Linear Algebra
Practice Problems for Unit 2 Exam

Definitions

In addition to the vocabulary from chapter 1, you should know the definitions of the following vocabulary terms from chapters 2, 3 and 5, although you will not be asked to give any of these definitions specifically on the exam.

1. diagonal entries of a matrix, diagonal matrix
2. $m \times n$ zero matrix, $n \times n$ identity matrix I_n
3. sum $A + B$, product AB of matrices A and B (when defined)
4. scalar multiple kA, transpose A^T of a matrix A
5. invertible matrix, inverse A^{-1} of an invertible matrix A
6. The Invertible Matrix Theorem
7. a subspace H of $\mathbb{R}^n$
8. zero subspace $\{\vec{0}\}$
9. Column space $\mathrm{Col}(A)$ and nullspace $\mathrm{Nul}(A)$ of matrix A
10. a basis for a subspace H
11. the coordinate vector relative to a basis for a subspace
12. the dimension of a subspace
13. the rank of a matrix A
14. the Rank Theorem, the Basis Theorem
15. determinant, ij-minor, ij-cofactor of a matrix A
16. Cramer's Rule
17. Adjugate (or Classical Adjoint) matrix Adj A.
18. cofactor expansion along row i or column j
19. eigenvector $\vec{x}$ of eigenvalue λ for matrix A
20. eigenspaces E_λ for matrix A
21. characteristic polynomial, characteristic equation of A
22. algebraic and geometric multiplicity of an eigenvalue
23. matrix A is similar to matrix B
24. matrix A is diagonalizable
25. complex eigenvalues, $\mathbb{C}^2$, Re $\vec{x}$, Im$\vec{x}$, Arg λ
26. For 2×2 matrices with a complex eigenvalue λ, factorization of A as $A = PCP^{-1}$.

TRUE or FALSE

Expect roughly five questions like these to appear on the exam. You will be asked to justify all your answers!

1. If A is a 5×5 matrix whose columns span $\mathbb{R}^5$, then the columns of A^5 also span $\mathbb{R}^5$
2. If A is a 7×7 matrix and the columns of A^{10} form a basis for $\mathbb{R}^7$, then the columns for A itself also form a basis for $\mathbb{R}^7$.
3. If A is an $n \times n$ matrix for which the equation $A\vec{x} = \vec{b}$ has at least one solutions for every $\vec{b} \in \mathbb{R}^n$, then the equation $A^3\vec{x} = \vec{0}$ has only the trivial solution.
4. If X is the set of vectors with only integer entries:
$$X = \left\{ \begin{bmatrix} m \\ n \end{bmatrix} \in \mathbb{R}^2 \,|\, m, n \in \mathbb{Z} \right\},$$
then X is a subspace of $\mathbb{R}^2$.
5. If A is the following matrix, then the columns of A^{25} are linearly independent.
$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 1 \\ 2 & 3 & 1 & 5 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$
6. For all $n \times n$ matrices A, $\det(AA^T) \geq 0$.
7. There exists a 2×3 matrix A and a 3×2 matrix B such that the product AB is invertible.
8. Every 2×2 matrix has at least one real eigenvalue.
9. Every symmetric (that is, $A = A^T$) 2×2 matrix always has two real eigenvalues.
10. If A is similar to λI for some scalar λ, then $A = \lambda I$.
11. If A is a 3×3 matrix with eigenvalues 1,2 and 3, then $\det(A) = 6$.
12. If $A^n = 0$ and λ is an eigenvalue of A, then $\lambda = 0$.
13. If λ is an eigenvalue of A and μ is an eigenvalue of B, then the product $\lambda\mu$ must be an eigenvalue of AB.
14. The identity matrix is not similar to any other matrix except itself.
15. If A and B are nonzero matrices but $AB = 0$, then 0 must be an eigenvalue of both A and B.
16. For all square matrices A, $\det(\text{A}) = \det(A^T)$.
17. If λ is an eigenvalue, then the dimension of the corresponding eigenspace E_λ must be less than or equal to the algebraic multiplicity of λ.
18. If A is a 2×2 real matrix with complex eigenvalues $\lambda = 1 \pm i$ then $A^4 = I_2$.
19. Every 3×3 matrix with real entries will have at least one real eigenvalue.

ALWAYS, SOMETIMES, or NEVER

Expect roughly five questions like these to appear on the exam. You will be asked to justify all your answers!

1. If A and B are 5×5 matrices, then rank(AB) =rank$(A)\cdot$ rank(B).
2. If $\{\vec{a}_1, \ldots, \vec{a}_p\}$ is a linearly dependent set of vectors, then some subset of these vectors forms a basis for Span$\{\vec{a_1}, \ldots, \vec{a}_p\}$.
3. Suppose H is a four-dimensional subspace of $\mathbb{R}^7$. Then any set of four linearly independent vectors in H will span H.
4. If A and B are 5×5 matrices and $\dim(\mathrm{Nul}(B)) = 4$, then $\mathrm{Col}(BA)$ is either a point or a line in $\mathbb{R}^5$.
5. If $A\vec{x} = \lambda\vec{x}$, then $\vec{x}$ is an eigenvector of A.
6. If A and B are similar matrices, and $\vec{x}$ is an eigenvector for A, then $\vec{x}$ is also an eigenvector for B.
7. If $A^2 = A$, then 2 is not an eigenvalue of A.
8. If A is invertible and $\vec{x}$ is an eigenvector for A, then $\vec{x}$ is also an eigenvector for A^{-1}.
9. If A and B are $n \times n$ matrices, then $\det(AB) = \det(BA)$.
10. If A and B are both $n \times n$ and $\det(A) = \det(B)$, then A and B are similar.
11. If A and B are both $n \times n$, then $\det(A + B) = \det(A) + \det(B)$

Short Answer

Expect roughly five questions like these to appear on the exam. Be sure to read each question carefully.

1. For all invertible matrices A, what is $\det(A^{-1})$ and what is $\det(\mathrm{Adj}A)$ in terms of $\det(A)$?

2. Use determinants to final all values of x making the vectors linearly **dependent**.

$$\left\{ \begin{bmatrix} x \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ x \\ 1 \end{bmatrix} \right\}$$

3. Suppose we know $\det(A)$=9, where A is the matrix given below.

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \quad B = \begin{bmatrix} a+6g & b+6h & c+6i \\ d-4g & e-4h & f-4i \\ g & h & i \end{bmatrix} \quad C = \begin{bmatrix} 7g & 7h & 7i \\ d & e & f \\ a & b & c \end{bmatrix}$$

   (a) Find $\det(B)$.
   (b) Find $\det(C)$.
   (c) Find $\det(BC)$.
   (d) Find $\det(ABC)$.

4. Suppose that

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 0 \\ 4 & 3 & 6 & 5 \end{bmatrix}$$

   Find a nonzero vector $\vec{x}$ such that $A^{1000}\vec{x} = \vec{0}$. Justify your answer.

5. Let

$$A = \begin{bmatrix} 3 & -2 \\ 4 & -3 \end{bmatrix}, \vec{b} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}.$$

   Solve the equation $A^{2011}\vec{x} = \vec{b}$.

6. Suppose that A is a nonzero matrix, but that $A^2 = 0$. Show that A can not be diagonalizable.

7. Suppose that A and B are both $n \times n$ and there is some $n \times n$ invertible matrix P such that $P^{-1}AP$ is a diagonal matrix and $P^{-1}BP$ is also a diagonal matrix. Show that $AB = BA$.

8. Suppose that H and K are two subspaces of R^{10}, and that

$$\mathcal{B} = \{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4, \vec{u}_5\}$$

   is a basis of H, and

$$\mathcal{C} = \{\vec{v}_1, \vec{v}_2\}$$

   is a basis of K. Suppose that $T : \mathbb{R}^{10} \to \mathbb{R}^{10}$ is a linear transformation and we know that for all vectors $\vec{x} \in H$, the image $T(\vec{x}) \in K$.

   (a) Let $A = [\vec{u}_1\ \vec{u}_2\ \vec{u}_3\ \vec{u}_4\ \vec{u}_5]$. Explain why, for all vectors $\vec{v} \in \mathbb{R}^5, A\vec{v} \in H$.
   (b) Explain why, for all vectors $\vec{v} \in R^5, \vec{v} = [A\vec{v}]_{\mathcal{B}}$.
   (c) For $\vec{y} \in K$, let $[\vec{y}]_{\mathcal{C}}$ be the vector of the $\mathcal{C}$-coordinate of $\vec{y}$. Define a linear transformation $S : \mathbb{R}^5 \to \mathbb{R}^2$ by letting

$$S(\vec{v}) = [T(A\vec{v})]_{\mathcal{C}}.$$

   Show that if $\vec{v} \in \mathrm{Nul}(S)$, then $A\vec{v} \in \mathrm{Nul}(T)$.

9. Suppose A is the following (invertible) matrix:

$$A = \begin{bmatrix} 1 & 3 & 5 & 7 & 1 \\ 2 & 4 & 2 & 1 & 2 \\ 3 & 1 & 1 & 4 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 5 & 4 & 3 & 2 & 1 \end{bmatrix}.$$

   (a) Which of the following vectors is a solution to $A^{-1}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 2 \\ 3 \end{pmatrix}$? (circle your answer)

$$\vec{u}_1 = \begin{pmatrix} 17 \\ 12 \\ 4 \\ 11 \\ 27 \end{pmatrix}, \vec{u}_2 = \begin{pmatrix} 31 \\ 24 \\ 23 \\ 8 \\ 33 \end{pmatrix}, \vec{u}_3 = \begin{pmatrix} 6 \\ 7 \\ 9 \\ 1 \\ 2 \end{pmatrix}, \vec{u}_4 = \begin{pmatrix} 28 \\ 24 \\ 13 \\ 8 \\ 33 \end{pmatrix}, \vec{u}_5 = \begin{pmatrix} 17 \\ 35 \\ 27 \\ 29 \\ 21 \end{pmatrix}$$

   (b) Find the solution set to the equation $A^{-1}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 2 \\ 3 \end{pmatrix}$.

10. Compute the determinant of the following matrix two ways, first using cofactor expansion and then using row reduction.

$$\begin{bmatrix} 4 & -1 & 3 \\ 2 & 5 & -1 \\ -8 & 2 & -6 \end{bmatrix}$$

11. Suppose A is the matrix

$$A = \begin{bmatrix} 3 & 7 \\ -1 & -2 \end{bmatrix}.$$

   (a) Find a rotation matrix

$$C = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

   so that $A = PCP^{-1}$ for some invertible matrix P. You do not have to find P!

   (b) Explain, without doing any matrix multiplications, why $A^3 = -I_2$.