Comment

Nonlinear response functions that can be linearized by a transformation are sometimes called *intrinsically linear* response functions. For example, the exponential response function:

$$f(\mathbf{X}, \boldsymbol{\gamma}) = \gamma_0[\exp(\gamma_1 X)]$$

is an intrinsically linear response function because it can be linearized by the logarithmic transformation:

$$\log_e f(\mathbf{X}, \boldsymbol{\gamma}) = \log_e \gamma_0 + \gamma_1 X$$

This transformed response function can be represented in the linear model form:

$$g(\mathbf{X}, \boldsymbol{\gamma}) = \beta_0 + \beta_1 X$$

where $g(\mathbf{X}, \boldsymbol{\gamma}) = \log_e f(\mathbf{X}, \boldsymbol{\gamma})$, $\beta_0 = \log_e \gamma_0$, and $\beta_1 = \gamma_1$.

Just because a nonlinear response function is intrinsically linear does not necessarily imply that linear regression is appropriate. The reason is that the transformation to linearize the response function will affect the error term in the model. For example, suppose that the following exponential regression model with normal error terms that have constant variance is appropriate:

$$Y_i = \gamma_0 \exp(\gamma_1 X_i) + \varepsilon_i$$

A logarithmic transformation of Y to linearize the response function will affect the normal error term ε_i so that the error term in the linearized model will no longer be normal with constant variance. Hence, it is important to study any nonlinear regression model that has been linearized for appropriateness; it may turn out that the nonlinear regression model is preferable to the linearized version. ■

Estimation of Regression Parameters

Estimation of the parameters of a nonlinear regression model is usually carried out by the method of least squares or the method of maximum likelihood, just as for linear regression models. Also as in linear regression, both of these methods of estimation yield the same parameter estimates when the error terms in nonlinear regression model (13.12) are independent normal with constant variance.

Unlike linear regression, it is usually not possible to find analytical expressions for the least squares and maximum likelihood estimators for nonlinear regression models. Instead, numerical search procedures must be used with both of these estimation procedures, requiring intensive computations. The analysis of nonlinear regression models is therefore usually carried out by utilizing standard computer software programs.

Example

To illustrate the fitting and analysis of nonlinear regression models in a simple fashion, we shall use an example where the model has only two parameters and the sample size is reasonably small. In so doing, we shall be able to explain the concepts and procedures without overwhelming the reader with details.

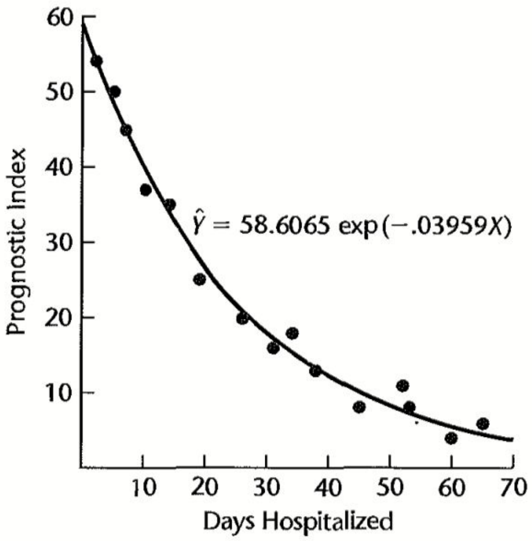
A hospital administrator wished to develop a regression model for predicting the degree of long-term recovery after discharge from the hospital for severely injured patients. The predictor variable to be utilized is number of days of hospitalization (X), and the response variable is a prognostic index for long-term recovery (Y), with large values of the index reflecting a good prognosis. Data for 15 patients were studied and are presented in Table 13.1. A scatter plot of the data is shown in Figure 13.2. Related earlier studies reported in the literature found the relationship between the predictor variable and the response variable to be exponential. Hence, it was decided to investigate the appropriateness of the two-parameter nonlinear exponential regression model (13.6):

$$Y_i = \gamma_0 \exp(\gamma_1 X_i) + \varepsilon_i \tag{13.13}$$

TABLE 13.1 Data—Severely Injured Patients Example.

Patient i	Days Hospitalized X_i	Prognostic Index Y_i
1	2	54
2	5	50
3	7	45
4	10	37
5	14	35
6	19	25
7	26	20
8	31	16
9	34	18
10	38	13
11	45	8
12	52	11
13	53	8
14	60	4
15	65	6

FIGURE 13.2 Scatter Plot and Fitted Nonlinear Regression Function—Severely Injured Patients Example.



where the ε_i are independent normal with constant variance. If this model is appropriate, it is desired to estimate the regression parameters γ_0 and γ_1.

13.2 Least Squares Estimation in Nonlinear Regression

We noted in Chapter 1 that the method of least squares for simple linear regression requires the minimization of the criterion Q in (1.8):

$$Q = \sum_{i=1}^{n} [Y_i - (\beta_0 + \beta_1 X_i)]^2 \tag{13.14}$$