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Partial Differential Equations in Action

From Modelling to Theory

Springer

5

Waves and Vibrations

General Concepts – Transversal Waves in a String – The One-dimensional Wave Equation – The d'Alembert Formula – Second Order Linear Equations – Hyperbolic Systems With Constant Coefficients – The Multi-dimensional Wave Equation ($n > 1$) – Two Classical Models – The Cauchy Problem – Linear Water Waves

5.1 General Concepts

5.1.1 Types of waves

Our dayly experience deals with sound waves, electromagnetic waves (as radio or light waves), deep or surface water waves, elastic waves in solid materials. Oscillatory phenomena manifest themselves also in contexts and ways less macroscopic and known. This is the case, for instance, of rarefaction and shock waves in traffic dynamics or of electrochemical waves in human nervous system and in the regulation of the heart beat. In quantum physics, everything can be described in terms of wave functions, at a sufficiently small scale.

Although the above phenomena share many similarities, they show several differences as well. For example, progressive water waves propagate a disturbance, while standing waves do not. Sound waves need a supporting medium, while electromagnetic waves do not. Electrochemical waves interact with the supporting medium, in general modifying it, while water waves do not.

Thus, it seems too hard to give a general definition of *wave*, capable of covering all the above cases, so that we limit ourselves to introducing some terminology and general concepts, related to specific types of waves. We start with one-dimensional waves.

a. **Progressive** or **travelling** waves are disturbances described by a function of the following form:

$$u(x,t) = g(x - ct).$$

For $t = 0$, we have $u(x,0) = g(x)$, which is the "initial" profile of the perturbation. This profile propagates without change of shape with speed $|c|$, in the positive

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(negative) $x-$direction if $c > 0$ $(c < 0)$. We have already met this kind of waves in Chapters 2 and 4.

b. **Harmonic** waves are particular progressive waves of the form

$$u(x,t) = A \exp\{i(kx - \omega t)\}, \qquad A, k, \omega \in \mathbb{R}. \tag{5.1}$$

It is understood that only the *real part* (or the imaginary part)

$$A \cos(kx - \omega t)$$

is of interest, but the complex notation may often simplify the computations. In (5.1) we distinguish, considering for simplicity ω and k positive:

- The wave *amplitude* $|A|$;
- The *wave number* k, which is the number of complete oscillations in the space interval $[0, 2\pi]$, and the *wavelength*

$$\lambda = \frac{2\pi}{k}$$

  which is the distance between successive maxima (*crest*) or minima (*troughs*) of the waveform;
- The *angular frequency* ω, and the *frequency*

$$f = \frac{\omega}{2\pi}$$

  which is the number of complete oscillations in one second (Hertz) at a fixed space position;
- The *wave or phase speed*

$$c_p = \frac{\omega}{k}$$

  which is the crests (or troughs) speed;

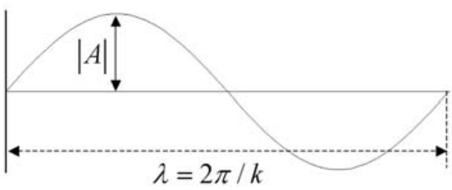


Fig. 5.1. Sinusoidal wave

c. **Standing** waves are of the form

$$u(x,t) = B \cos kx \cos \omega t.$$

In these disturbances, the basic sinusoidal wave, $\cos kx$, is modulated by the time dependent oscillation $B \cos \omega t$. A standing wave may be generated, for instance, by superposing two harmonic waves with the same amplitude, propagating in opposite directions:

$$A \cos(kx - \omega t) + A \cos(kx + \omega t) = 2A \cos kx \cos \omega t. \tag{5.2}$$

Consider now waves in dimension $n > 1$.

d. **Plane waves**. *Scalar* plane waves are of the form

$$u(\mathbf{x},t) = f(\mathbf{k} \cdot \mathbf{x} - \omega t).$$

The disturbance propagates in the direction of $\mathbf{k}$ with speed $c_p = \omega / |\mathbf{k}|$. The planes of equation

$$\theta(\mathbf{x},t) = \mathbf{k} \cdot \mathbf{x} - \omega t = \text{constant}$$

constitute the *wave-fronts.*

Harmonic or monochromatic plane waves have the form

$$u(\mathbf{x}, t) = A \exp\{i(\mathbf{k} \cdot \mathbf{x} - \omega t)\}.$$

Here $\mathbf{k}$ is the *wave number* vector and ω is the *angular frequency.* The vector $\mathbf{k}$ is orthogonal to the wave front and $|\mathbf{k}|/2\pi$ gives the number of waves per unit length. The scalar $\omega/2\pi$ still gives the number of complete oscillations in one second (Hertz) at a fixed space position.

e. **Spherical waves** are of the form

$$u(\mathbf{x},t) = v(r,t)$$

where $r = |\mathbf{x} - \mathbf{x}_0|$ and $\mathbf{x}_0 \in \mathbb{R}^n$ is a fixed point. In particular $u(\mathbf{x},t) = e^{i\omega t} v(r)$ represents a stationary spherical wave, while $u(\mathbf{x},t) = v(r - ct)$ is a progressive wave whose wavefronts are the spheres $r - ct = \text{constant}$, moving with speed $|c|$ (outgoing if $c > 0$, incoming if $c < 0$).

5.1.2 Group velocity and dispersion relation

Many oscillatory phenomena can be modelled by linear equations whose solutions are superpositions of harmonic waves with angular frequency depending on the wave number:

$$\omega = \omega(k). \tag{5.3}$$

A typical example is the wave system produced by dropping a stone in a pond.

If ω is linear, e.g. $\omega(k) = ck$, $c > 0$, the crests move with speed c, independent of the wave number. However, if $\omega(k)$ is not proportional to k, the crests move with speed $c_p = \omega(k)/k$, that *depends* on the wave number. In other words, the crests move at different speeds for different wavelengths. As a consequence, the various components in a wave packet given by the superposition of harmonic waves of different wavelengths will eventually separate or *disperse*. For this reason, (5.3) is called **dispersion relation**.

In the theory of dispersive waves, the **group velocity**, given by

$$c_g = \omega'(k)$$

is a central notion, mainly for the following three reasons.

1. *It is the speed at which an isolated wave packet moves as a whole.* A wave packet may be obtained by the superposition of dispersive harmonic waves, for instance through a Fourier integral of the form

$$u(x,t) = \int_{-\infty}^{+\infty} a(k)\, e^{i[kx-\omega(k)t]} dk \tag{5.4}$$

where the real part only has a physical meaning. Consider a localized wave packet, with wave number $k \approx k_0$, almost constant, and with amplitude slowly varying with x. Then, the packet contains a large number of crests and the amplitudes $|a(k)|$ of the various Fourier components are negligible except that in a small neighborhood of k_0, $(k_0 - \delta, k_0 + \delta)$, say.

Figure 5.2 shows the initial profile of a Gaussian packet,

$$\operatorname{Re} u(x,0) = \frac{3}{\sqrt{2}} \exp\left\{-\frac{x^2}{32}\right\} \cos 14x,$$

slowly varying with x, with $k_0 = 14$, and its Fourier transform:

$$a(k) = 6 \exp\{-8(k-14)^2\}.$$

As we can see, the amplitudes $|a(k)|$ of the various Fourier components are negligible except when k is near k_0.

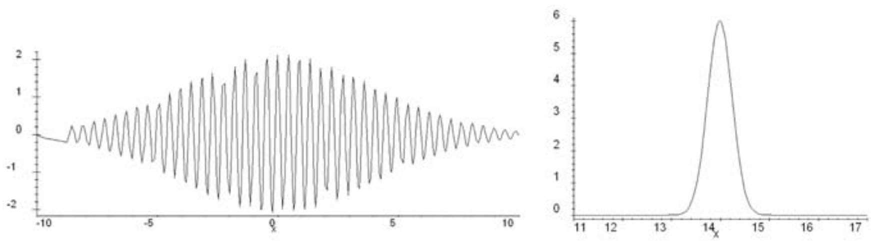


Fig. 5.2. Wave packet and its Fourier transform

Then we may write

$$\omega(k) \approx \omega(k_0) + \omega'(k_0)(k-k_0) = \omega(k_0) + c_g(k-k_0)$$

and

$$u(x,t) \approx e^{i\{k_0 x - \omega(k_0)t\}} \int_{k_0-\delta}^{k_0+\delta} a(k)\, e^{i(k-k_0)(x-c_g t)} dk. \tag{5.5}$$

Thus, u turns out to be well approximated by the product of two waves. The first one is a pure harmonic wave with relatively short wavelength $2\pi/k_0$ and phase speed $\omega(k_0)/k_0$. The second one depends on x, t through the combination $x - c_g t$, and is a superposition of waves of very small wavenumbers $k - k_0$, which correspond to very large wavelengths. We may interpret the second factor as a sort of envelope of the short waves of the packet, that is the packet as a whole, which therefore moves with the group speed.

2. *An observer that travels at the group velocity sees constantly waves of the same wavelength* $2\pi/k$, after the transitory effects due to a localized initial perturbation (e.g. a stone thrown into a pond). In other words, c_g is the propagation speed of the wave numbers.

Imagine dropping a stone into a pond. At the beginning, the water perturbation looks complicated, but after a sufficiently long time, the various Fourier components will be quite dispersed and the perturbation will appear as a slowly modulated wave train, almost sinusoidal near every point, with a *local wave number* $k(x,t)$ and a *local frequency* $\omega(x,t)$. If the water is deep enough, we expect that, at each fixed time t, the wavelength increases with the distance from the stone (longer waves move faster, see subsection 5.10.4) and that, at each fixed point x, the wavelength tends to decrease with time.

Thus, the essential features of the wave system can be observed at a relatively long distance from the location of the initial disturbance and after some time has elapsed.

Let us assume that the free surface displacement u is given by a Fourier integral of the form (5.4). We are interested on the behavior of u for $t \gg 1$. An important tool comes from the method of stationary phase[1] which gives an asymptotic formula for integrals of the form

$$I(t) = \int_{-\infty}^{+\infty} f(k)\, e^{it\varphi(k)} dk \tag{5.6}$$

as $t \to +\infty$. We can put u into the form (5.6) by writing

$$u(x,t) = \int_{-\infty}^{+\infty} a(k)\, e^{it[k\frac{x}{t} - \omega(k)]} dk,$$

then by moving from the origin at a fixed speed V (thus $x = Vt$) and defining

$$\varphi(k) = kV - \omega(k).$$

Assume for simplicity that φ has only one stationary point k_0, that is

$$\omega'(k_0) = V,$$

[1] See subsection 5.10.6

and that $\omega''(k_0) \neq 0$. Then, according to the *method of stationary phase*, we can write

$$u(Vt,t) = \sqrt{\frac{\pi}{|\omega''(k_0)|}} \frac{a(k_0)}{\sqrt{t}} \exp\{it[k_0 V - \omega(k_0)]\} + O\left(t^{-1}\right). \tag{5.7}$$

Thus, if we allow errors of order t^{-1}, moving with speed $V = \omega'(k_0) = c_g$, *the same wave number* k_0 *always appears* at the position $x = c_g t$. Note that the amplitude decreases like $t^{-1/2}$ as $t \to +\infty$. This is an important attenuation effect of dispersion.

3. *Energy is transported at the group velocity by waves of wavelength* $2\pi/k$. In a wave packet like (5.5), the energy is proportional to[2]

$$\int_{k_0-\delta}^{k_0+\delta} |a(k)|^2 \, dk \simeq 2\delta \, |a(k_0)|^2$$

so that it moves at the same speed of k_0, that is c_g.

Since the energy travels at the group velocity, there are significant differences in the wave system according to the sign of $c_g - c_p$, as we will see in Section 10.

5.2 Transversal Waves in a String

5.2.1 The model

We derive a classical model for the small transversal vibrations of a tightly stretched horizontal string (e.g. a string of a guitar). We assume the following hypotheses:

1. *Vibrations of the string have small amplitude.* This entails that the changes in the slope of the string from the horizontal equilibrium position are very small.
2. *Each point of the string undergoes vertical displacements only.* Horizontal displacements can be neglected, according to 1.
3. *The vertical displacement of a point depends on time and on its position on the string.* If we denote by u the vertical displacement of a point located at x when the string is at rest, then we have $u = u(x,t)$ and, according to 1, $|u_x(x,t)| \ll 1$.
4. *The string is perfectly flexible.* This means that it offers no resistance to bending. In particular, the stress at any point on the string can be modelled by a tangential[3] force $\mathbf{T}$ of magnitude τ, called *tension*. Figure 5.3 shows how the forces due to the tension acts at the end points of a small segment of the string.
5. *Friction is negligible.*

Under the above assumptions, the equation of motion of the string can be derived from *conservation of mass* and *Newton law*.

[2] See *A. Segel*, 1987.

[3] Consequence of absence of distributed moments along the string.

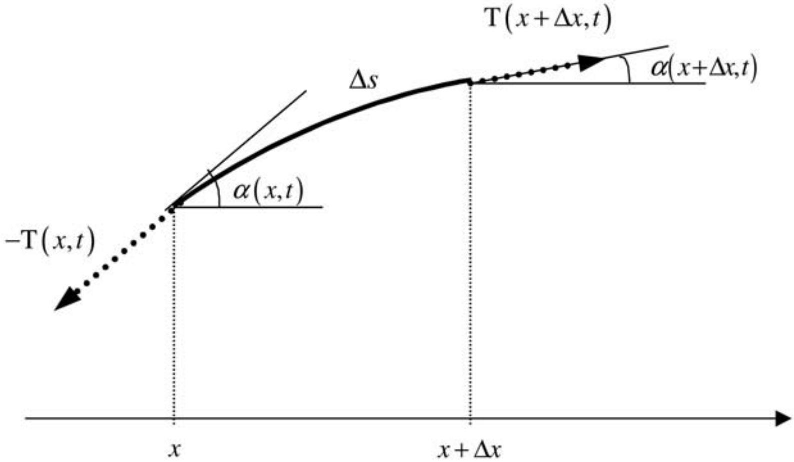


Fig. 5.3. Tension at the end points of a small segment of a string

Let $\rho_0 = \rho_0(x)$ be the linear density of the string at rest and $\rho = \rho(x,t)$ be its density at time t. Consider an arbitrary part of the string between x and $x + \Delta x$ and denote by Δs the corresponding length element at time t. Then, conservation of mass yields

$$\rho_0(x)\,\Delta x = \rho(x,t)\,\Delta s. \tag{5.8}$$

To write Newton law of motion we have to determine the forces acting on our small piece of string. Since the motion is vertical, the horizontal forces have to balance. On the other hand they come from the tension only, so that if $\tau(x,t)$ denotes the magnitude of the tension at x at time t, we can write (Fig. 5.3):

$$\tau(x+\Delta x,t)\cos\alpha(x+\Delta x,t) - \tau(x,t)\cos\alpha(x,t) = 0.$$

Dividing by Δx and letting $\Delta x \to 0$, we obtain

$$\frac{\partial}{\partial x}\left[\tau(x,t)\cos\alpha(x,t)\right] = 0$$

from which

$$\tau(x,t)\cos\alpha(x,t) = \tau_0(t) \tag{5.9}$$

where $\tau_0(t)$ is *positive*[4].

The vertical forces are given by the vertical component of the tension and by body forces such as gravity and external loads.

Using (5.9), the scalar vertical component of the tension at x, at time t, is given by:

$$\tau_{vert}(x,t) = \tau(x,t)\sin\alpha(x,t) = \tau_0(t)\tan\alpha(x,t) = \tau_0(t)\,u_x(x,t).$$

[4] It is the magnitude of a force.

Therefore, the (scalar) vertical component of the force acting on our small piece of string, due to the tension, is

$$\tau_{vert}(x+\Delta x,t)-\tau_{vert}(x,t)=\tau_0(t)\left[u_x(x+\Delta x,t)-u_x(x,t)\right].$$

Denote by $f(x,t)$ the magnitude of the (vertical) body forces per unit mass. Then, using (5.8), the magnitude of the body forces acting on the string segment is given by:

$$\int_x^{x+\Delta x}\rho(y,t)\,f(y,t)\,ds=\int_x^{x+\Delta x}\rho_0(y)\,f(y,t)\,dy.$$

Thus, using (5.8) again and observing that u_{tt} is the (scalar) vertical acceleration, Newton law gives:

$$\int_x^{x+\Delta x}\rho_0(y)\,u_{tt}(y,t)\,dy=\tau_0(t)\left[u_x(x+\Delta x,t)-u_x(x,t)\right]+\int_x^{x+\Delta x}\rho_0(y)\,f(y,t)\,dy.$$

Dividing by Δx and letting $\Delta x\to 0$, we obtain the equation

$$u_{tt}-c^2(x,t)\,u_{xx}=f(x,t) \tag{5.10}$$

where $c^2(x,t)=\tau_0(t)/\rho_0(x)$.

If the string is homogeneous then ρ_0 is constant. If moreover it is **perfectly elastic**[5] then τ_0 is constant as well, since the horizontal tension is nearly the same as for the string at rest, in the horizontal position. We shall come back to equation (5.10) shortly.

5.2.2 Energy

Suppose that a *perfectly flexible and elastic* string has length L at rest, in the horizontal position. We may identify its initial position with the segment $[0,L]$ on the x axis. Since $u_t(x,t)$ is the vertical velocity of the point at x, the expression

$$E_{cin}(t)=\frac{1}{2}\int_0^L\rho_0u_t^2\,dx \tag{5.11}$$

represents the total **kinetic energy during the vibrations**. The string stores **potential energy** too, due to the work of elastic forces. These forces stretch an element of string of length Δx at rest by[6]

$$\Delta s-\Delta x=\int_x^{x+\Delta x}\sqrt{1+u_x^2}\,dx-\Delta x=\int_x^{x+\Delta x}\left(\sqrt{1+u_x^2}-1\right)dx\approx\frac{1}{2}u_x^2\Delta x$$

[5] For instance, guitar and violin strings are nearly homogeneous, perfectly flexible and elastic.

[6] Recall that, at first order, if $\varepsilon\ll 1$, $\sqrt{1+\varepsilon}-1\simeq\varepsilon/2$.

since $|u_x| \ll 1$. Thus, the work done by the elastic forces on that string element is

$$dW = \frac{1}{2}\tau_0 u_x^2 \Delta x.$$

Summing all the contributions, the total **potential energy** is given by:

$$E_{pot}(t) = \frac{1}{2}\int_0^L \tau_0 u_x^2 \, dx. \tag{5.12}$$

From (5.11) and (5.12) we find, for the total energy:

$$E(t) = \frac{1}{2}\int_0^L \left[\rho_0 u_t^2 + \tau_0 u_x^2\right] dx. \tag{5.13}$$

Let us compute the variation of E. Taking the time derivative under the integral, we find (remember that $\rho_0 = \rho_0(x)$ and τ_0 is constant),

$$\dot{E}(t) = \int_0^L \left[\rho_0 u_t u_{tt} + \tau_0 u_x u_{xt}\right] dx.$$

By an integration by parts we get

$$\int_0^L \tau_0 u_x u_{xt} \, dx = \tau_0[u_x(L,t)\, u_t(L,t) - u_x(0,t)\, u_t(0,t)] - \tau_0 \int_0^L u_t u_{xx} dx$$

whence

$$\dot{E}(t) = \int_0^L [\rho_0 u_{tt} - \tau_0 u_{xx}] u_t dx + \tau_0[u_x(L,t)\, u_t(L,t) - u_x(0,t)\, u_t(0,t)].$$

Using (5.10), we find:

$$\dot{E}(t) = \int_0^L \rho_0 f u_t \, dx + \tau_0[u_x(L,t)\, u_t(L,t) - u_x(0,t)\, u_t(0,t)]. \tag{5.14}$$

In particular, if $f = 0$ and u is constant at the end points 0 and L (therefore $u_t(L,t) = u_t(0,t) = 0$) we deduce $\dot{E}(t) = 0$. This implies

$$E(t) = E(0)$$

which expresses the *conservation of energy.*

5.3 The One-dimensional Wave Equation

5.3.1 Initial and boundary conditions

Equation (5.10) is called the *one-dimensional wave equation.* The coefficient c has the dimensions of a speed and in fact, we will shortly see that it represents the wave propagation speed along the string. When $f \equiv 0$, the equation is *homogeneous* and the *superposition principle holds:* if u_1 and u_2 are solutions of

$$u_{tt} - c^2 u_{xx} = 0 \tag{5.15}$$

and a, b are (real or complex) scalars, then $au_1 + bu_2$ is a solution as well. More generally, if $u_k(\mathbf{x},t)$ is a family of solutions depending on the parameter k (integer or real) and $g = g(k)$ is a function rapidly vanishing at infinity, then

$$\sum_{k=1}^{\infty} u_k(\mathbf{x},t)\, g(k) \quad \text{and} \quad \int_{-\infty}^{+\infty} u_k(\mathbf{x},t)\, g(k)\, dk$$

are still solutions of (5.15).

Suppose we are considering the space-time region $0 < x < L$, $0 < t < T$. In a well posed problem for the (one-dimensional) heat equation it is appropriate to assign the initial profile of the temperature, because of the presence of a first order time derivative, and a boundary condition at both ends $x = 0$ and $x = L$, because of the second order space derivative.

By analogy with the Cauchy problem for second order ordinary differential equations, the second order time derivative in (5.10) suggests that not only the initial profile of the string but the initial velocity has to be assigned as well.

Thus, our initial (or Cauchy) data are

$$u(x,0) = g(x), \qquad u_t(x,0) = h(x), \qquad x \in [0, L].$$

The boundary data are formally similar to those for the heat equation. Typically:

Dirichlet data describe the displacement of the end points of the string:

$$u(0,t) = a(t), \qquad u(L,t) = b(t), \qquad t > 0.$$

If $a(t) = b(t) \equiv 0$ (homogeneous data), both ends are fixed, with zero displacement.

Neumann data describe the applied (scalar) vertical tension at the end points. As in the derivation of the wave equation, we may model this tension by $\tau_0 u_x$ so that the Neumann conditions take the form

$$\tau_0 u_x(0,t) = a(t), \qquad \tau_0 u_x(L,t) = b(t), \qquad t > 0.$$

In the special case of homogeneous data, $a(t) = b(t) \equiv 0$, both ends of the string are attached to a frictionless sleeve and are free to move vertically.

Robin data describe a linear elastic attachment at the end points. One way to realize this type of boundary condition is to attach an end point to a linear spring[7] whose other end is fixed. This translates into assigning

$$\tau_0 u_x(0,t) = ku(0,t), \qquad \tau_0 u_x(L,t) = -ku(L,t), \qquad t > 0,$$

where k (positive) is the elastic constant of the spring.

[7] Which obeys Hooke's law: the strain is a linear function of the stress.

In several concrete situations, *mixed conditions* have to be assigned. For instance, Robin data at $x = 0$ and Dirichlet data at $x = L$.

Global Cauchy problem. We may think of a string of infinite length and assign only the initial data

$$u(x,0) = g(x), \qquad u_t(x,0) = h(x), \qquad x \in \mathbb{R}.$$

Although physically unrealistic, it turns out that the solution of the global Cauchy problem is of fundamental importance. We shall solve it in Section 5.4.

Under reasonable assumptions on the data, the above problems are well posed. In the next section we use separation of variables to show it for a Cauchy-Dirichlet problem.

Remark 5.1. Other kinds of problems for the wave equation are the so called *Goursat problem* and the *characteristic Cauchy problem*. Some examples are given in Problems 5.9, 5.10.

5.3.2 Separation of variables

Suppose that the vibration of a violin chord is modelled by the following Cauchy-Dirichlet problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < L, t > 0 \\ u(0,t) = u(L,t) = 0 & t \geq 0 \\ u(x,0) = g(x), u_t(x,0) = h(x) & 0 \leq x \leq L \end{cases} \tag{5.16}$$

where $c^2 = \tau_0/\rho_0$ is constant.

We want to check whether this problem is *well posed*, that is, whether a solution exists, is unique and it is stable (i.e. it depends "continuously" on the data g and h). For the time being we proceed formally, without worrying too much about the correct hypotheses on g and h and the regularity of u.

• *Existence*. Since the boundary conditions are homogeneous[8], we try to construct a solution using separation of variables.

Step **1**. We start looking for solutions of the form

$$U(x,t) = w(t)\, v(x)$$

with $v(0) = v(L) = 0$. Inserting U into the wave equation we find

$$0 = U_{tt} - c^2 U_{xx} = w''(t)\, v(x) - c^2 w(t)\, v''(x)$$

or, separating the variables,

$$\frac{1}{c^2}\frac{w''(t)}{w(t)} = \frac{v''(x)}{v(x)}. \tag{5.17}$$

[8] Remember that this is essential for using separation of variables.

We have reached a familiar situation: (5.17) is an identity between two functions, one depending on t only and the other one depending on x only. Therefore the two sides of (5.17) must be both equal to the same constant, say λ. Thus, we are lead to the equation

$$w''(t) - \lambda c^2 w(t) = 0 \tag{5.18}$$

and to the *eigenvalue problem*

$$v''(x) - \lambda v(x) = 0 \tag{5.19}$$

$$v(0) = v(L) = 0. \tag{5.20}$$

Step **2**. Solution of the eigenvalue problem. There are three possibilities for the general integral of (5.19).

a) If $\lambda = 0$, then $v(x) = A + Bx$ and (5.20) imply $A = B = 0$.

b) If $\lambda = \mu^2 > 0$, then $v(x) = Ae^{-\mu x} + Be^{\mu x}$ and again (5.20) imply $A = B = 0$.

c) If $\lambda = -\mu^2 < 0$, then $v(x) = A \sin \mu x + B \cos \mu x$. From (5.20) we get

$$\begin{aligned} v(0) &= B = 0 \\ v(1) &= A \sin \mu L + B \cos \mu L = 0 \end{aligned}$$

whence

$$A \text{ arbitrary}, \; B = 0, \; \mu L = m\pi, \; m = 1, 2, \dots .$$

Thus, in case c) only we find non trivial solutions, of the form

$$v_m(x) = A_m \sin \mu_m x, \qquad \mu_m = \frac{m\pi}{L}. \tag{5.21}$$

Step **3**. Insert $\lambda = -\mu_m^2 = -m^2\pi^2/L^2$ into (5.18). Then, the general solution is

$$w_m(t) = C_m \cos(\mu_m ct) + D_m \sin(\mu_m ct). \tag{5.22}$$

From (5.21) and (5.22) we construct the family of solutions

$$U_m(x,t) = [a_m \cos(\mu_m ct) + b_m \sin(\mu_m ct)] \sin \mu_m x, \qquad m = 1, 2, ...$$

where a_m and b_m are arbitrary constants.

U_m is called the m^{th}–**normal mode** of vibration or m^{th} – *harmonic,* and is a *standing wave* with frequency $m/2L$. The first harmonic and its frequency $1/2L$, the lowest possible, are said to be *fundamental.* All the other frequencies are *integral multiples* of the fundamental one. Because of this reason it seems that a violin chord produces good quality tones, pleasant to the ear (this is not so, for instance, for a vibrating membrane like a drum, as we will see shortly).

Step **4**. If the initial conditions are

$$u(x,0) = a_m \sin \mu_m x \qquad u_t(x,0) = cb_m \mu_m \sin \mu_m x$$

then the solution of our problem is exactly U_m and the string vibrates at its $m^{th}-$mode. In general, the solution is constructed by superposing the harmonics U_m through the formula

$$u(x,t) = \sum_{m=1}^{\infty} [a_m \cos(\mu_m ct) + b_m \sin(\mu_m ct)] \sin \mu_m x, \tag{5.23}$$

where the coefficients a_m and b_m have to be chosen such that the initial conditions

$$u(x,0) = \sum_{m=1}^{\infty} a_m \sin \mu_m x = g(x) \tag{5.24}$$

and

$$u_t(x,0) = \sum_{m=1}^{\infty} c\mu_m b_m \sin \mu_m x = h(x) \tag{5.25}$$

are satisfied, for $0 \le x \le L$.

Looking at (5.24) and (5.25), it is natural to assume that both g and h have an expansion in Fourier sine series in the interval $[0, L]$. Let

$$\hat{g}_m = \frac{2}{L}\int_0^L g(x)\sin \mu_m x\, dx \quad \text{ and } \quad \hat{h}_m = \frac{2}{L}\int_0^L h(x)\sin \mu_m x\, dx$$

be the Fourier sine coefficients of g and h. If we choose

$$a_m = \hat{g}_m, \qquad b_m = \frac{\hat{h}_m}{\mu_m c}, \tag{5.26}$$

then (5.23) becomes

$$u(x,t) = \sum_{m=1}^{\infty} \left[\hat{g}_m \cos(\mu_m ct) + \frac{\hat{h}_m}{\mu_m c} \sin(\mu_m ct) \right] \sin \mu_m x \tag{5.27}$$

and satisfies (5.24) and (5.25).

Although every U_m is a smooth solution of the wave equation, in principle (5.27) is only a formal solution, unless we may differentiate term by term twice with respect to both x and t, obtaining

$$(\partial_{tt} - c^2\partial_{xx}^2)u(x,t) = \sum_{m=1}^{\infty} (\partial_{tt} - c^2\partial_{xx}^2)U_m(x,t) = 0. \tag{5.28}$$

This is possible if $\hat{g}_m$ and $\hat{h}_m$ vanish sufficiently fast as $m \to +\infty$. In fact, differentiating term by term twice, we have

$$u_{xx}(x,t) = -\sum_{m=1}^{\infty} \left[\mu_m^2 \hat{g}_m \cos(\mu_m ct) + \frac{\mu_m \hat{h}_m}{c} \sin(\mu_m ct) \right] \sin \mu_m x \tag{5.29}$$

and

$$u_{tt}(x,t) = -\sum_{m=1}^{\infty}\left[\mu_m^2 \hat{g}_m c^2 \cos(\mu_m ct) + \mu_m \hat{h}_m c \sin(\mu_m ct)\right] \sin \mu_m x. \tag{5.30}$$

Thus, if, for instance,

$$|\hat{g}_m| \le \frac{C}{m^4} \quad \text{and} \quad \left|\hat{h}_m\right| \le \frac{C}{m^3}, \tag{5.31}$$

then

$$\left|\mu_m^2 \hat{g}_m \cos(\mu_m ct)\right| \le \frac{C\pi^2}{L^2 m^2}, \quad \text{and} \quad \left|\mu_m \hat{h}_m c \sin(\mu_m ct)\right| \le \frac{cC}{Lm^2}$$

so that, by the Weierstrass test, the series in (5.29), (5.30) converge uniformly in $[0, L] \times [0, +\infty)$. Since also the series (5.27) is clearly uniformly convergent in $[0, L] \times [0, +\infty)$, differentiation term by term is allowed and u is a C^2 solution of the wave equation.

Under which assumptions on g and h do the (5.31) hold?

Let $g \in C^4([0, L])$, $h \in C^3([0, L])$ and assume the following compatibility conditions:

$$g(0) = g(L) = g''(0) = g''(L) = 0$$
$$h(0) = h(L) = 0.$$

Then (5.31) hold[9].

Moreover, under the same assumptions, it is not difficult to check that

$$u(y,t) \to g(x), \; u_t(y,t) \to h(x), \qquad \text{as } (y,t) \to (x,0) \tag{5.32}$$

for every $x \in [0, L]$ and we conclude that u is a smooth solution of (5.16).

• *Uniqueness.* To show that (5.27) is the unique solution of problem (5.16), we use conservation of energy. Let u and v be solutions of (5.16). Then $w = u - v$ is a solution of the same problem with zero initial and boundary data. We want to show that $w \equiv 0$.

Formula (5.13) gives, for the total mechanical energy,

$$E(t) = E_{cin}(t) + E_{pot}(t) = \frac{1}{2}\int_0^L [\rho_0 w_t^2 + \tau_0 w_x^2]\, dx$$

[9] It is an exercise on integration by parts. For instance, if $f \in C^4([0, L])$ and $f(0) = f(L) = f''(0) = f''(L) = 0$, then, integrating by parts four times, we have

$$\hat{f}_m = \int_0^L f(x) \sin\left(\frac{m\pi}{L}\right) dx = \frac{1}{m^4}\int_0^L f^{(4)}(x) \sin\left(\frac{m\pi}{L}\right) dx$$

and

$$\left|\hat{f}_m\right| \le \max\left|f^{(4)}\right| \frac{L}{m^4}.$$

and in our case we have

$$\dot{E}(t) = 0$$

since $f = 0$ and $w_t(L,t) = w_t(0,t) = 0$, whence

$$E(t) = E(0)$$

for every $t \geq 0$. Since, in particular, $w_t(x,0) = w_x(x,0) = 0$, we have

$$E(t) = E(0) = 0$$

for every $t \geq 0$. On the other hand, $E_{cin}(t) \geq 0$, $E_{pot}(t) \geq 0$, so that we deduce

$$E_{cin}(t) = 0,\; E_{pot}(t) = 0$$

which force $w_t = w_x = 0$. Therefore w is constant and since $w(x,0) = 0$, we conclude that $w(x,t) = 0$ for every $t \geq 0$.

• *Stability.* We want to show that if the data are slightly perturbed, the corresponding solutions change only a little. Clearly, we need to establish how we intend to measure the distance for the data and for the corresponding solutions. For the initial data, we use the *least square distance*, given by[10]

$$\|g_1 - g_2\|_0 = \left(\int_0^L |g_1(x) - g_2(x)|^2 \, dx \right)^{1/2}.$$

For functions depending also on time, we define

$$\|u - v\|_{0,\infty} = \sup_{t>0} \left(\int_0^L |u(x,t) - v(x,t)|^2 \, dx \right)^{1/2}$$

which measures the maximum in time of the *least squares distance* in space.

Now, let u_1 and u_2 be solutions of problem (5.16) corresponding to the data g_1, h_1 and g_2, h_2, respectively. Their difference $w = u_1 - u_2$ is a solution of the same problem with Cauchy data $g = g_1 - g_2$ and $h = h_1 - h_2$. From (5.27) we know that

$$w(x,t) = \sum_{m=1}^{\infty} \left[\hat{g}_m \cos(\mu_m ct) + \frac{\hat{h}_m}{\mu_m c} \sin(\mu_m ct) \right] \sin \mu_m x.$$

From Parseval's identity[11] and the elementary inequality $(a+b)^2 \leq 2(a^2 + b^2)$, $a, b \in \mathbb{R}$, we can write

$$\begin{aligned}
\int_0^L |w(x,t)|^2 \, dx &= \frac{L}{2} \sum_{m=1}^{\infty} \left[\hat{g}_m \cos(\mu_m ct) + \frac{\hat{h}_m}{\mu_m c} \sin(\mu_m ct) \right]^2 \\
&\leq L \sum_{m=1}^{\infty} \left[\hat{g}_m^2 + \left(\frac{\hat{h}_m}{\mu_m c} \right)^2 \right].
\end{aligned}$$

[10] The symbol $\|g\|$ denotes a *norm* of g. See Chapter 6.

[11] Appendix A.

Since $\mu_m \geq \pi/L$, using Parseval's equality again, we obtain

$$\int_0^L |w(x,t)|^2 \, dx \leq L \max\left\{1, \left(\frac{L}{\pi c}\right)^2\right\} \sum_{m=1}^{\infty} \left[\hat{g}_m^2 + \hat{h}_m^2\right]$$
$$= 2 \max\left\{1, \left(\frac{L}{\pi c}\right)^2\right\} \left[\|g\|_0^2 + \|h\|_0^2\right]$$

whence the stability estimate

$$\|u_1 - u_2\|_{0,\infty}^2 \leq 2 \max\left\{1, \left(\frac{L}{\pi c}\right)^2\right\} \left[\|g_1 - g_2\|_0^2 + \|h_1 - h_2\|_0^2\right]. \tag{5.33}$$

Thus, "close" data produce "close" solutions.

Remark 5.2. From (5.27), the chord vibration is given by the superposition of harmonics corresponding to the non-zero Fourier coefficients of the initial data. The complex of such harmonics determines a particular feature of the emitted sound, known as the *timbre*, a sort of signature of the musical instrument!

Remark 5.3. The hypotheses we have made on g and h are unnaturally restrictive. For example, if we pluck a violin chord at a point, the initial profile is continuous but has a corner at that point and cannot be even C^1. A physically realistic assumption for the initial profile g is *continuity.*

Similarly, if we are willing to model the vibration of a chord set into motion by a strike of a little hammer, we should allow discontinuity in the initial velocity. Thus it is realistic to assume h *bounded.*

Under these weak hypotheses the separation of variables method does not work. On the other hand, we have already faced a similar situation in Chapter 4, where the necessity to admit discontinuous solutions of a conservation law has lead to a more general and flexible formulation of the initial value problem. Also for the wave equation it is possible to introduce suitable *weak* formulations of the various initial-boundary value problems, in order to include realistic initial data and solutions with a low degree of regularity. A first attempt is shown in subsection 5.4.2. A weak formulation more suitable for numerical methods is treated in Chapter 9.

5.4 The d'Alembert Formula

5.4.1 The homogeneous equation

In this section we establish the celebrated formula of d'Alembert for the solution of the following global Cauchy problem:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & x \in \mathbb{R},\, t > 0 \\ u(x,0) = g(x), \quad u_t(x,0) = h(x) & x \in \mathbb{R}. \end{cases} \tag{5.34}$$

To find the solution, we first factorize the wave equation in the following way:

$$(\partial_t - c\partial_x)(\partial_t + c\partial_x) u = 0. \tag{5.35}$$

Now, let

$$v = u_t + cu_x. \tag{5.36}$$

Then v solves the linear transport equation

$$v_t - cv_x = 0$$

whence

$$v(x,t) = \psi(x + ct)$$

where ψ is a differentiable arbitrary function. From (5.36) we have

$$u_t + cu_x = \psi(x + ct)$$

and formula (4.10) in subsection 4.2.2 yields

$$u(x,t) = \int_0^t \psi(x - c(t-s) + cs)\ ds + \varphi(x - ct),$$

where φ is another arbitrary differentiable function.

Letting $x - ct + 2cs = y$, we find

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\, dy + \varphi(x - ct). \tag{5.37}$$

To determine ψ and φ we impose the initial conditions:

$$u(x,0) = \varphi(x) = g(x) \tag{5.38}$$

and

$$u_t(x,0) = \psi(x) - c\varphi'(x) = h(x)$$

whence

$$\psi(x) = h(x) + cg'(x). \tag{5.39}$$

Inserting (5.39) and (5.38) into (5.37) we get:

$$\begin{aligned} u(x,t) &= \frac{1}{2c}\int_{x-ct}^{x+ct} [h(y) + cg'(y)]\ dy + g(x - ct) \\ &= \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)\, dy + \frac{1}{2}[g(x+ct) - g(x-ct)] + g(x-ct) \end{aligned}$$

and finally the **d'Alembert** formula

$$u(x,t) = \frac{1}{2}[g(x+ct) + g(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)\, dy. \tag{5.40}$$

If $g \in C^2(\mathbb{R})$ and $h \in C^1(\mathbb{R})$, formula (5.40) defines a C^2-solution in the half-plane $\mathbb{R}\times[0,+\infty)$. On the other hand, a C^2-solution u in $\mathbb{R}\times[0,+\infty)$ has to be given by (5.40), just because of the procedure we have used to solve the Cauchy problem. Thus the solution is *unique.* Observe however, that *no regularizing effect* takes place here: the solution u remains no more than C^2 for any $t > 0$. Thus, there is a striking difference with diffusion phenomena, governed by the heat equation.

Furthermore, let u_1 and u_2 be the solutions corresponding to the data g_1, h_1 and g_2, h_2, respectively. Then, the d'Alembert formula for $u_1 - u_2$ yields, for every $x \in \mathbb{R}$ and $t \in [0,T]$,

$$|u_1(x,t) - u_2(x,t)| \leq \|g_1 - g_2\|_\infty + T\,\|h_1 - h_2\|_\infty$$

where

$$\|g_1 - g_2\|_\infty = \sup_{x\in\mathbb{R}} |g_1(x) - g_2(x)|, \qquad \|h_1 - h_2\|_\infty = \sup_{x\in\mathbb{R}} |h_1(x) - h_2(x)|.$$

Therefore, we have stability in *pointwise uniform sense*, at least for finite time.

Rearranging the terms in (5.40), we may write u in the form[12]

$$u(x,t) = F(x+ct) + G(x-ct) \tag{5.41}$$

which gives u as a *superposition of two progressive waves moving at constant speed c in the negative and positive $x-$direction*, respectively. Thus, these waves are not dispersive.

The two terms in (5.41) are respectively constant along the two families of straight lines γ^+ and γ^- given by

$$x + ct = \text{constant}, \qquad x - ct = \text{constant}.$$

These lines are called *characteristics*[13] and carry important information, as we will see in the next subsection.

An interesting consequence of (5.41) comes from looking at figure 5.4. Consider the *characteristic parallelogram* with vertices at the point A, B, C, D. From (5.41) we have

$$\begin{aligned} F(A) = F(C), \quad G(A) &= G(B) \\ F(D) = F(B), \quad G(D) &= G(C). \end{aligned}$$

[12] For instance:

$$F(x+ct) = \frac{1}{2}g(x+ct) + \frac{1}{2c}\int_0^{x+ct} h(y)\,dy$$

and

$$G(x-ct) = \frac{1}{2}g(x-ct) + \frac{1}{2c}\int_{x-ct}^{0} h(y)\,dy.$$

[13] In fact they are the *characteristics* for the two first order factors in the factorization (5.35).

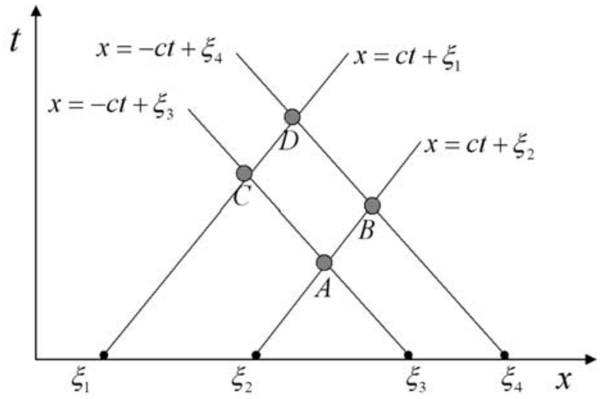


Fig. 5.4. Characteristic parallelogram

Summing these relations we get

$$[F(A) + G(A)] + [F(D) + G(D)] = [F(C) + G(C)] + [F(B) + G(B)]$$

which is equivalent to

$$u(A) + u(D) = u(C) + u(B). \tag{5.42}$$

Thus, knowing u at three points of a characteristic parallelogram, we can compute u at the fourth one.

From d'Alembert formula it follows that the value of u at the point (x, t) depends on the values of g at the points $x - ct$ e $x + ct$ and on the values of h over the whole interval $[x - ct, x + ct]$. This interval is called **domain of dependence of** (x, t) (Fig. 5.5).

From a different perspective, the values of g and h at a point z affect the value of u at the points (x, t) in the sector

$$z - ct \leq x \leq z + ct,$$

which is called **range of influence** of z (Fig. 5.5). This entails that a disturbance initially localized at z is not felt at a point x until time

$$t = \frac{|x - z|}{c}.$$

Remark 5.4. Differentiating the last term in (5.40) with respect to time we get:

$$\begin{aligned}\frac{\partial}{\partial t}\frac{1}{2c}\int_{x-ct}^{x+ct} h(y)\,dy &= \frac{1}{2c}[ch(x+ct) - (-c)h(x-ct)] \\ &= \frac{1}{2}[h(x+ct) + h(x-ct)]\end{aligned}$$

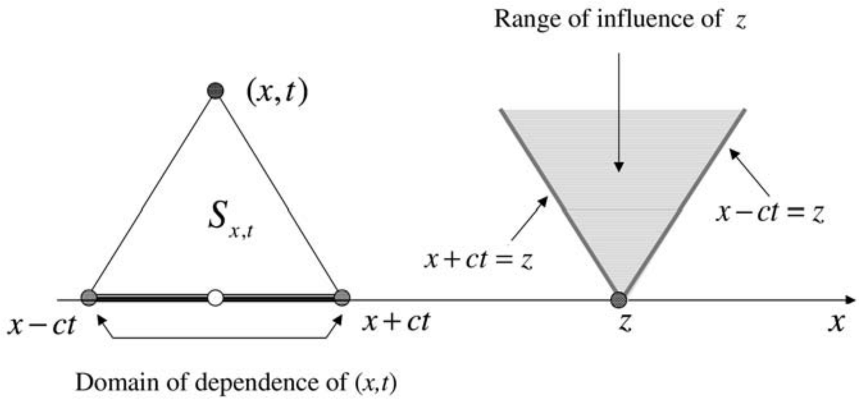


Fig. 5.5. Domain of dependence and range of influence

which has the form of the first term with g replaced by h. It follows that if w_h denotes the solution of the problem

$$\begin{cases} w_{tt} - c^2 w_{xx} = 0 & x \in \mathbb{R},\, t > 0 \\ w(x,0) = 0, w_t(x,0) = h(x). & x \in \mathbb{R} \end{cases} \tag{5.43}$$

then, d'Alembert formula can be written in the form

$$u(x,t) = \frac{\partial}{\partial t} w_g(x,t) + w_h(x,t). \tag{5.44}$$

Actually, (5.44), can be established without reference to d'Alembert formula, as we will see later.

5.4.2 Generalized solutions and propagation of singularities

In Remark 5.3 we have emphasized the necessity of a weak formulation to include physically realistic data. On the other hand, observe that d'Alembert formula makes perfect sense even for g *continuous* and h *bounded.* The question is in which sense the resulting function satisfies the wave equation, since, in principle, it is not even differentiable, only continuous. There are several ways to weaken the notion of solution to include this case; here, for instance, we mimic what we did for conservation laws.

Assuming for the moment that u is a smooth solution of the global Cauchy problem, we multiply the wave equation by a C^2-*test function* v, defined in $\mathbb{R} \times [0, +\infty)$ and compactly supported. Integrating over $\mathbb{R} \times [0, +\infty)$ we obtain

$$\int_0^\infty \int_{\mathbb{R}} [u_{tt} - c^2 u_{xx}] v \; dxdt = 0.$$

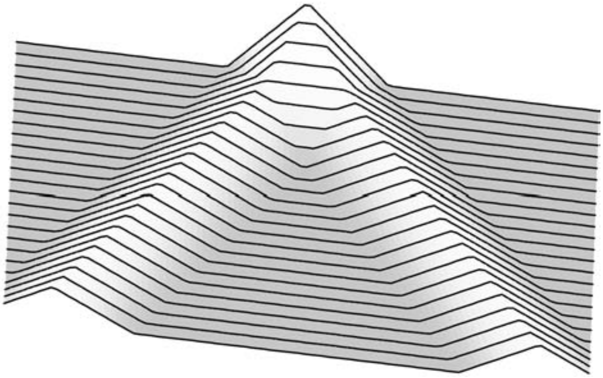


Fig. 5.6. Chord plucked at the origin ($c = 1$)

Now we integrate by parts both terms twice, to transfer all the derivatives from u to v. This yields, being v zero outside a compact subset of $\mathbb{R} \times [0, +\infty)$,

$$\int_0^\infty \int_{\mathbb{R}} c^2 u_{xx} v \; dxdt = \int_0^\infty \int_{\mathbb{R}} c^2 u v_{xx} \; dxdt$$

and

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}} u_{tt} v \; dxdt &= -\int_{\mathbb{R}} u_t(x,0)\, v(x,0)\, dx - \int_0^\infty \int_{\mathbb{R}} u_t v_t \; dxdt \\ &= -\int_{\mathbb{R}} [u_t(x,0)\, v(x,0) - u(x,0)\, v_t(x,0)]\, dx + \int_0^\infty \int_{\mathbb{R}} u v_{tt} dxdt.\end{aligned}$$

Using the Cauchy data $u(x,0) = g(x)$ and $u_t(x,0) = h(x)$, we arrive to the integral equation

$$\int_0^\infty \int_{\mathbb{R}} u[v_{tt} - c^2 v_{xx}] \; dxdt - \int_{\mathbb{R}} [h(x)\, v(x,0) - g(x)\, v_t(x,0)]\, dx = 0. \qquad (5.45)$$

Note that (5.45) makes perfect sense for u continuous, g continuous and h bounded, only. Conversely, if u is a C^2 function that satisfies (5.45) **for every** test function v, then it turns out[14] that u is a solution of problem (5.34).

Thus we may adopt the following definition.

Definition 5.1. *Let $g \in C(\mathbb{R})$ and h be bounded in $\mathbb{R}$. We say that $u \in C(\mathbb{R} \times [0, +\infty))$ is a **generalized** solution of problem (5.34) if (5.45) holds **for every** test function v.*

If g is continuous and h is bounded, it can be shown that formula (5.41) constitutes precisely a generalized solution.

[14] Check it.

Figure 5.6 shows the wave propagation along a chord of infinite length, plucked at the origin and originally at rest, modelled by the solution of the problem

$$\begin{cases} u_{tt} - u_{xx} = 0 & x \in \mathbb{R},\, t > 0 \\ u(x,0) = g(x)\,,\; u_t(x,0) = 0 & x \in \mathbb{R} \end{cases}$$

where g has a triangular profile. As we see, this generalized solution displays lines of discontinuities of the first derivatives, while outside these lines it is smooth.

We want to show that these lines are *characteristics*. More generally, consider a region $G \subset \mathbb{R}\times(0,+\infty)$, divided into two domains $G^{(1)}$ e $G^{(2)}$ by a smooth curve Γ of equation $x = s(t)$, as in figure 5.7. Let

$$\boldsymbol{\nu} = \nu_1 \mathbf{i} + \nu_2 \mathbf{j} = \frac{1}{\sqrt{1 + (\dot{s}(t))^2}} \left(-\mathbf{i} + \dot{s}(t)\,\mathbf{j}\right) \tag{5.46}$$

be the unit normal to Γ, pointing inward to $G^{(1)}$.

Given any function f defined in G, we denote by

$$f^{(1)} \text{ and } f^{(2)}$$

its restriction to the closure of $G^{(1)}$ and $G^{(2)}$, respectively, and we use the symbol

$$[f(s(t),t)] = f^{(1)}(s(t),t) - f^{(2)}(s(t),t)\,.$$

for the jump of f across Γ, or simply $[f]$ when there is no risk of confusion.

Now, let u be a generalized solution of our Cauchy problem, of class C^2 both in the closure[15] of $G^{(1)}$ and $G^{(2)}$, whose first derivatives undergo a jump discontinuity on Γ. We want to prove that:

Proposition 5.1. *Γ is a characteristic.*

Proof. First of all observe that, from our hypotheses, we have $[u] = 0$ and $[u_x], [u_t] \neq 0$. Moreover, the jumps $[u_x]$ and $[u_t]$ are continuous along Γ.

By analogy with conservation laws, we expect that the integral formulation (5.45) should imply a sort of Rankine-Hugoniot condition, relating the jumps of the derivatives with the slope of Γ and expressing the balance of linear momentum across Γ.

In fact, let v be a test function with compact support in G. Inserting v into (5.45), we can write

$$0 = \int_G \left(c^2 u v_{xx} - u v_{tt}\right)\, dxdt = \int_{G^{(2)}} (...)\, dxdt + \int_{G^{(1)}} (...)\, dxdt. \tag{5.47}$$

[15] That is, the first and second derivatives of u extend continuously up to Γ, from both sides, separately.

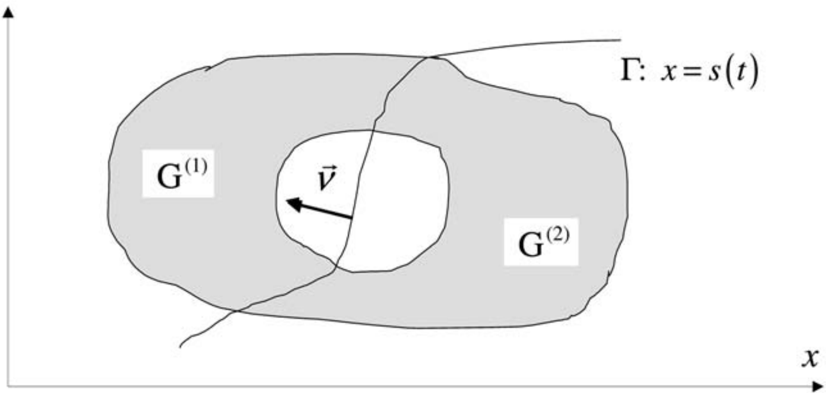


Fig. 5.7. Line of discontinuity of first derivatives

Integrating by parts, since $v = 0$ on ∂G (dl denotes arc length on Γ),

$$\begin{aligned}
&\int_{G^{(2)}} \left(c^2 u^{(2)} v_{xx} - u^{(2)} v_{tt}\right) dxdt \\
&= \int_{\Gamma} (\nu_1 c^2 u^{(2)} v_x - \nu_2 u^{(2)} v_t)\ dl - \int_{G^{(2)}} (c_x^2 u^{(2)} v_x - u_t^{(2)} v_t)\ dxdt \\
&= \int_{\Gamma} \left(\nu_1 c^2 v_x - \nu_2 v_t\right) u^{(2)}\ dl - \int_{\Gamma} (\nu_1 c^2 u_x^{(2)} - \nu_2 u_t^{(2)}) v\ dl,
\end{aligned}$$

because $\int_{G^{(2)}} [c^2 u_{xx}^{(2)} - u_{tt}^{(2)}] v\ dxdt = 0$. Similarly,

$$\begin{aligned}
&\int_{G^{(1)}} (c^2 u^{(1)} v_{xx} - u^{(1)} v_{tt})\ dxdt \\
&= -\int_{\Gamma} \left(\nu_1 c^2 v_x - \nu_2 v_t\right) u^{(1)}\ dl + \int_{\Gamma} (\nu_1 c^2 u_x^{(1)} - \nu_2 u_t^{(1)}) v\ dl,
\end{aligned}$$

because $\int_{G^{(2)}} [c^2 u_{xx}^{(1)} - u_{tt}^{(1)}] v\ dxdt = 0$ as well.

Thus, since $[u] = 0$ on Γ, or more explicitly $[u\,(s\,(t)\,, t)] \equiv 0$, (5.47) yields

$$\int_{\Gamma} \left(c^2 \left[u_x\right] \nu_1 - \left[u_t\right] \nu_2\right) v\ dl = 0.$$

Due to the arbitrariness of v and the continuity of $[u_x]$ and $[u_t]$ on Γ, we deduce

$$c^2 \left[u_x\right] \nu_1 - \left[u_t\right] \nu_2 = 0, \qquad \text{on } \Gamma,$$

or, recalling (5.46),

$$\dot{s} = -c^2 \frac{[u_x]}{[u_t]} \qquad \text{on } \Gamma, \tag{5.48}$$

which is the analogue of the Rankine-Hugoniot condition for conservation laws.

On the other hand, differentiating $[u(s(t), t)] \equiv 0$ we obtain

$$\frac{d}{dt}[u(s(t), t)] = [u_x(s(t), t)]\dot{s}(t) + [u_t(s(t), t)] \equiv 0$$

or

$$\dot{s} = -\frac{[u_t]}{[u_x]} \qquad \text{on } \Gamma. \tag{5.49}$$

Equations (5.48) and (5.49) entail

$$\dot{s}(t) = \pm c$$

which yields

$$s(t) = \pm ct + \text{constant}$$

showing that Γ is a characteristic. □

5.4.3 The fundamental solution

It is rather instructive to solve the global Cauchy problem with $g \equiv 0$ and a special h: *the Dirac delta at a point* ξ, that is $h(x) = \delta(x - \xi)$. For instance, this models the vibrations of a violin string generated by a *unit impulse localized at* ξ (a strike of a sharp hammer). The corresponding solution is called **fundamental solution** and plays the same role of the fundamental solution for the diffusion equation.

Certainly, the *Dirac delta* is a quite unusual data, out of reach of the theory we have developed so far. Therefore, we proceed formally.

Thus, let $K = K(x, \xi, t)$ denote our fundamental solution and apply d'Alembert formula; we find

$$K(x, \xi, t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \delta(y - \xi)\, dy \tag{5.50}$$

which at first glance looks like a mathematical *UFO*.

To get a more explicit formula, we first compute $\int_{-\infty}^{x} \delta(y)\, dy$. To do it, recall that (subsection 2.3.3), if $\mathcal{H}$ is the *Heaviside* function and

$$I_\varepsilon(y) = \frac{\mathcal{H}(y+\varepsilon) - \mathcal{H}(y-\varepsilon)}{2\varepsilon} = \begin{cases} \frac{1}{2\varepsilon} & -\varepsilon \le y < \varepsilon \\ 0 & \text{everywhere else} \end{cases} \tag{5.51}$$

is the unit impulse of extent ε, then $\lim_{\varepsilon \downarrow 0} I_\varepsilon(y) = \delta(y)$. Then it seems appropriate to compute $\int_{-\infty}^{x} \delta(y)\, dy$ by means of the formula

$$\int_{-\infty}^{x} \delta(y)\, dy = \lim_{\varepsilon \downarrow 0} \int_{-\infty}^{x} I_\varepsilon(y)\, dy.$$

Now, we have:

$$\int_{-\infty}^{x} I_\varepsilon(y)\, dy = \begin{cases} 0 & x \le -\varepsilon \\ (x+\varepsilon)/2\varepsilon & -\varepsilon < x < \varepsilon \\ 1 & x \ge \varepsilon. \end{cases}$$

Letting $\varepsilon \to 0$ we deduce that (the value at zero is irrelevant)

$$\int_{-\infty}^{x} \delta(y)\, dy = \mathcal{H}(x), \tag{5.52}$$

which actually is not surprising, if we remember that $\mathcal{H}' = \delta$. Everything works nicely.

Let us go back to our mathematical *UFO*, by now ... identified; we write

$$\int_{x-ct}^{x+ct} \delta(y-\xi)\, dy = \lim_{\varepsilon \downarrow 0} \int_{-\infty}^{x+ct} I_\varepsilon(y-\xi)\, dy - \lim_{\varepsilon \downarrow 0} \int_{-\infty}^{x-ct} I_\varepsilon(y-\xi)\, dy.$$

Then, using (5.50), (5.51) and (5.52), we conclude:

$$K(x,\xi,t) = \frac{1}{2c}\left\{\mathcal{H}(x-\xi+ct) - \mathcal{H}(x-\xi-ct)\right\}. \tag{5.53}$$

Figure 5.8 shows the graph of $K(x,\xi,t)$, with $c=1$

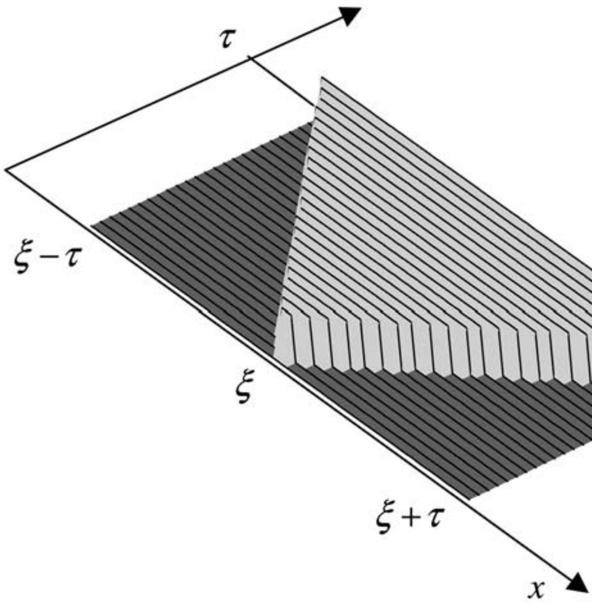


Fig. 5.8. The fundamental solution $K(x,\xi,t)$

Note how the initial discontinuity at $x = \xi$ propagates along the characteristics

$$x = \xi \pm t.$$

We have found the fundamental solution (5.53) through d'Alembert formula. Conversely, using the fundamental solution we may derive d'Alembert formula.

Namely, consider the solution w_h of the Cauchy problem (5.43), that is with data (see Remark 5.4)

$$w(x,0) = 0, \quad w_t(x,0) = h(x), \qquad x \in \mathbb{R}.$$

We may write

$$h(x) = \int_{-\infty}^{+\infty} \delta(x-\xi)\, h(\xi)\, d\xi$$

looking at $h(x)$ as a superposition of impulses $\delta(x-\xi)\, h(\xi)$, concentrated at ξ. Then, we may construct w_h by superposing the solutions of the same problem with data $\delta(x-\xi)\, h(\xi)$ instead of h. But these solutions are given by

$$K(x,\xi,t)\, h(\xi)$$

and therefore we obtain

$$w_h(x,t) = \int_{-\infty}^{+\infty} K(x,\xi,t)\, h(\xi)\, d\xi.$$

More explicitly, from (5.53):

$$\begin{aligned} w_h(x,t) &= \frac{1}{2c}\int_{-\infty}^{+\infty} \{H(x-\xi+ct) - H(x-\xi-ct)\}\, h(\xi)\, d\xi \\ &= \frac{1}{2c}\int_{-\infty}^{x+ct} h(\xi)\, d\xi - \frac{1}{2c}\int_{-\infty}^{x-ct} h(\xi)\, d\xi \\ &= \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)\, dy. \end{aligned}$$

At this point, (5.44) yields d'Alembert formula.

We shall use this method to construct the solution of the global Cauchy problem in dimension 3.

5.4.4 Non homogeneous equation. Duhamel's method

To solve the nonhomogeneous problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) & x \in \mathbb{R},\, t > 0 \\ u(x,0) = 0,\; u_t(x,0) = 0 & x \in \mathbb{R}. \end{cases} \tag{5.54}$$

we use the Duhamel's method (see subsection 2.2.8). For $s \geq 0$ fixed, let $w = w(x,t;s)$ be the solution of problem

$$\begin{cases} w_{tt} - c^2 w_{xx} = 0 & x \in \mathbb{R},\, t \geq s \\ w(x,s;s) = 0,\; w_t(x,s;s) = f(x,s) & x \in \mathbb{R}. \end{cases} \tag{5.55}$$

Since the wave equation is invariant under (time) translations, from (5.40) we get

$$w(x,t;s) = \frac{1}{2c}\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy.$$

Then, the solution of (5.54) is given by

$$u(x,t) = \int_0^t w(x,t;s)\;ds = \frac{1}{2c}\int_0^t ds \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy.$$

In fact, $u(x,0) = 0$ and

$$u_t(x,t) = w(x,t;t) + \int_0^t w_t(x,t;s)\;ds = \int_0^t w_t(x,t;s)\;ds$$

since $w(x,t;t) = 0$. Thus $u_t(x,0) = 0$. Moreover,

$$u_{tt}(x,t) = w_t(x,t;t) + \int_0^t w_{tt}(x,t;s)\;ds = f(x,t) + \int_0^t w_{tt}(x,t;s)\;ds$$

and

$$u_{xx}(x,t) = \int_0^t w_{xx}(x,t;s)\;ds.$$

Therefore, since $w_{tt} - c^2 w_{xx} = 0$,

$$\begin{aligned} u_{tt}(x,t) - c^2 u_{xx}(x,t) &= f(x,t) + \int_0^t w_{tt}(x,t;s)\;ds - c^2\int_0^t w_{xx}(x,t;s)\;ds \\ &= f(x,t). \end{aligned}$$

Everything works and gives the *unique* solution in $C^2(\mathbb{R}\times[0,+\infty))$, under rather natural hypotheses on f: we require f and f_x be continuous in $\mathbb{R}\times[0,+\infty)$.

Finally note that the value of u at the point (x,t) depends on the values of the forcing term f *in all the triangular sector* $S_{x,t}$ in figure 5.5.

5.4.5 Dissipation and dispersion

Dissipation and dispersion effects are quite important in wave propagation phenomena. Let us go back to our model for the vibrating string, assuming that its weight is negligible and that there are no external loads.

• *External damping.* External factors of dissipation like friction due to the medium may be included into the model through some empirical constitutive law. We may assume, for instance, a *linear law* of friction expressing a force proportional to the speed of vibration. Then, a force given by $-k\rho_0 u_t \Delta x\mathbf{j}$, where $k>0$ is a damping constant, acts on the segment of string between x and $x+\Delta x$. The final equation takes the form

$$\rho_0 u_{tt} - \tau_0 u_{xx} + k\rho_0 u_t = 0. \tag{5.56}$$

For a string with fixed end points, the same calculations in subsection 5.2.2 yield

$$\dot{E}(t) = -\int_0^L k\rho_0 u_t^2 \, dx = -kE_{cin}(t) \leq 0 \tag{5.57}$$

which shows a rate of energy dissipation proportional to the kinetic energy.

For equation (5.56), the usual initial-boundary value problems are still well posed under reasonable assumptions on the data. In particular, the uniqueness of the solution follows from (5.57), since $E(0) = 0$ implies $E(t) = 0$ for all $t > 0$.

• *Internal damping.* The derivation of the wave equation in subsection 5.2.1 leads to

$$\rho_0 u_{tt} = (\tau_{vert})_x$$

where τ_{vert} is the (scalar) vertical component of the tension. The hypothesis of vibrations of small amplitude corresponds to taking

$$\tau_{vert} \simeq \tau_0 u_x, \tag{5.58}$$

where τ_0 is the (scalar) horizontal component of the tension. In other words, we assume that the vertical forces due to the tension at two end points of a string element are proportional to the relative displacement of these points. On the other hand, the string vibrations convert kinetic energy into heat, because of the friction among the particles. The amount of heat increases with the speed of vibration while, at the same time, the vertical tension decreases. Thus, the vertical tension depends not only on the relative displacements u_x, but also on how fast these displacements change with time[16]. Hence, we modify (5.58) by inserting a term proportional to u_{xt}:

$$\tau_{vert} = \tau u_x + \gamma u_{xt} \tag{5.59}$$

where γ is a *positive* constant. The positivity of γ follows from the fact that energy dissipation lowers the vertical tension, so that the slope u_x decreases if $u_x > 0$ and increases if $u_x < 0$. Using the law (5.59) we derive the third order equation

$$\rho_0 u_{tt} - \tau u_{xx} - \gamma u_{xxt} = 0. \tag{5.60}$$

In spite of the presence of the term u_{xxt}, the usual initial-boundary value problems are again well posed under reasonable assumptions on the data. In particular, uniqueness of the solution follows once again from dissipation of energy, since, in this case[17],

$$\dot{E}(t) = -\int_0^L \gamma\rho_0 u_{xt}^2 \leq 0.$$

• *Dispersion.* When the string is under the action of a vertical elastic restoring force proportional to u, the equation of motion becomes

$$u_{tt} - c^2 u_{xx} + \lambda u = 0 \qquad (\lambda > 0) \tag{5.61}$$

[16] In the movie *The Legend of 1900* there is a spectacular demo of this phenomenon.
[17] Check it.

known as the *linearized Klein-Gordon equation.* To emphasize the effect of the zero order term λu, let us seek for *harmonic waves solutions* of the form

$$u(x,t) = Ae^{i(kx-\omega t)}.$$

Inserting u into (5.61) we find the *dispersion relation*

$$\omega^2 - c^2k^2 = \lambda \qquad \Longrightarrow \qquad \omega(k) = \pm\sqrt{c^2k^2+\lambda}.$$

Thus, this waves are dispersive with phase and group velocities given respectively by

$$c_p(k) = \frac{\sqrt{c^2k^2+\lambda}}{|k|}, \quad c_g = \frac{d\omega}{dk} = \frac{c^2|k|}{\sqrt{c^2k^2+\lambda}}.$$

Observe that $c_g < c_p$.

A wave packet solution can be obtained by an integration over all possible wave numbers k :

$$u(x,t) = \int_{-\infty}^{+\infty} A(k)\, e^{i[kx-\omega(k)t]}dk \tag{5.62}$$

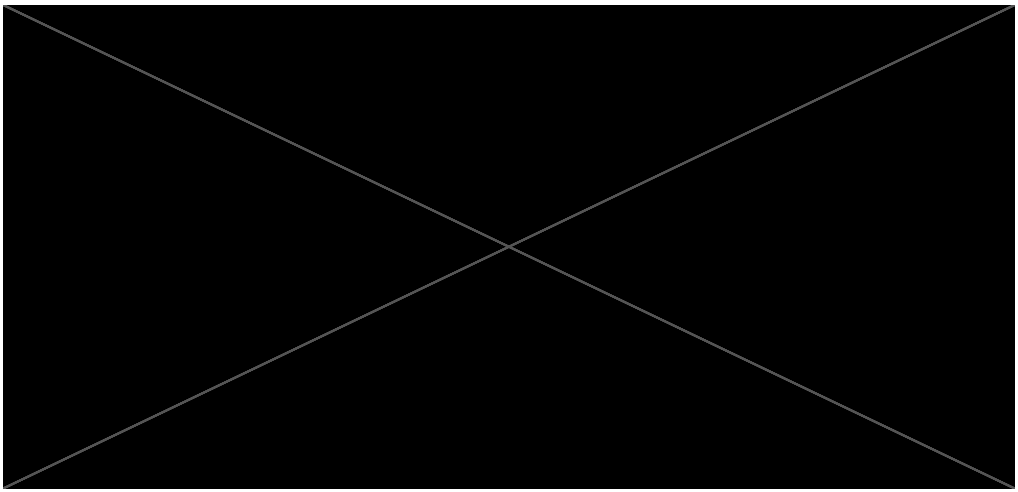
where $A(k)$ is the Fourier transform of the initial condition:

$$A(k) = \int_{-\infty}^{+\infty} u(x,0)\, e^{-ikx}dx.$$

This entails that, even if the initial condition is *localized* inside a small interval, *all* the wavelength contribute to the value of u. Although we have seen in subsection 5.1.2 that we observe a decaying in amplitude of order $t^{-1/2}$ (see formula (5.7)), these dispersive waves do not dissipate energy. For example, if the ends of the string are fixed, the total mechanical energy is given by

$$E(t) = \frac{\rho_0}{2}\int_0^L \left(u_t^2 + c^2u_x^2 + \lambda u^2\right) dx$$

and one may check that $\dot{E}(t) = 0$, $t > 0$.



5.7 The Multi-dimensional Wave Equation ($n > 1$)

5.7.1 Special solutions

The wave equation

$$u_{tt} - c^2 \Delta u = f, \tag{5.89}$$

constitutes a basic model for describing a remarkable number of oscillatory phenomena in dimension $n > 1$. Here $u = u(\mathbf{x},t)$, $\mathbf{x} \in \mathbb{R}^n$ and, as in the one-dimensional case, c is the *speed of propagation.* If $f \equiv 0$, the equation is said *homogeneous* and the *superposition principle holds.* Let us examine some relevant solutions of (5.89).

- *Plane waves.* If $\mathbf{k} \in \mathbb{R}^n$ and $\omega^2 = c^2 |\mathbf{k}|^2$, the function

$$u(\mathbf{x},t) = w(\mathbf{x} \cdot \mathbf{k} - \omega t)$$

is a solution of the homogeneous (5.89). Indeed,

$$u_{tt}(\mathbf{x},t) - c^2 \Delta u(\mathbf{x},t) = \omega^2 w''(\mathbf{x} \cdot \mathbf{n} - \omega t) - c^2 |\mathbf{k}|^2 w''(\mathbf{x} \cdot \mathbf{n} - \omega t) = 0.$$

We have already seen in subsection 5.1.1 that the planes

$$\mathbf{x} \cdot \mathbf{k} - \omega t = \text{constant}$$

constitute the wave fronts, moving at speed $c_p = \omega / |\mathbf{k}|$ in the $\mathbf{k}$ direction. The scalar $\lambda = 2\pi / |\mathbf{k}|$ is the wavelength. If $w(z) = Ae^{iz}$, the wave is said *monochromatic* or *harmonic.*

- *Cylindrical waves* $(n = 3)$ are of the form

$$u(\mathbf{x},t) = w(r,t)$$

where $\mathbf{x} = (x_1, x_2, x_3)$, $r = \sqrt{x_1^2 + x_2^2}$. In particular, solutions like $u(\mathbf{x},t) = e^{i\omega t} w(r)$ represent stationary cylindrical waves, that can be found solving the homogeneous version of equation (5.89) using the separation of variables, in axially symmetric domains.

If the axis of symmetry is the x_3 axis, it is appropriate to use the cylindrical coordinates $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, x_3. Then, the wave equation becomes[23]

$$u_{tt} - c^2 \left(u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} + u_{x_3 x_3} \right) = 0.$$

Looking for standing waves of the form $u(r,t) = e^{i\lambda ct} w(r)$, $\lambda \geq 0$, we find, after dividing by $c^2 e^{i\lambda ct}$,

$$w''(r) + \frac{1}{r} w' + \lambda^2 w = 0.$$

This is a Bessel equation of zero order. We know that the only solutions bounded at $r = 0$ are

$$w(r) = aJ(\lambda r), \qquad a \in \mathbb{R}$$

[23] Appendix C.

where, we recall,

$$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{x}{2}\right)^{2k}$$

is the Bessel function of first kind of zero order. In this way we obtain waves of the form

$$u(r,t) = aJ_0(\lambda r) e^{i\lambda ct}.$$

- *Spherical waves* $(n = 3)$ are of the form

$$u(\mathbf{x},t) = w(r,t)$$

where $\mathbf{x} = (x_1, x_2, x_3)$, $r = |\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2}$. In particular $u(\mathbf{x},t) = e^{i\omega t} w(r)$ represent standing spherical waves and can be determined by solving the homogeneous version of equation (5.89) using separation of variables in spherically symmetric domains. In this case, spherical coordinates

$$x_1 = r\cos\theta\sin\psi,\ x_2 = r\sin\theta\sin\psi,\ x_3\cos\psi,$$

are appropriate and the wave equation becomes[24]

$$\frac{1}{c^2}u_{tt} - u_{rr} - \frac{2}{r}u_r - \frac{1}{r^2}\left\{\frac{1}{(\sin\psi)^2}u_{\theta\theta} + u_{\psi\psi} + \frac{\cos\psi}{\sin\psi}u_\psi\right\} = 0. \tag{5.90}$$

Let us look for solutions of the form $u(r,t) = e^{i\lambda ct}w(r)$, $\lambda \geq 0$. We find, after simplifying out $c^2 e^{i\lambda ct}$,

$$w''(r) + \frac{2}{r}w' + \lambda^2 w = 0$$

which can be written[25]

$$(rw)'' + \lambda^2 rw = 0.$$

Thus, $v = rw$ is solution of

$$v'' + \lambda^2 v = 0$$

which gives $v(r) = a\cos(\lambda r) + b\sin(\lambda r)$ and hence the attenuated spherical waves

$$w(r,t) = ae^{i\lambda ct}\frac{\cos(\lambda r)}{r}, \qquad w(r,t) = be^{i\lambda ct}\frac{\sin(\lambda r)}{r}. \tag{5.91}$$

Let us now determine the general form of a spherical wave in $\mathbb{R}^3$. Inserting $u(\mathbf{x},t) = w(r,t)$ into (5.90) we obtain

$$w_{tt} - c^2\left\{w_{rr}(r) + \frac{2}{r}w_r\right\} = 0$$

[24] Appendix C.

[25] Thanks to the miraculous presence of the factor 2 in the coefficient of w'!

which can be written in the form

$$(rw)_{tt} - c^2 (rw)_{rr} = 0. \tag{5.92}$$

Then, formula (5.41) gives

$$w(r,t) = \frac{F(r+ct)}{r} + \frac{G(r-ct)}{r} \equiv w_i(r,t) + w_o(r,t) \tag{5.93}$$

which represents the superposition of two attenuated progressive spherical waves. The wave fronts of u_o are the spheres $r - ct = k$, expanding as time goes on. Hence, w_o represents an *outgoing wave.* On the contrary, the wave w_i is *incoming,* since its wave fronts are the contracting spheres $r + ct = k$.

5.7.2 Well posed problems. Uniqueness

The well posed problems in dimension one, are still well posed in any number of dimensions. Let

$$Q_T = \Omega \times (0,T)$$

a *space-time cylinder*, where Ω is a bounded C^1−domain[26] in $\mathbb{R}^n$. A solution $u(\mathbf{x},t)$ is uniquely determined by assigning initial data and appropriate boundary conditions on the boundary $\partial\Omega$ of Ω.

More specifically, we may pose the following problems: *Determine* $u = u(\mathbf{x},t)$ *such that:*

$$\begin{cases} u_{tt} - c^2 \Delta u = f & \text{in } Q_T \\ u(\mathbf{x},0) = g(\mathbf{x}), \ u_t(\mathbf{x},0) = h(\mathbf{x}) & \text{in } \Omega \\ + \text{ boundary conditions} & \text{on } \partial\Omega \times [0,T) \end{cases} \tag{5.94}$$

where the boundary conditions are:

(*a*) $u = h$ (Dirichlet),

(*b*) $\partial_{\boldsymbol{\nu}} u = h$ (Neumann),

(*c*) $\partial_{\boldsymbol{\nu}} u + \alpha u = h$ ($\alpha > 0$, Robin),

(*d*) $u = h_1$ on $\partial_D\Omega$ and $\partial_\nu u = h_2$ on $\partial_N\Omega$ (mixed problem) with $\partial_N\Omega$ a relatively open subset of $\partial\Omega$ and $\partial_D\Omega = \partial\Omega \backslash \partial_N\Omega$.

The *global Cauchy problem*

$$\begin{cases} u_{tt} - c^2 \Delta u = f & \mathbf{x} \in \mathbb{R}^n, t > 0 \\ u(\mathbf{x},0) = g(\mathbf{x}), \ u_t(\mathbf{x},0) = h(\mathbf{x}) & \mathbf{x} \in \mathbb{R}^n \end{cases} \tag{5.95}$$

is quite important also in dimension $n > 1$. We will examine it with some details later on. Particularly relevant are the different features that the solutions exhibit for $n = 2$ and $n = 3$.

---

[26] As usual we can afford corner points (e.g. a triangle or a cone) and also some edges (e.g. a cube or a hemisphere).

Under rather natural hypotheses on the data, problem (5.94) has at most one solution. To see it, we may use once again the conservation of energy, which is proportional to:

$$E(t) = \frac{1}{2}\int_{\Omega}\left\{u_t^2 + c^2|\nabla u|^2\right\} d\mathbf{x}.$$

The growth rate is:

$$\dot{E}(t) = \int_{\Omega}\left\{u_t u_{tt} + c^2\nabla u_t\cdot\nabla u\right\} d\mathbf{x}.$$

Integrating by parts, we have

$$\int_{\Omega} c^2\nabla u_t\cdot\nabla u \; d\mathbf{x} = c^2\int_{\partial\Omega} u_{\boldsymbol{\nu}} u_t \; d\sigma - \int_{\Omega} c^2 u_t \Delta u \; d\mathbf{x}$$

whence, since $u_{tt} - c^2\Delta u = f$,

$$\dot{E}(t) = \int_{\Omega}\left\{u_{tt} - c^2\Delta u\right\} u_t \; d\mathbf{x} + c^2\int_{\partial\Omega} u_{\boldsymbol{\nu}} u_t \; d\sigma = \int_{\Omega} f u_t \; d\mathbf{x} + c^2\int_{\partial\Omega} u_{\boldsymbol{\nu}} u_t \; d\sigma.$$

Now it is easy to prove the following result, where we use the symbol $C^{h,k}(D)$ to denote the set of functions h times continuously differentiable with respect to space and k times with respect to time in D.

Theorem 5.1. *Problem (5.94), coupled with one of the boundary conditions (a)–(d) above, has at most one solution in* $C^{2,2}(Q_T)\cap C^{1,1}\left(\overline{Q}_T\right)$.

Proof. Let u_1 and u_2 be solutions of the same problem, sharing the same data. Their difference $w = u_1 - u_2$ is a solution of the homogeneous equation, with zero data. We show that $w(\mathbf{x},t)\equiv 0$.

In the case of Dirichlet, Neumann and mixed conditions, since either $w_{\boldsymbol{\nu}} = 0$ or $w_t = 0$ on $\partial\Omega\times[0,T)$, we have $\dot{E}(t) = 0$. Thus, since $E(0) = 0$, we infer:

$$E(t) = \frac{1}{2}\int_{\Omega}\left\{w_t^2 + c^2|\nabla w|^2\right\} d\mathbf{x} = 0, \qquad \forall t > 0.$$

Therefore, for each $t > 0$, both w_t and $|\nabla w(\mathbf{x},t)|$ vanish so that $w(\mathbf{x},t)$ is constant. Then $w(\mathbf{x},t)\equiv 0$, since $w(\mathbf{x},0) = 0$.

For the Robin problem

$$\dot{E}(t) = -c^2\int_{\partial\Omega}\alpha w w_t \; d\sigma = -\frac{c^2}{2}\frac{d}{dt}\int_{\partial\Omega}\alpha w^2 \; d\sigma$$

that is

$$\frac{d}{dt}\left\{E(t) + \frac{c^2}{2}\int_{\partial\Omega}\alpha w^2 \; d\sigma\right\} = 0.$$

Hence,

$$E(t) + \frac{c^2}{2}\int_{\partial\Omega}\alpha w^2 d\sigma = \text{ constant}$$

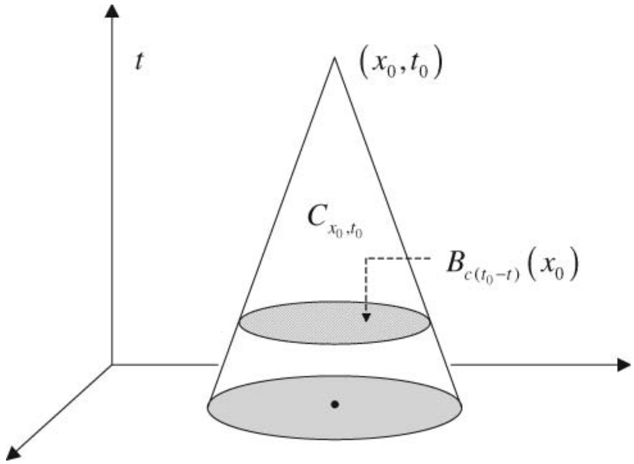


Fig. 5.9. Retrograde cone

and, being zero initially, it is zero for all $t > 0$. Since $\alpha > 0$, we again conclude that $w \equiv 0$. $\square$

Uniqueness for the global Cauchy problem follows from another energy inequality, with more interesting consequences.

First a remark. For sake of clarity, let $n = 2$. Suppose that a disturbance governed by the homogeneous wave equation ($f = 0$) is felt at $\mathbf{x}_0$ at time t_0. Since the disturbances travel with speed c, $u(x_0, t_0)$ is, in principle, only affected by the values of the initial data in the circle $B_{ct_0}(\mathbf{x}_0)$. More generally, at time $t_0 - t$, $u(\mathbf{x}_0, t_0)$ is determined by those values in the circle $B_{c(t_0-t)}(\mathbf{x}_0)$. As t varies from 0 to t_0, the union of the circles $B_{c(t_0-t)}(\mathbf{x}_0)$ in the $\mathbf{x}$,t space coincides with the so called *backward or retrograde cone with vertex at* $(\mathbf{x}_0, t_0)$ and opening $\theta = \tan^{-1} c$, given by (see Fig. 5.9):

$$C_{\mathbf{x}_0,t_0} = \{(\mathbf{x},t) \colon \ |\mathbf{x} - \mathbf{x}_0| \le c(t_0 - t),\ 0 \le t \le t_0\}.$$

Thus, given a point $\mathbf{x}_0$, it is natural to introduce an energy associated with its backward cone by the formula

$$e(t) = \frac{1}{2} \int_{B_{c(t_0-t)}(\mathbf{x}_0)} (u_t^2 + c^2 |\nabla u|^2) d\mathbf{x}.$$

It turns out that $e(t)$ is a decreasing function. Namely:

Lemma 5.1. *Let u be a C^2-solution of the homogeneous wave equation in $\mathbb{R}^n \times [0, +\infty)$. Then*

$$\dot{e}(t) \le 0.$$

Proof. We may write

$$e(t) = \frac{1}{2} \int_0^{c(t_0-t)} dr \int_{\partial B_r(x_0)} (u_t^2 + c^2 |\nabla u|^2) d\sigma$$

so that

$$\dot{e}\left(t\right)=-\frac{c}{2}\int_{\partial B_{c(t_0-t)}(x_0)}(u_t^2+c^2\left|\nabla u\right|^2)d\sigma+\int_{B_{c(t_0-t)}(x_0)}\left(u_tu_{tt}+c^2\nabla u\cdot\nabla u_t\right)d\mathbf{x}.$$

An integration by parts yields

$$\int_{B_{c(t_0-t)}(x_0)}\nabla u\cdot\nabla u_t\;d\mathbf{x}=\int_{\partial B_{c(t_0-t)}(x_0)}u_tu_{\boldsymbol{\nu}}\;d\sigma-\int_{B_{c(t_0-t)}(x_0)}u_t\Delta u\;d\mathbf{x}$$

whence

$$\begin{aligned}\dot{e}\left(t\right)&=\int_{B_{c(t_0-t)}(x_0)}u_t(u_{tt}-c^2\Delta u)d\mathbf{x}+\frac{c}{2}\int_{\partial B_{c(t_0-t)}(x_0)}(2cu_tu_{\boldsymbol{\nu}}-u_t^2-c^2\left|\nabla u\right|^2)d\sigma\\&=\frac{c}{2}\int_{\partial B_{c(t_0-t)}(x_0)}(2cu_tu_{\boldsymbol{\nu}}-u_t^2-c^2\left|\nabla u\right|^2)d\sigma\end{aligned}$$

Now

$$\left|u_tu_{\boldsymbol{\nu}}\right|\leq\left|u_t\right|\left|\nabla u\right|$$

so that

$$2cu_tu_{\boldsymbol{\nu}}-u_t^2-c^2\left|\nabla u\right|^2\leq2c\left|u_t\right|\left|\nabla u\right|-u_t^2-c^2\left|\nabla u\right|^2=-\left(u_t-c\left|\nabla u\right|\right)^2\leq0$$

and therefore $\dot{e}\left(t\right)\leq0$. $\square$

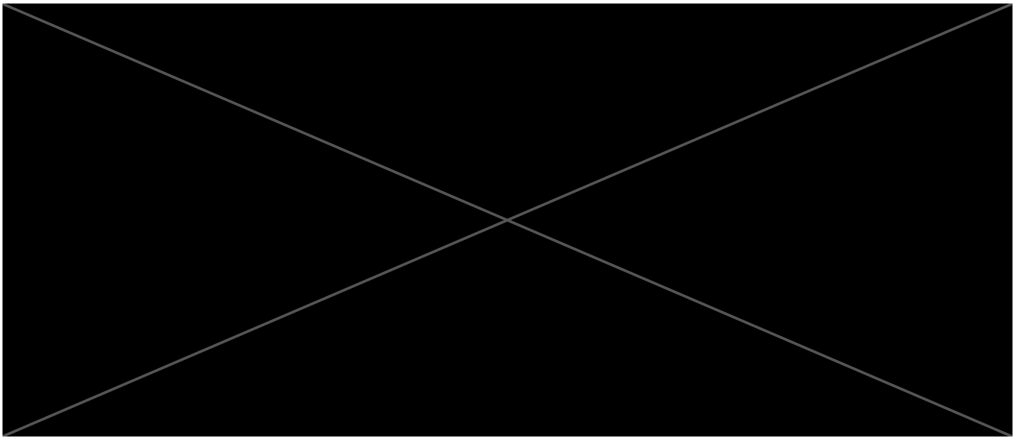
Two almost immediate consequences are stated in the following theorem:

Theorem 5.2. *Let* $u\in C^2\left(\mathbb{R}^n\times[0,+\infty)\right)$ *be a solution of the Cauchy problem (5.95). Then:*

(*a*) *If* $g\equiv h\equiv0$ *in* $B_{ct_0}\left(\mathbf{x}_0\right)$ *and* $f\equiv0$ *in* $C_{\mathbf{x}_0,t_0}$ *then* $u\equiv0$ *in* $C_{\mathbf{x}_0,t_0}$.

(*b*) *Problem (5.95) has at most one solution in* $C^2\left(\mathbb{R}^n\times[0,+\infty)\right)$.

[redacted]



9.4 The Wave Equation

9.4.1 Hyperbolic Equations

The wave propagation in a nonhomogeneous and anisotropic medium leads to second order *hyperbolic* equations. With the same notations of section 9.1, an equation in *divergence form* of the type

$$u_{tt} - \operatorname{div}(\mathbf{A}(\mathbf{x},t)\nabla u) + \mathbf{b}(\mathbf{x},t)\cdot\nabla u + c(\mathbf{x},t)\,u = f(\mathbf{x},t) \tag{9.50}$$

or in *non-divergence form* of the type

$$u_{tt} - \operatorname{tr}(\mathbf{A}(\mathbf{x},t)\,D^2 u) + \mathbf{b}(\mathbf{x},t)\cdot\nabla u + c(\mathbf{x},t)\,u = f(\mathbf{x},t) \tag{9.51}$$

is called **hyperbolic** in $Q_T = \Omega \times (0,T)$ if

$$\mathbf{A}(\mathbf{x},t)\,\boldsymbol{\xi}\cdot\boldsymbol{\xi} > 0 \qquad \text{a.e. } (\mathbf{x},t)\in Q_T, \forall\boldsymbol{\xi}\in\mathbb{R}^n,\ \boldsymbol{\xi}\neq\mathbf{0}.$$

The typical problems for hyperbolic equations are those already considered for the wave equation. Given f in Q_T, we want to determine a solution u of (9.50) or (9.51) satisfying the *initial* conditions

$$u(\mathbf{x},0) = g(\mathbf{x}), \quad u_t(\mathbf{x},0) = h(\mathbf{x}) \qquad \text{in } \Omega$$

and one of the usual boundary conditions (*Dirichlet, Neumann, mixed or Robin*) on the lateral boundary $S_T = \partial\Omega \times [0,T]$.

Even if from the phenomenological point of view, the hyperbolic equations display substantial differences from the parabolic ones, for *divergence form* equations it is possible to give a similar weak formulation,which can be analyzed by means of Faedo-Galerkin method. We will limit ourselves to the Cauchy-Dirichlet problem for the wave equation. For general equations, the theory is more complicated, unless we assume that the coefficients a_{jk}, entries of the matrix $\mathbf{A}$, are continuously differentiable with respect to both $\mathbf{x}$ and t.

9.4.2 The Cauchy-Dirichlet problem

Consider the problem

$$\begin{cases} u_{tt} - c^2 \Delta u = f & \text{in } Q_T \\ u(\mathbf{x},0) = g(\mathbf{x}), \; u_t(\mathbf{x},0) = g^1(\mathbf{x}) & \mathbf{x} \in \Omega \\ u(\boldsymbol{\sigma},t) = 0 & (\boldsymbol{\sigma},t) \in S_T. \end{cases} \tag{9.52}$$

To find an appropriate weak formulation, multiply the wave equation by a function $v = v(\mathbf{x})$, vanishing at the boundary, and integrate over Ω. We find

$$\int_\Omega u_{tt}(\mathbf{x},t)\, v(\mathbf{x})\, d\mathbf{x} - c^2 \int_\Omega \Delta u(\mathbf{x},t)\, v(\mathbf{x})\, d\mathbf{x} = \int_\Omega f(\mathbf{x},t)\, v(\mathbf{x})\, d\mathbf{x}.$$

Integrating by parts the second term, we get

$$\int_\Omega u_{tt}(\mathbf{x},t)\, v(\mathbf{x})\, d\mathbf{x} + c^2 \int_\Omega \nabla u(\mathbf{x},t) \cdot \nabla v(\mathbf{x})\, d\mathbf{x} = \int_\Omega f(\mathbf{x},t)\, v(\mathbf{x})\, d\mathbf{x} \tag{9.53}$$

which becomes, in the notations of the previous sections,

$$\int_\Omega \ddot{u}(t)\, v \; d\mathbf{x} + \alpha \int_\Omega \nabla u(t) \cdot \nabla v \; d\mathbf{x} = \int_\Omega f(t)\, v \; d\mathbf{x}$$

where $\ddot{u}$ stays for u_{tt}. Again the natural space for u is $L^2\left(0,T;H_0^1(\Omega)\right)$. Thus, a.e. $t > 0$, $u(t) \in V = H_0^1(\Omega)$, and $\Delta u(t) \in V^* = H^{-1}(\Omega)$. On the other hand, from the wave equation we have

$$u_{tt} = c^2 \Delta u + f.$$

If $f \in L^2(0,T;H)$, with $H = L^2(\Omega)$, it is natural to require $\ddot{u} \in L^2(0,T;V^*)$.

Accordingly, a reasonable assumption for $\dot{u}$ is $\dot{u} \in L^2(0,T;H)$, an intermediate space between $L^2(0,T;V)$ and $L^2(0,T;V^*)$. Thus, we look for solutions u such that

$$u \in L^2(0,T;V), \quad \dot{u} \in L^2(0,T;H), \quad \ddot{u} \in L^2(0,T;V^*). \tag{9.54}$$

It can be shown[12] that, if u satisfies (9.54), then,

$$u \in C([0,T];V) \qquad \text{and} \qquad \dot{u} \in C([0,T];H).$$

Thus, it is reasonable to assume $u(0) = g \in V$, $\dot{u}(0) = g^1 \in H$.

The above considerations lead to the following weak formulation.

Given $f \in L^2(0,T;V^*)$ and $g \in V$, $g^1 \in H$, determine $u \in L^2(0,T;V)$ such that

$$\dot{u} \in L^2(0,T;H), \quad \ddot{u} \in L^2(0,T;V^*)$$

and that:

[12] *Lions-Magenes*, Chapter 3, 1972.

1. *for all* $v \in V$ *and a.e.* $t \in [0,T]$,

$$\langle \ddot{u}(t), v\rangle_* + c^2 (\nabla u(t), \nabla v)_0 = (f(t), v)_0, \tag{9.55}$$

2. $u(0) = g$, $\dot{u}(0) = g^1$

Remark 9.7. Equation (9.55) may be interpreted in the sense of distributions in $\mathcal{D}'(0,T)$. First observe that, for every $v \in V$, the real function

$$w(t) = \langle \ddot{u}(t), v\rangle_*$$

is a distribution $\mathcal{D}'(0,T)$ and

$$w(t) = \frac{d^2}{dt^2}(u(t), v)_0 \qquad \text{in } \mathcal{D}'(0,T). \tag{9.56}$$

This means that for every $\varphi \in \mathcal{D}(0,T)$ we have

$$\int_0^T w(t)\varphi(t)\,dt = \int_0^T (u(t), v)\,\ddot{\varphi}(t)\,dt.$$

In fact, since $u(t) \in V^*$, we may write, thanks to Bochner's Theorem,

$$\begin{aligned}\int_0^T w(t)\varphi(t)\,dt &= \int_0^T \langle \ddot{u}(t), v\rangle_* \varphi(t)\,dt = \left\langle \int_0^T \ddot{u}(t)\varphi(t)\,dt, v\right\rangle_* \\ &= \left(\int_0^T u(t)\ddot{\varphi}(t)\,dt\ , v\right)_0 = \int_0^T (u(t), v))_0\ddot{\varphi}(t)\,dt.\end{aligned}$$

for all $\varphi \in \mathcal{D}(0,T)$. Since the last integral is well defined, $w \in L^1_{loc}(0,T)$ and therefore $w \in \mathcal{D}'(0,T)$. Moreover, by definition,

$$\int_0^T (u(t), v)\,\ddot{\varphi}(t)\,dt = \int_0^T \frac{d^2}{dt^2}(u(t), v)\ \ \varphi(t)\,dt$$

in $\mathcal{D}'(0,T)$, which is (9.56). As a consequence, (9.55) may be written in the form

$$\frac{d^2}{dt^2}(u(t), v)_0 + c^2 (\nabla w(t), \nabla v)_0 = (f(t), v)_0 \tag{9.57}$$

for all $v \in V$ *and in the sense of distributions in* $[0,T]$.

Remark 9.8. We leave it to the reader to check that if a *weak* solution u is *smooth*, i.e. $u \in C^2\left(\overline{Q}_T\right)$, then u is a classical solution.

9.4.3 Faedo-Galerkin method (III)

We want to show that problem (9.52) has a unique weak solution, which continuously depends on the data, in appropriate norms. Once more, we are going to use the method of Faedo-Galerkin, so that we briefly review the main steps, emphasizing the differences with the parabolic case.

1. We select a sequence of smooth functions $\{w_k\}_{k=1}^{\infty}$ constituting

an *orthogonal basis in* V

and

an *orthonormal basis in* H.

In particular, we can write

$$g = \sum_{k=1}^{\infty} g_k w_k, \qquad g^1 = \sum_{k=1}^{\infty} g_k^1 w_k$$

where $g_k = (g, w_k)_0$, $g_k^1 = \left(g^1, w_k\right)_0$, with the series converging in H.

2. Let

$$V_m = \text{span}\{w_1, w_2, ..., w_m\}$$

and

$$u_m(t) = \sum_{k=1}^{m} r_k(t)\, w_k, \qquad G_m = \sum_{k=1}^{m} g_k w_k, \qquad G_m^1 = \sum_{k=1}^{m} g_k^1 w_k.$$

We construct the sequence of Galerkin approximations u_m by solving the following *projected* problem:

Determine $u_m \in H^2(0,T;V)$ *such that, for all* $s = 1, ..., m$,

$$\begin{cases} (\ddot{u}_m(t), w_s)_0 + c^2 (\nabla u_m(t), \nabla w_s)_0 = (f(t), w_s)_0, & 0 \le t \le T \\ u_m(0) = G_m, \quad \dot{u}_m(0) = G_m^1. \end{cases} \tag{9.58}$$

Note that the differential equation in (9.58) is true for each element of the basis w_s, $s = 1, ..., m$, if and only if it is true for every $v \in V_m$. Moreover, since $u_m \in H^2(0,T;V)$ we have $\ddot{u}_m \in L^2(0,T;V)$, so that

$$(\ddot{u}_m(t), v)_0 = \langle \ddot{u}_m(t), v \rangle_* .$$

3. We show that $\{u_m\}$, $\{\dot{u}_m\}$ and $\{\ddot{u}_m\}$ are bounded in $L^2(0,T;V)$, $L^2(0,T;H)$ and $L^2(0,T;V^*)$, respectively (*energy estimates*). Then, the weak compactness Theorem 6.11 implies that a subsequence $\{u_{m_k}\}$ converges weakly in $L^2(0,T;V)$ to u, while $\{\dot{u}_{m_k}\}$ and $\{\ddot{u}_{m_k}\}$ converge weakly in $L^2(0,T;H)$ and $L^2(0,T;V^*)$ to $\dot{u}$ and $\ddot{u}$.

4. We prove that u in step **3** is the unique weak solution of problem (9.52).

9.4.4 Solution of the approximate problem

The following lemma holds.

Lemma 9.2. *For all $m \geq 1$, there exists a unique solution to problem (9.58). In particular, since $u_m \in H^2(0,T;V)$, we have $u_m \in C^1([0,T];V)$.*

Proof. Observe that, since w_1, w_2, ...,w_m are orthonormal in H,

$$(\ddot{u}_m(t), w_s)_0 = \left(\sum_{k=1}^m \ddot{r}_k(t) w_k, w_s\right)_0 = \ddot{r}_k(t)$$

and since they are orthogonal in V,

$$c^2 \left(\sum_{k=1}^m r_k(t) \nabla w_k, \nabla w_s\right)_0 = c^2 (\nabla w_s, \nabla w_s)_0 r_s(t) = c^2 \|\nabla w_s\|_0^2 r_s(t).$$

Set

$$F_s(t) = (f(t), w_s), \qquad \mathbf{F}(t) = (F_1(t), ..., F_m(t))$$

and

$$\mathbf{R}_m(t) = (r_1(t), ..., r_m(t)), \quad \mathbf{g}_m = (g_1, ..., g_m), \quad \mathbf{g}_m^1 = \left(g_1^1, ..., g_m^1\right).$$

If we introduce the diagonal matrix

$$\mathbf{W} = \text{diag}\left\{\|\nabla w_1\|_0^2, \|\nabla w_2\|_0^2, ..., \|\nabla w_m\|_0^2\right\}$$

of order m, problem (9.58) is equivalent to the following system of m *uncoupled* linear ordinary differential equations, with constant coefficients:

$$\ddot{\mathbf{R}}_m(t) = -c^2 \mathbf{W} \mathbf{R}_m(t) + \mathbf{F}_m(t), \qquad \text{a.e. } t \in [0,T] \tag{9.59}$$

with initial conditions

$$\mathbf{R}_m(0) = \mathbf{g}_m. \quad \dot{\mathbf{R}}_m(0) = \mathbf{g}_m^1.$$

Since $F_s \in L^2(0,T)$, for all $s = 1, ..., m$, system (9.59) has a unique solution $\mathbf{R}_m(t) \in H^2(0,T;\mathbb{R}^m)$. From

$$u_m(t) = \sum_{k=1}^m r_k(t) w_k,$$

we deduce $u_m \in H^2(0,T;V)$. □

9.4.5 Energy estimates

We want to show that from the sequence of Galerkin approximations $\{u_m\}$ it is possible to extract a subsequence converging to the weak solution of the original problem. As in the parabolic case, we are going to prove that the relevant Sobolev norms of u_m can be controlled by the norms of the data, **in a way that does not depend on** m. Moreover, the estimates must be powerful enough in order to pass to the limit as $m \to +\infty$ in the approximating equation.

$$(\ddot{u}_m(t), v)_0 + c^2 (\nabla u_m(t), \nabla v)_0 = (f(t), v)_0 .$$

In this case we can give a bound of the norms of u_m in $L^\infty(0,T;V)$, of $\dot{u}_m$ in $L^\infty(0,T;H)$ and of $\ddot{u}$ in $L^2(0,T;V^*)$, that is the norms

$$\max_{t\in[0,T]} \|u_m\|_1 , \qquad \max_{t\in[0,T]} \|\dot{u}_m\|_0 \quad \text{and} \quad \int_0^T \|\ddot{u}_m(t)\|_*^2 \, dt.$$

For the proof, we shall use the following elementary but very useful lemma.

Lemma 9.3. (Gronwall). *Let Ψ, G be continuous in $[0,T]$, with G nondecreasing and $\gamma > 0$. If*

$$\Psi(t) \le G(t) + \gamma \int_0^t \Psi(s)\, ds, \qquad \textit{for all } t \in [0,T]$$

then

$$\Psi(t) \le G(t)\, e^{\gamma t}, \qquad \textit{for all } t \in [0,T] .$$

Proof. Let

$$R(s) = \gamma \int_0^s \Psi(r)\, dr.$$

Then, for all $t \in [0,T]$,

$$R'(s) = \gamma \Psi(s) \le \gamma \left[G(s) + \gamma \int_0^s \Psi(r)\, dr \right] = \gamma \left[G(s) + R(s) \right].$$

Multiplying both sides by $\exp(-\gamma t)$, we can write the above inequality in the form

$$\frac{d}{ds} \left[R(s) \exp(-\gamma t) \right] \le \gamma G(s) \exp(-\gamma t) .$$

Integrating over $(0,t)$ gives $(R(0) = 0)$:

$$R(t) \le \gamma \int_0^t G(s)\, e^{\gamma(t-s)} ds \le G(t)\, e^{\gamma t}, \qquad \textit{for all } t \in [0,T] .$$

□

Theorem 9.10. (Estimate of $u_m, \dot{u}_m$). *Let u_m be the solution of problem (9.58). Then*

$$\max_{t\in[0,T]} \left\{ \|\dot{u}_m(t)\|_0^2 + 2c^2 \|u_m(t)\|_1^2 \right\} \le e^T \left\{ \|g\|_1^2 + \left\|g^1\right\|_0^2 + \|f\|_{L^2(0,T;H)}^2 \right\}. \quad (9.60)$$

Proof. Since $u_m \in H^2(0,T;V)$, we may choose $v = \dot{u}_m(t)$ as a test function in (9.58). We find

$$(\ddot{u}_m(t), \dot{u}_m(t))_0 + c^2 (\nabla u_m(t), \nabla \dot{u}_m(t))_0 = (f(t), \dot{u}_m(t))_0 \quad (9.61)$$

for a.e. $t \in [0,T]$. Observe that

$$(\ddot{u}_m(t), \dot{u}_m(t))_0 = \frac{1}{2}\frac{d}{dt} \|\dot{u}_m(t)\|_0^2, \qquad \text{a.e. } t \in (0,T)$$

and

$$(\nabla u_m(t), \nabla \dot{u}_m(t))_0 = c^2 \frac{d}{dt} \|\nabla u_m(t)\|_0^2.$$

By Schwarz's inequality,

$$(f(t), \dot{u}_m(t))_0 \le \|f(t)\|_0 \|\dot{u}_m(t)\|_0 \le \frac{1}{2}\|f(t)\|_0^2 + \frac{1}{2}\|\dot{u}_m(t)\|_0^2$$

so that, from (9.61) we deduce

$$\frac{d}{dt}\left\{ \|\dot{u}_m(t)\|_0^2 + 2c^2 \|u_m(t)\|_1^2 \right\} \le \|f(t)\|_0^2 + \|\dot{u}_m(t)\|_0^2.$$

Integrating over $(0,t)$ we get (Remark 7.34 applied to $\dot{u}_m$ and ∇u_m)

$$\begin{aligned} &\|\dot{u}_m(t)\|_0^2 + 2c^2 \|u_m(t)\|_1^2 \\ &\le \|G_m\|_1^2 + \left\|G_m^1\right\|_0^2 + \int_0^t \|f(s)\|_0^2 \, ds + \int_0^t \|\dot{u}_m(s)\|_0^2 \, ds \\ &\le \|g\|_1^2 + \left\|g^1\right\|_0^2 + \int_0^t \|f(s)\|_0^2 \, ds + \int_0^t \|\dot{u}_m(s)\|_0^2 \, ds, \end{aligned}$$

since

$$\|G_m\|_1^2 \le \|g\|_1^2, \quad \left\|G_m^1\right\|_0^2 \le \left\|g^1\right\|_0^2.$$

Let

$$\Psi(t) = \|\dot{u}_m(t)\|_0^2 + 2c^2 \|u_m(t)\|_1^2, \quad G(t) = \|g\|_1^2 + \left\|g^1\right\|_0^2 + \int_0^t \|f(s)\|_0^2 \, ds.$$

Note that both Ψ and G are continuous in $[0,T]$. Then

$$\Psi(t) \le G(t) + \int_0^t \Psi(s) \, ds$$

and Gronwall Lemma yields, for every $t \in [0, T]$,

$$\|\dot{u}_m (t)\|_0^2 + 2c^2 \|u_m (t)\|_1^2 \leq e^t \left\{ \|g\|_1^2 + \|h\|_0^2 + \int_0^t \|f\|_0^2 \, ds \right\}$$

□

We now give a control of the norm of $\ddot{u}_m$ in $L^2 (0, T; V^*)$.

Theorem 9.11. (Estimate of $\ddot{u}_m$). *Let u_m be the solution of problem (9.58). Then*

$$\int_0^T \|\ddot{u}_m (t)\|_*^2 \, dt \leq C(c, T) \left\{ \|g\|_1^2 + \left\|g^1\right\|_0^2 + \int_0^T \|f (s)\|_0^2 \, ds. \right\} \tag{9.62}$$

Proof. Let $v \in V$ and write

$$v = w + z$$

with $w \in V_m = \operatorname{span}\{w_1, w_2, ..., w_m\}$ and $z \in V_m^\perp$. Since $w_1, ..., w_k$ are orthogonal in V, we have

$$\|w\|_1 \leq \|v\|_1 .$$

Choosing w as a test function in problem (9.58), we obtain

$$(\ddot{u}_m (t) , v)_0 = (\ddot{u}_m (t) , w)_0 = -c^2 (\nabla u_m (t) , \nabla w)_0 + (f (t) ,w)_0 .$$

Since

$$|(\nabla u_m (t) , \nabla w)_0| \leq \|u_m (t)\|_1 \|w\|_1$$

we may write

$$\begin{aligned} |(\ddot{u}_m (t) , v)_0| &\leq \left\{c^2 \|u_m (t)\|_1 + \|f (t)\|_0\right\} \|w\|_1 \\ &\leq \left\{c^2 \|u_m (t)\|_1 + \|f (t)\|_0\right\} \|v\|_1 . \end{aligned}$$

Thus, by the definition of norm in V^*, we infer

$$\|\ddot{u}_m (t)\|_* \leq c^2 \|u_m (t)\|_1 + \|f (t)\|_0 .$$

Squaring and integrating over $(0, T)$ we obtain

$$\int_0^T \|\ddot{u}_m (t)\|_*^2 \, dt \leq 2c^4 \int_0^T \|u_m (t)\|_1^2 \, dt + 2 \int_0^T \|f (t)\|_0^2 \, dt$$

and Theorem 9.10 gives (9.62). □

9.4.6 Existence, uniqueness and stability

Theorem 9.10 shows that the sequence $\{u_m\}$ of Galerkin approximations is bounded in $L^\infty(0,T;V)$, hence, in particular, in $L^2(0,T;V)$, while the sequence $\{\ddot{u}_m\}$ is bounded in $L^2(0,T;V^*)$.

Theorem 6.11 implies that there exists a subsequence, which for simplicity we still denote by $\{u_m\}$, such that, as $m \to \infty$,

$$\begin{aligned} u_m &\rightharpoonup u \quad \textit{weakly in } L^2(0,T;V) \\ \dot{u}_m &\rightharpoonup \dot{u} \quad \textit{weakly in } L^2(0,T;H) \\ \ddot{u}_m &\rightharpoonup \ddot{u} \quad \textit{weakly in } L^2(0,T;V^*). \end{aligned}$$

The following theorem holds:

Theorem 9.12. *Let $f \in L^2(0,T;H)$, $g \in V$, $g^1 \in H$. Then u is the unique weak solution of problem (9.52). Moreover,*

$$\|u\|^2_{L^\infty(0,T;V)} + \|\dot{u}\|^2_{L^\infty(0,T;H)} + \|\ddot{u}\|^2_{L^2(0,T;V^*)} \le C\left\{\|f\|^2_{L^2(0,T;H)} + \|g\|_1^2 + \left\|g^1\right\|_0^2\right\}$$

with $C = C(c,T)$.

Proof. **Existence**. We know that:

$$\int_0^T (\nabla u_m(t), \nabla v(t))_0 \; dt \to \int_0^T (\nabla u(t), \nabla v(t))_0 \; dt$$

for all $v \in L^2(0,T;V)$,

$$\int_0^T (\dot{u}_m(t), w(t))_0 \, dt \to \int_0^T (\dot{u}(t), w(t))_0 \, dt$$

for all $w \in L^2(0,T;H)$, and

$$\int_0^T (\ddot{u}_m(t), v(t))_0 = \int_0^T \langle \ddot{u}_m(t), v(t)\rangle_* \, dt \to \int_0^T \langle \ddot{u}(t), v(t)\rangle_* \, dt$$

for all $v \in L^2(0,T;V)$,

We want to use these properties to pass to the limit as $m \to +\infty$ in problem (9.58), keeping in mind that the test functions have to be chosen in V_m. Fix $v \in L^2(0,T;V)$; we may write

$$v(t) = \sum_{k=1}^{\infty} b_k(t)\, w_k$$

where the series converges in V for a.e. $t \in [0,T]$. Let

$$v_N(t) = \sum_{k=1}^{N} b_k(t)\, w_k \tag{9.63}$$

and keep N fixed, for the time being. If $m \geq N$, then $v_N \in L^2(0,T;V_m)$. Multiplying equation (9.58) by $b_k(t)$ and summing for $k = 1, ..., N$, we get

$$(\ddot{u}_m(t), v_N(t))_0 + c^2 (\nabla u_m, \nabla v_N)_0 = (f(t), v_N(t))_0 .$$

An integration over $(0,T)$ yields

$$\int_0^T \left\{ (\ddot{u}_m, v_N)_0 + c^2 (\nabla u_m, \nabla v_N)_0 \right\} dt = \int_0^T (f, v_N)_0 \, dt. \tag{9.64}$$

Thanks to the weak convergence of u_m and $\ddot{u}_m$ in their respective spaces, we can let $m \to +\infty$. Since

$$(\ddot{u}_m(t), v_N(t))_0 = \langle \ddot{u}_m(t), v_N(t) \rangle_* \to \langle \ddot{u}(t), v_N(t) \rangle_* ,$$

we obtain

$$\int_0^T \left\{ \langle \ddot{u}, v_N \rangle_* + c^2 (\nabla u, \nabla v_N)_0 \right\} dt = \int_0^T (f, v_N)_0 \, dt.$$

Now, let $N \to \infty$, observing that $v_N \to v$ in $L^2(0,T;V)$ and, in particular, weakly in this space as well. We obtain

$$\int_0^T \left\{ \langle \ddot{u}(t), v(t) \rangle_* + c^2 (\nabla u(t), \nabla v(t))_0 \right\} dt = \int_0^T (f(t), v(t))_0 \, dt. \tag{9.65}$$

Then, (9.65) is valid for all $v \in L^2(0,T;V)$. This entails, in particular (see footnote 7),

$$\langle \ddot{u}(t), v \rangle_* + c^2 (\nabla u(t), \nabla v)_0 \, dt = (f(t), v)_0$$

for all $v \in V$ and a.e. $t \in [0,T]$. Therefore u satisfies (9.55) and we know that $u \in C([0,T];V)$, $\dot{u} \in C([0,T];H)$.

To check the initial conditions, we proceed as in Theorem 9.3. We choose any function $v \in C^2([0,T];V)$, with $v(T) = \dot{v}(T) = 0$. Integrating by parts twice in (9.65), we find

$$\int_0^T \left\{ \langle u(t), \ddot{v}(t) \rangle_* + c^2 (\nabla u(t), \nabla v(t))_0 \right\} dt \tag{9.66}$$
$$= \int_0^T (f(t), v(t))_0 \, dt + (\dot{u}(0), \dot{v}(0)) - (u(0), v(0)).$$

On the other hand, integrating by parts twice in (9.64), and letting first $m \to +\infty$, then $N \to \infty$, we deduce

$$\int_0^T \left\{ \langle u(t), \ddot{v}(t) \rangle_* + c^2 (\nabla u(t), \nabla v(t))_0 \right\} dt \tag{9.67}$$
$$= \int_0^T (f(t), v(t))_0 \, dt + \left(g^1, \dot{v}(0)\right) - (g, v(0)).$$

Comparing (9.66) and (9.67), we conclude

$$(\dot{u}(0), \dot{v}(0)) - (u(0), v(0)) = \left(g^1, \dot{v}(0)\right) - (g, v(0))$$

for every $v \in C^2([0,T];V)$, with $v(T) = \dot{v}(T) = 0$. The arbitrariness of $\dot{v}(0)$ and $v(0)$ gives

$$\dot{u}(0) = g^1 \quad \text{and} \quad u(0) = g.$$

Uniqueness. Assume $g = g^1 \equiv 0$ and $f \equiv 0$. We want to show that $u \equiv 0$. The proof would be easy if we could choose $\dot{u}$ as a test function in (9.55), but $\dot{u}(t)$ does not belong to V. Thus, for fixed s, set[13]

$$v(t) = \begin{cases} \int_t^s u(r)\,dr & \text{if } 0 \le t \le s \\ 0 & \text{if } s \le t \le T. \end{cases}$$

We have $v(t) \in V$ for all $t \in [0,T]$, so that we may insert it into (9.55). After an integration over $(0,T)$, we deduce

$$\int_0^s \left\{ \langle \ddot{u}(t), v(t) \rangle_* + c^2 (\nabla u(t), \nabla v(t))_0 \right\} dt = 0. \tag{9.68}$$

An integration by parts yields

$$\begin{aligned} \int_0^s \langle \ddot{u}(t), v(t) \rangle_* \,dt &= -\int_0^s (\dot{u}(t), \dot{v}(t))_0 \,dt = \int_0^s (\dot{u}(t), u(t))_0 \,dt \\ &= \frac{1}{2} \int_0^s \frac{d}{dt} \|u(t)\|_0^2 \,dt \end{aligned}$$

since $v(s) = \dot{u}(0) = 0$ and $\dot{v}(t) = -u(t)$ if $0 < t < s$. On the other hand,

$$\int_0^s (\nabla u(t), \nabla v(t))_0 \,dt = -\int_0^s (\nabla \dot{v}(t), \nabla v(t))_0 \,dt = -\frac{1}{2} \int_0^s \frac{d}{dt} \|\nabla v(t)\|_0^2 \,dt.$$

Hence, from (9.68),

$$\int_0^s \frac{d}{dt} \left\{ \|u(t)\|_0^2 - c^2 \|\nabla v(t)\|_0^2 \right\} dt = 0$$

or

$$\|u(s)\|_0^2 + c^2 \|\nabla v(0)\|_0^2 = 0$$

which entails $u(s) \equiv 0$.

Stability. To prove the estimate in Theorem 9.12, use Proposition 7.16. to pass to the limit as $m \to \infty$ in (9.60). This gives the estimates for u and $\dot{u}$. The estimate for $\ddot{u}$ follows from the weak lower semicontinuity of the norm in $L^2(0,T;V^*)$. □

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[13] We follow *Evans*, 1998.