Trees

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These notes cover trees and induction on trees.

1 Why trees?

Trees are the central structure for storing and organizing data in computer science. Examples of trees include

- Trees which show the organization of real-world data: family/geneaology trees, taxonomies (e.g. animal subspecies, species, genera, families)
- Data structures for efficiently storing and retrieving data. The basic idea is the same one we saw for binary search within an array: sort the data, so that you can repeatedly cut your search area in half.
- Parse trees, which show the structure of a piece of (for example) computer program, so that the compiler can correctly produce the corresponding machine code.
- Decision trees, which classify data by asking a series of questions. Each tree node contains a question, whose answer directs you to one of the node's children.

For example, here's a parse tree for the arithmetic expression

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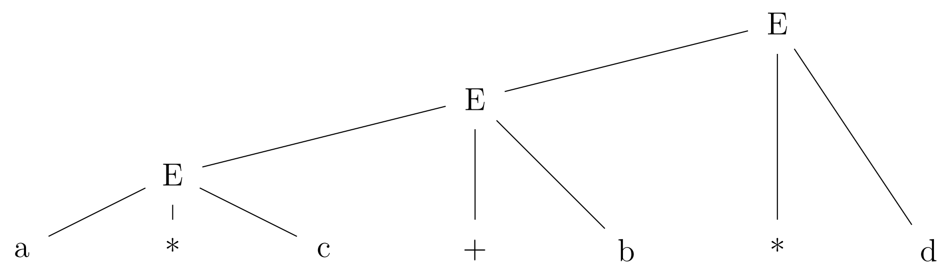
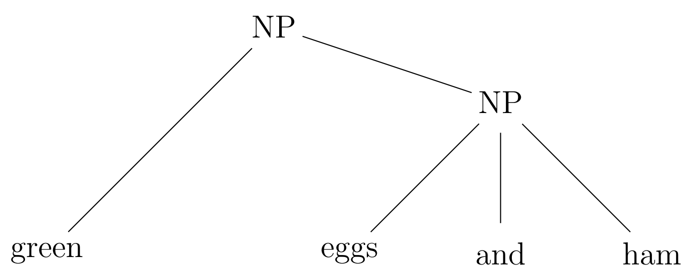
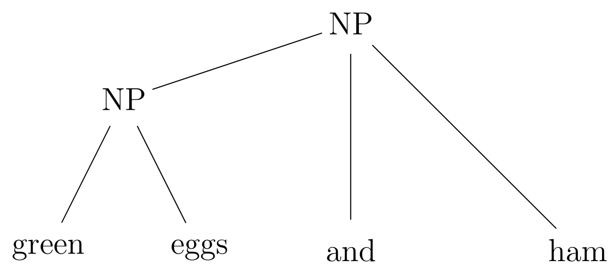
((a * c) + b) * d)
```

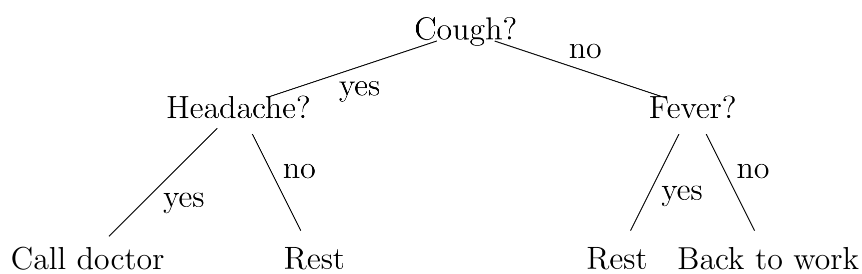


Computer programs that try to understand natural language use parse trees to represent the structure of sentences. For example, here are two possible structures for the phrase "green eggs and ham." In the first, but not the second, the ham would be green.

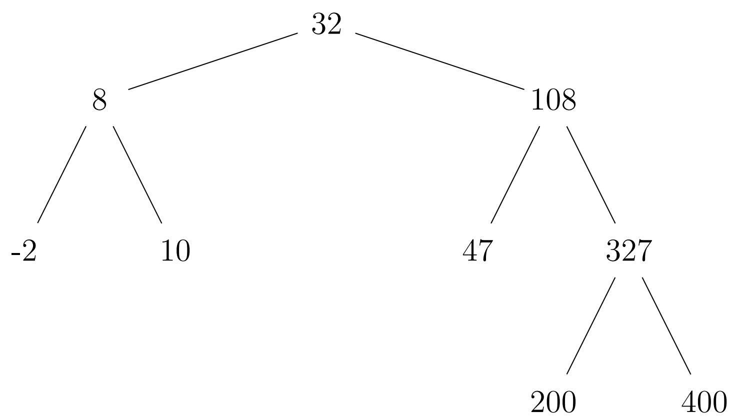




Here's what a medical decision tree might look like. Decision trees are also used for engineering classification problems, such as transcribing speech waveforms into the basic sounds of a natural language.



And here is a tree storing the set of number $\{-2, 8, 10, 32, 47, 108, 200, 327, 400\}$



2 Defining trees

Formally, a tree is a undirected graph with a special node called the *root*, in which every node is connected to the root by exactly one path. When a pair of nodes are neighbors in the graph, the node nearest the root is called the *parent* and the other node is its *child*. By convention, trees are drawn with the root at the top. Two children of the same parent are known as *siblings*.

To keep things simple, we will assume that the set of nodes in a tree is finite. We will also assume that each set of siblings is ordered from left to right, because this is common in computer science applications.

A **leaf node** is a node that has no children. A node that does have children is known as an **internal node**. The root is an internal node, except in the special case of a tree that consists of just one node (and no edges).

The nodes of a tree can be organized into **levels**, based on how many edges away from the root they are. The root is defined to be level 0. Its children are level 1. Their children are level 2, and so forth. The **height** of a tree is the maximum level of any of its nodes or, equivalently, the maximum level of any of its leaves or, equivalently, the maximum length of a path from the root to a leaf.

If you can get from x to g by following one or more parent links, then g is an **ancestor** of x and x is a **descendent** of g. We will treat x as an ancestor/descendent of itself. The ancestors/descendents of x other than x

itself are its **proper** ancestors/descendents. If you pick some random node a in a tree T, the **subtree rooted at** a consists of a (its root), all of a's descendents, and all the edges linking these nodes.

3 m-ary trees

Many applications restrict how many children each node can have. A binary tree (very common!) allows each node to have at most two children. An m-ary tree allows each node to have up to m children. Trees with "fat" nodes with a large bound on the number of children (e.g. 16) occur in some storage applications.

Important special cases involve trees that are nicely filled out in some sense. In a **full** m-ary tree, each node has either zero or m children. Never an intermediate number. So in a full 3-ary tree, nodes can have zero or three children, but not one child or two children.

In a **complete** m-ary tree, all leaves are at the same height. Normally, we'd be interested only in **full and complete** m-ary trees, where this means that the whole bottom level is fully populated with leaves.

For restricted types of trees like this, there are strong relationships between the numbers of different types of notes. for example:

Claim 1 *A full m-ary tree with i internal nodes has $mi + 1$ nodes total.*

To see why this is true, notice that there are two types of nodes: nodes with a parent and nodes without a parent. A tree has exactly one node with no parent. We can count the nodes with a parent by taking the number of parents in the tree (i) and multiplying by the branching factor m.

Therefore, the number of leaves in a full m-ary tree with i internal nodes is $(mi + 1) - i = (m - 1)i + 1$.

4 Height vs number of nodes

Suppose that we have a binary tree of height h. How many nodes and how many leaves does it contain? This clearly can't be an exact formula, since some trees are more bushy than others. But we can give useful upper and lower bounds.

To minimize the node counts, consider a tree of height h that has just one leaf. It contains $h+1$ nodes connected into a straight line by h edges. So the minimum number of leaves is 1 (regardless of h) and the minimum number of nodes is $h+1$.

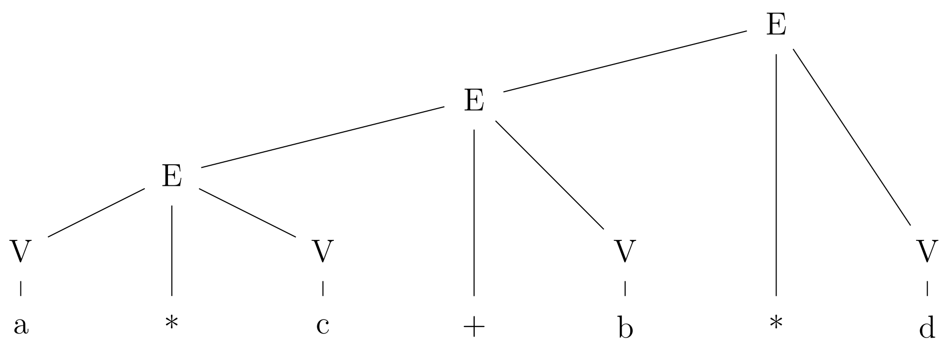
The node counts are maximized by a tree which is full and complete. For these trees, the number of leaves is 2^h. More generally, the number of nodes at level L is 2^L. So the total number of nodes n is $\sum_{L=0}^{h} 2^L$. The closed form for this summation is $2^{h+1}-1$. So, for full and complete binary trees, the height is proportional to $\log_2 n$.

In a **balanced** m-ary tree of height h, all leaves are either at height h or at height $h-1$. Balanced trees are useful when you want to store n items (where n is some random natural number that might not be a power of 2) while keeping all the leaves at approximately the same height. Balanced trees aren't as rigid as full binary trees, but they also have height proportional to $\log_2 n$. This means that all the leaves are fairly close to the root, which leads to good behavior from algorithms trying to store and find things in the tree.

5 Context-free grammars

Applications involving languages, both human languages and computer languages, frequently involve parse trees that show the structure of a sequence of **terminals**: words or characters. Each node contains a label, which can be either a terminal or a **variable**. An internal node or a root node must be labelled with a variable; leaves (except for the root) may have either sort of label. To keep things simple, we'll consistently use uppercase letters for variables.

For example, the following tree shows one structure for the sequence of terminals: a*c+b*d. This structure corresponds to adding parentheses as follows: ((a*c)+b)*d). This tree uses two variables (E and V) and six different terminals: a, b, c, d, $+$, $*$.



This sort of labelled tree can be conveniently specified by a **context-free grammar**. A context-free grammar is a set of rules which specify what sorts of children are possible for a parent node with each type of label. The lefthand side of each rule gives a label for the parent and the righthand side shows one possible pattern for the labels on its children. If all the nodes of a tree T have children matching the rules of some grammar G, we say that T is **generated** by G.

For example, the following grammar contains four rules for an E node. The first rule allows an E node to have three children, the leftmost one labelled E, the middle one labelled $+$ and the right one labelled V. According to these rules, a node labelled V can only have one child, labelled either a, b, c, or d. The tree shown above follows these grammar rules.

$$
\begin{array}{lcl}
E & \rightarrow & E + V \\
E & \rightarrow & E * V \\
E & \rightarrow & V + V \\
E & \rightarrow & V * V \\
V & \rightarrow & a \\
V & \rightarrow & b \\
V & \rightarrow & c \\
V & \rightarrow & d
\end{array}
$$

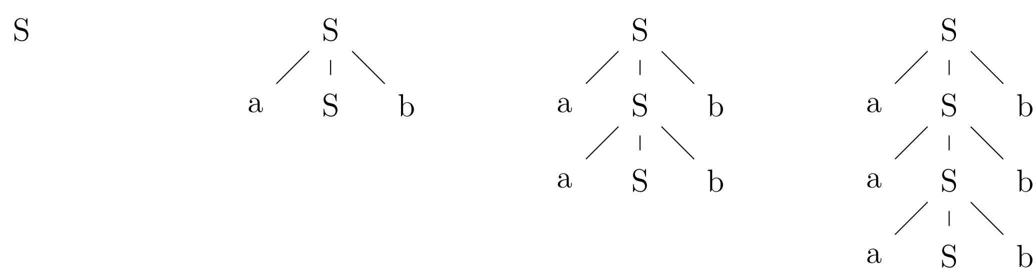
If a grammar contains several patterns for the children of a certain node label, we can pack them onto one line using a vertical bar to separate the options. E.g. we can write this grammar more compactly as

$$
\begin{aligned}
E &\rightarrow E + V \mid E * V \mid V + V \mid V * V \\
V &\rightarrow a \mid b \mid c \mid d
\end{aligned}
$$

The grammar above allows the root of a tree to be labelled either E or V. Depending on the application, we may wish to restrict the root to have one specific label, called the **start symbol**. For example, if we say that E is the start symbol, then the root must be labelled E but other interior nodes may be labelled either V or E.

The symbol ϵ is often used for an empty string or sequence. So a rule with ϵ on its righthand side specifies that this type of node is allowed to have no children. For example, here is a grammar using ϵ and a few trees generated by it:

$$
\begin{aligned}
S &\rightarrow aSb \\
S &\rightarrow \epsilon
\end{aligned}
$$

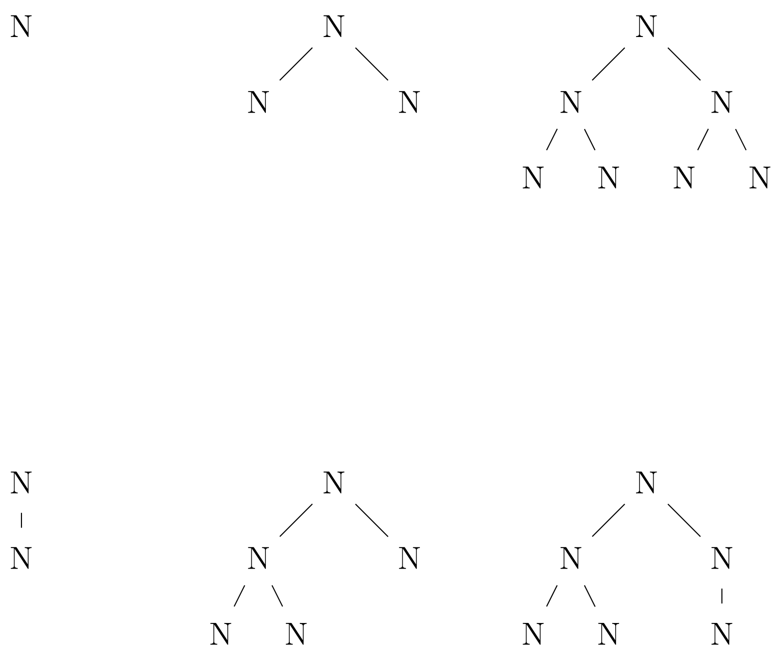


The sequences of terminals for the above trees are (left to right): empty sequence (no terminals at all), ab, aabb, aaabbb. The sequences from this grammar always have a number of a's followed by the same number of b's.

We can also use context-free grammars to define a generic sort of tree, where all nodes have the same generic label. For example, here's the grammar for binary trees. Its rules state that a node (with a generic label N) can have no children, one child, or two children. Its trees don't have corresponding terminal sequences because the grammar doesn't contain any terminals.

$$N \rightarrow \epsilon \mid N \mid NN$$

Here are a number of trees generated by these rules:

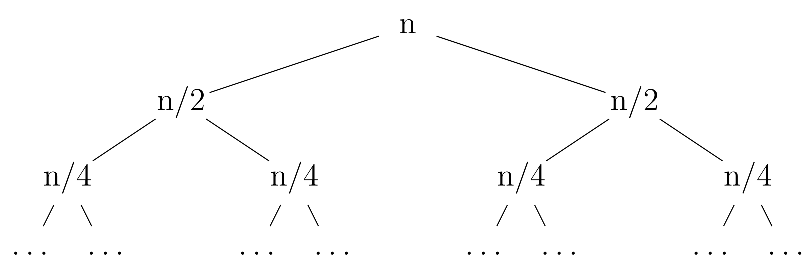


6 Recursion trees

One nice application for trees is visualizing the behavior of certain recursive definitions, especially those used to describe algorithm behavior. For example, consider the following definition, where c is some constant.

$$
\begin{aligned}
S(1) &= c \\
S(n) &= 2S(n/2) + n, \quad \forall n \geq 2 \;\; (n \text{ a power of } 2)
\end{aligned}
$$

We can draw a picture of this definition using a "recursion tree". The top node in the tree represents $S(n)$ and contains everything in the formula for $S(n)$ **except the recursive calls to** S. The two nodes below it represent two copies of the computation of $S(n/2)$. Again, the value in each node contains the non-recursive part of the formula for computing $S(n/2)$. The value of $S(n)$ is then the sum of the values in all the nodes in the recursion tree.



To sum everything in the tree, we need to ask:

- How high is this tree, i.e. how many levels do we need to expand before we hit the base case $n = 1$?
- For each level of the tree, what is the sum of the values in all nodes at that level?
- How many leaf nodes are there?

In this example, the tree has height $\log n$, i.e. there are $\log n$ non-leaf levels. At each level of the tree, the node values sum to n. So the sum for all non-leaf nodes is $n \log n$. There are n leaf nodes, each of which contributes c to the sum. So the sum of everything in the tree is $n \log n + cn$, which is the same closed form we found earlier for this recursive definition.

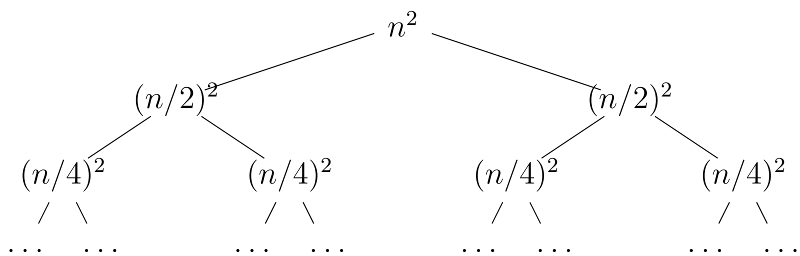
Recursion trees are particularly handy when you only need an approximate description of the closed form, e.g. is its leading term quadratic or cubic?

7 Another recursion tree example

Now, let's suppose that we have the following definition, where c is some constant.

$$\begin{aligned} P(1) &= c \\ P(n) &= 2P(n/2) + n^2, \qquad \forall n \geq 2 \quad (n \text{ a power of } 2) \end{aligned}$$

Its recursion tree is



The height of the tree is again $\log n$. The sums of all nodes at the top level is n^2. The next level down sums to $n^2/2$. And then we have sums: $n^2/4$, $n^2/8$, $n^2/16$, and so forth. So the sum of all nodes at level k is $n^2\frac{1}{2^k}$.

The lowest non-leaf nodes are at level $\log n - 1$. So the sum of all the non-leaf nodes in the tree is

$$\begin{aligned} P(n) &= \sum_{k=0}^{\log n - 1} n^2 \frac{1}{2^k} = n^2 \sum_{k=0}^{\log n - 1} \frac{1}{2^k} \\ &= n^2\left(2 - \frac{1}{2^{\log n - 1}}\right) = n^2\left(2 - \frac{2}{2^{\log n}}\right) = n^2\left(2 - \frac{2}{n}\right) = 2n^2 - 2n \end{aligned}$$

Adding cn to cover the leaf nodes, our final closed form is $2n^2 + (c-2)n$.

8 Tree induction

When doing induction on trees, we divide the tree up at the top. That is, we view a tree as consisting of a root node plus some number of subtrees rooted at its children. The induction variable is typically the height of the tree. The child subtrees have height less than that of the parent tree. So, in the inductive step, we'll be assuming that the claim is true for these shorter child subtrees and showing that it's true for the taller tree that contains them.

For example, we claimed above that

Claim 2 *Let T be a binary tree, with height h and n nodes. Then $n \leq 2^{h+1} - 1$.*

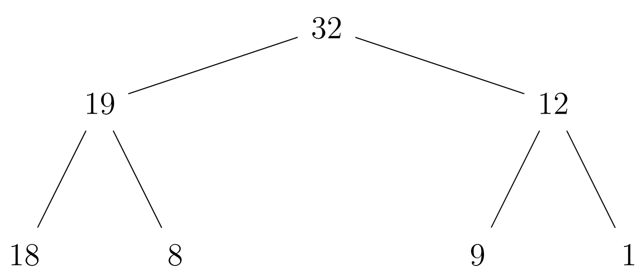
> Proof by induction on h, where h is the height of the tree.
>
> Base: The base case is a tree consisting of a single node with no edges. It has $h = 0$ and $n = 1$. Then we work out that $2^{h+1} - 1 = 2^1 - 1 = 1 = n$.
>
> Induction: Suppose that the claim is true for all binary trees of height $< h$. Let T be a binary tree of height h ($h > 0$).
>
> Case 1: T consists of a root plus one subtree X. X has height $h - 1$. So X contains at most $2^h - 1$ nodes. T only contains one more node (its root), so this means T contains at most 2^h nodes, which is less than $2^{h+1} - 1$.
>
> Case 2: T consists of a root plus two subtrees X and Y. X and Y have heights p and q, both of which have to be less than h, i.e. $\leq h - 1$. X contains at most $2^{p+1} - 1$ nodes and Y contains at most $2^{q+1} - 1$ nodes, by the inductive hypothesis. But, since p and q are less than h, this means that X and Y each contain $\leq 2^h - 1$ nodes.
>
> So the total number of nodes in T is the number of nodes in X plus the number of nodes in Y plus one (the new root node). This is $\leq 1 + (2^p - 1) + (2^q - 1) \leq 1 + 2(2^h - 1) = 1 + 2^{h+1} - 2 = 2^{h+1} - 1$
>
> So the total number of nodes in T is $\leq 2^{h+1} - 1$, which is what we needed to show. □

In writing such a proof, it's tempting to think that if the full tree has height h, the child subtrees must have height $h - 1$. This is only true if the tree is complete. For a tree that isn't necessarily complete, one of the subtrees must have height $h-1$ but the other subtree(s) might be shorter than $h-1$. So, for induction on trees, it is almost always necessary to use a strong inductive hypothesis.

In the inductive step, notice that we split up the big tree (T) at its root, producing two smaller subtrees (X) and (Y). Some students try to do induction on trees by grafting stuff onto the bottom of the tree. **Do not do this.** There are many claims, especially in later classes, for which this grafting approach will not work and it is essential to divide at the root.

9 Heap example

In practical applications, the nodes of a tree are often used to store data. Algorithm designers often need to prove claims about how this data is arranged in the tree. For example, suppose we store numbers in the nodes of a full binary tree. The numbers obey the *heap property* if, for every node X in the tree, the value in X is at least as big as the value in each of X's children. For example:



Notice that the values at one level aren't uniformly bigger than the values at the next lower level. For example, 18 in the bottom level is larger than 12 on the middle level. But values never decrease as you move along a path from a leaf up to the root.

Trees with the heap property are convenient for applications where you have to maintain a list of people or tasks with associated priorities. It's easy to retrieve the person or task with top priority: it lives in the root. And it's easy to restore the heap property if you add or remove a person or task.

I claim that:

Claim 3 *If a tree has the heap property, then the value in the root of the tree is at least as large as the value in any node of the tree.*

To keep the proof simple, let's restrict our attention to full binary trees:

Claim 4 *If a full binary tree has the heap property, then the value in the root of the tree is at least as large as the value in any node of the tree.*

Let's let $v(a)$ be the value at node a and let's use the recursive structure of trees to do our proof.

> Proof by induction on the tree height h.
>
> Base: $h = 0$. A tree of height zero contains only one node, so obviously the largest value in the tree lives in the root!
>
> Induction: Suppose that the claim is true for all full binary trees of height $< h$. Let T be a tree of height h ($h > 0$) which has the heap property. Since T is a full binary tree, its root r has two children p and q. Suppose that X is the subtree rooted at p and Y is the subtree rooted at q.
>
> Both X and Y have height $< h$. Moreover, notice that X and Y must have the heap property, because they are subtrees of T and the heap property is a purely local constraint on node values.
>
> Suppose that x is any node of T. We need to show that $v(r) \geq v(x)$. There are three cases:
>
> Case 1: $x = r$. This is obvious.
>
> Case 2: x is any node in the subtree X. Since X has the heap property and height $\leq h$, $v(p) \geq v(x)$ by the inductive hypothesis. But we know that $v(r) \geq v(p)$ because T has the heap property. So $v(r) \geq v(x)$.
>
> Case 3: x is any node in the subtree Y. Similar to case 2.
>
> So, for any node x in T, $v(r) \geq v(x)$.$\square$

10 Proof using grammar trees

Consider the following grammar G. I claim that all trees generated by G have the same number of nodes with label a as with label b.

$$\begin{aligned} S &\rightarrow ab \\ S &\rightarrow SS \\ S &\rightarrow aSb \end{aligned}$$

We can prove this by induction as follows:

> Proof by induction on the tree height h.
>
> Base: Notice that trees from this grammar always have height at least 1. The only way to produce a tree of height 1 is from the first rule, which generates exactly one a node and one b node.
>
> Induction: Suppose that the claim is true for all trees of height $< k$, where $k \geq 1$. Consider a tree T of height k. The root must be labelled S and the grammar rules give us three possibilities for what the root's children look like:
>
> Case 1: The root's children are labelled a and b. This is just the base case.
>
> Case 2: The root's children are both labelled S. The subtrees rooted at these children have height $< k$. So, by the inductive hypothesis, each subtree has equal numbers of a and b nodes. Say that the left tree has m of each type and the right tree has n of each type. Then the whole tree has $m + n$ nodes of each type.
>
> Case 3: The root has three children, labelled a, S, and b. Since the subtree rooted at the middle child has height $< k$, it has equal numbers of a and b nodes by the inductive hypothesis. Suppose it has m nodes of each type. Then the whole tree has $m+1$ nodes of each type.
>
> In all three cases, the whole tree T has equal numbers of a and b nodes.

11 Variation in terminology

There are actually two sorts of trees in the mathematical world. The ones commonly used in computer science are formally "rooted trees" with a left-to-right order. In graph theory, the term "tree" typically refers to "free trees," which are connected acyclic graphs. Free trees don't have a distinguished root vertex and don't have clear up/down or left-right directions. They often occur as subgraphs of some larger structure. We will see free trees towards the end of the term. Occasionally, but much less frequently, one sees intermediate possibilities, such as trees with roots but no left-to-right ordering.

Infinite trees occur occasionally in computer science applications. They are an obvious generalization of the finite trees discussed here.

Tree terminology varies a lot, apparently because trees are used for a wide range of applications with very diverse needs. In particular, some authors use the term "complete binary tree" to refer to a tree in which only the leftmost part of the bottom level is filled in. They then use the term "perfect" to refer to a tree whose bottom level is entirely filled. Some authors don't allow a node to be an ancestor or descendent of itself. A few authors don't consider the root node to be a leaf.