CS61B Lecture #15

Announcements:

- Please use `bug-submit` for code problems.
- Watch the newsgroup and class web site for updates, hints, useful new utilities, etc.

Readings for Today: *Data Structures (Into Java)*, Chapter 1;

Readings for next Topics: *Data Structures*, Chapter 2–4

What Are the Questions?

- Cost is a principal concern throughout engineering:

  "An engineer is someone who can do for a dime what any fool can do for a dollar."
- Cost can mean
  - Operational cost (for programs, time to run, space requirements).
  - Development costs: How much engineering time? When delivered?
  - Costs of failure: How robust? How safe?
- Is this program fast enough? Depends on:
  - *For what purpose;*
  - *What input data.*
- How much space (memory, disk space)?
  - Again depends on what input data.
- How will it scale, as input gets big?

Enlightening Example

Problem: Scan a text corpus (say 10^7 bytes or so), and find and print the 20 most frequently used words, together with counts of how often they occur.

- Solution 1 (Knuth): Heavy-Duty data structures
  - Hash Trie implementation, randomized placement, pointers galore, several pages long.
- Solution 2 (Doug McIlroy): UNIX shell script:

```
tr -c -s '[:alpha:]' '[\n*]' < FILE | \
sort | \
uniq -c | \
sort -n -r -k 1,1 | \
sed 20q
```

- Which is better?
  - #1 is much faster,
  - but #2 took 5 minutes to write and processes 20MB in 1 minute.
  - I pick #2.
- In most cases, anything will do: Keep It Simple.

Cost Measures (Time)

- Wall-clock or execution time
  - You can do this at home:

    `time java FindPrimes 1000`
  - Advantages: easy to measure, meaning is obvious.
  - Appropriate where time is critical (real-time systems, *e.g.*).
  - Disadvantages: applies only to specific data set, compiler, machine, etc.
- Number of times certain statements are executed:
  - Advantages: more general (not sensitive to speed of machine).
  - Disadvantages: doesn't tell you actual time, still applies only to specific data sets.
- Symbolic execution times:
  - That is, *formulas* for execution times or statement counts in terms of input size.
  - Advantages: applies to all inputs, makes scaling clear.
  - Disadvantage: practical formula must be approximate, may tell very little about actual time.

Asymptotic Cost

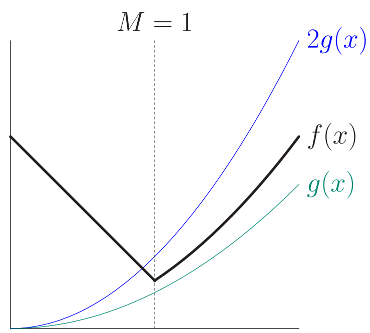
- Symbolic execution time lets us see *shape* of the cost function.
- Since we are approximating anyway, pointless to be precise about certain things:
  - Behavior on small inputs:
    * Can always pre-calculate some results.
    * Times for small inputs not usually important.
  - Constant factors (as in "off by factor of 2"):
    * Just changing machines causes constant-factor change.
- How to abstract away from (i.e., ignore) these things?

Handy Tool: Order Notation

- Idea: Don't try to produce *specific* functions that specify size, but rather *families* of *similar* functions.
- Say something like "f is bounded by g if it is in g's family."
- For any function $g(x)$, the functions $2g(x)$, $1000g(x)$, or for any $K > 0$, $K \cdot g(x)$, all have the same "shape". So put all of them into g's family.
- Any function $h(x)$ such that $h(x) = K \cdot g(x)$ for $x > M$ (for some constant M) has g's shape "except for small values." So put all of these in g's family.
- If we want upper limits, throw in all functions that are everywhere $\leq$ some other member of g's family. Call this family $O(g)$ or $O(g(n))$.
- Or, if we want lower limits, throw in all functions that are everywhere $\geq$ some other member of g's family. Call this family $\Omega(g)$.
- Finally, define $\Theta(g) = O(g) \cap \Omega(g)$—the set of functions *bracketed by* members of g's family.

Big Oh

- Goal: Specify bounding from above.



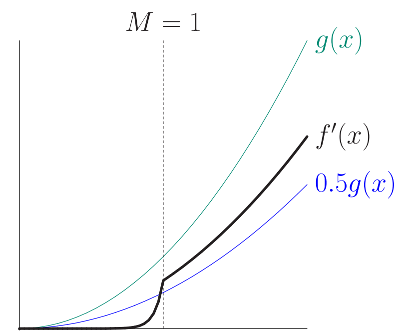
- Here, $f(x) \leq 2g(x)$ as long as $x > 1$,
- So $f(x)$ is in g's upper-bound family, written

$$f(x) \in O(g(x)),$$

- ... even though $f(x) > g(x)$ everywhere.

Big Omega

- Goal: Specify bounding from below:



- Here, $f'(x) \geq \frac{1}{2}g(x)$ as long as $x > 1$,
- So $f'(x)$ is in g's lower-bound family, written

$$f'(x) \in \Omega(g(x)),$$

- ... even though $f(x) < g(x)$ everywhere.
- In fact, we also have $f'(x) \in O(g(x))$ and $f(x) \in \Omega(g(x))$ and so we can also write

$$f(x), f'(x) \in \Theta(g(x)).$$

Why It Matters

- Computer scientists often talk as if constant factors didn't matter at all, only the difference of $\Theta(N)$ vs. $\Theta(N^2)$.
- In reality they do, but we still have a point: at some point, constants get swamped.

n	$16\lg n$	$\sqrt{n}$	n	$n\lg n$	n^2	n^3	2^n
2	16	1.4	2	2	4	8	4
4	32	2	4	8	16	64	16
8	48	2.8	8	24	64	512	256
16	64	4	16	64	256	$4,096$	$65,636$
32	80	5.7	32	160	1024	$32,768$	4.2×10^9
64	96	8	64	384	$4,096$	$262,144$	1.8×10^{19}
128	112	11	128	896	$16,384$	2.1×10^9	3.4×10^{38}
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
$1,024$	160	32	$1,024$	$10,240$	1.0×10^6	1.1×10^9	1.8×10^{308}
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
2^{20}	320	1024	1.0×10^6	2.1×10^7	1.1×10^{12}	1.2×10^{18}	$6.7\times 10^{315,652}$

Some Intuition on Meaning of Growth

- How big a problem can you solve in a given time?
- In the following table, left column shows time in microseconds to solve a given problem as a function of problem size N.
- Entries show the *size of problem* that can be solved in a second, hour, month (31 days), and century, for various relationships between time required and problem size.
- $N =$ problem size

Time (μsec) for problem size N	**Max N Possible in** 1 second	1 hour	1 month	1 century
$\lg N$	10^{300000}	$10^{1000000000}$	$10^{8\cdot 10^{11}}$	$10^{9\cdot 10^{14}}$
N	10^6	$3.6\cdot 10^9$	$2.7\cdot 10^{12}$	$3.2\cdot 10^{15}$
$N\lg N$	63000	$1.3\cdot 10^8$	$7.4\cdot 10^{10}$	$6.9\cdot 10^{13}$
N^2	1000	60000	$1.6\cdot 10^6$	$5.6\cdot 10^7$
N^3	100	1500	14000	150000
2^N	20	32	41	51

Using the Notation

- Can use this order notation for any kind of real-valued function.
- We will use them to describe cost functions. Example:

```
/** Find position of X in list L.  Return -1 if not found */
int find (List L, Object X) {
   int c;
   for (c = 0; L != null; L = L.next, c += 1)
      if (X.equals (L.head)) return c;
   return -1;
}
```

- Choose representative operation: number of `.equals` tests.
- If N is length of L, then loop does *at most* N tests: *worst-case time* is N tests.
- In fact, total # of instructions executed is roughly proportional to N in the worst case, so can also say worst-case time is $O(N)$, regardless of units used to measure.
- Use $N > M$ provision (in defn. of $O(\cdot)$) to handle empty list.

Careful!

- It's also true that the worst-case time is $O(N^2)$, since $N \in O(N^2)$ also: Big-Oh bounds are loose.
- The worst-case time is $\Omega(N)$, since $N \in \Omega(N)$, but that does *not* mean that the loop *always* takes time N, or even $K \cdot N$ for some K.
- Instead, we are just saying something about the *function* that maps N into the *largest possible* time required to process an array of length N.
- To say as much as possible about our worst-case time, we should try to give a Θ bound: in this case, we can: $\Theta(N)$.
- But again, that still tells us nothing about *best-case* time, which happens when we find `X` at the beginning of the loop. Best-case time is $\Theta(1)$.

Effect of Nested Loops

- Nested loops often lead to polynomial bounds:

```
for (int i = 0; i < A.length; i += 1)
   for (int j = 0; j < A.length; j += 1)
      if (i != j && A[i] == A[j])
         return true;
return false;
```

- Clearly, time is $O(N^2)$, where $N = $ `A.length`. *Worst-case time* is $\Theta(N^2)$.
- Loop is inefficient though:

```
for (int i = 0; i < A.length; i += 1)
   for (int j = i+1; j < A.length; j += 1)
      if (A[i] == A[j]) return true;
return false;
```

- Now worst-case time is proportional to

$$N-1+N-2+\ldots+1 = N(N-1)/2 \in \Theta(N^2)$$

(so asymptotic time unchanged by the constant factor).

Recursion and Recurrences: Fast Growth

- Silly example of recursion:

```
/** True iff X is a substring of S */
boolean occurs (String S, String X) {
  if (S.equals (X)) return true;
  if (S.length () <= X.length ()) return false;
  return
    occurs (S.substring (1), X) ||
    occurs (S.substring (0, S.length ()-1), X);
}
```

- In the worst case, both recursive calls happen.
- Define $C(N)$ to be the worst-case cost of `occurs(S,X)` for `S` of length N, `X` of fixed size N_0, measured in # of calls to `occurs`. Then

$$C(N) = \begin{cases} 1, & \text{if } N \le N_0, \\ 2C(N-1)+1 & \text{if } N > N_0 \end{cases}$$

- So $C(N)$ grows exponentially:

$$\begin{aligned} C(N) &= 2C(N-1)+1 = 2(2C(N-2)+1)+1 = \ldots = \underbrace{2(\cdots 2}_{N-N_0}\cdot 1+1)+\ldots+1 \\ &= 2^{N-N_0}+2^{N-N_0-1}+2^{N-N_0-2}+\ldots+1 = 2^{N-N_0+1}-1 \in \Theta(2^N) \end{aligned}$$

Binary Search: Slow Growth

```
/** True X iff is an element of S[L .. U]. Assumes
 *  S in ascending order, 0 <= L <= U-1 < S.length. */
boolean isIn (String X, String[] S, int L, int U) {
  if (L > U) return false;
  int M = (L+U)/2;
  int direct = X.compareTo (S[M]);
  if (direct < 0) return isIn (X, S, L, M-1);
  else if (direct > 0) return isIn (X, S, M+1, U);
  else return true;
}
```

- Here, worst-case time, $C(D)$, (as measured by # of string comparisons), depends on size $D = U - L + 1$.
- We eliminate `S[M]` from consideration each time and look at half the rest. Assume $D = 2^k - 1$ for simplicity, so:

$$\begin{aligned} C(D) &= \begin{cases} 0, & \text{if } D \le 0, \\ 1 + C((D-1)/2), & \text{if } D > 0. \end{cases} \\ &= \underbrace{1+1+\ldots+1}_{k} + 0 \\ &= k = \lceil \lg D \rceil \in \Theta(\lg D) \end{aligned}$$

Another Typical Pattern: Merge Sort

```
List sort (List L) {
  if (L.length () < 2) return L;
  Split L into L0 and L1 of about equal size;
  L0 = sort (L0);  L1 = sort (L1);
  return Merge of L0 and L1
}
```

Merge ("combine into a single, ordered list") takes time proportional to size of its result.

- Assuming that size of `L` is $N = 2^k$, worst-case cost function, $C(N)$, counting just merge time ($\propto$ # items merged):

$$\begin{aligned} C(N) &= \begin{cases} 1, & \text{if } N < 2; \\ 2C(N/2)+N, & \text{if } N \ge 2. \end{cases} \\ &= 2(2C(N/4)+N/2)+N \\ &= 4C(N/4)+N+N \\ &= 8C(N/8)+N+N+N \\ &= N\cdot 1 + \underbrace{N+N+\ldots+N}_{k=\lg N} \\ &= N + N\lg N \in \Theta(N \lg N) \end{aligned}$$

- In general, $\Theta(N \lg N)$ for arbitrary N (not just 2^k).

Amortization: Expanding Vectors

- When using array for expanding sequence, best to double size of array to grow it. Here's why.
- If array is size s, doubling its size and moving s elements to the new array takes time $\propto 2s$.
- Cost of inserting N items into array, doubling size as needed, starting with array size 1:

To Insert Item #	Resizing Cost	Cumulative Cost	Resizing Cost per Item	Array Size After Insertions
1	0	0	0	1
2	2	2	1	2
3 to 4	4	6	1.5	4
5 to 8	8	14	1.75	8
⋮	⋮	⋮	⋮	⋮
2^m+1 to 2^{m+1}	2^{m+1}	$2^{m+2}-2$	≈ 2	2^{m+1}

- If we spread out (*amortize*) the cost of resizing, we average about 2 time units on each item: "amortized insertion time is 2 units."
- So even though worst-case time for adding *one* element to array of N elements is $2N$, time to add N elements is $\Theta(N)$, ***not*** $\Theta(N^2)$.

Demonstrating Amortized Time: Potential Method

- To formalize the argument, associate a *potential*, $\Phi_i \geq 0$, to the i^{th} operation that keeps track of "saved up" time from cheap operations that we can "spend" on later expensive ones. Start with $\Phi_0 = 0$.
- Now define the *amortized cost* of the i^{th} operation as

$$a_i = c_i + \Phi_{i+1} - \Phi_i,$$

  where c_i is the real cost of the operation.
- On cheap operations, we artificially set $a_i > c_i$ and increase Φ ($\Phi_{i+1} > \Phi_i$).
- On expensive ones, we typically have $a_i \ll c_i$ and greatly decrease Φ (but don't let it go negative—may not be "overdrawn").
- We try to do all this so that a_i remains as we desired (e.g., $O(1)$ for expanding array), without allowing $\Phi_i < 0$.
- Requires that we choose a_i so that Φ_i always stays ahead of c_i.

Application to Expanding Arrays

- When adding to our array, the cost, c_i, of adding element #i when the array already has space for it is 1 unit.
- The array does not initially have space when adding items 1, 2, 4, 8, 16,...—in other words at item 2^n for all $n \geq 0$. So,
  - $c_i = 1$ if $i \geq 0$ and is not a power of 2; and
  - $c_i = 3i + 1$ (allocate $2i$ items, copy i items, and then add item #i) when i is a power of 2.
- So on each operation #2^n we're going to need to have saved up at least $3 \cdot 2^n$ units of potential to cover the expense, and we have the preceding 2^{n-1} operations to do it (the ones since the preceding doubling operation).
- To do so, just choose $a_i = 7$ (or could let $a_0 = 1, a_1 = 4$)
- Here's what happens:

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
c_i	1	4	7	1	13	1	1	1	25	1	1	1	1	1	1	1	49
a_i	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7
Φ_i	0	6	9	9	15	9	15	21	27	9	15	21	27	33	39	45	51