Chapter 3: Techniques from Group Cohomology

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Central Simple Algebras and Galois Cohomology

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Section 3.1

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- Morphism $A \longrightarrow B$ is morphism of abelian groups and G-linear

Cohomology basics

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$$\begin{tikzcd}
 & \ker\alpha \arrow[d] & \ker\beta \arrow[d] & \ker\gamma \arrow[d] & \\
 & A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & D \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & F \arrow[d] & \\
 & \mathrm{coker}\alpha & \ker\beta & \ker\gamma &
\end{tikzcd}$$

Projective Modules

$$\begin{tikzcd} & A \arrow[d, two heads, "\alpha"] \\ P \arrow[ur, dashed] \arrow[r, "f"] & B \end{tikzcd}$$

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$$... \longrightarrow P_3 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A$$

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- Short exact sequences of G-modules induce long exact sequences
- $H^i(G, A) := H^i(Hom_G(P_i, A))$

Lemma 1

$A^G \cong Hom_G(\mathbb{Z}, A) \cong \ker(Hom(P_0, A) \longrightarrow Hom(P_1, A)) := H^0(G, A)$

Proof

$\phi \in Hom_G(\mathbb{Z}, A) \iff \phi(1) \in A^G$ since $m = \phi(1) = \phi(g.1) = g.\phi(1) = g.m$

$$
\begin{tikzcd}
 & & & & \mathbb{Z} \arrow[lld, bend right, "i"'] \arrow[dl, Rightarrow] \\
\dots \arrow[r] & P_1 \arrow[r, "p_1"] & P_0 \arrow[r, "p_0"] \arrow[dr, dashed, "\lambda_0"] & \mathbb{Z} \arrow[d, "\phi"] & \\
 & & & A &
\end{tikzcd}
$$

$$Hom(P_0, A) \longrightarrow Hom(P_1, A) \longrightarrow Hom(P_2, A) \longrightarrow \dots \square$$

Lemma 2

$A \longrightarrow B$ induces $H^i(G, A) \longrightarrow H^i(G, B)$

Proof

$$
\begin{tikzcd}
\ldots \arrow[r] & P_0 \arrow[r] & \mathbb{Z} \arrow[d] & Hom_G(P_0, A) \arrow[r] \arrow[dd] & Hom_G(P_1, A) \arrow[r] \arrow[dd] & \ldots \\
 & & A \arrow[d] & & & \\
 & & B & Hom_G(P_0, B) \arrow[r] & Hom_G(P_1, B) \arrow[r] & \ldots
\end{tikzcd}
$$

$\square$

Corollary

A short exact sequence of G-modules

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

induce

$$\ldots \longrightarrow H^i(G, A) \longrightarrow H^i(G, B) \longrightarrow H^i(G, C) \longrightarrow H^{i+1}(G, A) \longrightarrow \ldots$$

Section 3.2

- More general

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- .. but not concrete enough

Standard Resolution of $\mathbb{Z}$

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- Define $\delta_i : \mathbb{Z}\left[G^{i+1}\right] \longrightarrow \mathbb{Z}\left[G^{i}\right]$ by $\delta_i = \sum_j \left(-1\right)^j s^i_j$ where

  $s^i_j\left(\sigma_0, ...\sigma_i\right) = \left(\sigma_0, ..., \widehat{\sigma}_j, ..., \sigma_i\right)$

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- For $i = 0$, we have $\delta_0 : \mathbb{Z}\left[G\right] \longrightarrow \mathbb{Z} = \langle \varnothing \rangle$ with $\delta_0\left(\sigma_0\right) = 1$
- $... \longrightarrow \mathbb{Z}\left[G^2\right] \xrightarrow{\delta_2} \mathbb{Z}\left[G^1\right] \xrightarrow{\delta_1} \mathbb{Z}\left[G^0\right] \xrightarrow{\delta_0} \mathbb{Z}$

Lemma 3

$\delta_i \circ \delta_{i+1} = 0$

Proof

$$\delta_i \circ \delta_{i+1} = \begin{cases} \sum_k \sum_j (-1)^k \, s_k^i \left((-1)^j \, s_j^{i+1} \right) & \text{with } j < k \\ + \sum_k \sum_j (-1)^k \, s_k^i \left((-1)^{j-1} \, s_j^{i+1} \right) & \text{with } k < j \end{cases}$$

Fact

$$
\begin{tikzcd}
\cdots \arrow[r] & C_{i+1} \arrow[r, "\partial_{i+1}"] \arrow[d, "f_{i+1},g_{i+1}"'] & C_i \arrow[r, "\partial_i"] \arrow[d, "f_i,g_i"'] \arrow[dl, "p_i"'] & C_{i-1} \arrow[r] \arrow[d, "f_{i-1},g_{i-1}"] \arrow[dl, "p_{i-1}"] & \cdots \\
\cdots \arrow[r] & D_{i+1} \arrow[r, "\partial'_{i+1}"] & D_i \arrow[r, "\partial'_i"] & D_{i-1} \arrow[r] & \cdots
\end{tikzcd}
$$

f is said to be **homotopic** to g if there exist maps p such that

$$\partial'_{i+1} \circ p_i + p_{i-1} \circ \partial_i = g_i - f_i$$

Chain homotopic maps preserve homology

- Now fix $\sigma \in \mathbb{Z}\left[G^{i+1}\right]$ and define $h_i : \mathbb{Z}\left[G^{i+1}\right] \longrightarrow \mathbb{Z}\left[G^{i+2}\right]$ by $h_i\left(\sigma_0, ...\sigma_i\right) = \left(\sigma, \sigma_0, ...\sigma_i\right)$
- Note that $\delta_{i+1} \circ h_i + h_{i-1} \circ \delta_i = id_{\mathbb{Z}[G^{i+1}]}$

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Proof

$$\left(\sum_k (-1)^k s_k^{i+1}\right) h_i\left(\sigma_0, ...\sigma_i\right) + h_{i-1} \circ \sum_j (-1)^j s_j^i\left(\sigma_0, ..., \sigma_i\right)$$

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$$= \left(\sum_k (-1)^k s_k^{i+1}\right)\left(\sigma, \sigma_0, \ldots \sigma_i\right) + h_{i-1} \sum_j (-1)^j \left(\sigma_0, \ldots, \widehat{\sigma}_j, \ldots, \sigma_i\right)$$

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$$= \left(\sigma_0, ..., \sigma_i\right)$$

Finally..

$$Hom_G\left(\mathbb{Z}\left[G\right], A\right) \longrightarrow Hom_G\left(\mathbb{Z}\left[G^2\right], A\right) \longrightarrow Hom_G\left(\mathbb{Z}\left[G^3\right], A\right) \longrightarrow ...$$

Finally..

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i-cochains $\in Hom_G\left(\mathbb{Z}\left[G^{i+1}\right], A\right)$

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i-cochains $\in Hom_G\left(\mathbb{Z}\left[G^{i+1}\right], A\right)$
i-cocyles $\in \ker\left(Hom_G\left(\mathbb{Z}\left[G^{i+1}\right], A\right) \longrightarrow Hom_G\left(\mathbb{Z}\left[G^{i+2}\right], A\right)\right)$

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i-coboundaries $\in \mathrm{Im}\left(Hom_G\left(\mathbb{Z}\left[G^{i}\right], A\right) \longrightarrow Hom_G\left(\mathbb{Z}\left[G^{i+1}\right], A\right)\right)$

Inhomogeneous Cochains

Alternatively.. use $\mathbb{Z}\left[G^{i+1}\right] = \langle e, \sigma_1, \sigma_1\sigma_2, \sigma_1\sigma_2\sigma_3, ..., \sigma_1 \cdots \sigma_i \rangle$

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Let $[\sigma_1, ..., \sigma_i] = (e, \sigma_1, \sigma_1\sigma_2, \sigma_1\sigma_2\sigma_3, ..., \sigma_1 \cdots \sigma_i)$

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Let $[\sigma_1, ..., \sigma_i] = (e, \sigma_1, \sigma_1\sigma_2, \sigma_1\sigma_2\sigma_3, ..., \sigma_1 \cdots \sigma_i)$
Note that

$$\begin{aligned}
& \delta_i [\sigma_1, ..., \sigma_i] \\
= \; & \delta_i (e, \sigma_1, \sigma_1\sigma_2, \sigma_1\sigma_2\sigma_3, ..., \sigma_1 \cdots \sigma_i) \\
= \; & \sum_j (-1)^j s^i_j (e, \sigma_1, \sigma_1\sigma_2, \sigma_1\sigma_2\sigma_3, ..., \sigma_1 \cdots \sigma_i) \\
= \; & (\sigma_1, \sigma_1\sigma_2, \sigma_1\sigma_2\sigma_3, ..., \sigma_1 \cdots \sigma_i) - (e, \sigma_1\sigma_2, \sigma_1\sigma_2\sigma_3, ..., \sigma_1 \cdots \sigma_i) + ... \\
& + (-1)^{i+1} (e, \sigma_1, \sigma_1\sigma_2, \sigma_1\sigma_2\sigma_3, ..., \sigma_1 \cdots \sigma_{i-1}) \\
= \; & \sigma_1 . (e, \sigma_2, \sigma_2\sigma_3, ..., \sigma_2 \cdots \sigma_i) - (e, \sigma_1\sigma_2, \sigma_1\sigma_2\sigma_3, ..., \sigma_1 \cdots \sigma_i) + ... \\
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= \; & \sigma_1 . [\sigma_2, ..., \sigma_i] + \sum_j (-1)^j [\sigma_1, ..., \sigma_j\sigma_{j+1}, ..., \sigma_i] + (-1)^{i+1} [\sigma_1, ..., \sigma_{i-1}]
\end{aligned}$$

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An i-cochain is then the function $[\sigma_1, ..., \sigma_i] \mapsto a_{\sigma_1,...,\sigma_i}$

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An i-cochain is then the function $\left[\sigma_1, \ldots, \sigma_i\right] \mapsto a_{\sigma_1, \ldots, \sigma_i}$

$\delta^i : C^{i-1}(G, A) \longrightarrow C^i(G, A)$ with

$$\delta^i\left(a_{\sigma_1, \ldots, \sigma_i}\right)=\sigma_1 . a_{\sigma_2, \ldots, \sigma_i}+\sum_j(-1)^j a_{\sigma_1, \ldots, \sigma_j \sigma_{j+1}, \ldots, \sigma_i}+(-1)^{i+1} a_{\sigma_1, \ldots, \sigma_{i-1}}$$

Inhomogeneous Cochains: Examples

- 1-cocycle
$\iff \delta^1(a_{\sigma_1}) = 0 \iff \sigma_1.a_{\sigma_1} - a_{\sigma_1\sigma_2} + a_{\sigma_1} = 0 \iff a_{\sigma_1\sigma_2} = \sigma_1.a_{\sigma_1} + a_{\sigma_1}$
- 1-coboundary $\iff \sigma \longmapsto \sigma a - a$ so $Z^1(G,A) = H^1(G,A) = Hom(G,A)$
- 2-cocyle $\iff \delta^2(a_{\sigma_1\sigma_2}) = 0 \iff \sigma_1.a_{\sigma_2\sigma_3} - a_{\sigma_1\sigma_2,\sigma_3} + a_{\sigma_1,\sigma_2\sigma_3} - a_{\sigma_1,\sigma_2} = 0$
- 2-coboundary $\iff a_{\sigma_1\sigma_2} = \delta^1(b) = \sigma_1.b_{\sigma_1} - b_{\sigma_1\sigma_2} + b_{\sigma_1}$

Normalized Resolution

- Same as $[\sigma_1, ..., \sigma_i]$, but $\sigma_j \neq 0$ for all j
- If $\sigma_j \sigma_{j+1} = e$, then set $a_{\sigma_1,\ldots,\sigma_j\sigma_{j+1},\ldots,\sigma_i} = 0$

Group Extensions

- E is an extension of G by A if

$$\{e\} \longrightarrow A \xrightarrow{\imath} E \xrightarrow{\pi} G \longrightarrow \{e\}$$

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- Aim: Want to define $E \mapsto c(E) \in H^2(G, A)$

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For $a \in A$ (abelian), $\sigma \in G$, and s such that $\pi \circ s = id_G$ and $s(e) = e$, define $\imath(\sigma.a) := \widehat{\sigma}\imath(a)\widehat{\sigma}^{-1} \in \imath(A)$ where $\widehat{\sigma} = s(\sigma) \in E$

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Let $\imath(b)\widehat{\sigma} = s(\sigma)$ for some $b \in A$. Then $\imath(b)\widehat{\sigma}\imath(a)\widehat{\sigma}^{-1}\imath\left(b^{-1}\right) = \widehat{\sigma}\imath(a)\widehat{\sigma}^{-1}$

Group Extensions

Theorem

Equivalent extensions define the same G-module structure on A

Proof

Let $\widehat{\sigma}' = \beta(\widehat{\sigma})$. Then $s(\widehat{\sigma}') = s(\widehat{\sigma}) = \sigma$ and $\imath(\sigma.a) = \imath(\sigma'.a)$

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Theorem

$a_{\sigma_1,\sigma_2} := s(\sigma_1)\, s(\sigma_2)\, s(\sigma_1\sigma_2)^{-1}$ is a normalized 2-cocycle

Proof

$\pi(a_{\sigma_1,\sigma_2}) = e$
$a_{\sigma_1,e} = a_{e,\sigma_2} = e$
$\forall \sigma_1, \sigma_2, \sigma_3 \in G$, $\sigma_1.a_{\sigma_2\sigma_3} - a_{\sigma_1\sigma_2,\sigma_3} + a_{\sigma_1,\sigma_2\sigma_3} - a_{\sigma_1,\sigma_2} = 0$

Group Extensions

- For $c^{-1}(E)$, given $a_{\sigma_1\sigma_2}$, define $E := A \times G$ with binary operation

$$(a_1, \sigma_1) * (a_2, \sigma_2) = (a_1 + \sigma_1(a_2) + a_{\sigma_1\sigma_2}, \sigma_1\sigma_2)$$

Unit $(0, e)$ and $(a, \sigma) * \left(-\sigma^{-1}(a) - \sigma^{-1}\left(a_{\sigma,\sigma^{-1}}\right), \sigma^{-1}\right) = (0, e)$

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- The induced map $\phi_* : H^2(G, A) \longrightarrow H^2(G, B)$ from G-map $\phi : A \longrightarrow B$ commutes with c where $\phi_*(E) := (B \times E)/N$ where N is the normal subgroup generated by $\left(\phi(a), \imath(a)^{-1}\right)$

Group Cohomology

Put $G = \mathbb{Z}$ and consider

$$0 \longrightarrow \mathbb{Z}[\mathbb{Z}] \xrightarrow{\alpha} \mathbb{Z}[\mathbb{Z}] \xrightarrow{\beta} \mathbb{Z} \longrightarrow 0$$

$\alpha(\sigma) = \sigma - 1$ is injective and $\beta(\sigma) = 1$ is surjective, where $\langle\sigma\rangle = \mathbb{Z}$ and $\beta \circ \alpha = 0$

$H^0(\mathbb{Z}, A) = A^\sigma$, $H^1(\mathbb{Z}, A) = A/\sigma - 1(A)$ and $H^i(\mathbb{Z}, A) = 0$ for $i \geq 2$

Group Cohomology

If G is a cyclic group of order n, let $\mathbb{Z}[\mathbb{Z}] \xrightarrow{\alpha} \mathbb{Z}[\mathbb{Z}]$ and $\mathbb{Z}[\mathbb{Z}] \xrightarrow{\beta} \mathbb{Z}[\mathbb{Z}]$ be such that

$$\alpha(a) = a\sigma - a \text{ and } \beta(a) = \sum_{i=0}^{n-1} \sigma^i a$$

Then $\ker\alpha = \mathrm{Im}\beta$ and $\mathrm{Im}\alpha = \ker\beta$

$$\ldots \xrightarrow{\beta} \mathbb{Z}[\mathbb{Z}] \xrightarrow{\alpha} \mathbb{Z}[\mathbb{Z}] \xrightarrow{\beta} \mathbb{Z}[\mathbb{Z}] \xrightarrow{\alpha} \mathbb{Z}[\mathbb{Z}] \xrightarrow{\beta} \mathbb{Z}[\mathbb{Z}] \xrightarrow{\alpha} \mathbb{Z}[\mathbb{Z}] \xrightarrow{\beta} \mathbb{Z} \longrightarrow$$

Extend $\alpha : A \longrightarrow A$ and $\beta : A \longrightarrow A$ and write ${}_N A = \ker\beta$. Then $H^0(G, A) = A^G$, $H^{2i+1}(G, A) =_N A/(\sigma - 1)A$ and $H^{2i+2}(G, A) = A^G/\beta(A)$

Galois Cohomology

Now let $A = K^\times$ where $K|k$ is a finite Galois extension with cyclic Galois group G. Then $H^1(G, K^\times) =_N K^\times / (\sigma - 1) K^\times = \{e\}$