

Differential Essential Dimension

Colloquium, Florida State University

August 26, 2022

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In this talk

Fix a field F of characteristic zero, e.g., $F = \mathbb{C}$.

All fields appearing here contain F .

How much can we simplify ...

Quadratic:

$$x^2 + ax + b \xrightarrow{x=y-a/2} y^2 + c \quad (c = b - a^2/4)$$

Cubic:

$$x^3 + ax^2 + bx + c \xrightarrow{x=y-a/3} y^3 + dy + e$$
$$\xrightarrow{y=(e/d)z} z^3 + fz + f$$

Quintic:

$$x^5 + ax^4 + bx^3 + cx^2 + dx + e \xrightarrow{\text{Hermite (1861)}} y^5 + fy^3 + gy + g$$

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... using Tschirnhaus transformations (\sim)?

$q(y)$ is a **Tschirnhaus transformation** of $p(x)$ over a field K and write $q(y) \sim p(x)$ if

$$\{\text{zeros of } p(x)\} \xrightarrow[\text{bijection}]{\exists \text{ polynomial transformation}} \{\text{zeros of } q(y)\}.$$

Equivalently:

$$K[x]/(p(x)) \cong K[y]/(q(y)).$$

Translation: $p(x-a) \sim p(x) \quad (a \in K).$

Scaling: $p(bx) \sim p(x) \quad (b \in K^\times)$

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Consider the **general polynomial** over F :

$$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \quad (a_i\text{'s algebraically independent over } F).$$

$\tau(n)$:= minimum number of algebraically independent coefficients of $q(y)$,
for all $q(y) \sim p(x)$ over $F(a_0, \dots, a_{n-1})$.

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... using Tschirnhaus transformations?

n	2	3	4	5	6	7
$\tau(n)$	1	1	(BR)	2 (Hermite and Klein)	3 (BR)	4 (A. Duncan, 2010)

J. Buhler and Z. Reichstein (1997) introduced essential dimension to prove:

$$\tau(4) = 2, \quad \lfloor n/2 \rfloor \leq \tau(n) \leq n - 3 \quad (n \geq 5).$$

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Counting parameters with essential dimension (ed)

$$\begin{array}{lll} \text{Polynomial:} & p(x) = x^2 + ax + b & \xrightarrow{\text{Tschirnhaus}} q(y) = y^2 + c \\ \text{Over:} & K = F(a, b) & \supset K_0 = F(c) \end{array}$$

For a polynomial $p(x)$ (up to Tschirnhaus transformation):

$$\text{ed}(p(x)) = \min_{K_0} \text{trdeg}(K_0/F)$$

over all fields $K_0 \supset F$ such that some $q(y) \sim p(x)$ has coefficients in K_0 .

$$\Rightarrow \tau(n) = \text{ed}(\text{the general degree } n \text{ polynomial}).$$

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For an “algebraic object” X over a field K up to some relation \sim :

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Example. For non-degenerate quadratic forms of degree n up to linear change of variables,

$$q = \sum_{1 \leq i \leq j \leq n} a_{ij} x_i x_j \xrightarrow{\text{diagonalize}} \sum_{i=1}^n b_i y^2$$

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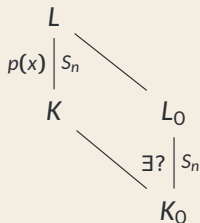
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Lower bound on $\tau(n)$

Know: general polynomial $p(x)$ \rightsquigarrow

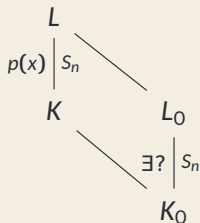


Know: $S_n \supset \langle (1\ 2), (3\ 4), \dots \rangle \cong (\mathbb{Z}/2\mathbb{Z})^{\lfloor n/2 \rfloor}$.

$$\begin{aligned} \Rightarrow \quad \tau(n) &= \text{ed}(p(x)) = \text{ed}(L/K) \\ &\geq \text{ed}(\text{the } (\mathbb{Z}/2\mathbb{Z})^{\lfloor n/2 \rfloor}\text{-subextension}) \\ &= \lfloor n/2 \rfloor. \end{aligned}$$

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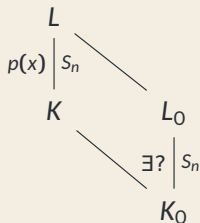


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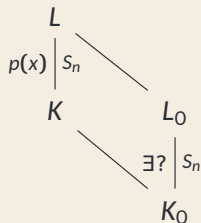


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Differential equations

Differential fields

A **differential field** is a field F with a derivation $\partial : F \rightarrow F$.

Like $(F, \partial) = \left(\mathbb{C}(x), \frac{d}{dx} \right)$.

Simplifying matrix differential equation (DE)

$Z' = BZ$ is a **gauge transformation** of $Y' = AY$ over F and write

$$Z' = BZ \sim Y' = AY$$

if $Z = PY$ for some $P \in \mathrm{GL}_n(F)$.

Consider the **general matrix DE**

$$Y' = AY$$

with the matrix entries A_{ij} and their higher derivatives algebraically independent over F .

$\gamma(n) :=$ how few parameters $Y' = AY$ simplifies to using \sim .

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$$Y' = AY \sim$$

$$Z' = BZ, \quad B = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -b_0 & -b_1 & -b_2 & \cdots & -b_{n-1} \end{pmatrix}.$$

So $\gamma(n) \leq n$.

Theorem (T.)

$$\gamma(n) = n.$$

Where does this $Z' = BZ$ come from?

$$Z' = BZ \quad \longleftrightarrow \quad z^{(n)} + b_{n-1}z^{(n-1)} + \cdots + b_0z = 0.$$

Meaning of $\gamma(n) = n$:

“Homogeneous linear DE’s are the most compact way to write DE’s if you know nothing about the coefficients of your DE.”

Differential Galois theory

Example.

$Y' = Y$ has solution $y = e^x$ in $\mathbb{C}((x))$.

$$K = \mathbb{C}(x, e^x)$$

$$\Bigg| \mathbb{G}_m$$

$$F = \mathbb{C}(x)$$

$$\begin{aligned} \{\text{Differential automorphism of } K/F\} &= \{e^x \mapsto C \cdot e^x, C \in \mathbb{C}^\times\} \\ &\cong \mathbb{C}^\times = \mathbb{G}_m(\mathbb{C}). \end{aligned}$$

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Analogies

	Polynomial of degree n	Linear or matrix DE, order n
Zero set	Finite set	\mathbb{C} -Vector space
Extension L/K	Galois extension	Picard-Vessiot extension
Galois group	$\leq S_n$	$\leq \mathrm{GL}_n(\mathbb{C})$ (Linear algebraic group)

Can define differential essential dimension $\mathrm{ed}^\partial(\bullet)$ similarly.

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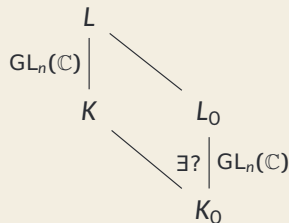
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Bound on $\gamma(n)$

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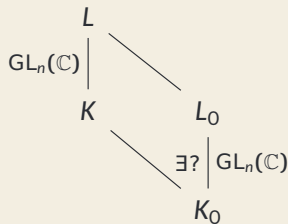


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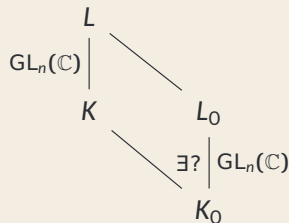


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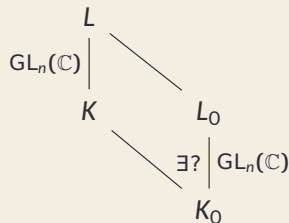


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How much more can we simplify using more general transformations?

$$\begin{array}{l}
 p(y) = y^{(n)} + a_{n-1}y^{(n-1)} + a_{n-2}y^{(n-2)} + \dots + a_0y \\
 \underbrace{y = e^{-\frac{1}{n} \int a_{n-1} z}}_{\text{wavy arrow}} \quad q(z) = z^{(n)} + 0 \cdot z^{(n-1)} + b_{n-2}z^{(n-2)} + \dots + b_0z
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Analogue of allowing $\sqrt[n]{\bullet}$.

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Thanks! Questions?