2.7 A basic exact sequence

Let K/F be Galois field extension with gp G.

Saw:

$$\left\{ \begin{array}{c} \text{iso classes of objects } / F \text{ that} \\ \text{become } \cong \text{ object } V/_F \text{ over } K \end{array} \right\} \overset{1:1}{\longleftrightarrow} H^1(\underset{\substack{\| \\ \mathrm{Gal}(K/F)}}{G}, \underset{\substack{\| \\ \mathrm{Aut}(V \otimes_F K)}}{A})$$

Eg the CSA $M_n(K)$ has $\mathrm{Aut}_{K\text{-alg}}(M_n(K)) \cong PGL_n(K)$

(conjugate by B) $\longleftarrow$ A

$$\Rightarrow \left\{ \begin{array}{c} \text{Degree } n \text{ CSA's } /F \text{ that} \\ \text{become isomorphic to } M_n(K) \end{array} \right\} /\cong \overset{1:1}{\longleftrightarrow} H^1(G, PGL_n(K))$$

Prop 2.7.1 (Short exact sequence (SES) induce ^not so long exact sequence (LES)).

Let G be a gp.

Let $1 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 1$ be SES of gps with G-equivariant maps.

Then get LES of pointed sets

$$1 \longrightarrow A^G \xrightarrow{\varphi} B^G \xrightarrow{\psi} C^G \xrightarrow{\delta} H^1(G, A) \longrightarrow H^1(G,B) \longrightarrow H^1(G,C)$$

$$(G \to A) \mapsto (G \to A \xrightarrow{\varphi} B)$$

What do the words in this Prop mean?

- Group homos $A \xrightarrow{f} B \xrightarrow{g} C$ is exact at B if $\mathrm{im}(f) = \ker(g)$.
- $A^G = \{ a \in A \mid g \cdot a = a \}$.
- A pointed set is just a set with a distinguished element : $1_A \in A$, $1_A \in A^G$, $\left[\begin{array}{l} G \to A \\ \sigma \mapsto 1_A \end{array} \right] \in H^1(G, A)$.
- A map of pointed sets between (A, a) and (B, b) is a map of sets $f : A \to B : a \mapsto b$.
- Maps of pointed sets $(A, a) \xrightarrow{f} (B, b) \xrightarrow{g} (C, c)$ is exact at B if $\mathrm{im}(f)$ equals the set $\ker(g) := \{ b \in B \mid g(b) = c \}$.

What is δ?

① Let $c \in C^G$. ψ surjects $\Rightarrow \exists\, b \longmapsto c$.
$B \to C$

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① Let $c \in C^G$. ψ surjects $\Rightarrow$ $\begin{array}{c} B \to C \\ \exists\, b \mapsto c \end{array}$.

② For $\sigma \in G$, $\begin{array}{ccc} B & \longrightarrow & C \\ b\sigma(b)^{-1} & \longmapsto & c\cdot\sigma(c)^{-1} \end{array}$ $\overset{\boxed{c \in C^G}}{=} c\cdot c^{-1} = 1$ $\Rightarrow$ $b\sigma(b)^{-1} \in \ker(\psi) = \mathrm{im}(\varphi)$ "$\in A$".

③ Define
$$\delta : C^G \longrightarrow H^1(G,A)$$
$$c \longmapsto \left[\begin{array}{c} G \longrightarrow A \\ \sigma \longmapsto b\cdot\sigma(b)^{-1} \end{array}\right].$$

Thm 2.7.2 (Skolem-Noether) All automorphisms of a CSA are conjugation maps.
(Saw this for $M_n(F)$ in Lemma 2.4.1).

pf $A :=$ degree n, CSA/F

$K :=$ a splitting field of A, so $A \otimes_F K \cong M_n(K)$.

Lemma 2.4.1 gives SES

$$1 \to K^\times \to \underbrace{(\underbrace{A\otimes_F K}_{M_n(K)})^\times}_{GL_n(K)} \twoheadrightarrow \underbrace{\mathrm{Aut}_{K\text{-alg}}(A\otimes_F K)}_{PGL_n(K)} \to 1$$

$G = \mathrm{Gal}(K/F)$ acts on this sequence via K, so by SES $\Rightarrow$ LES, get

$$1 \to \underbrace{(K^\times)^G}_{F^\times} \to \underbrace{((A\otimes_F K)^\times)^G}_{(A\otimes_F F)^\times = A^\times} \to \underbrace{(\mathrm{Aut}_{K\text{-alg}}(A\otimes_F K))^G}_{\mathrm{Aut}_{F\text{-alg}}(A)} \xrightarrow{\delta} \underbrace{H^1(G,K^\times)}_{= * \text{ by Hilbert thm 90}}$$

$\Rightarrow$ $1 \to F^\times \to A^\times \twoheadrightarrow \mathrm{Aut}_{F\text{-alg}}(A) \to 1$ is exact as pointed sets.

$$\underbrace{a \longmapsto \left(\begin{array}{c} A \longrightarrow A \\ b \longmapsto aba^{-1} \end{array}\right)}_{\text{surjects, so any automorphism of } A/F \text{ is conjugation}}$$

Can generalize above:

Lemma 2.7.4 For A, K, G as above, $H^1(G,(A\otimes_F K)^\times) = 1$. ($\approx$ Hilbert thm 90)

Let $SL_1(A) := \{a \in A \mid \mathrm{Nrd}(a) = 1_A\}$. Get SES

$$1 \to SL_1(A\otimes_F K) \to (A\otimes_F K)^\times \xrightarrow{\mathrm{Nrd}} K^\times \to 1$$

Apply cohomology & Lemma 2.7.4, get

Prop 2.7.3 $H^1(G, SL_1(A\otimes_F K)) \cong F^\times / \mathrm{Nrd}(A^\times)$.

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