

# Finite Element Solution of the 2D Wave Equation

## Fully Distributed MPI Implementation with $\theta$ -Family and Newmark- $\beta$

NM-PDE Project

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## 1 Introduction and Problem Statement

We solve the wave-equation problem: find  $u(x, t)$  such that

$$u_{tt} - c^2 \Delta u = f \quad \text{in } \Omega \times (0, T], \quad (1)$$

with initial conditions

$$u(\cdot, 0) = u_0, \quad u_t(\cdot, 0) = u_1, \quad (2)$$

and Dirichlet boundary data on  $\partial\Omega$ .

The implementation is a config-driven MPI C++ solver using deal.II and Trilinos, with simplicial finite elements on triangular Gmsh meshes, two time-discretization families ( $\theta$  and Newmark- $\beta$ ), ParaView output, and automated convergence studies.

## 2 Continuous Model and Weak Formulation

### 2.1 Strong Form and Functional Setting

Let  $\Omega \subset \mathbb{R}^2$  be bounded. The weak formulation is posed in Sobolev spaces  $H^1(\Omega)$  and  $H_0^1(\Omega)$  [2]. For homogeneous Dirichlet data, test functions belong to  $V = H_0^1(\Omega)$ .

### 2.2 Homogeneous Dirichlet Weak Form and Semi-Discrete System

For  $v \in V$ , multiplying by  $v$  and integrating by parts gives

$$\langle u_{tt}, v \rangle + c^2 (\nabla u, \nabla v) = (f, v), \quad \forall v \in V. \quad (3)$$

Using a simplicial FE subspace  $V_h = \text{span}\{\varphi_i\}_{i=1}^N$  and  $u_h(x, t) = \sum_{j=1}^N U_j(t)\varphi_j(x)$ , one obtains the matrix ODE

$$M \ddot{\mathbf{U}}(t) + c^2 K \mathbf{U}(t) = \mathbf{F}(t), \quad (4)$$

with

$$M_{ij} = \int_{\Omega} \varphi_i \varphi_j \, dx, \quad K_{ij} = \int_{\Omega} \nabla \varphi_i \cdot \nabla \varphi_j \, dx, \quad F_i(t) = \int_{\Omega} f(\cdot, t) \varphi_i \, dx. \quad (5)$$

### 2.3 Non-Homogeneous Dirichlet Data via Lifting

Following the standard lifting argument [1, 2], for boundary data  $u = g$  on  $\partial\Omega$ , write

$$u = w + \tilde{g}, \quad w \in H_0^1(\Omega), \quad \tilde{g}|_{\partial\Omega} = g. \quad (6)$$

Then  $w$  solves

$$\langle w_{tt}, v \rangle + c^2 (\nabla w, \nabla v) = (f, v) - \langle \tilde{g}_{tt}, v \rangle - c^2 (\nabla \tilde{g}, \nabla v). \quad (7)$$

In matrix form:

$$M\ddot{\mathbf{W}} + c^2 K \mathbf{W} = \mathbf{F} - M\ddot{\mathbf{G}} - c^2 K \mathbf{G}. \quad (8)$$

In the code, Dirichlet values are imposed strongly at each step on all boundary IDs via constraints and boundary-value elimination. This is algebraically equivalent to working with lifted unknowns and interior DoFs.

### 3 Space Discretization and Fully Distributed MPI Implementation

The space discretization follows the project requirements and simplicial-FEM theory [1].

#### 3.1 Mesh and FE Space

- Mesh input: Gmsh triangle meshes (`.msh`) generated from `mesh/*.geo`.
- Triangulation: `parallel::fullydistributed::Triangulation<2>`.
- FE: runtime-selectable `FE_SimplexP<2>(degree)` (default degree 1).
- Quadrature: only `QGaussSimplex<2>`.
- DoFs/constraints: `DoFHandler<2>` + hanging-node constraints + Dirichlet BC on all boundary IDs with configured boundary function values.

#### 3.2 Distributed Mesh Import Pipeline

The code uses this pipeline:

1. Read Gmsh mesh into serial `Triangulation<2>` with `GridIn`.
2. Partition with `GridTools::partition_triangulation`.
3. Build a `TriangulationDescription`.
4. Create the fully distributed triangulation from the description.

#### 3.3 Assembly and Linear Algebra

Mass and stiffness matrices are assembled once. The forcing vector  $\mathbf{F}(t)$  is assembled each step. All vectors/matrices are distributed Trilinos MPI objects. Implicit systems are solved with Krylov iterations (CG) and AMG-style preconditioning (Trilinos `PreconditionAMG`).

### 4 Time Discretization

The report peer examples (2.pdf, 5.pdf) emphasized explicit update formulas and stability/dispersion implications. The present report includes those details for both families.

## 4.1 $\theta$ -Method Family

Introduce  $\mathbf{v} = \dot{\mathbf{u}}$ , giving the first-order system

$$\dot{\mathbf{u}} = \mathbf{v}, \quad M\dot{\mathbf{v}} + c^2 K \mathbf{u} = \mathbf{F}(t). \quad (9)$$

With step  $\Delta t$  and weight  $\theta \in [0, 1]$ :

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} = \theta \mathbf{v}^{n+1} + (1 - \theta) \mathbf{v}^n, \quad (10)$$

$$M \frac{\mathbf{v}^{n+1} - \mathbf{v}^n}{\Delta t} + c^2 K (\theta \mathbf{u}^{n+1} + (1 - \theta) \mathbf{u}^n) = \mathbf{F}_\theta^{n+1}, \quad (11)$$

where  $\mathbf{F}_\theta^{n+1} = \theta \mathbf{F}^{n+1} + (1 - \theta) \mathbf{F}^n$ . For  $\theta > 0$ , eliminating  $\mathbf{v}^{n+1}$  gives an SPD displacement solve with effective matrix

$$A_\theta = \frac{1}{\Delta t^2} M + c^2 \theta^2 K. \quad (12)$$

Special cases:

- $\theta = 0$ : forward Euler-type explicit update (conditionally stable).
- $\theta = 0.5$ : Crank–Nicolson (second-order, low dissipation, phase error dominates when  $\Delta t$  is large).
- $\theta = 1$ : backward Euler (first-order, strongly dissipative).

## 4.2 Newmark- $\beta$ Family

The Newmark update acts directly on  $M\ddot{\mathbf{u}} + c^2 K \mathbf{u} = \mathbf{F}$ :

$$\mathbf{u}^{n+1} = \mathbf{u}^n + \Delta t \mathbf{v}^n + \Delta t^2 \left( \frac{1}{2} - \beta \right) \mathbf{a}^n + \beta \Delta t^2 \mathbf{a}^{n+1}, \quad (13)$$

$$\mathbf{v}^{n+1} = \mathbf{v}^n + \Delta t (1 - \gamma) \mathbf{a}^n + \Delta t \gamma \mathbf{a}^{n+1}. \quad (14)$$

For  $\beta > 0$ , the implementation solves for  $\mathbf{u}^{n+1}$  through

$$\left( \frac{1}{\beta \Delta t^2} M + c^2 K \right) \mathbf{u}^{n+1} = \mathbf{r}^{n+1}, \quad (15)$$

then reconstructs  $\mathbf{a}^{n+1}$  and  $\mathbf{v}^{n+1}$ . For  $\beta = 0$  (central-difference/leapfrog setting with  $\gamma = 0.5$ ), the code uses explicit displacement prediction and a mass solve for acceleration.

Important presets:

- $(\beta, \gamma) = (0.25, 0.5)$  average-acceleration: second-order, unconditionally stable, minimal algorithmic dissipation.
- $(\beta, \gamma) = (0, 0.5)$  central difference style: explicit/conditionally stable, CFL-restricted.

## 4.3 Dissipation and Dispersion (Concise Discussion)

Consistent with standard references [1, 2] and peer-report observations:

- Strongly implicit first-order damping schemes (e.g., backward Euler) introduce noticeable numerical dissipation.
- Crank–Nicolson and average-acceleration Newmark preserve amplitudes better but can show phase drift (dispersion) when  $\Delta t$  is not small enough.
- Explicit schemes can be low-dissipative in stable CFL regimes, but instability appears quickly once CFL is violated.

## 5 Software Architecture and Config Workflow

### 5.1 Source Organization

- `src/wave_solver.*`: FE setup, assembly, MPI vectors/matrices, output, solver interface.
- `src/integrators/theta_integrator.cpp + src/theta_integrator.hpp`.
- `src/integrators/newmark_integrator.cpp + src/newmark_integrator.hpp`.
- `src/config.*`: strict key=value parser with method-aware validation.
- `src/convergence_runner.*`: spatial/temporal sweep driver and CSV writer.
- `src/function_factory.*`: manufactured solutions / forcing / boundary scenarios.

### 5.2 CLI and Config

CLI is intentionally minimal:

```
wave-equation --config path/to/file.cfg
```

Key config blocks:

- Core run: `mode`, `method`, `mesh_file`, `fe_degree`, `wave_speed`, `dt`, `n_steps`.
- Output/scenario: `output_interval`, `output_dir`, `scenario_u0`, `scenario_u1`, `scenario_f`, `scenario_bc`.
- Method-specific: `theta` for the  $\theta$  family; `beta` and `gamma` for Newmark.
- Convergence: `convergence_mesh_files`, `convergence_dt_values`, `convergence_csv_space`, `convergence_csv_time`.

Unknown keys and invalid ranges are rejected during parsing.

## 6 Convergence Methodology and Results

### 6.1 Manufactured Reference and Error Norms

To measure error, we use the standing-wave exact solution on  $\Omega = [0, 1]^2$ :

$$u_{\text{exact}}(x, y, t) = \cos(c\sqrt{2}\pi t) \sin(\pi x) \sin(\pi y), \quad (16)$$

with compatible initial/boundary data and zero forcing.

At final time  $t_f$ , errors are computed in:

- $L^2$  norm,
- full  $H^1$  norm via `VectorTools::H1_norm` (as requested).

### 6.2 Automated Studies and CSV Output

Implemented modes:

- `convergence_space`: vary mesh hierarchy, fixed  $\Delta t$ .
- `convergence_time`: vary  $\Delta t$ , fixed fine mesh.
- `convergence_both`: run both and write two CSV files.

Each CSV row stores:

```
study,method,fe_degree,theta,beta,gamma,mesh_file,dt,n_steps,t_final,h,ndofs,  
l2_error,h1_error,observed_order_l2,observed_order_h1
```

### 6.3 Observed Behavior from Current Runs

A representative run with implicit Newmark ( $\beta = 0.25, \gamma = 0.5$ ), degree-1 FE, and generated unit-square mesh hierarchy gives:

- Spatial study: observed  $L^2$  orders around  $\approx 1.7\text{--}2.6$  and  $H^1$  around  $\approx 0.87\text{--}1.35$ , consistent with expected  $\mathcal{O}(h^2)$  and  $\mathcal{O}(h)$  asymptotics for  $P1$ .
- Temporal study on a fixed mesh: error is nearly saturated (very small slope), indicating spatial error dominates for the selected mesh/time range.

Figure 1 is produced by `scripts/plot_convergence.py` from the generated CSV files.

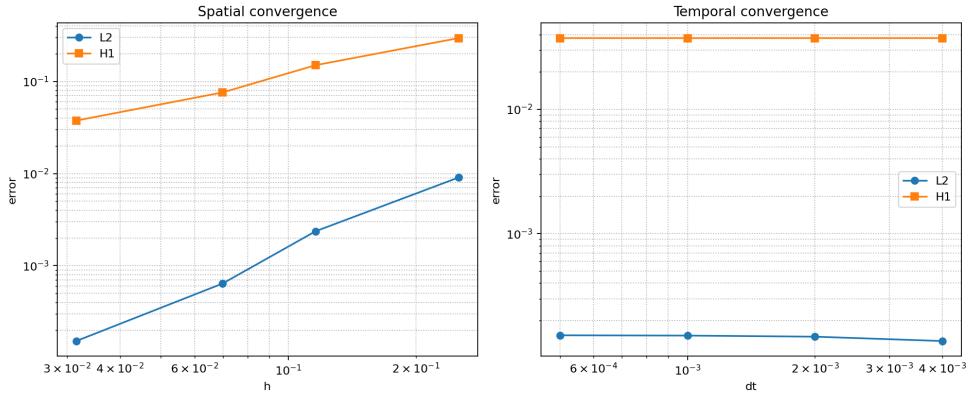


Figure 1: Log–log convergence plot automatically generated from CSV outputs.

## 7 How to Build and Run

### 7.1 Build

```
cmake -S . -B cmake-build-release -DCMAKE_BUILD_TYPE=Release
cmake --build cmake-build-release -j
```

### 7.2 Mesh Generation

```
./scripts/generate_mesh.sh
```

### 7.3 Solve Mode

```
mpirun -n 4 ./cmake-build-release/wave-equation \
--config configs/theta_crank_nicolson.cfg
```

### 7.4 Convergence + Plot

```
mpirun -n 4 ./cmake-build-release/wave-equation \
--config configs/convergence_both.cfg
python3 scripts/plot_convergence.py \
--space-csv results/convergence_space.csv \
--time-csv results/convergence_time.csv \
--output results/convergence.png
```

ParaView files (.pvtu/.vtu) are written to `solution/`.

## 8 Conclusions

The project now provides a fully distributed simplicial-FEM wave solver aligned with the requested workflow: separate  $\theta$  and Newmark integrators, strict config-driven execution, AMG-preconditioned Krylov implicit solves, ParaView export, and automated convergence CSV/plot generation. The report includes full weak-form context, method formulas, and implementation-to-theory mapping.

## References

- [1] Alfio Quarteroni. *Numerical Models for Differential Problems*. Springer, 2009.
- [2] Sandro Salsa. *Partial Differential Equations in Action: From Modelling to Theory*. Springer, 2008.