

# Chapter 3: Techniques from Group Cohomology

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Central Simple Algebras and Galois Cohomology

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## Section 3.1

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- $A$  is trivial if  $G$  acts trivially.
- $A^G := \{m \in A : g.m = m\}$
- Morphism  $A \longrightarrow B$  is morphism of abelian groups and  $G$ -linear

# Cohomology basics

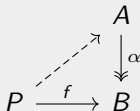
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# Cohomology basics

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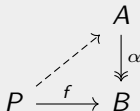
$$\begin{array}{ccccccc} & \ker\alpha & & \ker\beta & & \ker\gamma & \\ & \downarrow & & \downarrow & & \downarrow & \\ & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow 0 \\ & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & \\ 0 & \longrightarrow & D & \longrightarrow & E & \longrightarrow & F \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \operatorname{coker}\alpha & & \ker\beta & & \ker\gamma \end{array}$$

# Projective Modules



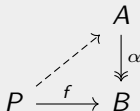


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$$\dots \longrightarrow P_3 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A$$

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- Short exact sequences of  $G$ -modules induce long exact sequences
- $H^i(G, A) := H^i(\text{Hom}_G(P_i, A))$

## Lemma 1

$$A^G \cong \operatorname{Hom}_G(\mathbb{Z}, A) \cong \ker(\operatorname{Hom}(P_0, A) \longrightarrow \operatorname{Hom}(P_1, A)) := H^0(G, A)$$

## Proof

$$\phi \in \operatorname{Hom}_G(\mathbb{Z}, A) \iff \phi(1) \in A^G \text{ since } m = \phi(1) = \phi(g.1) = g.\phi(1) = g.m$$

$$\begin{array}{ccccccc}
 & & & & & & \mathbb{Z} \\
 & & & & & \swarrow & \\
 & & & & & i & \\
 \dots & \longrightarrow & P_1 & \xrightarrow{p_1} & P_0 & \xrightarrow{p_0} & \mathbb{Z} \\
 & & & & \searrow \lambda_0 & \downarrow \phi & \\
 & & & & & A & 
 \end{array}$$

$$\operatorname{Hom}(P_0, A) \longrightarrow \operatorname{Hom}(P_1, A) \longrightarrow \operatorname{Hom}(P_2, A) \longrightarrow \dots \square$$

## Lemma 2

$$A \longrightarrow B \text{ induces } H^i(G, A) \longrightarrow H^i(G, B)$$

## Proof

$$\begin{array}{ccccccc} \dots & \longrightarrow & P_0 & \longrightarrow & \mathbb{Z} & & \\ & & & & \downarrow & & \\ & & & & A & & \\ & & & & \downarrow & & \\ & & & & B & & \\ & & & & & & \\ & & & & & & \\ Hom_G(P_0, A) & \longrightarrow & Hom_G(P_1, A) & \longrightarrow & \dots & & \\ & & \downarrow & & \downarrow & & \\ Hom_G(P_0, B) & \longrightarrow & Hom_G(P_1, B) & \longrightarrow & \dots & \square & \end{array}$$



## Corollary

A short exact sequence of  $G$ -modules

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

induce

$$\dots \longrightarrow H^i(G, A) \longrightarrow H^i(G, B) \longrightarrow H^i(G, C) \longrightarrow H^{i+1}(G, A) \longrightarrow \dots$$

## Section 3.2

- More general

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- .. but not concrete enough

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- Define  $\delta_i : \mathbb{Z}[G^{i+1}] \longrightarrow \mathbb{Z}[G^i]$  by  $\delta_i = \sum_j (-1)^j s_j^i$  where  $s_j^i(\sigma_0, \dots, \sigma_i) = (\sigma_0, \dots, \hat{\sigma}_j, \dots, \sigma_i)$

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- $\dots \longrightarrow \mathbb{Z}[G^2] \xrightarrow{\delta_2} \mathbb{Z}[G^1] \xrightarrow{\delta_1} \mathbb{Z}[G^0] \xrightarrow{\delta_0} \mathbb{Z}$

### Lemma 3

$$\delta_i \circ \delta_{i+1} = 0$$

### Proof

$$\delta_i \circ \delta_{i+1} = \begin{cases} \sum_k \sum_j (-1)^k s_k^i \left( (-1)^j s_j^{i+1} \right) & \text{with } j < k \\ + \sum_k \sum_j (-1)^k s_k^i \left( (-1)^{j-1} s_j^{i+1} \right) & \text{with } k < j \end{cases}$$

# Fact

$$\begin{array}{ccccccc}
 \dots & \longrightarrow & C_{i+1} & \xrightarrow{\partial_{i+1}} & C_i & \xrightarrow{\partial_i} & C_{i-1} \longrightarrow \dots \\
 & & \downarrow f_{i+1}, g_{i+1} & \swarrow p_i & \downarrow f_i, g_i & \swarrow p_{i-1} & \downarrow f_{i-1}, g_{i-1} \\
 \dots & \longrightarrow & D_{i+1} & \xrightarrow{\partial'_{i+1}} & D_i & \xrightarrow{\partial'_i} & D_{i-1} \longrightarrow \dots
 \end{array}$$

$f$  is said to be **homotopic** to  $g$  if there exist maps  $p$  such that

$$\partial'_{i+1} \circ p_i + p_{i-1} \circ \partial_i = g_i - f_i$$

Chain homotopic maps preserve homology

- Now fix  $\sigma \in \mathbb{Z} [G^{i+1}]$  and define  $h_i : \mathbb{Z} [G^{i+1}] \longrightarrow \mathbb{Z} [G^{i+2}]$  by  
 $h_i (\sigma_0, \dots \sigma_i) = (\sigma, \sigma_0, \dots \sigma_i)$
- Note that  $\delta_{i+1} \circ h_i + h_{i-1} \circ \delta_i = id_{\mathbb{Z}[G^{i+1}]}$

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$$\left( \sum_k (-1)^k s_k^{i+1} \right) h_i (\sigma_0, \dots \sigma_i) + h_{i-1} \circ \sum_j (-1)^j s_j^i (\sigma_0, \dots, \sigma_i)$$

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 &= (\sigma_0, \dots, \sigma_i)
 \end{aligned}$$



Finally..

$$\operatorname{Hom}_G(\mathbb{Z}[G], A) \longrightarrow \operatorname{Hom}_G(\mathbb{Z}[G^2], A) \longrightarrow \operatorname{Hom}_G(\mathbb{Z}[G^3], A) \longrightarrow \dots$$

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$$i\text{-coboundaries} \in \text{Im}(\text{Hom}_G(\mathbb{Z}[G^i], A) \longrightarrow \text{Hom}_G(\mathbb{Z}[G^{i+1}], A))$$

# Inhomogeneous Cochains

Alternatively.. use  $\mathbb{Z}[G^{i+1}] = \langle e, \sigma_1, \sigma_1\sigma_2, \sigma_1\sigma_2\sigma_3, \dots, \sigma_1 \cdots \sigma_i \rangle$

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Let  $[\sigma_1, \dots, \sigma_i] = (e, \sigma_1, \sigma_1\sigma_2, \sigma_1\sigma_2\sigma_3, \dots, \sigma_1 \cdots \sigma_i)$

Note that

$$\begin{aligned} & \delta_i [\sigma_1, \dots, \sigma_i] \\ &= \delta_i (e, \sigma_1, \sigma_1\sigma_2, \sigma_1\sigma_2\sigma_3, \dots, \sigma_1 \cdots \sigma_i) \\ &= \sum_j (-1)^j s_j^i (e, \sigma_1, \sigma_1\sigma_2, \sigma_1\sigma_2\sigma_3, \dots, \sigma_1 \cdots \sigma_i) \\ &= (\sigma_1, \sigma_1\sigma_2, \sigma_1\sigma_2\sigma_3, \dots, \sigma_1 \cdots \sigma_i) - (e, \sigma_1\sigma_2, \sigma_1\sigma_2\sigma_3, \dots, \sigma_1 \cdots \sigma_i) + \dots \\ & \quad + (-1)^{i+1} (e, \sigma_1, \sigma_1\sigma_2, \sigma_1\sigma_2\sigma_3, \dots, \sigma_1 \cdots \sigma_{i-1}) \\ &= \sigma_1 \cdot (e, \sigma_2, \sigma_2\sigma_3, \dots, \sigma_2 \cdots \sigma_i) - (e, \sigma_1\sigma_2, \sigma_1\sigma_2\sigma_3, \dots, \sigma_1 \cdots \sigma_i) + \dots \\ & \quad + (-1)^{i+1} (e, \sigma_1, \sigma_1\sigma_2, \sigma_1\sigma_2\sigma_3, \dots, \sigma_1 \cdots \sigma_{i-1}) \\ &= \sigma_1 \cdot [\sigma_2, \dots, \sigma_i] + \sum_j (-1)^j [\sigma_1, \dots, \sigma_j \sigma_{j+1}, \dots, \sigma_i] + (-1)^{i+1} [\sigma_1, \dots, \sigma_{i-1}] \end{aligned}$$

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That is,

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$\delta^i : C^{i-1}(G, A) \longrightarrow C^i(G, A)$  with

$$\delta^i(a_{\sigma_1, \dots, \sigma_i}) = \sigma_1 \cdot a_{\sigma_2, \dots, \sigma_i} + \sum_j (-1)^j a_{\sigma_1, \dots, \sigma_j \sigma_{j+1}, \dots, \sigma_i} + (-1)^{i+1} a_{\sigma_1, \dots, \sigma_{i-1}}$$

# Inhomogeneous Cochains: Examples

- 1-cocycle

$$\iff \delta^1(a_{\sigma_1}) = 0 \iff \sigma_1 \cdot a_{\sigma_1} - a_{\sigma_1 \sigma_2} + a_{\sigma_1} = 0 \iff a_{\sigma_1 \sigma_2} = \sigma_1 \cdot a_{\sigma_1} + a_{\sigma_1}$$

- 1-coboundary  $\iff \sigma \mapsto \sigma a - a$  so  $Z^1(G, A) = H^1(G, A) = \text{Hom}(G, A)$

- 2-cocycle  $\iff \delta^2(a_{\sigma_1 \sigma_2}) = 0 \iff \sigma_1 \cdot a_{\sigma_2 \sigma_3} - a_{\sigma_1 \sigma_2, \sigma_3} + a_{\sigma_1, \sigma_2 \sigma_3} - a_{\sigma_1, \sigma_2} = 0$

- 2-coboundary  $\iff a_{\sigma_1 \sigma_2} = \delta^1(b) = \sigma_1 \cdot b_{\sigma_1} - b_{\sigma_1 \sigma_2} + b_{\sigma_1}$

# Normalized Resolution

- Same as  $[\sigma_1, \dots, \sigma_i]$ , but  $\sigma_j \neq 0$  for all  $j$
- If  $\sigma_j \sigma_{j+1} = e$ , then set  $a_{\sigma_1, \dots, \sigma_j \sigma_{j+1}, \dots, \sigma_i} = 0$

# Group Extensions

- $E$  is an extension of  $G$  by  $A$  if

$$\{e\} \longrightarrow A \xrightarrow{\iota} E \xrightarrow{\pi} G \longrightarrow \{e\}$$

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$\{\text{Equivalence classes of extensions of } G \text{ by } A\} \cong H^2(G, A)$

- Aim: Want to define  $E \mapsto c(E) \in H^2(G, A)$

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## Theorem

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For  $a \in A$  (abelian),  $\sigma \in G$ , and  $s$  such that  $\pi \circ s = id_G$  and  $s(e) = e$ , define  $\iota(\sigma.a) := \hat{\sigma}\iota(a)\hat{\sigma}^{-1} \in \iota(A)$  where  $\hat{\sigma} = s(\sigma) \in E$

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Let  $\iota(b)\hat{\sigma} = s(\sigma)$  for some  $b \in A$ . Then  $\iota(b)\hat{\sigma}\iota(a)\hat{\sigma}^{-1}\iota(b^{-1}) = \hat{\sigma}\iota(a)\hat{\sigma}^{-1}$

# Group Extensions

## Theorem

Equivalent extensions define the same  $G$ -module structure on  $A$

## Proof

Let  $\hat{\sigma}' = \beta(\hat{\sigma})$ . Then  $s(\hat{\sigma}') = s(\hat{\sigma}) = \sigma$  and  $\iota(\sigma.a) = \iota(\sigma'.a)$

## Theorem

## Proof

# Group Extensions

## Theorem

Equivalent extensions define the same  $G$ -module structure on  $A$

## Proof

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## Theorem

$a_{\sigma_1, \sigma_2} := s(\sigma_1) s(\sigma_2) s(\sigma_1 \sigma_2)^{-1}$  is a normalized 2-cocycle

## Proof

$$\pi(a_{\sigma_1, \sigma_2}) = e$$

$$a_{\sigma_1, e} = a_{e, \sigma_2} = e$$

$$\forall \sigma_1, \sigma_2, \sigma_3 \in G, \sigma_1.a_{\sigma_2 \sigma_3} - a_{\sigma_1 \sigma_2, \sigma_3} + a_{\sigma_1, \sigma_2 \sigma_3} - a_{\sigma_1, \sigma_2} = 0$$

# Group Extensions

- For  $c^{-1}(E)$ , given  $a_{\sigma_1\sigma_2}$ , define  $E := A \times G$  with binary operation

$$(a_1, \sigma_1) * (a_2, \sigma_2) = (a_1 + \sigma_1(a_2) + a_{\sigma_1\sigma_2}, \sigma_1\sigma_2)$$

$$\text{Unit } (0, e) \text{ and } (a, \sigma) * (-\sigma^{-1}(a) - \sigma^{-1}(a_{\sigma, \sigma^{-1}}), \sigma^{-1}) = (0, e)$$

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- The induced map  $\phi_* : H^2(G, A) \longrightarrow H^2(G, B)$  from  $G$ -map  $\phi : A \longrightarrow B$  commutes with  $c$  where  $\phi_*(E) := (B \times E) / N$  where  $N$  is the normal subgroup generated by  $(\phi(a), \iota(a)^{-1})$



# Group Cohomology

Put  $G = \mathbb{Z}$  and consider

$$0 \longrightarrow \mathbb{Z}[\mathbb{Z}] \xrightarrow{\alpha} \mathbb{Z}[\mathbb{Z}] \xrightarrow{\beta} \mathbb{Z} \longrightarrow 0$$

$\alpha(\sigma) = \sigma - 1$  is injective and  $\beta(\sigma) = 1$  is surjective, where  $\langle \sigma \rangle = \mathbb{Z}$  and  $\beta \circ \alpha = 0$

$H^0(\mathbb{Z}, A) = A^\sigma$ ,  $H^1(\mathbb{Z}, A) = A/\sigma - 1(A)$  and  $H^i(\mathbb{Z}, A) = 0$  for  $i \geq 2$

# Group Cohomology

If  $G$  is a cyclic group of order  $n$ , let  $\mathbb{Z}[\mathbb{Z}] \xrightarrow{\alpha} \mathbb{Z}[\mathbb{Z}]$  and  $\mathbb{Z}[\mathbb{Z}] \xrightarrow{\beta} \mathbb{Z}[\mathbb{Z}]$  be such that

$$\alpha(a) = a\sigma - a \text{ and } \beta(a) = \sum_{i=0}^{n-1} \sigma^i a$$

Then  $\ker \alpha = \operatorname{Im} \beta$  and  $\operatorname{Im} \alpha = \ker \beta$

$$\dots \xrightarrow{\beta} \mathbb{Z}[\mathbb{Z}] \xrightarrow{\alpha} \mathbb{Z}[\mathbb{Z}] \xrightarrow{\beta} \mathbb{Z}[\mathbb{Z}] \xrightarrow{\alpha} \mathbb{Z}[\mathbb{Z}] \xrightarrow{\beta} \mathbb{Z}[\mathbb{Z}] \xrightarrow{\alpha} \mathbb{Z}[\mathbb{Z}] \xrightarrow{\beta} \mathbb{Z} \longrightarrow$$

Extend  $\alpha : A \longrightarrow A$  and  $\beta : A \longrightarrow A$  and write  ${}_N A = \ker \beta$ . Then

$$H^0(G, A) = A^G, \quad H^{2i+1}(G, A) = {}_N A / (\sigma - 1)A \text{ and } H^{2i+2}(G, A) = A^G / \beta(A)$$

# Galois Cohomology

Now let  $A = K^\times$  where  $K|k$  is a finite Galois extension with cyclic Galois group  $G$ . Then  $H^1(G, K^\times) = {}_N K^\times / (\sigma - 1) K^\times = \{e\}$