

## § Representation of finite gps

Let  $k$  be a field,  $G$  a finite group.

A  $G$ -representation is a finite-dimensional  $k$ -vector space  $V$  with a linear  $G$ -action, or equivalently, a  $k[G]$ -module where  $k[G]$  is the group ring of  $G$ .

Theorem (Maschke)  $k[G]$  is a semisimple  $k$ -algebra.

Thus  $V$  and  $k[G]$  are semisimple  $k[G]$ -modules, decomposing into simple submods  $V_i$ 's and  $W_j$ 's, resp.:

$$V = \bigoplus_i V_i^{\neq 0}, \quad k[G] = \bigoplus_j W_j^{\neq 0}$$

↑  
irreducible subrepresentations (irreps)

Let  $v \in V_i$ . Then

$$\bigoplus_j W_j = k[G] \xrightarrow{f} V_i \quad \Rightarrow \quad W_j \xrightarrow{f \cdot v} V_i$$

$f \mapsto f \cdot v$

homo of simple  $k[G]$ -mods.

is a  $k[G]$ -mod homo,  $\exists j$  st.  $W_j \rightarrow V_i$  is nonzero map,

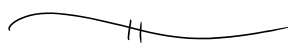
By Schur's lemma,  $W_j \cong V_i$ .

Conclusion The irreps of  $G \equiv$  indecomposable direct summands of  $k[G]$ .

Theorem (Artin-Wedderburn)  $R = \text{Simple ring} \Rightarrow R \cong M_n(D)$   
for some  $n \geq 1$  and division ring  $D$ .

### Conclusion

To classify  $G$ -reps over  $k$ , first classify the finite-dimensional division algebras  $/k \iff$  knowing  $\text{Br}(k')$  for each finite field extension  $k'/k$ .



Organize this as follows:

$$\{ \text{division algebras over } k \text{ with center } k \} / \cong$$

Organize this as follows:

$$\text{Br}(k) := \underbrace{\left\{ \text{division algebra over } k \text{ with center } k \right\}}_{\text{central division alg (CDA)}/k} / \cong.$$

Note

$$(*) \quad D_1 \otimes_k D_2 \cong M_n(D_3) \quad \text{some } n \geq 1, \text{ division alg } D_3 \text{ with center } k.$$

Thus  $\text{Br}(k)$  becomes a gp under  $[D_1][D_2] := [D_3]$  given by  $(*)$

Nicer defn of Br gp.

A central simple algebra / k (CSA/k) is a simple alg  $A$  with  $\dim_k A < \infty$  &  $Z(A) = k$ .

Thus  $A \cong M_n(D)$ ,  $Z(D) = k$ .

Two CSA/k  $A$  and  $B$  are Brauer equivalent if there exist  $m, n > 0$ :

$$M_m(A) \cong M_n(B) \text{ as } k\text{-algebras.}$$

Then

$$\text{Br}(k) = \{ \text{CSA}/k \} / \text{Brauer equivalence, a gp under } \otimes.$$

Exs  $\text{Br}(\mathbb{C}) = \{ [\mathbb{C}] \}$

$$\text{Br}(\mathbb{R}) = \{ [\mathbb{R}], [\mathbb{H}] \}$$

Here  $\mathbb{H}$  is the  $\mathbb{R}$ -algebra with presentation  $\langle i, j \mid i^2 = j^2 = -1, ij = -ji \rangle$

(§2.5)

### § Structure of CSA's / Br gp

$$\begin{matrix} \parallel \\ \langle -1, -1 \rangle_{-1} \end{matrix}$$

Assume  $m \in k^\times$ ,  $k$  has primitive  $m$ th rt of unity  $\omega$ .

Define the  $k$ -alg with presentation

"cyclic algebra"  $(a, b)_\omega := \langle x, y \mid x^m = a, y^m = b, xy = \omega yx \rangle.$

want these as "building blocks" for CSA.

Naive conjecture any CSA/k is  $\cong \otimes$  cyclic algs.

False (exercise: deduce this from discussion preceding Thm 1.5.8 in Gille-Szamuely)  
But true up to Brauer equivalence!

(Thm 2.5.7)

### Theorem (Merkurjev - Suslin)

Under hypothesis at start of this §,

if  $A$  is a CSA/ $k$  whose class in  $Br(k)$  has order  $m$ , then

$$A \sim (a_1, b_1) \otimes_k \dots \otimes_k (a_i, b_i) \otimes \dots \otimes_k (a_m, b_m) \sim \text{some cyclic algs of order } m.$$

Brauer equivalent

(§4.6)

### § Norm - residue isomorphism thm

Assume  $m \in k^\times$ .

$$K_n^M(k) \leftarrow n^{\text{th}} \text{ Milnor } K\text{-gp}$$

$$H^n(k, \mu_m^{\otimes n}) \leftarrow \text{Galois cohomology gp}$$

$$\textcircled{+} \quad \exists \text{ gp homo } K_n^M(k)/m \longrightarrow H^n(k, \mu_m^{\otimes n})$$

Thm (Norm - residue ... , Bloch - Kato conjecture; Voevodsky).

$\textcircled{+}$  is an iso.

Thm (Merkurjev - Suslin) Above thm holds for  $n = 2$ .

§ 4.7 explains why this version of M-S  $\Rightarrow$  previous version of M-S.