3.3 Relation to subgroups — Part 2

November 5, 2021

Last time

Main results for today

Relating $H^\bullet(H, A)$**,** $H^\bullet(G/H, A^H)$**,** $H^\bullet(G, A)$

Main results for today

Relating $H^\bullet(H, A)$, $H^\bullet(G/H, A^H)$, $H^\bullet(G, A)$

Let G be a group, A a G-module, and H a subgroup of G. Then there exist group homomorphisms, called **restriction maps**,

$$\mathrm{Res} : H^i(G, A) \to H^i(H, A) \qquad \text{for all} \quad i \geq 0.$$

Main results for today

Relating $H^\bullet(H, A)$**,** $H^\bullet(G/H, A^H)$**,** $H^\bullet(G, A)$

Let G be a group, A a G-module, and H a subgroup of G. Then there exist group homomorphisms, called **restriction maps**,

$$\mathrm{Res} : H^i(G, A) \to H^i(H, A) \qquad \text{for all} \quad i \geq 0.$$

If $[G : H] = n$ is finite, there exist group homomorphisms, called **corestriction maps**,

$$\mathrm{Cor} : H^i(H, A) \to H^i(G, A) \qquad \text{for all} \quad i \geq 0.$$

Main results for today

Relating $H^\bullet(H, A)$, $H^\bullet(G/H, A^H)$, $H^\bullet(G, A)$

Let G be a group, A a G-module, and H a subgroup of G. Then there exist group homomorphisms, called **restriction maps**,

$$\mathrm{Res} : H^i(G, A) \to H^i(H, A) \qquad \text{for all} \quad i \geq 0.$$

If $[G : H] = n$ is finite, there exist group homomorphisms, called **corestriction maps**,

$$\mathrm{Cor} : H^i(H, A) \to H^i(G, A) \qquad \text{for all} \quad i \geq 0.$$

If instead H is a normal subgroup of G, there exist group homomorphisms, called **inflation maps**,

$$\mathrm{Inf} : H^i(G/H, A) \to H^i(G, A) \qquad \text{for all} \quad i \geq 0.$$

Main results for today

Relating Res**,** Cor**,** Inf

Main results for today

Relating Res**,** Cor**,** Inf

Theorem A. Let G be a group, A a G-module, and H a subgroup of finite index n in G. Then

$$\mathrm{Cor} \circ \mathrm{Res} : H^i(G, A) \to H^i(G, A)$$

is given by multiplication by n for all $i \geq 0$.

Main results for today

Theorem B. Let G be a group, A a G-module, and H a normal subgroup of G. There exists a map

$$\tau : H^1(H, A)^{G/H} \to H^2(G/H, A^H),$$

called the **transgression map**, fitting into an exact sequence

$$0 \longrightarrow H^1(G/H, A^H) \xrightarrow{\text{Inf}} H^1(G, A) \xrightarrow{\text{Res}} H^1(G/H, A)^{G/H}$$

$$\xrightarrow{\tau} H^2(G/H, A^H) \xrightarrow{\text{Inf}} H^2(G, A)$$

The tool to help define Res, Cor, Inf:

Shapiro's lemma

Shapiro's lemma

Let $H < G$ and A be a H-module. Form the **co-induced module**

$$\mathrm{CoInd}_H^G(A) := \mathrm{Hom}_H(\mathbb{Z}[G], A)$$

Shapiro's lemma

Let $H < G$ and A be a H-module. Form the **co-induced module**

$$\mathrm{CoInd}_H^G(A) := \mathrm{Hom}_H(\mathbb{Z}[G], A)$$

with G-action given by taking

$$\phi : \mathbb{Z}[G] \to A$$

Shapiro's lemma

Let $H < G$ and A be a H-module. Form the **co-induced module**

$$\mathrm{CoInd}_H^G(A) := \mathrm{Hom}_H(\mathbb{Z}[G], A)$$

with G-action given by taking

$$\phi : \mathbb{Z}[G] \to A$$

to

$$g' \cdot \phi : \mathbb{Z}[G] \to A : g \mapsto \phi(gg')$$

for any $g' \in G$.[1]

[1] Check the map $g'\phi$ is again H-equivariant.

Shapiro's lemma

Let $H < G$ and A be a H-module. Form the **co-induced module**

$$\mathrm{CoInd}_H^G(A) := \mathrm{Hom}_H(\mathbb{Z}[G], A).$$

Lemma. (Shapiro) Given subgroup H of a group G and an H-module A, there are canonical isomorphisms

$$H^i(G, \mathrm{CoInd}_H^G(A)) \cong H^i(H, A) \qquad \text{for all} \quad i \geq 0.$$

Shapiro's lemma

Let $H < G$ and A be a H-module. Form the **co-induced module**

$$\mathrm{CoInd}_H^G(A) := \mathrm{Hom}_H(\mathbb{Z}[G], A).$$

Lemma. (Shapiro) Given subgroup H of a group G and an H-module A, there are canonical isomorphisms

$$H^i(G, \mathrm{CoInd}_H^G(A)) \cong H^i(H, A) \qquad \text{for all} \quad i \geq 0.$$

Proof.

Shapiro's lemma

Let $H < G$ and A be a H-module. Form the **co-induced module**

$$\mathrm{CoInd}_H^G(A) := \mathrm{Hom}_H(\mathbb{Z}[G], A).$$

Lemma. (Shapiro) Given subgroup H of a group G and an H-module A, there are canonical isomorphisms

$$H^i(G, \mathrm{CoInd}_H^G(A)) \cong H^i(H, A) \qquad \text{for all} \quad i \geq 0.$$

Proof. Fix a projective $\mathbb{Z}[G]$-resolution $P_\bullet$ of $\mathbb{Z}$.

Shapiro's lemma

Let $H < G$ and A be a H-module. Form the **co-induced module**

$$\mathrm{CoInd}_H^G(A) := \mathrm{Hom}_H(\mathbb{Z}[G], A).$$

Lemma. (Shapiro) Given subgroup H of a group G and an H-module A, there are canonical isomorphisms

$$H^i(G, \mathrm{CoInd}_H^G(A)) \cong H^i(H, A) \qquad \text{for all} \quad i \geq 0.$$

Proof. Fix a projective $\mathbb{Z}[G]$-resolution $P_\bullet$ of $\mathbb{Z}$. Since $\mathbb{Z}[G]$ is free as a $\mathbb{Z}[H]$-module,

Shapiro's lemma

Let $H < G$ and A be a H-module. Form the **co-induced module**

$$\mathrm{CoInd}_H^G(A) := \mathrm{Hom}_H(\mathbb{Z}[G], A).$$

Lemma. (Shapiro) Given subgroup H of a group G and an H-module A, there are canonical isomorphisms

$$H^i(G, \mathrm{CoInd}_H^G(A)) \cong H^i(H, A) \qquad \text{for all} \quad i \geq 0.$$

Proof. Fix a projective $\mathbb{Z}[G]$-resolution $P_\bullet$ of $\mathbb{Z}$. Since $\mathbb{Z}[G]$ is free as a $\mathbb{Z}[H]$-module, $P_\bullet$ is a projective $\mathbb{Z}[H]$-resolution of $\mathbb{Z}$.

Shapiro's lemma

Let $H < G$ and A be a H-module. Form the **co-induced module**

$$\mathrm{CoInd}_H^G(A) := \mathrm{Hom}_H(\mathbb{Z}[G], A).$$

Lemma. (Shapiro) Given subgroup H of a group G and an H-module A, there are canonical isomorphisms

$$H^i(G, \mathrm{CoInd}_H^G(A)) \cong H^i(H, A) \qquad \text{for all} \quad i \geq 0.$$

Proof. Fix a projective $\mathbb{Z}[G]$-resolution $P_\bullet$ of $\mathbb{Z}$. Since $\mathbb{Z}[G]$ is free as a $\mathbb{Z}[H]$-module, $P_\bullet$ is a projective $\mathbb{Z}[H]$-resolution of $\mathbb{Z}$. The book gives isomorphisms

$$\mathrm{Hom}_{\mathbb{Z}[G]}(P_i, \mathrm{CoInd}_H^G(A)) \cong \mathrm{Hom}_{\mathbb{Z}[H]}(P_i, A).$$

Shapiro's lemma

Let $H < G$ and A be a H-module. Form the **co-induced module**

$$\mathrm{CoInd}_H^G(A) := \mathrm{Hom}_H(\mathbb{Z}[G], A).$$

Lemma. (Shapiro) Given subgroup H of a group G and an H-module A, there are canonical isomorphisms

$$H^i(G, \mathrm{CoInd}_H^G(A)) \cong H^i(H, A) \qquad \text{for all} \quad i \geq 0.$$

Proof. Fix a projective $\mathbb{Z}[G]$-resolution $P_\bullet$ of $\mathbb{Z}$. Since $\mathbb{Z}[G]$ is free as a $\mathbb{Z}[H]$-module, $P_\bullet$ is a projective $\mathbb{Z}[H]$-resolution of $\mathbb{Z}$. The book gives isomorphisms

$$\mathrm{Hom}_{\mathbb{Z}[G]}(P_i, \mathrm{CoInd}_H^G(A)) \cong \mathrm{Hom}_{\mathbb{Z}[H]}(P_i, A).$$

Taking cohomology yields the lemma.

Shapiro's lemma

Lemma. (Shapiro) Given subgroup H of a group G and an H-module A, there are canonical isomorphisms

$$H^i(G, \mathrm{CoInd}_H^G(A)) \cong H^i(H, A) \qquad \text{for all} \quad i \geq 0.$$

Shapiro's lemma

Lemma. (Shapiro) Given subgroup H of a group G and an H-module A, there are canonical isomorphisms

$$H^i(G, \mathrm{CoInd}_H^G(A)) \cong H^i(H, A) \qquad \text{for all} \quad i \geq 0.$$

Let $\mathrm{CoInd}^G(A) := \mathrm{CoInd}_H^G(A)$ if $H = *$.

Shapiro's lemma

Lemma. (Shapiro) Given subgroup H of a group G and an H-module A, there are canonical isomorphisms

$$H^i(G, \mathrm{CoInd}_H^G(A)) \cong H^i(H, A) \qquad \text{for all} \quad i \geq 0.$$

Let $\mathrm{CoInd}^G(A) := \mathrm{CoInd}_H^G(A)$ if $H = *$.

Since $H^i(*, A) = 0$ for $i > 0$, we have:

Shapiro's lemma

Lemma. (Shapiro) Given subgroup H of a group G and an H-module A, there are canonical isomorphisms

$$H^i(G, \mathrm{CoInd}_H^G(A)) \cong H^i(H, A) \qquad \text{for all} \quad i \geq 0.$$

Let $\mathrm{CoInd}^G(A) := \mathrm{CoInd}_H^G(A)$ if $H = *$.

Since $H^i(*, A) = 0$ for $i > 0$, we have:

Corollary. $H^i(G, \mathrm{CoInd}^G(A)) = 0$ for $i > 0$.

Shapiro's lemma

Lemma. (Shapiro) Given subgroup H of a group G and an H-module A, there are canonical isomorphisms

$$H^i(G, \mathrm{CoInd}_H^G(A)) \cong H^i(H, A) \qquad \text{for all} \quad i \geq 0.$$

Let $\mathrm{CoInd}^G(A) := \mathrm{CoInd}_H^G(A)$ if $H = *$.

Since $H^i(*, A) = 0$ for $i > 0$, we have:

Corollary. $H^i(G, \mathrm{CoInd}^G(A)) = 0$ for $i > 0$.

If $H < G$, $\mathrm{CoInd}^G(A) = \bigoplus \mathrm{CoInd}^H(A)$

Shapiro's lemma

Lemma. (Shapiro) Given subgroup H of a group G and an H-module A, there are canonical isomorphisms

$$H^i(G, \mathrm{CoInd}_H^G(A)) \cong H^i(H, A) \qquad \text{for all} \quad i \geq 0.$$

Let $\mathrm{CoInd}^G(A) := \mathrm{CoInd}_H^G(A)$ if $H = *$.

Since $H^i(*, A) = 0$ for $i > 0$, we have:

Corollary. $H^i(G, \mathrm{CoInd}^G(A)) = 0$ for $i > 0$.

If $H < G$, $\mathrm{CoInd}^G(A) = \bigoplus \mathrm{CoInd}^H(A)$ and so

$$H^i(H, \mathrm{CoInd}^G(A)) = \oplus H^i(H, \mathrm{CoInd}^H(A)) = 0.$$

The maps Res, Cor, Inf

Restriction maps

G group, $H < G$, A a G-module.

We define maps $\mathrm{Res} : H^i(G, A) \to H^i(H, A)$ inductively on i.

Restriction maps

G group, $H < G$, A a G-module.

We define maps $\mathrm{Res} : H^i(G, A) \to H^i(H, A)$ inductively on i.

$\underline{i = 0.}$ The map is the inclusion $A^G \hookrightarrow A^H$.

Restriction maps

G group, $H < G$, A a G-module.

We define maps $\mathrm{Res} : H^i(G, A) \to H^i(H, A)$ inductively on i.

$\underline{i = 0.}$ The map is the inclusion $A^G \hookrightarrow A^H$.

$\underline{i \Rightarrow i + 1.}$ We do the case $i \geq 1$. The case $i = 0$ is similar.

Restriction maps

G group, $H < G$, A a G-module.

We define maps $\mathrm{Res} : H^i(G, A) \to H^i(H, A)$ inductively on i.

$\underline{i = 0.}$ The map is the inclusion $A^G \hookrightarrow A^H$.

$\underline{i \Rightarrow i + 1.}$ We do the case $i \geq 1$. The case $i = 0$ is similar.

The SES

$$0 \to A \to \mathrm{CoInd}^G(A) \to A' \to 0$$

Restriction maps

G group, $H < G$, A a G-module.

We define maps $\mathrm{Res} : H^i(G, A) \to H^i(H, A)$ inductively on i.

$\underline{i = 0.}$ The map is the inclusion $A^G \hookrightarrow A^H$.

$\underline{i \Rightarrow i + 1.}$ We do the case $i \geq 1$. The case $i = 0$ is similar.

The SES

$$0 \to A \to \mathrm{CoInd}^G(A) \to A' \to 0$$

induces the LES

$$H^i(G, \mathrm{CoInd}^G(A)) \to H^i(G, A') \to H^{i+1}(G, A) \to H^{i+1}(G, \mathrm{CoInd}^G(A)).$$

Restriction maps

G group, $H < G$, A a G-module.

We define maps $\mathrm{Res} : H^i(G, A) \to H^i(H, A)$ inductively on i.

$\underline{i = 0.}$ The map is the inclusion $A^G \hookrightarrow A^H$.

$\underline{i \Rightarrow i+1.}$ We do the case $i \geq 1$. The case $i = 0$ is similar.

The SES

$$0 \to A \to \mathrm{CoInd}^G(A) \to A' \to 0$$

induces the LES

$$H^i(G, \mathrm{CoInd}^G(A)) \to H^i(G, A') \to H^{i+1}(G, A) \to H^{i+1}(G, \mathrm{CoInd}^G(A)).$$

Since $i > 0$, we can use Shapiro's lemma.

Restriction maps

G group, $H < G$, A a G-module.

We define maps $\mathrm{Res} : H^i(G, A) \to H^i(H, A)$ inductively on i.

$\underline{i = 0.}$ The map is the inclusion $A^G \hookrightarrow A^H$.

$\underline{i \Rightarrow i + 1.}$ We do the case $i \geq 1$. The case $i = 0$ is similar.

The SES

$$0 \to A \to \mathrm{CoInd}^G(A) \to A' \to 0$$

induces the LES

$$0 \to H^i(G, A') \to H^{i+1}(G, A) \to 0.$$

Since $i > 0$, we can use Shapiro's lemma.

Restriction maps

G group, $H < G$, A a G-module.

We define maps $\mathrm{Res} : H^i(G, A) \to H^i(H, A)$ inductively on i.

$\underline{i = 0.}$ The map is the inclusion $A^G \hookrightarrow A^H$.

$\underline{i \Rightarrow i + 1.}$ We do the case $i \geq 1$. The case $i = 0$ is similar.

The SES

$$0 \to A \to \mathrm{CoInd}^G(A) \to A' \to 0$$

induces

$$H^i(G, A') \cong H^{i+1}(G, A).$$

Since $i > 0$, we can use Shapiro's lemma.

Restriction maps

G group, $H < G$, A a G-module.

We define maps $\mathrm{Res} : H^i(G, A) \to H^i(H, A)$ inductively on i.

$\underline{i = 0.}$ The map is the inclusion $A^G \hookrightarrow A^H$.

$\underline{i \Rightarrow i + 1.}$ We do the case $i \geq 1$. The case $i = 0$ is similar.

The SES

$$0 \to A \to \mathrm{CoInd}^G(A) \to A' \to 0$$

induces

$$H^i(G, A') \cong H^{i+1}(G, A).$$

Define $\mathrm{Res} : H^{i+1}(G, A) \to H^{i+1}(H, A)$ as
$\mathrm{Res} : H^i(G, A') \to H^i(H, A')$.

Restriction maps

Explicitly,

$$\text{Res} : H^i(G, A) \to H^i(H, A)$$

takes a cochain

$$f : G^i \to A$$

to the the same function with its domain restricted to H^i:

$$\text{Res}(f) : H^i \to A.$$

Corestriction maps

G group, H subgroup of finite index in G, A a G-module.

We define maps $\mathrm{Cor} : H^i(G, A) \to H^i(H, A)$ inductively on i.

Corestriction maps

G group, H subgroup of finite index in G, A a G-module.

We define maps $\mathrm{Cor} : H^i(G, A) \to H^i(H, A)$ inductively on i.

$\underline{i = 0.}$ The map "averages over the coset representatives $g_1, \ldots, g_n$ of G/H":

$$A^H \to A^G : a \mapsto \sum_{j=1}^{n} g_j \cdot a$$

Corestriction maps

G group, H subgroup of finite index in G, A a G-module.

We define maps $\mathrm{Cor} : H^i(G, A) \to H^i(H, A)$ inductively on i.

$\underline{i = 0.}$ The map "averages over the coset representatives $g_1, \ldots, g_n$ of G/H":

$$A^H \to A^G : a \mapsto \sum_{j=1}^{n} g_j \cdot a$$

$\underline{i \Rightarrow i + 1.}$ Do the same "dimension shifting" argument using Shapiro's lemma.

Restriction-Corestriction

Theorem A. Let G be a group, A a G-module, and H a subgroup of finite index n in G. Then

$$\mathrm{Cor} \circ \mathrm{Res} : H^i(G, A) \to H^i(G, A)$$

is given by multiplication by n for all $i \geq 0$.

Restriction-Corestriction

Theorem A. Let G be a group, A a G-module, and H a subgroup of finite index n in G. Then

$$\mathrm{Cor} \circ \mathrm{Res} : H^i(G, A) \to H^i(G, A)$$

is given by multiplication by n for all $i \geq 0$.

Proof.

$$H^i(G, A) \xrightarrow{\mathrm{Res}} H^i(H, A) \xrightarrow{\mathrm{Cor}} H^i(G, A)$$

Restriction-Corestriction

Theorem A. Let G be a group, A a G-module, and H a subgroup of finite index n in G. Then

$$\mathrm{Cor} \circ \mathrm{Res} : H^i(G, A) \to H^i(G, A)$$

is given by multiplication by n for all $i \geq 0$.

Proof. By dimension shifting, it suffices to prove the case $i = 0$.

$$H^i(G, A) \xrightarrow{\mathrm{Res}} H^i(H, A) \xrightarrow{\mathrm{Cor}} H^i(G, A)$$

Restriction-Corestriction

Theorem A. Let G be a group, A a G-module, and H a subgroup of finite index n in G. Then

$$\mathrm{Cor} \circ \mathrm{Res} : H^i(G, A) \to H^i(G, A)$$

is given by multiplication by n for all $i \geq 0$.

Proof. By dimension shifting, it suffices to prove the case $i = 0$.

$$H^0(G, A) \xrightarrow{\mathrm{Res}} H^0(H, A) \xrightarrow{\mathrm{Cor}} H^0(G, A)$$

Restriction-Corestriction

Theorem A. Let G be a group, A a G-module, and H a subgroup of finite index n in G. Then

$$\mathrm{Cor} \circ \mathrm{Res} : H^i(G, A) \to H^i(G, A)$$

is given by multiplication by n for all $i \geq 0$.

Proof. By dimension shifting, it suffices to prove the case $i = 0$.

$$A^G \xrightarrow{\mathrm{Res}} A^H \xrightarrow{\mathrm{Cor}} A^G$$

Restriction-Corestriction

Theorem A. Let G be a group, A a G-module, and H a subgroup of finite index n in G. Then

$$\mathrm{Cor} \circ \mathrm{Res} : H^i(G, A) \to H^i(G, A)$$

is given by multiplication by n for all $i \geq 0$.

Proof. By dimension shifting, it suffices to prove the case $i = 0$.

$$A^G \xrightarrow{\mathrm{Res}} A^H \xrightarrow{\mathrm{Cor}} A^G$$

$$a \longmapsto a$$

Restriction-Corestriction

Theorem A. Let G be a group, A a G-module, and H a subgroup of finite index n in G. Then

$$\mathrm{Cor} \circ \mathrm{Res} : H^i(G, A) \to H^i(G, A)$$

is given by multiplication by n for all $i \geq 0$.

Proof. By dimension shifting, it suffices to prove the case $i = 0$.

$$A^G \xrightarrow{\mathrm{Res}} A^H \xrightarrow{\quad\mathrm{Cor}\quad} A^G$$

$$a \longmapsto a \longmapsto \textstyle\sum_{j=1}^{n} g_j \cdot a$$

Restriction-Corestriction

Theorem A. Let G be a group, A a G-module, and H a subgroup of finite index n in G. Then

$$\mathrm{Cor} \circ \mathrm{Res} : H^i(G, A) \to H^i(G, A)$$

is given by multiplication by n for all $i \geq 0$.

Proof. By dimension shifting, it suffices to prove the case $i = 0$.

$$A^G \xrightarrow{\mathrm{Res}} A^H \xrightarrow{\mathrm{Cor}} A^G$$

$$a \longmapsto a \longmapsto \textstyle\sum_{j=1}^{n} a$$

Restriction-Corestriction

Theorem A. Let G be a group, A a G-module, and H a subgroup of finite index n in G. Then

$$\mathrm{Cor} \circ \mathrm{Res} : H^i(G, A) \to H^i(G, A)$$

is given by multiplication by n for all $i \geq 0$.

Proof. By dimension shifting, it suffices to prove the case $i = 0$.

$$A^G \xrightarrow{\mathrm{Res}} A^H \xrightarrow{\mathrm{Cor}} A^G$$

$$a \longmapsto a \longmapsto n \cdot a$$

Inflation maps

For a group G, normal subgroup H, and G-module A, we define maps

$$\mathrm{Inf} : H^i(G/H, A^H) \to H^i(G, A)$$

on degree zero and use dimension shifting to extend the definition to all $i \geq 0$.

Inflation maps

For a group G, normal subgroup H, and G-module A, we define maps

$$\mathrm{Inf} : H^i(G/H, A^H) \to H^i(G, A)$$

on degree zero and use dimension shifting to extend the definition to all $i \geq 0$.

Explicitly, Inf takes a cochain

$$f : (G/H)^i \to A^H$$

to the cochain

$$G^i \to (G/H)^i \xrightarrow{f} A^H \to A$$

Inflation-Restriction

Theorem B. Let G be a group, A a G-module, and H a normal subgroup of G. There exists a map

$$\tau : H^1(H, A)^{G/H} \to H^2(G/H, A^H),$$

called the **transgression map**, fitting into an exact sequence

$$0 \longrightarrow H^1(G/H, A^H) \xrightarrow{\text{Inf}} H^1(G, A) \xrightarrow{\text{Res}} H^1(G/H, A)^{G/H}$$

$$\xrightarrow{\tau} H^2(G/H, A^H) \xrightarrow{\text{Inf}} H^2(G, A)$$

Proof. Consequence of the Hochschild-Serre spectral sequence.

Inflation-Restriction

Proposition 3.3.17. In Theorem B, let $i > 1$ and assume $H^j(H, A) = 0$ for $1 \leq j \leq i-1$. Then there is a natural map

$$\tau_{i,A} : H^i(H, A)^{G/H} \to H^{i+1}(G/H, A^H)$$

fitting into an exact sequence

$$0 \longrightarrow H^i(G/H, A^H) \xrightarrow{\text{Inf}} H^i(G, A) \xrightarrow{\text{Res}} H^i(G/H, A)^{G/H}$$

$$\xrightarrow{\tau} H^{i+1}(G/H, A^H) \xrightarrow{\text{Inf}} H^{i+1}(G, A)$$

Proof. Follows from Theorem B and dimension shifting.

References

Section 3.3 from Gille-Szamuely, Central Simple Algebras and Galois Cohomology.

Sharifi's notes on group and Galois cohomology, https://www.math.ucla.edu/~sharifi/groupcoh.pdf

For spectral sequences, see e.g. Vakil's notes on algebraic geometry, section 1.7, http://math.stanford.edu/~vakil/216blog/FOAGnov1817public.pdf