3.4 Cup products

November 5, 2021

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making this into a graded *ring*: $dx \wedge dy = -dy \wedge dx$.

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This multiplication will be compatible with the maps Res, Cor, and Inf.

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$$(f \cup f')(g_1, g_2, \ldots, g_{i+j}) := f(g_1, \ldots, g_i) \otimes g_1 g_2 \cdots g_i f'(g_{i+1}, \ldots, g_{i+j}).$$

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This defines a map

$$C^i(G, A) \otimes C^j(G, B) \xrightarrow{\cup} C^{i+j}(G, A \otimes B).$$

Properties of the cup product

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For $f \in C^i(G, A)$ and $f' \in C^j(G, B)$, the cup product satisfies

$$d^{i+j}_{A\otimes B}(f \cup f') = d^i_A(f) \cup f' + (-1)^j f \cup d^i_B(f').$$

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For $i = j = 0$, this is just the map

$$A^G \otimes B^G \to (A \otimes B)^G$$

induced by the identity on $A \otimes B$.

Remark 1

The cup product can be better defined starting with projective resolutions of $\mathbb{Z}$ as $\mathbb{Z}[G]$-modules. See first few pages of Section 3.4 in Gille-Szamuely.

Remark 2

The map

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More generally, given morphism of G-modules $A \times B \to C$, the composite

$$H^i(G, A) \otimes H^j(G, B) \to H^{i+j}(G, A \otimes B) \to H^{i+j}(G, C).$$

is called a **cup-product** map.

Properties

On cohomology, the cup product is associative and graded-commutative:

$$a \cup b = (-1)^{ij}(b \cup a).$$

Technical propositions for later chapters — 1

Given an exact sequence

$$A_\bullet : \quad 0 \to A_1 \to A_2 \to A_3 \to 0$$

of G-modules such that $A_\bullet \otimes B$, with B a G-module, is also exact, then for $a \in H^i(G, A_3)$ and $b \in H^j(G, B)$,

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$$\delta(a) \cup b = \delta(a \cup b) \quad \in \; H^{i+j+1}(G, A_1 \otimes B)$$

holds, where the δ are the connecting maps in the associated long exact sequences.

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$$a \cup \delta(b) = (-1)^i \delta(a \cup b) \quad \in \ H^{i+j+1}(G, A \otimes B_1)$$

holds, where the δ are the connecting maps in the associated long exact sequences.

Technical propositions for later chapters — 2

See Proposition 3.4.9 on page 75 of Gille-Szamuely.

Cup product and Res, Cor, Inf

Given G-modules A and B, we have:

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1. For $a \in H^i(G, A)$ and $b \in H^j(G, B)$,

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2. For $H \trianglelefteq G$, $a \in H^i(G/H, A^H)$ and $b \in H^j(G/H, B^H)$,

$$\operatorname{Inf}(a \cup b) = \operatorname{Inf}(a) \cup \operatorname{Inf}(b).$$

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$$\mathrm{Inf}(a \cup b) = \mathrm{Inf}(a) \cup \mathrm{Inf}(b).$$
3. For $[G : H] < \infty$, $a \in H^i(G, A)$ and $b \in H^j(G, B)$,
$$\mathrm{Cor}(a \cup \mathrm{Res}(b)) = \mathrm{Cor}(a) \cup b.$$

Cohomology of finite cyclic groups

Let G be a finite cyclic group of order n and let $\chi \leftrightarrow \mathrm{id}$ in $H^1(G, \mathbb{Z}/n\mathbb{Z}) \cong \mathrm{Hom}(G, \mathbb{Z}/n\mathbb{Z})$.

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1. Consider the boundary map $\delta : H^1(G, \mathbb{Z}/n\mathbb{Z}) \to H^2(G, \mathbb{Z})$ coming from the SES

$$0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

   Then $\delta(\chi)$ is a generator of the cyclic group $H^2(G, \mathbb{Z})$.

Proof of 1. See text.

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2 The isomorphisms $H^i(G, A) \cong H^{i+2}(G, A)$ of §2 are induced by cup product with $\delta(\chi)$ for all $i > 0$.

Proof of 2. See text.

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Let G be a finite cyclic group of order n and let $\chi \leftrightarrow \mathrm{id}$ in $H^1(G, \mathbb{Z}/n\mathbb{Z}) \cong \mathrm{Hom}(G, \mathbb{Z}/n\mathbb{Z})$.

3 The isomorphism $A^G/NA \cong H^2(G, A)$ is induced by $a \mapsto a \cup \delta(\chi)$.

Proof of 3. See text.

References

Section 3.4 from Gille-Szamuely, Central Simple Algebras and Galois Cohomology.

Sharifi's notes on group and Galois cohomology, `https://www.math.ucla.edu/~sharifi/groupcoh.pdf`