UNIT-5 GRAPH THEORY

Graph: A graph G is an ordered triple (V, E, ϕ) consists of non-empty set of vertices V, set of edges E and a mapping ϕ from $E \to V$ (unordered pairs of vertices) It is simply denoted by G

Ex: Let $V = \{v_1, v_2, v_3, v_4, v_5\}$

$E = \{e_1, e_2, e_3, e_4\}$

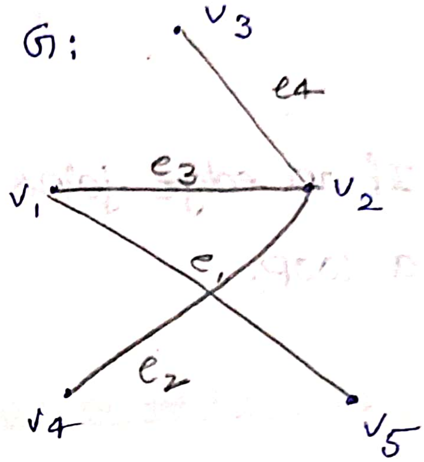
$\phi : E \to V$ defined as $\phi(e_1) = \{v_1, v_5\}$

$\phi(e_2) = \{v_2, v_4\}$

$\phi(e_3) = \{v_1, v_2\}$

$\phi(e_4) = \{v_2, v_3\}$

G:



- Any pair of vertices which are connected by an edge ~~is~~ are called ~~edges and~~ adjacent vertices

  Ex: $v_1, v_2 \to$ adjacent

  $v_1, v_4 \to$ not adjacent

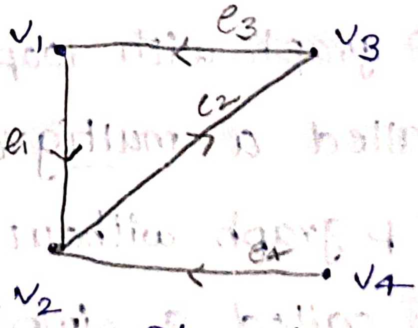
- A Graph $G(V, E)$ which is associated with ordered pair of vertices is called a directed graph or digraph.

Ex: $\phi(e_1) = (v_1, v_2)$

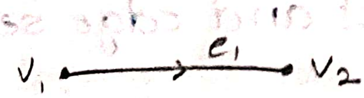
$\phi(e_2) = (v_2, v_3)$

$\phi(e_3) = (v_3, v_1)$

$\phi(e_4) = (v_4, v_2)$



Digraph

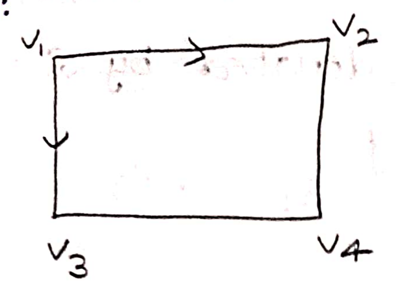


$v_1 \to$ initial vertex

$v_2 \to$ terminal / end vertex

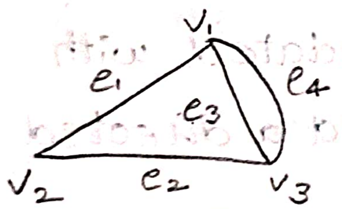
- If some edges are directed and some are undirected in a graph, then graph is called <u>mixed graph</u>.

Ex:



- If an edge joins a vertex to itself, it is called a <u>loop</u>.

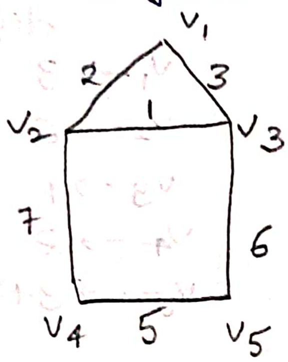
- In a graph if some pair of vertices are joined by more than one edge, such edges are called as <u>parallel edges</u>.



- A graph with loops and parallel edges is called a <u>multigraph</u>.
- A p graph without loops and parrallel edges is called a <u>simple graph</u>.
- A graph is finite vertex set and edge set is called a <u>finite graph</u>
- A graph with only one vertex and no edges is called <u>trivial graph</u>.

- A vertex which is not adjacent to any other vertex is called an <u>isolated vertex</u>.
- A graph with only isolated vertices is called a <u>null graph</u>.
- A graph in which weights are assigned to edges is called a <u>weighted graph</u>.

Ex:

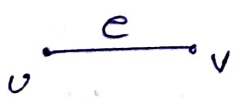


- In a finite graph $|V|$ denotes no. of vertices of G. known as <u>order</u> of G, $|E|$ denotes no. of edges, known as <u>size</u> of G
- A graph obtained by deleting all the loops and parallel edges is called <u>underlying simple graph</u>
- A graph obtained by ignoring directions, then a diagraph G is called <u>underlying undirected graph</u>.

<u>Incidence:</u>

Let $G(V,E)$ be a graph

$e \in E$ and $e = \{u,v\}$



then e is called <u>edge</u>

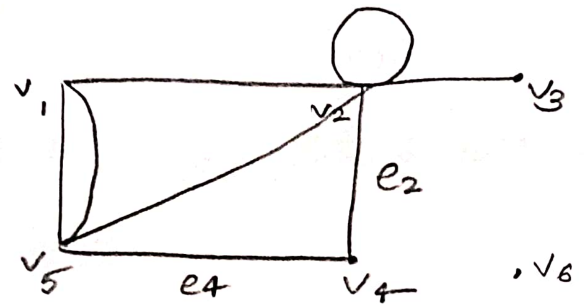
- e is said to be <u>incident</u> on the vertices u and v
- u,v are said to be <u>incident</u> with edge e.

Degree of a vertex:

Let v be a vertex in a graph G, then degree of the vertex v is the no. of edges that are incident with v, here loop is considered twice.

- Degree of the vertex is denoted by $d_G(v)$

Ex:



degree

$v_1 \to 3$

$v_2 \to 6$

$v_3 \to 1$

$v_4 \to 2$

$v_5 \to 3+1 = 4$

- A vertex with degree 1 is called pendent vertex

  Ex: v_3

- A vertex with degree 0 is called isolated vertex

  Ex: v_6

- Two adjacent edges are said to be in series if their common vertex is of degree 2

  Ex: e_2, e_4 are in series

Fundamental theorem of graph theory:

Let G be an undirected graph with |E| edges and |V| = n vertices

- Then $\sum_{i=1}^{n} \deg(v_i) = 2|E|$

- Proof: Let G be an undirected graph with |V| edges. Let v_1, v_2 be the vertices in G. When we sum the degree of all vertices, we consider each edge twice.

- For instance, e is an edge between U and v. Then once we count the edge 'e' for degree of 'u' and again we cound edge e for degree of v.

  ∴ edge e is counted twice.

- Hence this concludes that sum of degrees of all the vertices is twice the no. of edges.

Note:

In any graph, the no. of odd degree vertices is even

Degree Sequence Equation

If $v_1, v_2, \ldots$ are n vertices of G, then the sequence $(d_1, d_2, \ldots d_n)$; where $d_1 = deg(v_1) \ldots$ is the degree sequence of G.

- A sequence is graphic if there is a simple undirected graph with degree sequence.

Q] Is there a graph with degree sequence (1, 3, 3, 3, 5, 6, 6)?

Sol: No, it is not graphic; since in any graph the no. of odd degree vertices is even

→ Degree for directed graph:

- Let G be a directed graph with $e = (u, v)$.
- The edge 'e' is said to be incident out of initial vertex 'u'. The edge 'e' is incident into the vertex 'v'.

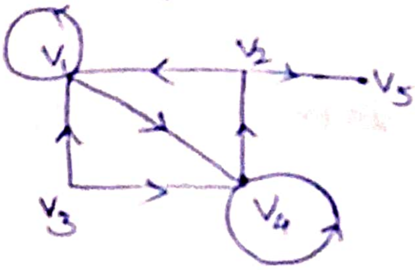
$$u \xrightarrow{e} v$$

- The number of edges that are incident out of the vertex V_i is the 'out degree' of V_i and is denoted by $deg^-_G(v_i)$
- The no. of edges that are incident into the vertex V_i is the in-degree of V_i and is denoted by $deg^+_G(V_i)$

- The sum of out degree and in degree of vertex 'v_i' is the degree of the vertex v_i and is denoted by $deg_G(v_i)$
- In a directed graph G:

$$\boxed{\sum_{i=1}^{n} deg_G^-(v_i) = \sum_{i=1}^{n} deg_G^+(v_i) = |E|}$$

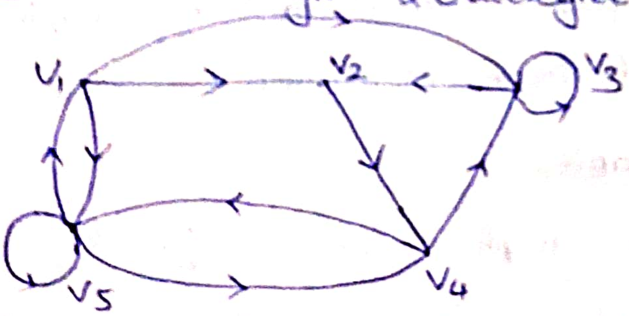
Q] Find indegree & outdegree of the following diagraph.



Sol

Vertex	Indegree [$deg^+(v_i)$]	Outdegree [$deg^-(v_i)$]
v_1	3	2
v_2	1	2
v_3	0	2
v_4	3	2
v_5	1	0
	8	8

Q] Find the indegree & outdegree of following diagraph:



Sol

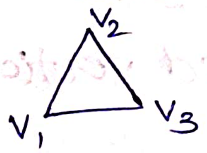
Vertex	Indegree ($deg^+(v_i)$)	Outdegree ($deg^-(v_i)$)
v_1	1	3
v_2	2	1
v_3	3	2
v_4	2	2
v_5	3	3
	11	11

- The minimum of all degrees of the vertices of graph G is denoted by $\delta(G)$
- The maximum of all degrees of the vertices of graph G is denoted by $\Delta(G)$

Note:
$$\boxed{\delta(G) \leq \frac{2|E|}{|V|} \leq \Delta(G)}$$

- If $\delta(G) = \Delta(G) = K$, if each vertex of graph G has degree K, then G is said to be K-regular graph (or) regular graph of degree K.

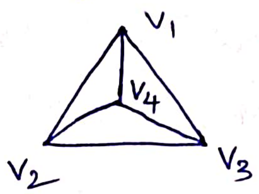
Ex:



$\Rightarrow$ 2-regular graph

- A simple graph in which every pair of distinct vertices are adjacent is called a complete graph.
- If G has 'n' vertices, then the complete graph is denoted by K_n.

Ex:



K_4 - complete graph.

- Path: In a non-directed graph G, a sequence of 0 or more edges of the form $V_0 - V_1 - V_2 - \cdots - V_n$ is called a path from V_0 to V_n

  V_0 is initial vertex and

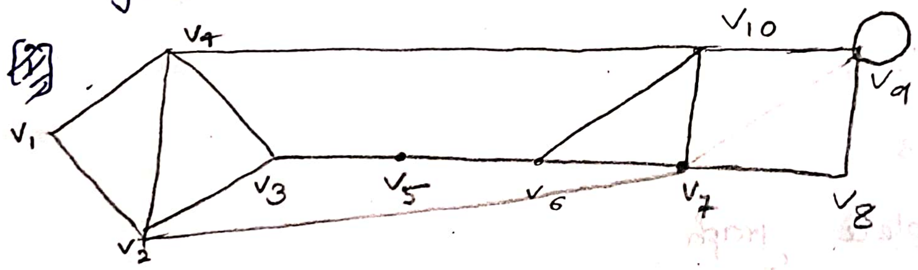
  V_n is terminal vertex of the path

  we denote the path P as $V_0 - V_n$

- If $V_0 = V_n$ then the path P is a closed path otherwise it is an open path

- The no. of edges appearing in the sequence of the path is called the length of the path.
- If the length of the path is zero (0), then it is trivial path.
- A path P is said to be simple if all edges and vertices in the path are distinct
- A path of length ≥ 1 with no repeated edges and end vertices same is called a circuit
- A cycle is a circuit with no repeated vertices, except its end vertices.

Q) Identify whether the given path from graph G is open, closed, simple, circuit, cycle and its length.



[i] Path $P_1 = v_1 - v_4 - v_3 - v_5 - v_6 - v_{10} - v_4 - v_1$

Sol P_1 is closed

length of P_1 is 7

P_1 is not a circuit ($\because v_1 - v_4$ is repeated)

P_1 is not a cycle

P_1 is not simple

[ii] $P_2 = v_2 - v_3 - v_5 - v_6 - v_7 - v_{10} - v_4 - v_2$

P_2 is closed

length of P_2 is 7

P_2 is a circuit

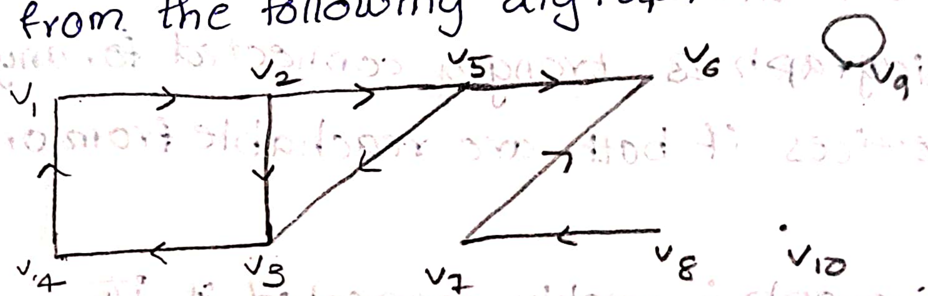
P_2 is a cycle

P_2 is not simple

<u>Reachability</u>: A vertex v of a simple digraph is said to be reachable from vertex u from same digraph if there exists a path from u to v

- The set of all vertices which are reachable from the given vertex v is said to be the rechable set of v, denoted by R(v)

Q] Find the reachable set for all the vertices from the following digraph



$R(v_1) = \{v_2, v_3, v_4, v_5, v_6, v_1\}$

$R(v_2) = \{v_3, v_4, v_1, v_5, v_6, v_2\}$

$R(v_3) = \{v_4, v_1, v_2, v_5, v_6, v_3\}$

$R(v_4) = \{v_1, v_2, v_3, v_5, v_6, v_4\}$

$R(v_5) = \{v_3, v_4, v_2, v_1; v_6\}$

$R(v_6) = \cancel{\{\}} \cancel{\phi} \{v_6\}$

$R(v_7) = \{v_6, v_7\}$

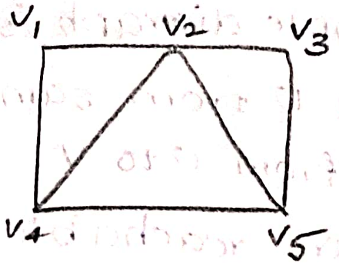
$R(v_8) = \{v_7; v_6, v_8\}$

$R(v_9) = \{v_9\}$

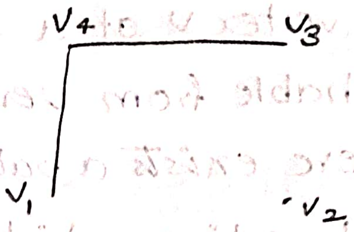
$R(v_{10}) = \cancel{\phi} \{v_{10}\}$

Q] A grap

• A graph G is said to be connected if there is a path between any two vertices



Connected

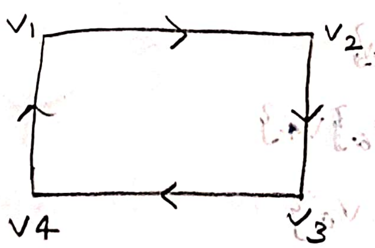


Not Connected

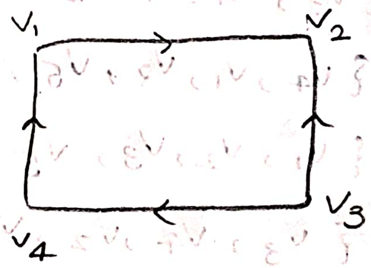
• A simple digraph is said to be unilaterally connected if for any pair atleast one of the vertices is reachable from the other vertices

• A simple diagraph is strongly connected for any pair of vertices if both are reachable from one another.

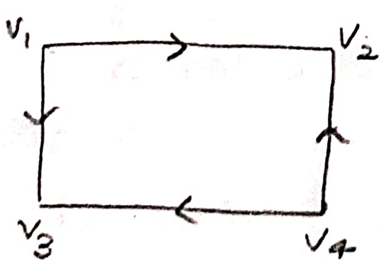
• A simple diagraph is weakly connected if its underlying graph is connected.



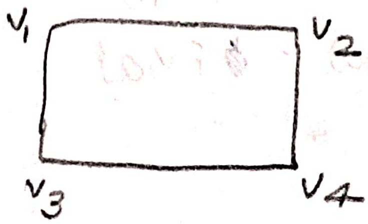
strongly connected



Unilaterally connected



underlying graph of given diagraph is



It is connected

∴ Given digraph is weakly connected

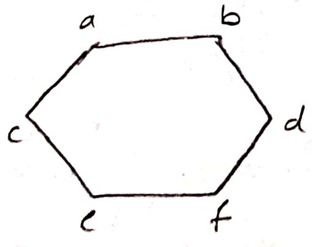
cut-vertex: Let G be a connected graph and v be a vertex in G such that $G-v$ is disconnected, then v is called cut-vertex.

Ex:

Special Graphs:

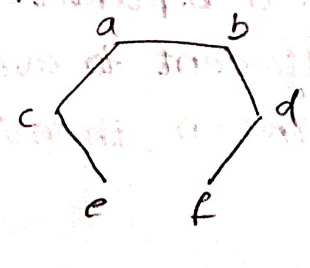
1. **Cycle Graph**: A graph G of order n is a cycle graph if its edges form a cycle of order 'n' denoted by C_n

Ex:



C_6 - cycle graph

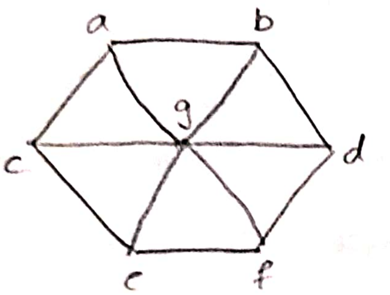
2. **Path Graph**: A path graph of order 'n' is obtained by removing one edge from cycle graph denoted by P_n



P_6 - Path graph

Null Graph : A graph of order n with n vertices and no edges is called a null graph denoted by N_n

Wheel Graph : A graph of order n obtained by joining a vertex called 'Hub' to each vertex of a cycle graph of order $n-1$ is called a wheel graph denoted by W_n



W_6 - wheel graph

'g' → Hub

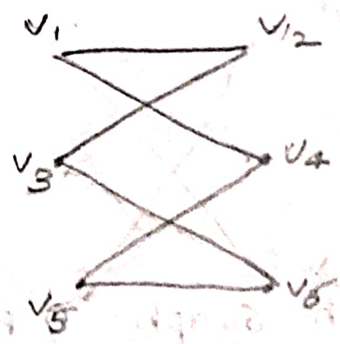
Bipartite Graph: A Bipartite graph is an undirected graph whose set of vertices can be partitioned into two sets M and N in such a way that each edge joins a vertex in M to a vertex in N and no edge joins either two vertices in M or two vertices in N

$V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$.

$M = \{v_1, v_3, v_5\}$

$N = \{v_2, v_4, v_6\}$

$M \cup N = V$

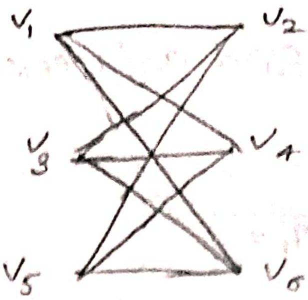


Complete Bipartite Graph:

A complete Bipartite graph is a bipartite graph in which every vertex of M is adjacent to every vertex of N and if $|M| = n$, $|N| = n$, then it is denoted by $K_{m,n}$

Complete Bipartite Graph

$K_{3,3}$

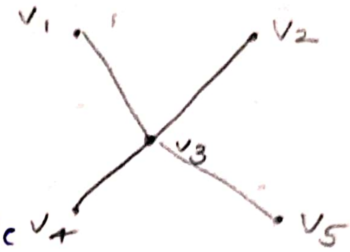


Any graph that is $K_{1,n}$ is a star graph

$V = \{v_1, v_2, v_3, v_4, v_5\}$

$M = \{v_3\}$

$N = \{v_1, v_2, v_4, v_5\}$



It is also a complete bipartite graph (special case).

Matrix Representation of Graph

1. Adjacency Matrix:

Let $G(E, V)$ be a simple graph with n vertices v_1 to v_n, then the adjacency matrix of G is given by: $A = [a_{ij}]_{n \times n}$

where $a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ adjacent to } v_j \\ 0 & \text{otherwise} \end{cases}$

Properties:

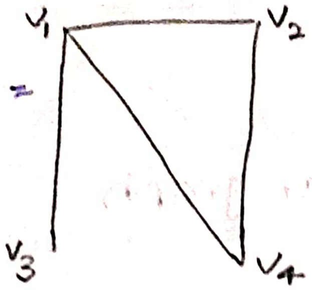
1. An adjacency matrix completely defines a simple graph.

2. Every element of adjacency matric is either 0 or 1, hence it is also called Boolean or Bit Matrix.

3. In adjacency matric, the degree of v_i is the no. of one's in the ith row of A

Q) Find adjacency matrix for the following graph.

(i) $G_1 =$



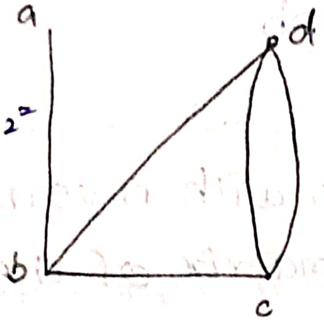
$A_{G_1} = [a_{ij}]_{4 \times 4}$

where $a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ adj to } v_j \\ 0, & \text{otherwise} \end{cases}$

$$\therefore A_{G_1} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

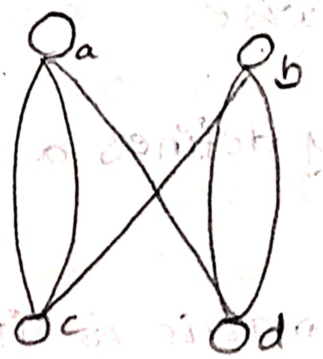
$\deg(v_3) = 1$

(ii) $G_2 =$



$$A_{G_2} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

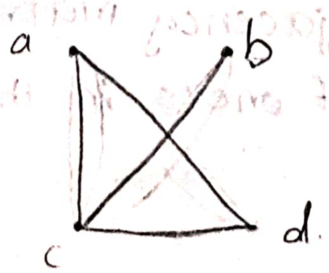
(iii). $G_3 =$



$$A_{G_3} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} \cancel{0}1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

Q) Draw the graph with the given adjacency matrix

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$



Adjacency Matrix for a Diagraph:

Let G be a diagraph with vertex set $v_1, v_2, \dots v_n$ and the vertices are assumed to be in order from v_1, to v_n

$$A = [a_{ij}]_{n \times n}$$

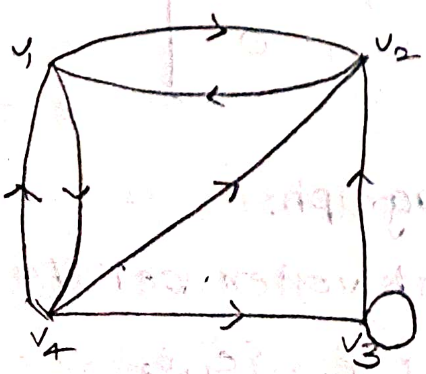
$$\text{where } a_{ij} = \begin{cases} 1, & (v_i, v_j) \in E \\ 0, & \text{otherwise} \end{cases}$$

then A is called adjacency matrix.

Observations:

1. The No. of one's (1's) in the ith row is the out degree of the vertex v_i
2. An adjacency matrics completely defines simple digraph.
3. If all the diagonal elements are 1, then digraph is reflexive.
4. If adjacency matrix is symmetric, then diagraph is symmetric.
5. If G has loop at every vertex and no other edges, then the adjacency matrix is Identity matrix

Q) Find adjacency matrix for the following digraph



$$A_G = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Incidence Matrix:

Let G be a graph with n vertices $v_1, v_2, \ldots v_n$ and m edges, $e_1, e_2 \ldots e_n$ define $n \times m$ matrix

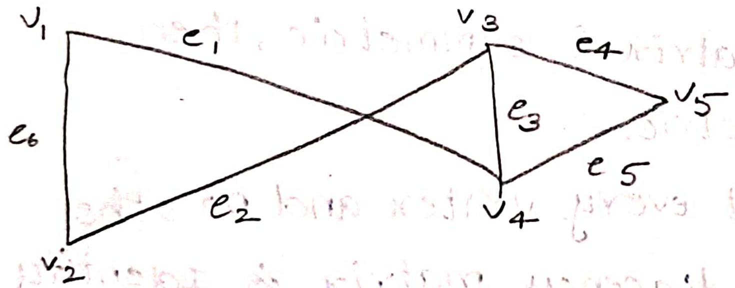
$$I_G = [a_{ij}]_{n \times m}$$

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ is incident on } e_j \\ 0, & \text{otherwise} \end{cases}$$

Observations:

1. Incidence Matrix contains 0 and 1.
2. No. of 1's in each row = degree of that vertex
3. A row with all zeroes represents isolated vertex.

Q) Find incidence matrix for the following graph



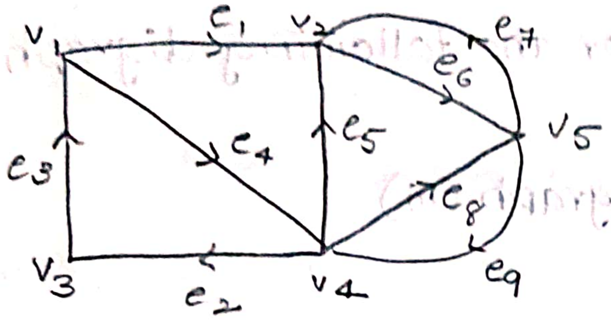
$$I_G = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Incidence Matrix for a diagraph:

Let G be a diagraph with vertex set $V = \{v_1, v_2, \ldots v_n\}$ and edge set $e = \{e_1, e_2, \ldots e_m\}$ and with no self loops then the incidence matrix of G is given by $I_G = [a_{ij}]_{m\times m}$

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ is initial vertex of } e_j \\ -1, & \text{if } v_i \text{ is terminal vertex of } e_j \\ 0, & \text{otherwise} \end{cases}$$

Q) Find incidence matrix for the following digraph



$$I_G = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 1 \end{bmatrix} \end{array}$$

Path matrix / Reachability matrix:

Let G be a simple diagraph with vertext set $V = \{v_1, v_2, \dots v_n\}$. An $n\times n$ matrix P

$$P = [P_{ij}]_{n\times n}$$

$$P_{ij} = \begin{cases} 1, & \text{if there is a path from } v_i \text{ to } v_j \\ 0, & \text{otherwise} \end{cases}$$

- To calculate Path matrix, we write

$$B = A + A^2 + \dots + A^n$$

then Path matrix $P = [P_{ij}]_{n\times n}$

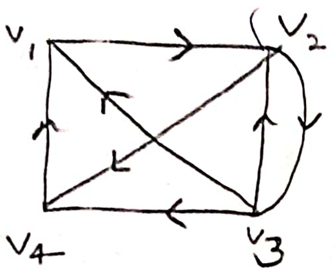
~~when~~ where $P_{ij} = \begin{cases} 1, & \text{if there is an non-zero element in } i^{th} \text{ row \& } j^{th} \text{ column of } B_n \\ 0, & \text{otherwise} \end{cases}$

where A_i is the adjacency matrix

<u>Observations:</u>

Path matrix will only show the existance of path but it does not give complete picture of graph.

Q) Find path matrix for the following digraph.



→ Digraph (G)

$$\text{Adjacency matrix } A = \begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & 0 & 1 & 0 & 0 \\ v_2 & 0 & 0 & 1 & 1 \\ v_3 & 1 & 1 & 0 & 1 \\ v_4 & 1 & 0 & 0 & 0 \end{array}$$

$$B = A + A^2 + A^3 + A^4$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 \\ 2 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$+ \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 2 & 2 & 3 \\ 3 & 3 & 2 & 3 \\ 2 & 1 & 0 & 1 \end{bmatrix}$$

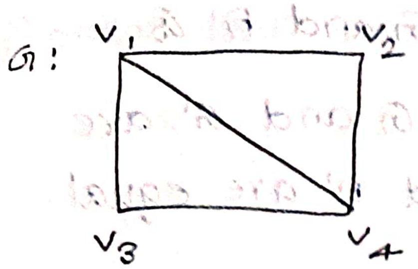
$$B_4 = \begin{bmatrix} 3 & 4 & 2 & 3 \\ 5 & 5 & 4 & 6 \\ 7 & 7 & 4 & 7 \\ 3 & 2 & 1 & 2 \end{bmatrix}$$

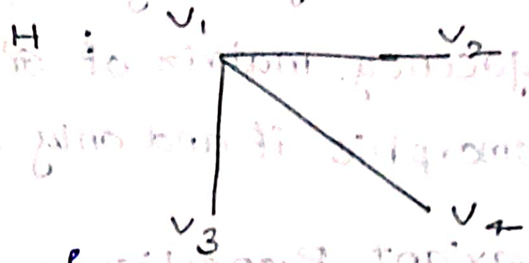
$$\text{Path matrix } P = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Sub-Graph :

Let $G(V, E)$ be a graph, the graph $H(V_1, E_1)$ is a sub-graph of G if $V_1 \subseteq V$

$E_1 \subseteq E$

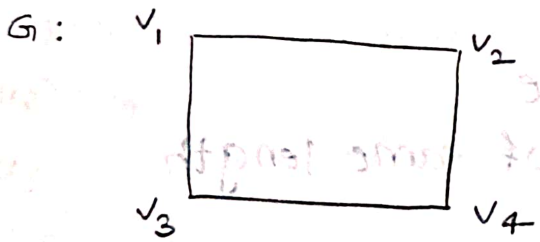


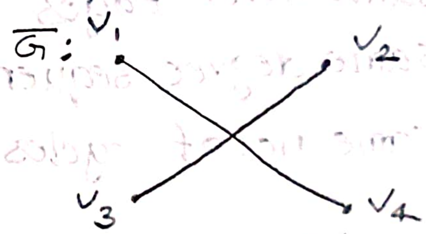


H is sub-graph of G

Complement of a Graph:

The complement of a graph G is the graph $\overline{G}$ with same no. of vertices as G and if two vertices are adjacent in G then they are not adjacent in $\overline{G}$





Isomorphic Graphs:

Two graphs G and G' are isomorphic if there exists a function $f : V(G) \rightarrow V(G')$ such that (where $V(G) \rightarrow$ vertex set of G).

1. f is one-one
2. f is onto
3. for every pair (u, v) of vertices $\{u, v\} \in E$ then $\{f(u), f(v)\} \in E'$

   (E is edge set of graph G)

$\downarrow$ u, v are adjacent

$(u, v) \rightarrow$ directed from u to v

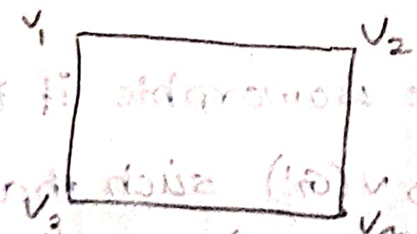
If the vertices u and v are adjacent in G
$\Leftrightarrow$ f(u) and f(v) are adjacent.

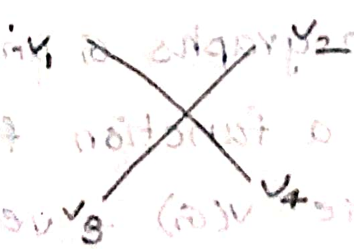
<u>Note</u>:
If A is adjacency matrix of G and A' is adjacency matrix of G', then G and G' are isomorphic if and only if A and A' are equal

<u>Invariant Properties</u>:
By showing that two graphs do not share a property, we conclude that the two graphs are not isomorphic.

<u>Properties</u>:
1. Same no. of vertices
2. same no. of edges
3. Same degree sequence
4. Same no. of cycles of same length.

Q] ~~Show that~~ Are the following graphs are isomorphic?

G:


G':


$|V| = 4$
$|E| = 4$
$d(G) = d(2,2,2,2)$

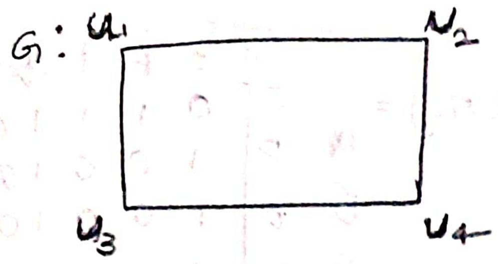
$|V| = 4$
$|E| = 2$

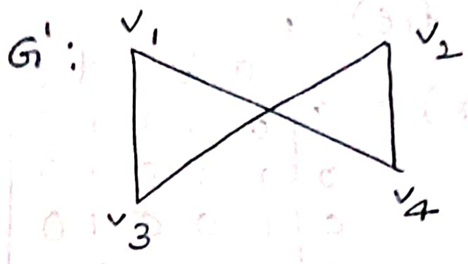
$\because$ they do not have same no. of edges, thus G, G' are not isomorphic

Note:

Two graphs G and G' are isomorphic if and only if their complements $\overline{G}$, $\overline{G'}$ are isomorphic

Q] Show that the following graphs are isomorphic





$|V| = 4$ $\quad$ $|V| = 4$

$|E| = 4$ $\quad$ $|E| = 4$

$d(G) = \{2,2,2,2\}$ $\quad$ $d(G) = \{2,2,2,2\}$

No. of cycles of length 4 = 1 $\quad$ No. of cycles of length 4 = 1

Define a function $f : V(G) \to V(G')$

$f(u_1) = v_1$

$f(u_2) = v_4$

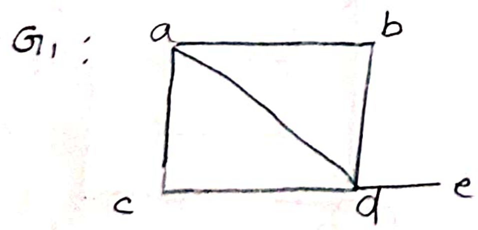
$f(u_3) = v_3$

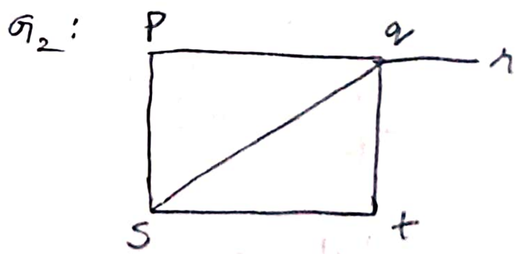
$f(u_4) = v_2$

$A(G) = A(G')$

$$\begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{array}\begin{array}{c} \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array} = \begin{array}{c} \\ v_1 \\ v_4 \\ v_3 \\ v_2 \end{array}\begin{array}{c} \begin{array}{cccc} v_1 & v_4 & v_3 & v_2 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Q] Show that the two graphs G_1, G_2 are isomorphic





Sol

$|V| = 5$ $\quad$ $|V| = 5$

$|E| = 6$ $\quad$ $|E| = 6$

$d(G_1) = ($ $\quad$ $d(G_2) = ($

No. of cycles of length 3 = 2 $\quad$ No. of cycles of length 3 = 2

No. of cycles of length 4 = 1 $\quad$ No. of cycles of length 4 = 1

Defining $f: V(G_1) \to V(G_2)$

$f(a) = s$

$f(b) = p$

$f(c) = t$

$f(d) = q$

$f(e) = r$

$$A(G_1) = \begin{array}{c|c} & \begin{matrix} a & b & c & d & e \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \\ e \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array} \qquad A(G_2) = \begin{array}{c|c} & \begin{matrix} s & p & t & q & r \end{matrix} \\ \begin{matrix} s \\ p \\ t \\ q \\ r \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

a	b	c	d	e
3	2	2	4	1

p	q	r	s	t
2	4	1	3	2

f is one-one

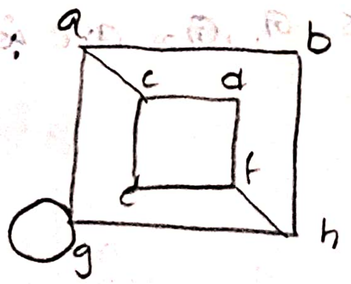
f is onto

f preserves adjacency

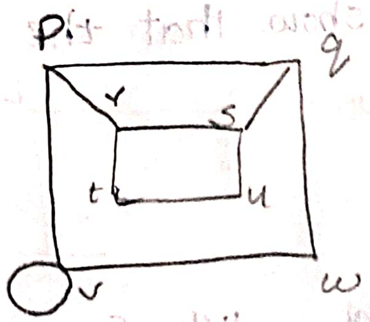
$\therefore$ G_1, G_2 are isomorphic

Q. Prove or disprove the following graphs are isomorphic

G_1:



G_2:



$|V| = 8$

$|E| = 10$

$|V| = 8$

$|E| = 10$

$d(G_1) = (3, 2, 3, 2, 2, 3, 3, 3)$

Length 6 = 2

Length 4 = 2

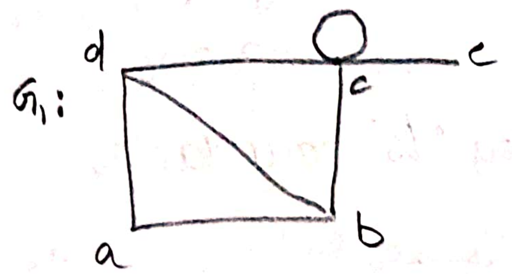
$d(G_2) = (3, 3, 3, 3, 2, 2, 3, 2)$

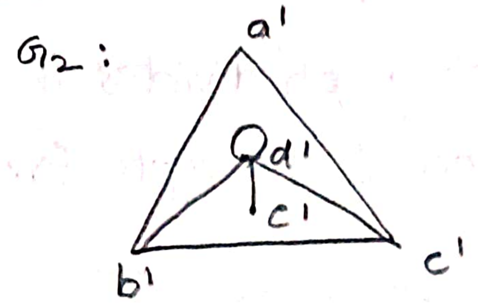
Length 6 = 2

Length 4 = 3

∴ They are not isomorphic, ∵ no. of cycles of length 4 are different.

Q] verify whether the following graphs are isomorphic or not.





$|V| = 5$ $\quad$ $|V| = 5$

$|E| = 6$ $\quad$ $|E| = 6$

$d(G_1) = (2,3,4,3,1)$ $\quad$ $d(G_2) = (2,3,3,4,1)$

length $3 = 2$ $\quad$ length $3 = 2$

$A(G_1) =$

$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

$A(G_2) =$

$$\begin{array}{c} \\ a' \\ b' \\ c' \\ d' \\ e' \end{array} \begin{array}{c} \begin{array}{ccccc} a' & b' & c' & d' & e' \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

$f(a) = a_1$

$f(b) = b_1$

$f(c) = c_1$

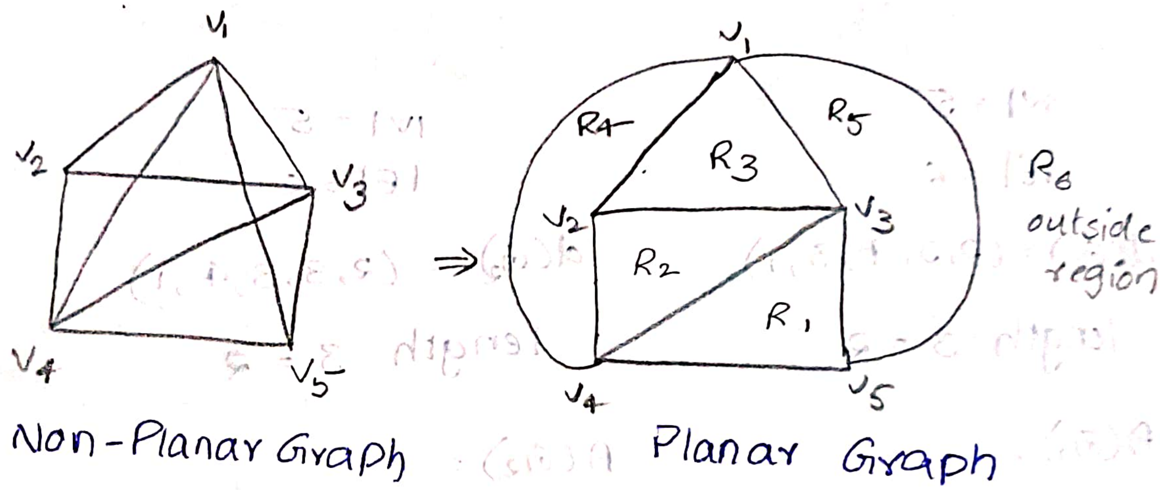
$f(d) = d_1$

$f(e) = e_1$

Planar Graphs:

A Graph G is said to be planar if it can be drawn in a plane without its edges crossing otherwise G is said to be non-planar graph.

A plane graph divides the plane into regions, a region is the cycle formed by its boundaries



Euler's Formula:

If 'G' is a connected plane graph then

$$\boxed{|V| - |E| + |R| = 2}$$

Eulerian Graphs:

Euler Path:

- A path in a multigraph that includes each edge of multigraph exactly once and includes or intersects each vertex of the multigraph atleast once.
- A multigraph is said to be traversal if it has an Euler path.

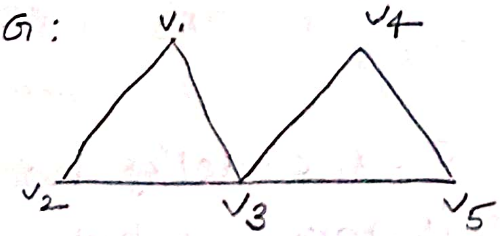
- An Euler circuit is an Eular path whose end points are identical.
- A multigraph is said to be an Eularian multigraph if it has an Euler circuit.
- A non-Eularian graph G is semi-Eularian if there exists a path of distinct edges containing every edge of G.
- A connected graph is semi-Eularian graph if and only if it has exactly two vertices of odd degree.

* • A Non-directed multigraph has an Euler path if and only if it is connected and has 0 or exactly two vertices of odd degree.

* • A Non-directed multigraph has an Euler circuit if it is connected and all of its vertices have even degree.

Q] Find Euler Path and Euler circuit for the following graph.

(i) G:



$deg(v_1) = 2$
$deg(v_2) = 2$
$deg(v_3) = 2$
$deg(v_4) = 2$
$deg(v_5) = 2$
$deg(v_6) = 2$

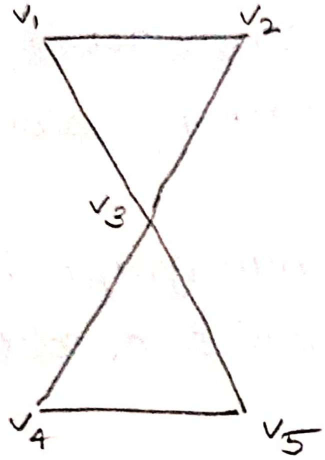
$\because$ all the vertices have even degree, the given graph has Euler circuit

Euler Circuit = $v_2 - v_3 - v_5 - v_4 - v_3 - v_1 - v_2$

(can start from any point/vertex)

$\therefore$ G is Eularian graph

(ii)

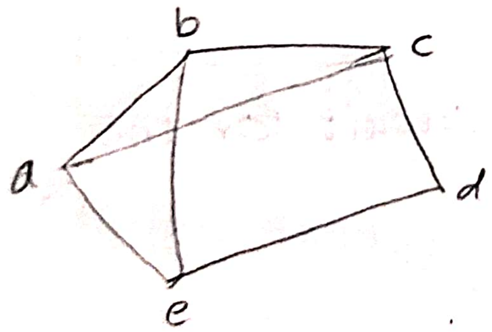


deg $(v_1) = 2$
deg $(v_2) = 2$
deg $(v_3) = 4$
deg $(v_4) = 2$
deg $(v_5) = 2$

$\therefore$ It has Euler circuit

Euler Circuit : $v_1 - v_2 - v_3 - v_4 - v_5 - v_3 - v_1$

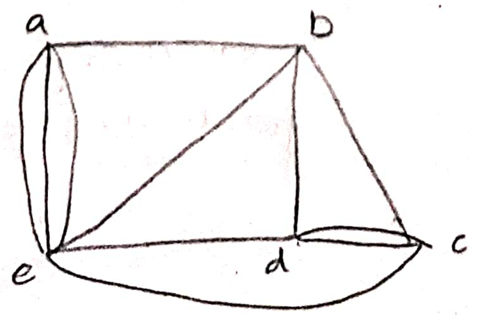
Q) Determine whether the graph has Euler path. construct such a path if it exists.



deg (a) = 3
deg (b) = 3
deg (c) = 3
deg (d) = 2
deg (e) = 3

$\because$ it does not have 0 or exactly two vertices of odd degree it does not have Euler path.

% it (ii)

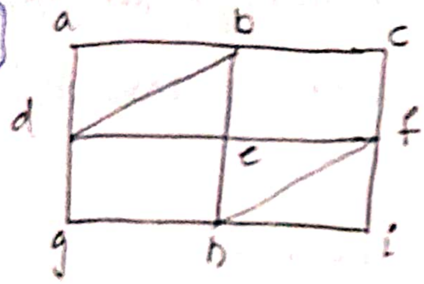


deg(a) = 4
deg(b) = 4
deg(c) = 4
deg(d) = 4
deg(e) = 6

∵ all the vertices have even degree it has Euler circuit

a - e - d - c - b - d - c - e - b - a - e - a

(iii)

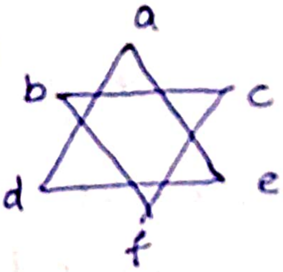


deg(a) = 2 deg(e) = 4
deg(b) = 4 deg(f) = 4
deg(c) = 2 deg(g) = 2
deg(d) = 4 deg(h) = 4
deg(i) = 2

∵ all the vertices have even degree, it has an Euler circuit.

a - b - e - h - i - f - h - g - d -
e - f - c - b - d - a

(iv)



deg(a) = 2
deg(b) = 2
deg(c) = 2
deg(d) = 2
deg(e) = 2
deg(f) = 2

It is not connected

∴ It doesn't have EC/EP

Hamiltonian graphs:

Hamiltonian Path: A simple path that contains each vertex of G exactly once is called hamiltonian path.

Hamiltonian Cycle (HC): A cycle in a graph G that contains each vertex of G exactly once except starting and end vertices which appear twice is known as HC.

- A graph containing HC is called Hamiltonian graphs.
- A non-Hamiltonian Graph is a semi-Hamiltonian graph if there exists a path passing through every vertex

Rules for constructing HP and HC

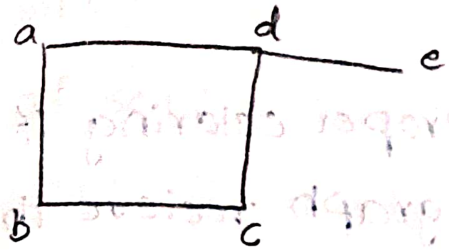
1. If G has 'n' vertices then HP must contain $(n-1)$ edges and HC must contain n edges.
2. If a vertex v has degree k then HP must contain atleast one edge incident on v and atmost two edges on v.
   HC must contain exactly 2 edges incident on v.
3. Once the HC we are building has passed through a vertex v, then all other unused edges incident on v can be deleted, because only two edges incident on v can be included in HC.

Note:

1. The complete graph K_n, $n \geq 1$, is Hamiltonion.
2. The complete Bipartite graph $K_{n,m}$ is Hamiltonian if and only if $m = n$, $n \geq 1$

Q] Find HP and HC for the following graph

G_1:

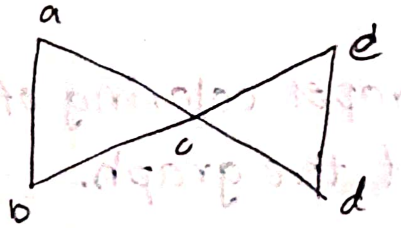


HP: e - d - a - b - c

HC: does not exist

∵ deg(e) = 1

2.



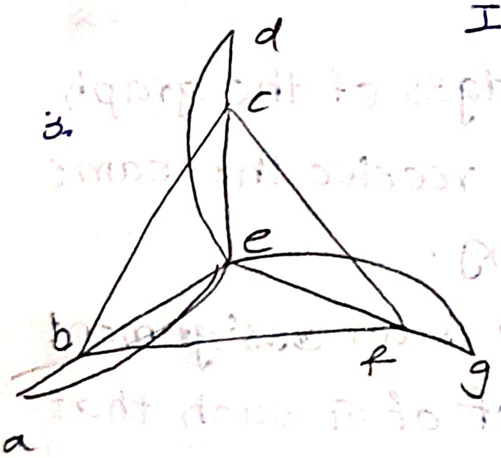
HP: e - d - c - a - b

HC: doesn't exist

∵ deg(c) = 4

It is not a complete graph

3.



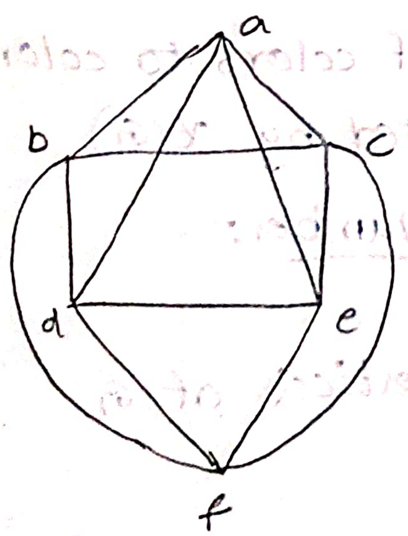
HP: d - c - e - g - f - b - a

HC: doesn't exist

∵ deg(e) = 6, deg(b) = 3 ...

It is not a complete graph

Find EC and HC for the following graph



EC: b - d - f - e - c - b - d - a - e - c - b - f - c

HC: a - b - f - c - e - d - a

Graph coloring:

vertex coloring:

- A vertex coloring is called proper coloring if no two adjacent vertices of the graph recieve the same color and the graph is then called properly-colored graph.
- The no. of colors required for proper coloring of a graph cant exceed the order of the graph.

Edge coloring:

- Assignment of colors to the edges of the graph so that no two adjacent edges receive the same colour is called Edge coloring.
- K Edge coloring of a graph is an assignment of K colors to the edges of of G such that no two edges of G receive same color.

Chromatic Number: The chromatic number of a graph G is the minimum no. of colors to color the vertices of G. It is denoted by $\chi(G)$

Rules for finding Chromatic Number:

1. $\chi(G) \le |V|$

   where $|V|$ is the no. of vertices of G

2. $\chi(K_n) = n$

   when K_n is complete graph with n vertices

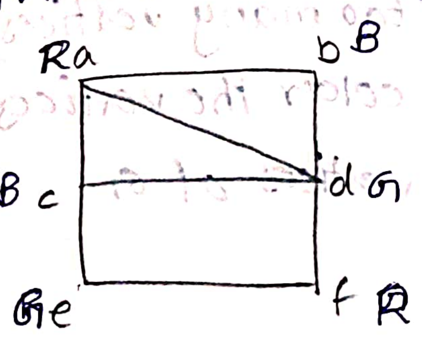
3. if $\deg(v) = d$ then atmost d colors are required to color the vertices adjacent to v.

4. $\chi(G) \leq 1 + \Delta(G)$

where $\Delta(G)$ is maximum degree of any vertex of G.

5. If some subgraph of G requires K colours then $\chi(G) \geq K$

Q) Find chromatic number for the following graph



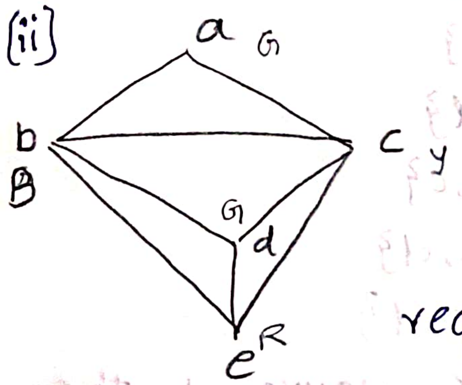
Considering 3 colors such that no 2 ~~vert~~ adjacent vertices have color.

R, B, G

$\therefore$ The min no. required for vertex coloring is 3

$\therefore$ chromatic number of given graph $\chi(G) = 3$

(ii)



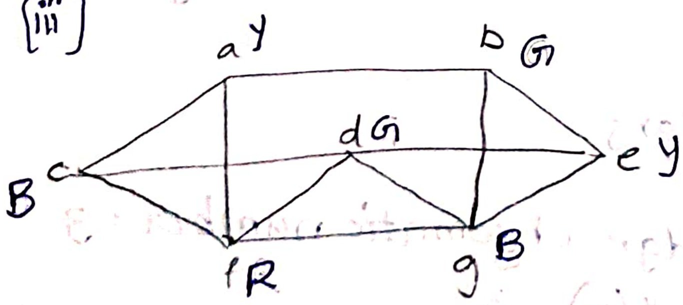
considering 4 colors

R, G, B, Y

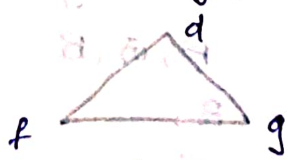
$\therefore$ The minimum no. of colors required for vertex coloring is 4

$\therefore \chi(G) = 4$

(iii)



considering a sub graph



K_2

$\therefore \chi(G) \geq 2$

$\therefore 2 \leq \chi(G) \leq 7$

In G, $\Delta G = 4$

we have $\chi(G) \leq 1 + \Delta G$

$\leq 1+4$

$\chi(G) \leq 5$

$\therefore 2 \leq \chi(G) \leq 5$

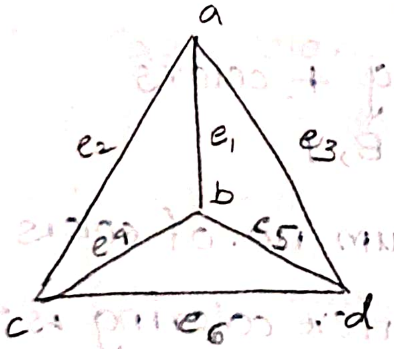
clearly 2 is not possible ($\because$ too many vertices)

3 is also not possible to color the vertices

4 is possible to color the vertices of G

$\therefore \chi(G) = 4$

Q] Find edge chromatic color for the following graph.



Let $e_1 = \{a,b\}$

$e_2 = \{a,c\}$

$e_3 = \{a,d\}$

$e_4 = \{b,c\}$

$e_5 = \{b,d\}$

$e_6 = \{c,d\}$

<u>Edge coloring</u>: Assignment of colours to the edges of the graph so that no two edges receive same colour

considering 3 colors

R, G, B

e_1 - R,

e_2 = G

e_3 = B

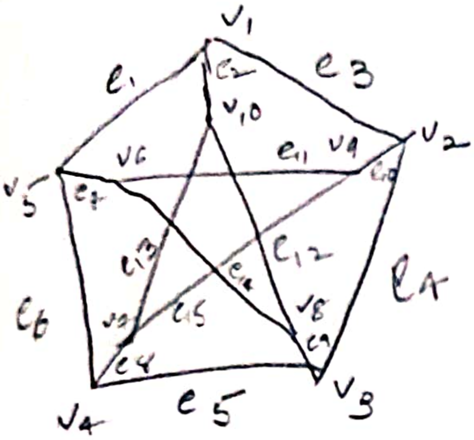
e_4 = B

e_5 = G

e_6 = R

$\therefore$ Edge chromatic number = 3

$\therefore \chi(G) = 3$



Adjacent edges

$e_1 \rightarrow e_2, e_7, e_6, e_3$

$e_2 \rightarrow e_1, e_3, e_{12}, e_{13}$

$e_3 \rightarrow e_2, e_{10}, e_4, e_1$

$e_4 \rightarrow e_3, e_5, e_{10}, e_9$

$e_5 \rightarrow e_4, e_6, e_9, e_8$

$e_6 \rightarrow e_5, e_1, e_8, e_7$

$e_7 \rightarrow$