Section 1.1

Monday, September 20, 2021 3:11 PM

$\{1, i, j, k\}$

$i^2 = -1$, $j^2 = -1$, $ij = -ji = k$

Note: Looking at the quaternions as a vectorspace over $\mathbb{R}$, we that it is in fact a division algebra.

- Meaning that each element has a two-sided multiplicative inverse.

Ex: $i \cdot (-i) = 1$ $\quad (-i) \cdot i = 1$

For $q = x + yi + zj + wk$ for $x, y, z, w \in \mathbb{R}$

the conjugate will be $\bar{q} = x - yi - zj - wk$

$$N(q) = q \cdot \bar{q} = (x + yi + zj + wk)(x - yi - zj - wk)$$
$$= x^2 + y^2 + z^2 + w^2$$

$$N(q) = 0 \iff q = 0$$

The inverse of q is given by $\bar{q} / N(q)$

Def: For $a, b \in K^\times$, we define the (generalized) quaternion algebra (a,b) as the 4th dimensional K-algebra with basis given by $\{1, i, j, ij\}$ and a multiplication given by:

$$i^2 = a, \quad j^2 = b, \quad ij = -ji$$

We call $\{1, i, j, ij\}$ the quaternion basis of (a,b).

Remark: The isomorphism class of (a,b) depends only on the classes of a, b in $K^\times / K^{\times 2}$

$$i \longmapsto ui \qquad j \longmapsto vj$$

$$(a,b) \xrightarrow{\sim} (u^2 a, v^2 b)$$

If $q = x + yi + zj + wij$

then $\bar{q} = x - yi - zj - wij$

and $\underline{N(q) = 1 \cdot x^2 - ay^2 - bz^2 + abw^2} \in K$

Note: $N : (a,b) \to K$ is a nondegenerate quadratic form

Reason $[N] = \begin{bmatrix} 1 & & & \\ & -a & & \\ & & -b & \\ & & & ab \end{bmatrix}$ has det $\neq 0$.

$$N(q_1 q_2) = q_1 q_2 \bar{q}_2 \bar{q}_1 = q_1 N(q_2) \bar{q}_1 = q_1 \bar{q}_1 N(q_2)$$
$$= N(q_1) N(q_2)$$

Lemma: An element q of the quaternion algebra (a,b)

Lemma: An element q of the quaternion algebra (a,b) is invertible iff it has non-zero norm.

Thus (a,b) is a division algebra iff

$N : (a,b) \to K$ does not vanish outside of zero.

Example: $M_2(K) \cong (1,b)$

$$i \longmapsto I := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad j \longmapsto J := \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}$$

$$I^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = Id_2$$

$$J^2 = \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} = b\,Id_2$$

$$IJ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b \\ -1 & 0 \end{pmatrix}$$

$$-JI = -\begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -\begin{pmatrix} 0 & -b \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & b \\ -1 & 0 \end{pmatrix}$$

Def: We say a quaternion algebra over K is split if it is isomorphic to $M_2(K)$ as a K-algebra.

Proposition: Let (a,b) be a quaternion algebra. TFAE:

(i) (a,b) is split

(ii) (a,b) is not a division algebra

(iii) The norm map $N : (a,b) \longrightarrow k$ has a nontrivial zero.

(iv) The element b is a norm from the field extension $k(\sqrt{a})/k$

(i) $\Rightarrow$ (ii)

Trivially

(ii) $\Rightarrow$ (iii)

By previous lemma

(iii) $\Rightarrow$ (iv)

Assume a is not a square in k

Take $q = x + yi + zj + wij \neq 0$

such that $N(q) = 0$

Then as $N(q) = x^2 - ay^2 - bz^2 + abw^2$

we see that $0 = x^2 - ay^2 - bz^2 + abw^2$

The norm over K is denoted $N_{K/k}$ where $K = k(\sqrt{a})$

$\Rightarrow (z^2 - aw^2)b = x^2 - ay^2$

However $z^2 - aw^2 = (z + w\sqrt{a})(z - w\sqrt{a}) \neq 0$

Thus $b = N_{K/k}(x + y\sqrt{a}) N_{K/k}(z + w\sqrt{a})^{-1}$

So (iii) $\Rightarrow$ (iv).