3.3 Relation to subgroups

November 1, 2021

Introduction

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Smooth manifold

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For $n > m \geq 0$, we have $S^n \not\simeq S^m$ since

$$H^n(S^n) \cong \mathbb{Z} \quad \text{but} \quad H^n(S^m) \cong 0.$$

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They also help classify things.

Nonabelian group cohomology (chapter 2)

group (nonabelian) group with compatible G-action

$H^0(G, A) = A^G$ is a group.

$H^1(G, A)$ is a pointed set.

H1 classifies twisted forms.

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Group cohomology (chapter 3)

group   abelian group with compatible G-action

$H^i(G, A)$ for $i \geq 0$.

$H^2(G, A)$ classifies group extensions of G by A
0-class corresponds to split extension (the semidirect product $A \rtimes G$ with the given G-action on A)

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Break the manifold X up into subspaces U and V and relate $H^\bullet(U)$ and $H^\bullet(V)$ to $H^\bullet(X)$. (Mayer-Vietoris sequence)

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How can we relate $H^\bullet(H, A)$ and $H^\bullet(G/H, A^H)$ to $H^\bullet(G, A)$?

Main results for section

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If instead H is a normal subgroup of G, there exist group homomorphisms, called **inflation maps**,

$$\text{Inf} : H^i(G/H, A) \to H^i(G, A) \qquad \text{for all} \quad i \geq 0.$$

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Theorem A. Let G be a group, A a G-module, and H a subgroup of finite index n in G. Then

$$\mathrm{Cor} \circ \mathrm{Res} : H^i(G, A) \to H^i(G, A)$$

is given by multiplication by n for all $i \geq 0$.

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Theorem B. Let G be a group, A a G-module, and H a normal subgroup of G. There exists a map

$$\tau : H^1(H, A)^{G/H} \to H^2(G/H, A^H),$$

called the **transgression map**, fitting into an exact sequence

$$0 \longrightarrow H^1(G/H, A^H) \xrightarrow{\text{Inf}} H^1(G, A) \xrightarrow{\text{Res}} H^1(G/H, A)^{G/H}$$

$$\xrightarrow{\tau} H^2(G/H, A^H) \xrightarrow{\text{Inf}} H^2(G, A)$$

Application

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Proof.

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So $\times n$ map is the $\times 0$ map.

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Let G be a group and A an abelian group. If $|G|$ and $|A|$ are finite and coprime, then any group extension of G by A is the semidirect product $A \rtimes G$.

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So $\gcd(|G|, |A|) \cdot H^2(G, A) = 0$.

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So $1 \cdot H^2(G, A) = 0$.