Why central simple algebras

§ Representation of finite gps

Let k be a field, G a finite group.

A G-representation is a finite-dimensional k-vector space V with a linear G-action, or equivalently, a $k[G]$-module where $k[G]$ is the group ring of G.

Theorem (Maschke) $k[G]$ is a semisimple k-algebra.

Thus V and $k[G]$ are semisimple $k[G]$-modules, decomposing into simple submods V_i's and W_j's, resp.:

$$V = \oplus \underbrace{V_i^{\neq 0}}_{\text{irreducible subrepresentations (irreps)}}, \qquad k[G] = \oplus W_j^{\neq 0}.$$

Let $v \in V_i$. Then

$$\bigoplus_j W_j = k[G] \longrightarrow V_i \qquad \Rightarrow \qquad W_j \longrightarrow V_i$$
$$f \longmapsto f \cdot v \qquad\qquad \text{homo of simple } k[G]\text{-mods.}$$

is a $k[G]$-mod homo, $\exists\, j$ st. $W_j \to V_i$ is nonzero map,

By Schur's lemma, $W_j \cong V_i$.

Conclusion The irreps of G $\equiv$ indecomposable direct summands of $k[G]$.

Theorem (Artin–Wedderburn) R = simple ring $\Rightarrow$ $R \cong M_n(D)$ for some $n \geq 1$ and division ring D.

Conclusion

To classify G-reps over k, first classify the finite-dimensional division algebras $/k$ $\longleftrightarrow$ knowing $Br(k')$ for each finite field extension k'/k.

Organize this as follows:

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$$\mathrm{Br}(k) := \{ \underbrace{\text{division algebra over } k \text{ with center } k}_{\text{central division alg (CDA)}/k.} \} / \cong .$$

Note

$$(*) \qquad D_1 \otimes_k D_2 \cong M_n(D_3) \quad \text{some } n \geq 1, \text{ division alg } D_3 \text{ with center } k.$$

Thus $\mathrm{Br}(k)$ becomes a gp under $[D_1]\cdot[D_2] := [D_3]$ given by $(*)$

Nicer defn of Br gp.

A central simple algebra / k (CSA/k) is a simple alg A with $\dim_k A < \infty$ & $Z(A) = k$.

Thus $A \cong M_n(D)$, $Z(D) = k$.

Two CSA/k A and B are Brauer equivalent if there exist $m, n > 0$:

$$M_m(A) \cong M_n(B) \quad \text{as } k\text{-algebras}.$$

Then

$$\mathrm{Br}(k) = \{ \text{CSA}/k \} / \text{Brauer equivalence}, \text{ a gp under } \otimes.$$

Exs $\quad \mathrm{Br}(\mathbb{C}) = \{ [\mathbb{C}] \}$

$\mathrm{Br}(\mathbb{R}) = \{ [\mathbb{R}], [\mathbb{H}] \}$

Here $\mathbb{H}$ is the $\mathbb{R}$-algebra with presentation $\langle i, j \mid i^2 = j^2 = -1, ij = -ji \rangle$
$= \langle -1, -1 \rangle_{-1}$

(§2.5) § Structure of CSA's / Br gp

Assume $m \in k^\times$, k has primitive m^{th} rt of unity ω.

Define the k-alg with presentation

"cyclic algebra" $$(a,b)_\omega := \langle x, y \mid x^m = a, y^m = b, xy = \omega yx \rangle.$$

Want these as "building blocks" for CSA.

Naive conjecture any CSA/k is $\cong \otimes$ cyclic algs.

False (exercise: deduce this from discussion preceding Thm 1.5.8 in Gille-Szamuely)

But true up to Brauer equivalence!

(Thm 2.5.7) **Theorem** (Merkurjev – Suslin)

Under hypothesis at start of this §,

if A is a CSA / k whose class in $Br(k)$ has order m, then

$$\underset{\text{Brauer equivalent}}{A \sim} (a_1, b_1)_\omega \otimes_k \cdots \otimes_k (a_i, b_i)_\omega \quad \text{some cyclic algs of order } m.$$

(§4.6) ## § Norm – residue isomorphism thm

Assume $m \in k^\times$.

$K_n^M(k)$ ← n^{th} Milnor K-gp

$H^n(k, \mu_m^{\otimes n})$ ← Galois cohomology gp

† $\exists$ gp homo $K_n^M(k)/m \longrightarrow H^n(k, \mu_m^{\otimes n})$

Thm (Norm-residue ..., Bloch-Kato conjecture; Voevodsky).

† is an iso.

Thm (Merkurjev – Suslin) Above thm holds for $n = 2$.

§ 4.7 explains why this version of M–S $\Rightarrow$ previous version of M-S.