{12}$ $n = 8$ or $\frac{9}{7}$, $n \neq \frac{9}{7}$ $\therefore$ $\underline{n=8}$

Combinations without repetitions :-

→ Let nC_r denotes the no of odd permutations combinations without repetition is nC_r or $C(n,r)$ is

$nC_r = \frac{n!}{r!(n-r)!}$

Note :-

→ The no of n combinations without repetitions is

$c(n,n) = \frac{n!}{n!\,0!} = 1$

Ex :- If $c(n,r) = 126$ then find n.

Sol Since $c(n,r)$ is a +ve integer we can take

$nC_r = 126 = \frac{9\times7\times8}{4}$

$= 63\times2 = \frac{9\times7\times8\times6}{6\times4}$

$= 9\times7\times2$

Q) Find n if $c(n,3) : c(n-1,4) = 8:5$, find n.

Q) Find n if $c(n,6) = c(n,10)$ then find $c(n,8)$

Sol $c(n,6) = c(n,10)$ $\quad c(n,8) = c(16,8)$

$6+10 = n$ $\quad = \frac{16!}{8!\,8!} = 12870$

$n = 16$

Q) How many ways can a hand of 5 cards be selected from a deck of 52 cards ?

Sol The no of ways of 5 combination of 52 cards is $52C_5$

Q) How many ways can a committee of 5 be chosen from 9 people ?

Sol $c(9,5) = 9C_5$

Q) How many committee's of 6 or more can be chosen from 9 people

Sol

Q) How manys ways can a committee of 4 teachers and 5 students be chosen from 9 teachers and 15 students ?

Sol The no of teachers can be selected in $9C_4$ ways.

The no of students can be selected in $15C_5$ ways.

∴ The committee can be formed in $9C_4 \times 15C_9$ ways.

Enumerations of Permutations and Combinations (With repetitions) :-

→ When repetitions are allowed, no of permutations of n distinct objects taken 'r' at a time is n^r.

Ex :- There are 25 true or false questions on an examination. How many different ways can a student do the examination if he/ she can also choose to leave the answer blank?

Sol $n = 3$, $r = 25$ ∴ 3^{25}.

Ex :- Find the no of different telephone numbers formed by taking 3 digits from 1,2,3,4,5.

Sol $n = 5$, $r = 3$

To form a telephone number with 3 digits from 1,2,3,4,5 is 5^3.

Combinations :-

→ When repetition is allowed then the no of combinations of n distinct objects taken 'r' at a time is $C(n-1+r, r)$.

Note :- $C(n-1+r, r) = (n-1+r, n-1)$

Ex :- The no of 3 combinations of 5 objects with unlimited repetitions are $C(5-1+3, 3) = {}^7C_3$.

Ex:- The no of non-negative integer solutions to $x_1+x_2+x_3+x_4+x_5 = 50$ is $C(5-1+50, 50) = C(54,50)$

$= \frac{54!}{4!\,50!}$

Ex:- The no of ways of placing 10 similar balls in 6 numbered boxes is $C(6-1+10, 10)$

Enumeration of Permutations with Constrained Reptitions :-

→ out of n objects, p objects are exactly alike of first kind, q objects are exactly alike of 2nd kind and r objects are exactly alike of 3rd kind and remaining are distinct. Then the no of permutations of n objects taken r at a time is $\frac{n!}{p!\,q!\,r!}$

Ex:- How many ways are there to arrange the 9 letter word "ALLAHABAD".

Sol: In the given word, the total no of letters = 9.

The word has 4 A's, 2 L's and the remaining letters are distinct.

∴ The no of ways of arranging the word is

$$\frac{9!}{2!\,4!}$$

2. ENGINEERING ⇒ Total = 11 ⇒ $\frac{11!}{3!\,3!\,2!\,2!}$

3. MISSISSIPPI ⇒ Total = 11 ⇒ $\frac{11!}{3!\,4!\,2!}$

Note :-

→ The no of permutations of the word "MISSISIPPI" can be written as {1'M, 4'I, 4'S, 2'P

* Prove that $nC_r + nC_{r-1} = (n+1)C_r$

Sol L'H'S = $nC_r + nC_{r-1}$

$$= \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!(n-(r-1))!}$$

$$= \frac{n!}{(r-1)!(n-r)!}\left[\frac{1}{r} + \frac{1}{n-(r-1)}\right]$$

$$= \frac{n!}{(r-1)!(n-r)!}\left[\frac{n-\cancel{r}+1+\cancel{r}}{r(n-(r-1))}\right]$$

$$= \frac{n!}{(r-1)!(n-r)!} \times \frac{(n+1)}{r(n-(r-1))}$$

$$= \frac{(n+1)!}{r!(n-r+1)!} = (n+1)C_r = R'H'S$$

Q) Prove that $nC_r = (n-1)C_{r-1} + (n-1)C_r$

Sol R'H'S = $(n-1)C_{r-1} + (n-1)C_r$

$$= \frac{(n-1)!}{(r-1)!(n-1-r+1)!} + \frac{(n-1)!}{r!(n-1-r)!}$$

$$= \frac{(n-1)!}{(r-1)!(n-r)!} + \frac{(n-1)!}{r!(n-1-r)!}$$

$$= \frac{(n-1)!}{(r-1)!(n-r-1)!}\left[\frac{1}{n-r} + \frac{1}{r}\right]$$

$$= \frac{(n-1)!}{(r-1)!(n-r-1)!}\left[\frac{r+n-r}{r(n-r)}\right]$$

$$= \frac{n(n-1)!}{r!(n-r)!} = \frac{n!}{r!(n-r)!} = nC_r = L'H'S$$