# Curvature of a discrete curve in 3D space

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Assume that the curve is given as list of x, y and z coordinates and let $\mathbf{P}_i = [x_i, y_i. z_i]$. The list is ordered so that moving along the list corresponds to moving along the curve. The circle passing through all three corners of the triangle formed by the neighboring points $\mathbf{P}_{i-1}$, $\mathbf{P}_i$ and $\mathbf{P}_{i+1}$ is called the circumscribed circle of the triangle. Let $\boldsymbol{M_i}$ be the center and $R_i$ the radius of this circle. The *curvature* of the curve at $\boldsymbol{P}_i$ is defined as:

$$\kappa_i = \frac{1}{R_i} \tag{1}$$

We define the *curvature vector* $\mathbf{k}_i$ as the vector of length $\kappa_i$ in the direction from $\mathbf{P}_i$ to $\boldsymbol{M_i}$. In the next paragraph we develop equations for the circumscribed circle for a general triangle ABC.

## Center of circle through three given points in 3D space

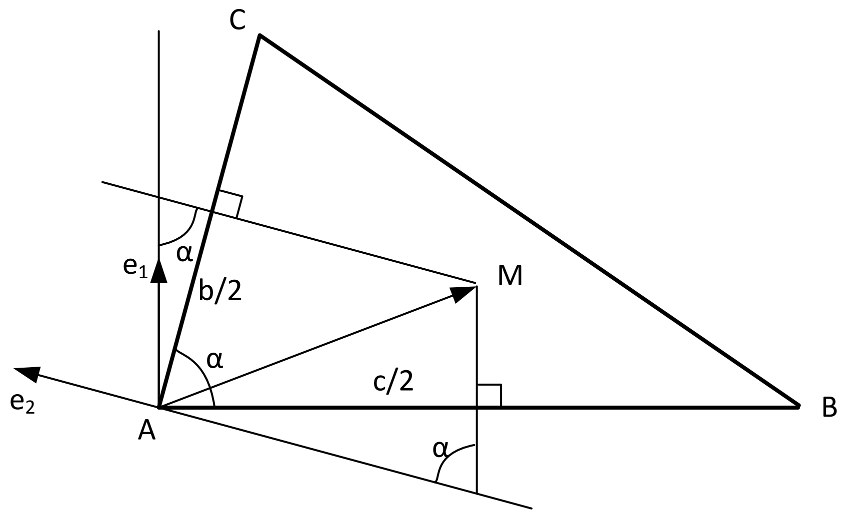


**Figure 1**

Consider a triangle in 3D space with corners A, B and C. The three points define a plane and Figure 1 shows the triangle in this plane. The center M of the circumscribed circle is equally distant from all corners and is therefore located at the intersection of the normals at the midpoint of sides AC and AB. The length of the side opposite to point B is denoted *b*- Likewise, *c* the length of side AB. Let α be the angle between sides AB and AC. Let the boldface symbols $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ denote the vectors to the corner points in some Cartesian coordinate system.

The cross product

$$\mathbf{D} = (\mathbf{B} - \mathbf{A}) \times (\mathbf{C} - \mathbf{A}) \tag{2}$$

is normal to the triangle plane, pointing towards the viewer. The norm of $\mathbf{D}$ is twice the area of the triangle:

$$\|\mathbf{D}\| = bc\sin(\alpha) \tag{3}$$

The cross product $\mathbf{E} = \mathbf{D} \times (\mathbf{B} - \mathbf{A})$ is normal to $\mathbf{D}$ and is thus in the plane of the triangle. It is also normal to AB and points upward in the drawing. It has norm:

$$||\mathbf{E}|| = bc^2 \sin(\alpha) \tag{4}$$

In the same manner, $\mathbf{F} = \mathbf{D} \times (\mathbf{C} - \mathbf{A})$ is in the plane of the triangle and normal to side AB, pointing to the left and upward in the drawing. The norm is:

$$||\mathbf{F}|| = b^2 c \sin(\alpha) \tag{5}$$

The unit vectors along $\mathbf{E}$ and $\mathbf{F}$ are shown in Figure 1:

$$\mathbf{e}_1 = \frac{\mathbf{E}}{||\mathbf{E}||} = \frac{\mathbf{E}}{bc^2 \sin(\alpha)} \tag{6}$$

$$\mathbf{e}_2 = \frac{\mathbf{F}}{||\mathbf{F}||} = \frac{\mathbf{F}}{b^2 c \sin(\alpha)} \tag{7}$$

The components of $\boldsymbol{G} = \boldsymbol{M} - \boldsymbol{A}$ along the two vectors are:

$$G_1 = \frac{b}{2\sin(\alpha)}\mathbf{e}_1 = \frac{\mathbf{E}}{\mathbf{2}c^2 \sin(\alpha)^2} = \frac{b^2\mathbf{E}}{2||\mathbf{D}||^2}$$

$$G_2 = \frac{-c}{2\sin(\alpha)}\mathbf{e}_1 = \frac{-\mathbf{F}}{\mathbf{2}b^2 \sin(\alpha)^2} = \frac{-\mathrm{c}^2\mathbf{F}}{2||\mathbf{D}||^2}$$

So:

$$\mathbf{G} = \frac{b^2\mathbf{E} - \mathrm{c}^2\mathbf{F}}{2||\mathbf{D}||^2} \tag{8}$$

Or:

$$\mathbf{M} = \mathbf{A} + \frac{\mathrm{b}^2\mathbf{D} \times (\mathbf{B} - \mathbf{A}) - c^2\mathbf{D} \times (\mathbf{C} - \mathbf{A})}{2||\mathbf{D}||^2} \tag{9}$$

The radius R of the circumscribed circle is:

$$\mathrm{R} = ||\mathbf{G}|| \tag{10}$$

Note that $\mathbf{G}$ depends only on the differences between the points $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ and not on the absolute vectors. $\mathbf{G}$ is therefore invariant under translations and rotations. It depends only on the proportions of the triangle and not on the absolute position or orientation.

**Alternative expression for R:**

Consider the triangle ABC in Figure 2. The line BD goes through the center M of the circumscribed circle. Angles BAC and BDC are equal, since they are inscribed angles covering the same arc BC. Therefore

$$\frac{a}{2R} = \sin(\alpha) \tag{11}$$

The area of ABC is

$$Area = \frac{1}{2} bc \sin(\alpha) = \frac{abc}{4R} \tag{12}$$

Or:

$$R = \frac{abc}{4 \cdot Area} = \frac{abc}{2\|\mathbf{D}\|} \tag{13}$$

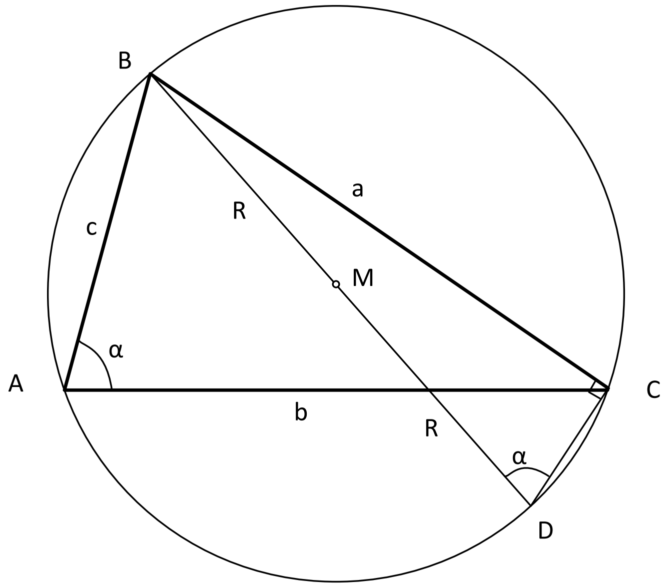


**Figure 2**