3.3 Relation to subgroups — Part 2

November 5, 2021

Last time

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If $[G : H] = n$ is finite, there exist group homomorphisms, called **corestriction maps**,

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$$\mathrm{Cor} : H^i(H, A) \to H^i(G, A) \qquad \text{for all} \quad i \geq 0.$$

If instead H is a normal subgroup of G, there exist group homomorphisms, called **inflation maps**,

$$\mathrm{Inf} : H^i(G/H, A) \to H^i(G, A) \qquad \text{for all} \quad i \geq 0.$$

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Theorem A. Let G be a group, A a G-module, and H a subgroup of finite index n in G. Then

$$\mathrm{Cor} \circ \mathrm{Res} : H^i(G, A) \to H^i(G, A)$$

is given by multiplication by n for all $i \geq 0$.

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Theorem B. Let G be a group, A a G-module, and H a normal subgroup of G. There exists a map

$$\tau : H^1(H, A)^{G/H} \to H^2(G/H, A^H),$$

called the **transgression map**, fitting into an exact sequence

$$0 \longrightarrow H^1(G/H, A^H) \xrightarrow{\text{Inf}} H^1(G, A) \xrightarrow{\text{Res}} H^1(G/H, A)^{G/H}$$

$$\xrightarrow{\ \tau\ } H^2(G/H, A^H) \xrightarrow{\text{Inf}} H^2(G, A)$$

The tool to help define Res, Cor, Inf:

Shapiro's lemma

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to

$$g' \cdot \phi : \mathbb{Z}[G] \to A : g \mapsto \phi(gg')$$

for any $g' \in G$.[1]

[1] Check the map $g'\phi$ is again H-equivariant.

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Lemma. (Shapiro) Given subgroup H of a group G and an H-module A, there are canonical isomorphisms

$$H^i(G, \mathrm{CoInd}_H^G(A)) \cong H^i(H, A) \qquad \text{for all} \quad i \geq 0.$$

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$$\mathrm{Hom}_{\mathbb{Z}[G]}(P_i, \mathrm{CoInd}_H^G(A)) \cong \mathrm{Hom}_{\mathbb{Z}[H]}(P_i, B).$$

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Taking cohomology yields the lemma.

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The maps Res, Cor, Inf

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Define $\mathrm{Res} : H^{i+1}(G, A) \to H^{i+1}(H, A)$ as
$\mathrm{Res} : H^i(G, A') \to H^i(H, A')$.

Restriction maps

Explicitly,

$$\mathrm{Res} : H^i(G, A) \to H^i(H, A)$$

takes a cochain

$$f : G^i \to A$$

to the the same function with its domain restricted to H^i:

$$\mathrm{Res}(f) : H^i \to A.$$

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$\underline{i = 0.}$ The map "averages over the coset representatives $g_1, ..., g_n$ of G/H":

$$A^H \to A^G : a \mapsto \sum_{j=1}^{n} g_j \cdot a$$

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$\underline{i \Rightarrow i + 1.}$ Do the same "dimension shifting" argument using Shapiro's lemma.

Restriction-Corestriction

Theorem A. Let G be a group, A a G-module, and H a subgroup of finite index n in G. Then

$$\mathrm{Cor} \circ \mathrm{Res} : H^i(G, A) \to H^i(G, A)$$

is given by multiplication by n for all $i \geq 0$.

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Proof.

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Inflation maps

For a group G, normal subgroup H, and G-module A, we define maps

$$\mathrm{Inf} : H^i(G/H, A^H) \to H^i(G, A)$$

on degree zero and use dimension shifting to extend the definition to all $i \geq 0$.

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Explicitly, Inf takes a cochain

$$f : (G/H)^i \to A^H$$

to the cochain

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Theorem B. Let G be a group, A a G-module, and H a normal subgroup of G. There exists a map

$$\tau : H^1(H, A)^{G/H} \to H^2(G/H, A^H),$$

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$$\xrightarrow{\tau} H^2(G/H, A^H) \xrightarrow{\text{Inf}} H^2(G, A)$$

Proof. Consequence of the Hochschild-Serre spectral sequence.

Inflation-Restriction

Proposition 3.3.17. In Theorem B, let $i > 1$ and assume $H^j(H, A) = 0$ for $1 \leq j \leq i-1$. Then there is a natural map

$$\tau_{i,A} : H^i(H, A)^{G/H} \to H^{i+1}(G/H, A^H)$$

fitting into an exact sequence

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$$\xrightarrow{\tau} H^{i+1}(G/H, A^H) \xrightarrow{\text{Inf}} H^{i+1}(G, A)$$

Proof. Follows from Theorem B and dimension shifting.

References

Section 3.3 from Gille-Szamuely, Central Simple Algebras and Galois Cohomology.

Sharifi's notes on group and Galois cohomology, https://www.math.ucla.edu/~sharifi/groupcoh.pdf

For spectral sequences, see e.g. Vakil's notes on algebraic geometry, section 1.7, http://math.stanford.edu/~vakil/216blog/FOAGnov1817public.pdf