Differential Essential Dimension

Colloquium, Florida State University

August 26, 2022

Man Cheung Tsui (2nd Year Postdoc)

In this talk

Fix a field F of characteristic zero, e.g., $F = \mathbb{C}$.

All fields appearing here contain F.

How much can we simplify ...

Quadratic:

$$x^2 + ax + b \xrightarrow{x=y-a/2} y^2 + c \quad (c = b - a^2/4)$$

Cubic:

$$x^3 + ax^2 + bx + c \xrightarrow{x=y-a/3} y^3 + dy + e$$

$$\xrightarrow{y=(e/d)z} z^3 + fz + f$$

Quintic:

$$x^5 + ax^4 + bx^3 + cx^2 + dx + e \xrightarrow{\text{Hermite (1861)}} y^5 + fy^3 + gy + g$$

How much can we simplify ...

Quadratic:

$$x^2 + ax + b \xrightarrow{x=y-a/2} y^2 + c \quad (c = b - a^2/4)$$

Cubic:

$$x^3 + ax^2 + bx + c \xrightarrow{x=y-a/3} y^3 + dy + e$$

$$\xrightarrow{y=(e/d)z} z^3 + fz + f$$

Quintic:

$$x^5 + ax^4 + bx^3 + cx^2 + dx + e \xrightarrow{\text{Hermite (1861)}} y^5 + fy^3 + gy + g$$

How much can we simplify ...

Quadratic:

$$x^2 + ax + b \xrightarrow{x=y-a/2} y^2 + c \quad (c = b - a^2/4)$$

Cubic:

$$x^3 + ax^2 + bx + c \xrightarrow{x=y-a/3} y^3 + dy + e$$

$$\xrightarrow{y=(e/d)z} z^3 + fz + f$$

Quintic:

$$x^5 + ax^4 + bx^3 + cx^2 + dx + e \xrightarrow{\text{Hermite (1861)}} y^5 + fy^3 + gy + g$$

… using Tschirnhaus transformations ($\sim$)?

$q(y)$ is a Tschirnhaus transformation of $p(x)$ over a field K and write $q(y) \sim p(x)$ if

$$\{\text{zeros of } p(x)\} \xrightarrow[\text{bijection}]{\exists \text{ polynomial transformation}} \{\text{zeros of } q(y)\}.$$

Equivalently:

$$K[x]/(p(x)) \cong K[y]/(q(y)).$$

Translation: $p(x-a) \sim p(x) \qquad (a \in K).$

Scaling: $p(bx) \sim p(x) \qquad (b \in K^\times)$

... using Tschirnhaus transformations ($\sim$)?

$q(y)$ is a Tschirnhaus transformation of $p(x)$ over a field K and write $q(y) \sim p(x)$ if

$$\{\text{zeros of } p(x)\} \xrightarrow[\text{bijection}]{\exists \text{ polynomial transformation}} \{\text{zeros of } q(y)\}.$$

Equivalently:

$$K[x]/(p(x)) \cong K[y]/(q(y)).$$

Translation: $p(x-a) \sim p(x)$ $(a \in K)$.

Scaling: $p(bx) \sim p(x)$ $(b \in K^\times)$

… using Tschirnhaus transformations ($\sim$)?

$q(y)$ is a Tschirnhaus transformation of $p(x)$ over a field K and write $q(y) \sim p(x)$ if

$$\{\text{zeros of } p(x)\} \xrightarrow[\text{bijection}]{\exists\text{ polynomial transformation}} \{\text{zeros of } q(y)\}.$$

Equivalently:

$$K[x]/(p(x)) \cong K[y]/(q(y)).$$

Translation: $p(x-a) \sim p(x) \qquad (a \in K).$

Scaling: $p(bx) \sim p(x) \qquad (b \in K^\times)$

… using Tschirnhaus transformations ($\sim$)?

Consider the general polynomial over F:

$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \quad (a_i\text{'s algebraically independent over } F).$

$\tau(n)$:= minimum number of algebraically independent coefficients of $q(y)$, for all $q(y) \sim p(x)$ over $F(a_0, \ldots, a_{n-1})$.

… using Tschirnhaus transformations ($\sim$)?

Consider the general polynomial over F:

$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \quad (a_i$'s algebraically independent over $F)$.

$\tau(n)$:= minimum number of algebraically independent coefficients of $q(y)$, for all $q(y) \sim p(x)$ over $F(a_0, \ldots, a_{n-1})$.

… using Tschirnhaus transformations?

n	2	3	4	5	6	7
$\tau(n)$	1	1	(BR)	2 (Hermite and Klein)	3 (BR)	4 (A. Duncan, 2010)

J. Buhler and Z. Reichstein (1997) introduced essential dimension to prove:

$$\tau(4) = 2, \qquad \lfloor n/2 \rfloor \leq \tau(n) \leq n - 3 \quad (n \geq 5).$$

… using Tschirnhaus transformations?

n	2	3	4	5	6	7
$\tau(n)$	1	1	≤ 2 (BR)	2 (Hermite and Klein)	3 (BR)	4 (A. Duncan, 2010)

J. Buhler and Z. Reichstein (1997) introduced essential dimension to prove:

$$\tau(4) = 2, \qquad \lfloor n/2 \rfloor \leq \tau(n) \leq n - 3 \quad (n \geq 5).$$

… using Tschirnhaus transformations?

n	2	3	4	5	6	7
$\tau(n)$	1	1	≤ 2 (BR)	2 (Hermite and Klein)	3 (BR)	4 (A. Duncan, 2010)

J. Buhler and Z. Reichstein (1997) introduced essential dimension to prove:

$$\tau(4) = 2, \qquad \lfloor n/2 \rfloor \leq \tau(n) \leq n - 3 \quad (n \geq 5).$$

… using Tschirnhaus transformations?

n	2	3	4	5	6	7
$\tau(n)$	1	1	2 (BR)	2 (Hermite and Klein)	3 (BR)	4 (A. Duncan, 2010)

J. Buhler and Z. Reichstein (1997) introduced essential dimension to prove:

$$\tau(4) = 2, \qquad \lfloor n/2 \rfloor \leq \tau(n) \leq n - 3 \quad (n \geq 5).$$

… using Tschirnhaus transformations?

n	2	3	4	5	6	7
$\tau(n)$	1	1	2 (BR)	2 (Hermite and Klein)	3 (BR)	4 (A. Duncan, 2010)

J. Buhler and Z. Reichstein (1997) introduced essential dimension to prove:

$$\tau(4) = 2, \qquad \lfloor n/2 \rfloor \leq \tau(n) \leq n - 3 \quad (n \geq 5).$$

Counting parameters with essential dimension (ed)

Polynomial:	$p(x) = x^2 + ax + b$	$\xrightarrow{\text{Tschirnhaus}}$	$q(y) = y^2 + c$
Over:	$K = F(a, b)$	$\supset$	$K_0 = F(c)$

For a polynomial $p(x)$ (up to Tschirnhaus transformation):

$$\mathrm{ed}(p(x)) = \min_{K_0} \mathrm{trdeg}(K_0/F)$$

over all fields $K_0 \supset F$ such that some $q(y) \sim p(x)$ has coefficients in K_0.

$\Rightarrow \tau(n) = \mathrm{ed}(\text{the general degree } n \text{ polynomial}).$

Counting parameters with essential dimension (ed)

Polynomial:	$p(x) = x^2 + ax + b$	$\overset{\text{Tschirnhaus}}{\rightsquigarrow}$	$q(y) = y^2 + c$
Over:	$K = F(a, b)$	$\supset$	$K_0 = F(c)$

For a polynomial $p(x)$ (up to Tschirnhaus transformation):

$$\mathrm{ed}(p(x)) = \min_{K_0} \mathrm{trdeg}(K_0/F)$$

over all fields $K_0 \supset F$ such that some $q(y) \sim p(x)$ has coefficients in K_0.

$\Rightarrow \tau(n) = \mathrm{ed}(\text{the general degree } n \text{ polynomial}).$

Counting parameters with essential dimension (ed)

Polynomial:	$p(x) = x^2 + ax + b$	$\xrightarrow{\text{Tschirnhaus}}$	$q(y) = y^2 + c$
Over:	$K = F(a, b)$	$\supset$	$K_0 = F(c)$

For a polynomial $p(x)$ (up to Tschirnhaus transformation):

$$\mathrm{ed}(p(x)) = \min_{K_0} \mathrm{trdeg}(K_0/F)$$

over all fields $K_0 \supset F$ such that some $q(y) \sim p(x)$ has coefficients in K_0.

$$\Rightarrow \tau(n) = \mathrm{ed}(\text{the general degree } n \text{ polynomial}).$$

Counting parameters with essential dimension (ed)

For an "algebraic object" X over a field K up to some relation $\sim$:

$$\mathrm{ed}(X) = \min_{K_0} \mathrm{trdeg}(K_0/F)$$

over all fields $K_0 \supset F$ such that some $Y \sim X$ has coefficients in K_0.

Example. For non-degenerate quadratic forms of degree n up to linear change of variables,

$$q = \sum_{1 \le i \le j \le n} a_{ij} x_i x_j \xrightarrow{\text{diagonalize}} \sum_{i=1}^{n} b_i y^2$$

so $\mathrm{ed}(q) \le n$.

Counting parameters with essential dimension (ed)

For an "algebraic object" X over a field K up to some relation $\sim$:

$$\operatorname{ed}(X) = \min_{K_0} \operatorname{trdeg}(K_0/F)$$

over all fields $K_0 \supset F$ such that some $Y \sim X$ has coefficients in K_0.

Example. For non-degenerate quadratic forms of degree n up to linear change of variables,

$$q = \sum_{1 \le i \le j \le n} a_{ij} x_i x_j \overset{\text{diagonalize}}{\rightsquigarrow} \sum_{i=1}^{n} b_i y^2$$

so $\operatorname{ed}(q) \le n$.

Lower bound on $\tau(n)$

Know: general polynomial $p(x)$ $\rightsquigarrow$

$$\begin{tikzcd} L \arrow[d, dash, "p(x)"', "S_n"] \arrow[dr, dash] & \\ K \arrow[dr, dash] & L_0 \arrow[d, dash, "\exists ?"', "S_n"] \\ & K_0 \end{tikzcd}$$

Know: $S_n \supset \langle (1\,2), (3\,4), \ldots \rangle \cong (\mathbb{Z}/2\mathbb{Z})^{\lfloor n/2 \rfloor}$.

$$\Rightarrow \qquad \begin{aligned} \tau(n) = \mathrm{ed}(p(x)) &= \mathrm{ed}(L/K) \\ &\geq \mathrm{ed}(\text{the } (\mathbb{Z}/2\mathbb{Z})^{\lfloor n/2 \rfloor}\text{-subextension}) \\ &= \lfloor n/2 \rfloor. \end{aligned}$$

Lower bound on $\tau(n)$

Know: general polynomial $p(x)$ $\rightsquigarrow$

$$\begin{tikzcd}
L \arrow[d, dash, "p(x)"', "S_n"] \arrow[dr, dash] & \\
K \arrow[dr, dash] & L_0 \arrow[d, dash, "\exists?"', "S_n"] \\
& K_0
\end{tikzcd}$$

Know: $S_n \supset \langle (1\,2), (3\,4), \ldots \rangle \cong (\mathbb{Z}/2\mathbb{Z})^{\lfloor n/2 \rfloor}$.

$$\Rightarrow \qquad \tau(n) = \mathrm{ed}(p(x)) = \mathrm{ed}(L/K)$$

$$\geq \mathrm{ed}(\text{the } (\mathbb{Z}/2\mathbb{Z})^{\lfloor n/2 \rfloor}\text{-subextension})$$

$$= \lfloor n/2 \rfloor.$$

Lower bound on $\tau(n)$

Know: general polynomial $p(x)$ $\rightsquigarrow$

$$\begin{tikzcd} L \arrow[d, dash, "p(x)"', "S_n"] \arrow[dr, dash] & \\ K \arrow[dr, dash] & L_0 \arrow[d, dash, "\exists ?"', "S_n"] \\ & K_0 \end{tikzcd}$$

Know: $S_n \supset \langle (1\,2), (3\,4), \ldots \rangle \cong (\mathbb{Z}/2\mathbb{Z})^{\lfloor n/2 \rfloor}$.

$$\begin{aligned} \Rightarrow \qquad \tau(n) = \mathrm{ed}(p(x)) &= \mathrm{ed}(L/K) \\ &\geq \mathrm{ed}(\text{the } (\mathbb{Z}/2\mathbb{Z})^{\lfloor n/2 \rfloor}\text{-subextension}) \\ &= \lfloor n/2 \rfloor. \end{aligned}$$

Lower bound on $\tau(n)$

Know: general polynomial $p(x)$ $\rightsquigarrow$

$$\begin{tikzcd}
L \arrow[d, dash, "p(x)"', "S_n"] \arrow[dr, dash] & \\
K \arrow[dr, dash] & L_0 \arrow[d, dash, "\exists ?"', "S_n"] \\
& K_0
\end{tikzcd}$$

Know: $S_n \supset \langle (1\,2), (3\,4), \ldots \rangle \cong (\mathbb{Z}/2\mathbb{Z})^{\lfloor n/2 \rfloor}$.

$$\Rightarrow \qquad \begin{aligned} \tau(n) = \mathrm{ed}(p(x)) &= \mathrm{ed}(L/K) \\ &\geq \mathrm{ed}(\text{the } (\mathbb{Z}/2\mathbb{Z})^{\lfloor n/2 \rfloor}\text{-subextension}) \\ &= \lfloor n/2 \rfloor. \end{aligned}$$

Differential equations

Differential fields

A differential field is a field F with a derivation $\partial : F \longrightarrow F$.

Like $(F, \partial) = \left(\mathbb{C}(x), \frac{d}{dx}\right)$.

Simplifying matrix differential equation (DE)

$Z' = BZ$ is a gauge transformation of $Y' = AY$ over F and write

$$Z' = BZ \sim Y' = AY$$

if $Z = PY$ for some $P \in \mathrm{GL}_n(F)$.

Consider the general matrix DE

$$Y' = AY$$

with the matrix entries A_{ij} and their higher derivatives algebraically independent over F.

$\gamma(n) :=$ how few parameters $Y' = AY$ simplifies to using $\sim$.

Simplifying matrix differential equation (DE)

$Z' = BZ$ is a gauge transformation of $Y' = AY$ over F and write

$$Z' = BZ \ \sim \ Y' = AY$$

if $Z = PY$ for some $P \in \mathrm{GL}_n(F)$.

Consider the general matrix DE

$$Y' = AY$$

with the matrix entries A_{ij} and their higher derivatives algebraically independent over F.

$\gamma(n) :=$ how few parameters $Y' = AY$ simplifies to using $\sim$.

Simplifying matrix differential equation (DE)

$Z' = BZ$ is a gauge transformation of $Y' = AY$ over F and write

$$Z' = BZ \sim Y' = AY$$

if $Z = PY$ for some $P \in \mathrm{GL}_n(F)$.

Consider the general matrix DE

$$Y' = AY$$

with the matrix entries A_{ij} and their higher derivatives algebraically independent over F.

$\gamma(n)$:= how few parameters $Y' = AY$ simplifies to using $\sim$.

Simplifying matrix differential equation (DE)

$Y' = AY \ \sim$

$$Z' = BZ, \quad B = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -b_0 & -b_1 & -b_2 & \cdots & -b_{n-1} \end{pmatrix}.$$

So $\gamma(n) \leq n$.

Theorem (T.)

$\gamma(n) = n$.

Where does this $Z' = BZ$ come from?

$$Z' = BZ \quad \longleftrightarrow \quad z^{(n)} + b_{n-1}z^{(n-1)} + \cdots + b_0 z = 0.$$

Meaning of $\gamma(n) = n$:

"Homogeneous linear DE's are the most compact way to write DE's if you know nothing about the coefficients of your DE."

Differential Galois theory

Example.

$Y' = Y$ has solution $y = e^x$ in $\mathbb{C}((x))$.

$$\boxed{\begin{array}{c} K = \mathbb{C}(x, e^x) \\ \Big| \; \mathbb{G}_m \\ F = \mathbb{C}(x) \end{array}}$$

$$\begin{aligned}\{\text{Differential automorphism of } K/F\} &= \{e^x \mapsto C \cdot e^x, C \in \mathbb{C}^\times\} \\ &\cong \mathbb{C}^\times = \mathbb{G}_m(\mathbb{C}).\end{aligned}$$

Differential Galois theory

Example.

$Y' = Y$ has solution $y = e^x$ in $\mathbb{C}((x))$.

$$\boxed{\begin{array}{c} K = \mathbb{C}(x, e^x) \\ \Big| \; \mathbb{G}_m \\ F = \mathbb{C}(x) \end{array}}$$

$$\begin{aligned} \{\text{Differential automorphism of } K/F\} &= \{e^x \mapsto C \cdot e^x, C \in \mathbb{C}^\times\} \\ &\cong \mathbb{C}^\times = \mathbb{G}_m(\mathbb{C}). \end{aligned}$$

Differential Galois theory

Example.

$Y' = Y$ has solution $y = e^x$ in $\mathbb{C}((x))$.

$$\boxed{\begin{array}{c} K = \mathbb{C}(x, e^x) \\ \Big| \; \mathbb{G}_m \\ F = \mathbb{C}(x) \end{array}}$$

$$\{\text{Differential automorphism of } K/F\} = \{e^x \mapsto C \cdot e^x, C \in \mathbb{C}^\times\}$$
$$\cong \mathbb{C}^\times = \mathbb{G}_m(\mathbb{C}).$$

Differential Galois theory

Example.

$Y' = Y$ has solution $y = e^x$ in $\mathbb{C}((x))$.

$$\boxed{\begin{array}{c} K = \mathbb{C}(x, e^x) \\ \Big| \ \mathbb{G}_m \\ F = \mathbb{C}(x) \end{array}}$$

$$\{\text{Differential automorphism of } K/F\} = \{e^x \mapsto C \cdot e^x, C \in \mathbb{C}^\times\}$$
$$\cong \mathbb{C}^\times = \mathbb{G}_m(\mathbb{C}).$$

Differential Galois theory

Example.

$Y' = Y$ has solution $y = e^x$ in $\mathbb{C}((x))$.

$$\boxed{\begin{array}{c} K = \mathbb{C}(x, e^x) \\ \Big| \; \mathbb{G}_m \\ F = \mathbb{C}(x) \end{array}}$$

$$\begin{aligned} \{\text{Differential automorphism of } K/F\} &= \{e^x \mapsto C \cdot e^x, C \in \mathbb{C}^\times\} \\ &\cong \mathbb{C}^\times = \mathbb{G}_m(\mathbb{C}). \end{aligned}$$

Analogies

	Polynomial of degree n	Linear or matrix DE, order n
Zero set	Finite set	$\mathbb{C}$-Vector space
Extension L/K	Galois extension	Picard-Vessiot extension
Galois group	$\leq S_n$	$\leq \mathrm{GL}_n(\mathbb{C})$
		(Linear algebraic group)

Can define differential essential dimension $\mathrm{ed}^{\partial}(\bullet)$ similarly.

Analogies

	Polynomial of degree n	Linear or matrix DE, order n
Zero set	Finite set	$\mathbb{C}$-Vector space
Extension L/K	Galois extension	Picard-Vessiot extension
Galois group	$\leq S_n$	$\leq \mathrm{GL}_n(\mathbb{C})$
		(Linear algebraic group)

Can define differential essential dimension $\mathrm{ed}^{\partial}(\bullet)$ similarly.

Analogies

	Polynomial of degree n	Linear or matrix DE, order n
Zero set	Finite set	$\mathbb{C}$-Vector space
Extension L/K	Galois extension	Picard-Vessiot extension
Galois group	$\leq S_n$	$\leq \mathrm{GL}_n(\mathbb{C})$ (Linear algebraic group)

Can define differential essential dimension $\mathrm{ed}^{\partial}(\bullet)$ similarly.

Analogies

	Polynomial of degree n	Linear or matrix DE, order n
Zero set	Finite set	$\mathbb{C}$-Vector space
Extension L/K	Galois extension	Picard-Vessiot extension
Galois group	$\leq S_n$	$\leq \mathrm{GL}_n(\mathbb{C})$
		(Linear algebraic group)

Can define differential essential dimension $\mathrm{ed}^{\partial}(\bullet)$ similarly.

Analogies

	Polynomial of degree n	Linear or matrix DE, order n
Zero set	Finite set	$\mathbb{C}$-Vector space
Extension L/K	Galois extension	Picard-Vessiot extension
Galois group	$\leq S_n$	$\leq \mathrm{GL}_n(\mathbb{C})$
		(Linear algebraic group)

Can define differential essential dimension $\mathrm{ed}^{\partial}(\bullet)$ similarly.

Analogies

	Polynomial of degree n	Linear or matrix DE, order n
Zero set	Finite set	$\mathbb{C}$-Vector space
Extension L/K	Galois extension	Picard-Vessiot extension
Galois group	$\leq S_n$	$\leq \mathrm{GL}_n(\mathbb{C})$
		(Linear algebraic group)

Can define differential essential dimension $\mathrm{ed}^{\partial}(\bullet)$ similarly.

Bound on $\gamma(n)$

Know: general DE $Y' = AY$ $\rightsquigarrow$

$$
\begin{tikzcd}
L \arrow[d, dash, "\mathrm{GL}_n(\mathbb{C})"'] \arrow[dr, dash] & \\
K \arrow[dr, dash] & L_0 \arrow[d, dash, "\exists?"', "\mathrm{GL}_n(\mathbb{C})"] \\
& K_0
\end{tikzcd}
$$

Know: $\mathrm{GL}_n \geq \mathbb{G}_m^n$.

$$
\begin{aligned}
\Rightarrow \qquad n \geq \gamma(n) = \mathrm{ed}^\partial(Y' = AY) &= \mathrm{ed}^\partial(L/K) \\
&\geq \mathrm{ed}^\partial(\text{the } \mathbb{G}_m^n\text{-subextension}) \\
&= n.
\end{aligned}
$$

Bound on $\gamma(n)$

Know: general DE $Y' = AY$ $\rightsquigarrow$

$$
\begin{tikzcd}
L \arrow[d, dash, "\mathrm{GL}_n(\mathbb{C})"'] \arrow[dr, dash] & \\
K \arrow[dr, dash] & L_0 \arrow[d, dash, "\exists ?"', "\mathrm{GL}_n(\mathbb{C})"] \\
& K_0
\end{tikzcd}
$$

Know: $\mathrm{GL}_n \geq \mathbb{G}_m^n$.

$$
\Rightarrow \qquad n \geq \gamma(n) = \mathrm{ed}^{\partial}(Y' = AY) = \mathrm{ed}^{\partial}(L/K)
$$

$$
\geq \mathrm{ed}^{\partial}(\text{the } \mathbb{G}_m^n\text{-subextension})
$$

$$
= n.
$$

Bound on $\gamma(n)$

Know: general DE $Y' = AY$ $\rightsquigarrow$

$$\begin{tikzcd} L \arrow[d, dash, "\mathrm{GL}_n(\mathbb{C})"'] \arrow[dr, dash] & \\ K \arrow[dr, dash] & L_0 \arrow[d, dash, "\mathrm{GL}_n(\mathbb{C})", "\exists ?"'] \\ & K_0 \end{tikzcd}$$

Know: $\mathrm{GL}_n \geq \mathbb{G}_m^n$.

$$\Rightarrow \qquad n \geq \gamma(n) = \mathrm{ed}^{\partial}(Y' = AY) = \mathrm{ed}^{\partial}(L/K)$$
$$\geq \mathrm{ed}^{\partial}(\text{the } \mathbb{G}_m^n\text{-subextension})$$
$$= n.$$

Bound on $\gamma(n)$

Know: general DE $Y' = AY$ $\rightsquigarrow$

$$\begin{tikzcd}[arrows=dash]
L \arrow[d, "\mathrm{GL}_n(\mathbb{C})"'] \arrow[dr] & \\
K \arrow[dr] & L_0 \arrow[d, "\exists?"', "\mathrm{GL}_n(\mathbb{C})"] \\
& K_0
\end{tikzcd}$$

Know: $\mathrm{GL}_n \geq \mathbb{G}_m^n$.

$$\begin{aligned}
\Rightarrow \qquad n \geq \gamma(n) = \mathrm{ed}^{\partial}(Y' = AY) &= \mathrm{ed}^{\partial}(L/K) \\
&\geq \mathrm{ed}^{\partial}(\text{the } \mathbb{G}_m^n\text{-subextension}) \\
&= n.
\end{aligned}$$

How much more can we simplify using more general transformations?

$$p(y) = y^{(n)} + a_{n-1}y^{(n-1)} + a_{n-2}y^{(n-2)} + \cdots + a_0 y$$

$$\xrightarrow{y = e^{-\frac{1}{n}\int a_{n-1}} z} \quad q(z) = z^{(n)} + 0 \cdot z^{(n-1)} + b_{n-2}z^{(n-2)} + \cdots + b_0 z$$

Analogue of allowing $\sqrt[n]{\bullet}$.

Tools: differential analogue of Galois cohomology (nonabelian form of Hochschild cohomology).

How much more can we simplify using more general transformations?

$$p(y) = y^{(n)} + a_{n-1}y^{(n-1)} + a_{n-2}y^{(n-2)} + \cdots + a_0 y$$

$$\xrightarrow{y = e^{-\frac{1}{n}\int a_{n-1}} z} \quad q(z) = z^{(n)} + 0 \cdot z^{(n-1)} + b_{n-2}z^{(n-2)} + \cdots + b_0 z$$

Analogue of allowing $\sqrt[n]{\bullet}$.

Tools: differential analogue of Galois cohomology (nonabelian form of Hochschild cohomology).

How much more can we simplify using more general transformations?

$$
\begin{aligned}
& p(y) = y^{(n)} + a_{n-1}y^{(n-1)} + a_{n-2}y^{(n-2)} + \cdots + a_0 y \\
\xrightsquigarrow{y=e^{-\frac{1}{n}\int a_{n-1}}z} \quad & q(z) = z^{(n)} + 0 \cdot z^{(n-1)} + b_{n-2}z^{(n-2)} + \cdots + b_0 z
\end{aligned}
$$

Analogue of allowing $\sqrt[n]{\bullet}$.

Tools: differential analogue of Galois cohomology (nonabelian form of Hochschild cohomology).

How much more can we simplify using more general transformations?

$$p(y) = y^{(n)} + a_{n-1}y^{(n-1)} + a_{n-2}y^{(n-2)} + \cdots + a_0 y$$

$$\overset{y=e^{-\frac{1}{n}\int a_{n-1}} z}{\rightsquigarrow} \quad q(z) = z^{(n)} + 0 \cdot z^{(n-1)} + b_{n-2}z^{(n-2)} + \cdots + b_0 z$$

Analogue of allowing $\sqrt[n]{\bullet}$.

Tools: differential analogue of Galois cohomology (nonabelian form of Hochschild cohomology).

Thanks! Questions?