09.17

Agenda:

- Everyone introduce themselves.
- Next Friday's time: ~~11 AM - 12 PM ok~~?
  Mark: 120-220 teach
  Alternatively 2:20 - 3:20 PM? -- conflicts
- **ASK MARK 4 - 5 PM Friday.**
- Introductory remarks and goal of seminar
- Assign talk topics

Active participants

- Man Cheung, Yidi, Yi, Abdullah, Marcus Lawson, Jonathan Cerq…
- Not present: Salash, Mark van Hoeij -- any assignment OK.

Mailing list of reminders:

- Active:
- Passive: Amod?

Emails:
<u>To all participants (passive, active):</u>
Friday time change.
First talk(s): Chapter 1: Quaternion algebra

§ Representation of finite gps

Let k be a field, G a finite group.

A G-representation is a finite-dimensional k-vector space V with a linear G-action, or equivalently, a $k[G]$-module where $k[G]$ is the group ring of G.

Theorem (Maschke) $k[G]$ is a semisimple k-algebra.

Thus V and $k[G]$ are semisimple $k[G]$-modules, decomposing into simple submods V_i's and W_j's, resp.:

$$V = \bigoplus \underbrace{V_i^{\neq 0}}_{\text{irreducible subrepresentations (irreps)}}, \qquad k[G] = \bigoplus W_j^{\neq 0}.$$

Let $v \in V_i$. Then

$$\bigoplus_j W_j = k[G] \longrightarrow V_i \qquad \Rightarrow \qquad W_j \longrightarrow V_i$$
$$f \longmapsto f\cdot v \qquad\qquad \text{homo of simple } k[G]\text{-mods.}$$

is a $k[G]$-mod homo, $\exists j$ st. $W_j \to V_i$ is nonzero map,

By Schur's lemma, $W_j \cong V_i$.

Conclusion The irreps of G $\equiv$ indecomposable direct summands of $k[G]$.

Theorem (Artin–Wedderburn) R = simple ring $\Rightarrow$ $R \cong M_n(D)$ for some $n \geq 1$ and division ring D.

Conclusion

To classify G-reps over k, first classify the finite-dimensional division algebras / k $\longleftrightarrow$ Knowing $Br(k')$ for each finite field extension k'/k.

Organize this as follows:

$Br(k) := \{$ division algebra over k with center $k \} / \cong$.

Organize this as follows:

$$\mathrm{Br}(k) := \{\underbrace{\text{division algebra over } k \text{ with center } k}_{\text{central division alg (CDA)}/k.}\}/\cong.$$

Note

(*) $$D_1 \otimes_k D_2 \cong M_n(D_3) \quad \text{some } n \geq 1, \text{ division alg } D_3 \text{ with center } k.$$

Thus $\mathrm{Br}(k)$ becomes a gp under $[D_1][D_2] := [D_3]$ given by (*)

Nicer defn of Br gp.

A central simple algebra / k (CSA/k) is a simple alg A with $\dim_k A < \infty$ & $Z(A) = k$.

Thus $A \cong M_n(D)$, $Z(D) = k$.

Two CSA/k A and B are Brauer equivalent if there exist $m, n > 0$:

$$M_m(A) \cong M_n(B) \quad \text{as } k\text{-algebras.}$$

Then

$$\mathrm{Br}(k) = \{\mathrm{CSA}/k\} / \text{Brauer equivalence}, \text{ a gp under } \otimes.$$

Exs
$$\mathrm{Br}(\mathbb{C}) = \{[\mathbb{C}]\}$$
$$\mathrm{Br}(\mathbb{R}) = \{[\mathbb{R}], [\mathbb{H}]\}$$

Here $\mathbb{H}$ is the $\mathbb{R}$-algebra with presentation $\langle i, j \mid i^2 = j^2 = -1, ij = -ji \rangle = \langle -1, -1 \rangle_{-1}$

(§2.5)

§ Structure of CSA's / Br gp

Assume $m \in k^\times$, k has primitive m^{th} rt of unity ω.

Define the k-alg with presentation

"cyclic algebra" $$(a,b)_\omega := \langle x, y \mid x^m = a, y^m = b, xy = \omega yx \rangle.$$

Want these as "building blocks" for CSA.

Naive conjecture any CSA/k is $\cong \otimes$ cyclic algs.

False (exercise: deduce this from discussion preceding Thm 1.5.8 in Gille-Szamuely)

But true up to Brauer equivalence!

(Thm 2.5.7) **Theorem** (Merkurjev – Suslin)

Under hypothesis at start of this §,
if A is a CSA$/k$ whose class in $Br(k)$ has order m, then

$$A \underset{\text{Brauer equivalent}}{\sim} (a_1, b_1)_\omega \otimes_k \cdots \otimes_k (a_i, b_i)_\omega$$ some cyclic algs of order m.

(§4.6) § Norm – residue isomorphism thm

Assume $m \in k^\times$.

$K_n^M(k)$ ← nth Milnor K-gp

$H^n(k, \mu_m^{\otimes n})$ ← Galois cohomology gp

(†) $\exists$ gp homo $K_n^M(k)/m \longrightarrow H^n(k, \mu_m^{\otimes n})$

Thm (Norm-residue..., Bloch-Kato conjecture; Voevodsky).

(†) is an iso.

Thm (Merkurjev – Suslin) Above thm holds for $n = 2$.

§ 4.7 explains why this version of M–S $\Rightarrow$ previous version of M–S.

Friday, September 17, 2021 2:13 PM

GroupMe

- OH for 5–6
- email suggesting ~~study gp~~ gp chat (gpme) for everyone.

$$\int t \sin^2 t \, dt = t\left[\frac{1}{2}t - \frac{1}{2}\cdot\frac{1}{2}\cdot \sin(2t)\right] - \underline{\int \frac{1}{2}t - \frac{1}{4}\sin(2t)\, dt}$$

$u = t$ $\quad v = \int \sin^2 t \, dt = \int \frac{1}{2} - \frac{1}{2}\cos(2t)\,dt = \int \frac{1}{2}\,dt - \frac{1}{2}\int \cos(2t)\,dt$

$du = dt$ $\quad dv = \sin^2 t \, dt.$ $\quad = \frac{1}{2}t - \frac{1}{2}\left[\frac{1}{2}\cdot \sin(2t)\right]$

$$= t\left[\frac{1}{2}t - \frac{1}{2}\cdot\frac{1}{2}\cdot \sin(2t)\right] - \left[\frac{1}{4}t^2 - \frac{1}{4}\left[\frac{-\cos(2t)}{2}\right]\right] + C.$$

$$\int t^1 \, dt = \frac{t^2}{2}$$

In general

$$\int t^n \, dt = \frac{t^{n+1}}{n+1} + C \quad (\text{for } n \neq -1) \quad \int \frac{1}{t}\,dt = \ln|t| + C.$$

$$\int t^2 \, dt = \frac{t^3}{3} + C$$

$$\int 1 \, dt = \int t^0 \, dt = \frac{t^1}{1} + C = t + C$$

$$\int \frac{1}{t^2}\,dt = \int t^{-2}\,dt = \frac{t^{-1}}{-1} + C = -\frac{1}{t} + C.$$

$$\int \frac{1}{\sqrt{t}}\,dt = \int \frac{1}{t^{1/2}}\,dt = \int t^{-1/2}\,dt = \frac{t^{1/2}}{1/2} + C = 2t^{1/2} + C.$$

$$\int c \cdot f(x) \cdot dx = c \int f(x)\,dx$$

$$\Rightarrow \int \frac{1}{2}\,dt = \frac{1}{2}\int 1 \cdot dt = \frac{1}{2}t + C.$$

09.20

Saturday, September 18, 2021 8:28 PM

Ask Mark and Salash if Friday 4 - 5 PM is OK

Want show

Q $A =$ (torsion) ab gp $\ni$ a order 2 elt

If $a = \sum a_i$ then a_i = order 2 elts ?

(excluding $a_1 + a_2 = 0$.)

$\exists$ 64-dim CDA $D \underset{-}{\cong} \not\exists \otimes a_i$ (a_i = 4-dim QA's).

$64 = 4^3 = 2^6$

partition even exponents: $6 = 2 + 2 + 2$] dim 64 = dim $2^2 \otimes$ dim $2^2 \otimes$ dim 2^2.

$= \underbrace{4}_{} + 2$] dim 64 =

cyclic alg of dim 2^4 = dim 16 ... nec of period 2 ?! Yes $\therefore$ biquaternion (Albert)

$a = b * c$ in $Br(k)$

$1 = a^2 = b^2 \underbrace{c^2}_{=1}$ a = ord 2, c = ord 2.

$\therefore b^2 = 1$ ✓

Or use ed argument ?!

$$D_1 \otimes_k D_2 \cong D_3 = CDA \overset{?}{\Rightarrow} D_1 = CDA ?$$

Sps not D not div alg so some elt $x \in D_1$ not invertible.

OR D_1 has nontrivial left prime id ? $\underset{?}{\Rightarrow}$ D_3 nontriv (left) prime id ?

OR

Consider dim 16 CSA/k. $\cong$ D = CDA, or $M_2(D)$ of $\dim_k 4 \cdot \dim_k D \Rightarrow \dim_k D = 4$ $\Rightarrow D = QA$.

~~or $M_3(D)$ $2^3 \dim_k D \Rightarrow$~~

or $M_4(D) \Rightarrow \dim_k D = 1 \Rightarrow D = k$

$\therefore$ dim 16 CSA is either (a) CDA/k $D \overset{\text{albert}}{\cong}$ biquaternion

(b) $M_2(D) = M_2(k) \otimes_k D$

(c) $M_4(k)$

But then (c) $D = M_4(k) \otimes_k D = M_4(D)$ = not div alg $\ni \begin{pmatrix} 1 & 0 \\ 0 & 0 \cdots \\ \cdots & \end{pmatrix}$ = 0-divisor.

$M_2(D) \otimes_k D' \ni$ zero divisor $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes 1$

$\cong M_4(k) \otimes_k D = M_4(D)$ = not div alg $\ni \begin{pmatrix} 0 & 0 & \cdots \\ \cdot & \cdot & \cdots \end{pmatrix}$

$M_2(D) \otimes_k D' \ni$ zero divisor $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes 1$

$\mathcal{D}$ = CDA$/k$ D

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes 1$

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \otimes 1$