Outline

Motivation:

- Corollary 2.2.6

Definitions:

- Twisted forms
- First group cohomology set

Main result:

- Twisted forms $\Longleftrightarrow$ first group cohomology set

Examples:

- Hilbert's Theorem 90
- Quadratic forms
- CSA

Special goals:

- Justify the term "descent"
- Use diagrams to illustrate cocycle condition and cohomologous condition (Man Cheung)

Other resources:

- Notes by Joshua Ruiter
- Notes by Keith Conrad
- Video by Daniel Krashen

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A finite dimensional k-algebra A is a CSA iff there exist an integer $n > 0$ and a finite Galois extension $K \mid k$ so that $A \otimes_k K$ is isomorphic to $M_n(K)$.

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Generalization of the Question

$K \mid k$ finite Galois extension. V is a k-object (k-vector space with or without additional structure). $V \otimes_k K$ is a K-object.

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- When does W descent?
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- Relation between A, B if $A \otimes K \cong B \otimes K \cong W$?

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Vector Spaces: Speiser Lemma

Lemma

$K \mid k$, finite Galois extension with Galois group G. V is a K-v.s. equipped with a G-action which is additive and K-semilinear.

$$\sigma(av) = \sigma(a)\sigma(v), \quad \sigma \in G, a \in K, v \in V.$$

Then the natural map

$$\lambda : V^G \otimes_k K \to V.$$

$$\begin{array}{ccc} K & & V \\ \uparrow & & \downarrow \\ k & & ? \end{array}$$

When G-action on V is semilinear, $? = V^G$.

k-objects

k-object:
V a k-v.s. equipped with a tensor Φ of type (p, q), meaning

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A k-isomorphism between (V, Φ) and (W, Ψ) is an isomorphism of v.s. $f : V \to W$ such that

$$\begin{tikzcd} V^{\otimes q} \arrow[r, "f^{\otimes q}"] \arrow[d, "\Phi"] & W^{\otimes q} \arrow[d, "\Psi"] \\ V^{\otimes p} \arrow[r, "f^{\otimes p}"] & W^{\otimes q} \end{tikzcd}$$

commutes.

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For $\sigma \in G$, it induces a k-automorphism of $V_K = V \otimes_k K$ (*not* a K-automorphism), still denoted by σ. For a K-linear map $f : V_K \to W_K$, σ induces a K-linear map $\sigma(f) = \sigma \circ f \circ \sigma^{-1}$.

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Easy to check: if f is a K-isomorphism $(V_K, \Phi_K) \to (W_K, \Psi_K)$, then so is $\sigma(f)$. $f \mapsto \sigma(f)$ preserves composition (and hence is a left action of G on $\mathrm{Aut}_K(\Phi)$?).

$$\begin{tikzcd}
(V_K, \Phi_K) \arrow[r, bend left, "\sigma(g)"] & (W_K, \Psi_K) \arrow[l, "g^{-1}"]
\end{tikzcd}$$

Given $g : (V_K, \Phi_K) \to (W_K, \Psi_K)$, we get a map $G \to \mathrm{Aut}_K(\Phi)$ associating $a_\sigma = g^{-1} \circ \sigma(g)$ to $\sigma \in G$.

$$\text{cocycle relation:} \qquad a_{\sigma\tau} = a_\sigma \cdot \sigma(a_\tau) \tag{1}$$

For another K-automorphism $b : (V_K, \Phi_K) \to (W_K, \Psi_K)$,

$$\text{coboundary relation:} \qquad a_\sigma = c^{-1} b_\sigma \sigma(c), \tag{2}$$

where $c = h^{-1} \circ g \in \mathrm{aut}_K(\Phi)$.

Definition

Given groups G, A. G acts on A (on the left).

$$\sigma(ab) = \sigma(a)\sigma(b), \quad \sigma(\tau(a)) = \sigma\tau(a).$$

A *1-cocycle* is a map $G \to A$ satisfying Equation 1 . Two cocycles a_σ, b_σ are called *equivalent* or *cohomologous* if they satisfy Equation 2.
$H_1(G, A)$, called *the first cohomology set*, is the set of cohomologous classes of 1-cocycles. It is a *pointed set*. The equivalence class of the trivial 1-cocycle $\sigma \mapsto 1$ is the *base point*.

Theorem

$$TF_K(V,\Phi) \to H^1(G, \mathrm{Aut_K}(\Phi)),$$

$$[W,\Psi] \mapsto [a_\sigma]$$

is a bijection preserving the base point.

Example: Hilbert's Theorem 90

$G = \mathrm{Gal}(K \mid k)$
V is a v.s. of dim n over k without additional structure ($\Phi = 0$).
Then

$$\mathrm{Aut}_K(\Phi) = \mathrm{GL}_n(K).$$

If two k-v.s. are K-isomorphic then they are k-isomorphic (Speiser Lemma). Consequently

$$TF_K(V, \Phi) \cong H^1(G, \mathrm{GL}_K(n)) = \{1\}.$$

When $n = 1$,

$$H^1(G, K^*) = 1.$$

Original form:
If G is cyclic and generated by σ, then for $x \in K$ with $N(x) = 1$, there exists $c \in K$ such that $x = \frac{\sigma(c)}{c}$.

Example: Quadratic Forms

$\mathrm{char}(k) \neq 2$

Φ is a tensor of type $(0,2)$. That is, $\Phi : V \otimes V \to k$. Φ defines a nondegenerate symmetric bilinear form $<,>$. $\mathrm{Aut}_K(\Phi)$ is the group $O_n(K)$ of matrices preserving $<,>$.

$$TF_K(V,\Phi) \cong H^1(G, O_n(K)))$$

Example: CSA

Corollary

A finite dimensional k-algebra A is a CSA iff there exist an integer $n > 0$ and a finite Galois extension $K \mid k$ so that $A \otimes_k K$ is isomorphic to $M_n(K)$.

$\mathrm{Aut}_K(GL_n(K)) = \mathrm{PGL}_n(K)$

$$CSA_K(n) \cong H^1(G, \mathrm{PGL}_n(K))$$

$CSA_K(n)$: k-isomorphism classes of CSA over k of degree n split by K