Deriving the Optimal Kelly Fraction for Gaussian Returns with Autocorrelation

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June 2025

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1 Introduction

The Kelly criterion provides a theoretically optimal approach to allocating capital across repeated investments with uncertain outcomes. When asset returns are independent and normally distributed, the Kelly fraction is relatively straightforward to compute. However, when returns are autocorrelated, as often observed in real-world financial time series, we must account for this serial dependence to avoid misestimating optimal leverage.

2 Theoretical Background

2.1 Expected Utility Theory

The Kelly criterion stems from expected utility theory, where an agent maximizes expected utility of final wealth. For log utility, this leads to maximizing the expected logarithmic return:

$$U(W_T) = \mathbb{E}[\log(W_T)]. \tag{1}$$

2.2 Capital Growth

Assume we invest a fraction f of wealth at each time t in an asset with return R_t. Then wealth evolves as:

$$W_{t+1} = W_t(1 + fR_t). \tag{2}$$

Cumulative log wealth over T steps is:

$$\log W_T = \sum_{t=1}^{T} \log(1 + fR_t). \tag{3}$$

2.3 Key Definitions

- **Return (R_t)**: The percentage change in price from t to $t + 1$, modeled as a stochastic process.
- **Kelly Fraction (f^*)**: Optimal fraction of wealth to bet that maximizes $\mathbb{E}[\log(1+fR)]$.
- **Autocorrelation**: Correlation of a time series with a lagged version of itself.
- **AR(1) Process**: First-order autoregressive process defined as $R_t = \mu+\phi(R_{t-1}-\mu)+\epsilon_t$.

3 Classical Kelly Criterion for i.i.d. Gaussian Returns

Assuming $R_t \sim \mathcal{N}(\mu, \sigma^2)$ and independent across t, we expand $\log(1 + fR_t)$ using a Taylor series:

$$\log(1 + fR_t) \approx fR_t - \frac{1}{2}f^2R_t^2. \tag{4}$$

Taking expectations, we get:

$$\mathbb{E}[\log(1 + fR)] \approx f\mu - \frac{1}{2}f^2(\mu^2 + \sigma^2). \tag{5}$$

Maximizing this with respect to f gives:

$$f^* = \frac{\mu}{\mu^2 + \sigma^2}. \tag{6}$$

For small μ, this simplifies to the familiar form:

$$f^* \approx \frac{\mu}{\sigma^2}. \tag{7}$$

4 Autocorrelated Gaussian Returns

4.1 Autoregressive (AR) Models

We consider an AR(1) model:

$$R_t = \mu + \phi(R_{t-1} - \mu) + \epsilon_t, \quad \epsilon_t \sim \mathcal{N}(0, \sigma_\epsilon^2). \tag{8}$$

4.2 Variance and Autocorrelation

For this process, the variance of R_t is:

$$\sigma_R^2 = \frac{\sigma_\epsilon^2}{1 - \phi^2}. \tag{9}$$

- $\phi = 0$: i.i.d. returns
- $\phi > 0$: positively autocorrelated
- $\phi < 0$: negatively autocorrelated

5 Expected Log Return under Autocorrelation

As before, we use the second-order expansion:

$$\log(1 + fR_t) \approx fR_t - \frac{1}{2}f^2R_t^2, \tag{10}$$

$$\mathbb{E}[\log(1 + fR_t)] \approx f\mu - \frac{1}{2}f^2(\mu^2 + \sigma_R^2). \tag{11}$$

6 Deriving the Optimal Kelly Fraction

We maximize the expected log return:

$$g(f) = f\mu - \frac{1}{2}f^2(\mu^2 + \sigma_R^2). \tag{12}$$

Setting derivative to zero:

$$g'(f) = \mu - f(\mu^2 + \sigma_R^2) = 0, \tag{13}$$

$$f^* = \frac{\mu}{\mu^2 + \sigma_R^2}. \tag{14}$$

7 Impact of Autocorrelation

From the previous section:

$$f^* = \frac{\mu}{\mu^2 + \frac{\sigma_\epsilon^2}{1-\phi^2}}. \tag{15}$$

As ϕ increases, $1 - \phi^2$ decreases, making the denominator larger and reducing f^*.

8 Python Simulation and Visualization

```
import numpy as np
import matplotlib.pyplot as plt

def kelly_fraction(mu=0.01, sigma_eps=0.02, phi=0.5):
    sigma_R2 = sigma_eps**2 / (1 - phi**2)
    return mu / (mu**2 + sigma_R2)

phis = np.linspace(0, 0.99, 100)
kelly_vals = [kelly_fraction(phi=phi) for phi in phis]

```

```
plt.plot(phis, kelly_vals)
plt.title("Effect of Autocorrelation on Optimal Kelly Fraction")
plt.xlabel("Autocorrelation Coefficient (phi)")
plt.ylabel("Optimal Kelly Fraction")
plt.grid(True)
plt.show()
```

Listing 1: Effect of Autocorrelation on Kelly Fraction

9 Conclusion

We derived the optimal Kelly leverage for autocorrelated Gaussian returns modeled by an AR(1) process. Positive autocorrelation inflates return variance, leading to lower optimal leverage. This understanding is essential when applying Kelly strategies in real markets.