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Linear Algebra Assignment 9

1) Determine diagonalizability. If so, find invertible matrix P and diagonal matrix D such that $A = PDP^{-1}$.

a) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ First we find λ_n: $\det(A - \lambda I) = 0$.

$$\begin{vmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)(1-\lambda) - 0 = 0. = \lambda^2 - 2\lambda + 1$$
$$\therefore (\lambda - 1)^2 = 0, \ \lambda_1 = 1 = \lambda_2$$

✓ There are enough eigenvalues (multiplicity 2).

Now we find eigenvectors:

$$\begin{bmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \begin{array}{l} x_1 = x_1 \\ x_2 = 0 \end{array} \ \bar{v} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$ eigenspace is span$\{e_1\}$

✗ There are not enough eigenvectors to diagonalize.

b) $\begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix}$. $\begin{vmatrix} 0-\lambda & -2 \\ 1 & 3-\lambda \end{vmatrix} = (-\lambda)(3-\lambda) + 2 = 0 = \lambda^2 - 3\lambda + 2$

$(\lambda-1)(\lambda-2) = 0, \ \lambda_1 = 1, \lambda_2 = 2$

✓ There are enough eigenvalues.

$$\begin{bmatrix} 0-1 & -2 \\ 1 & 3-1 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix}, \left[\begin{array}{cc|c} -1 & -2 & 0 \\ 1 & 2 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \begin{array}{l} x_1 = -2x_2 \\ x_2 = x_2 \end{array} \ \bar{v}_1 = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0-2 & -2 \\ 1 & 3-2 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 1 & 1 \end{bmatrix} = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \begin{array}{l} x_1 = -x_2 \\ x_2 = x_2 \end{array} \ \bar{v}_2 = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

✓ There are enough eigenvectors

✓ $A = PDP^{-1}$ is: $\begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}$.

c) $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$. $\begin{vmatrix} 0-\lambda & 0 & 1 \\ 1 & 0-\lambda & -1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = -\lambda(\lambda^2-\lambda+1) - \emptyset + 1$

$= -\lambda^3+\lambda^2-\lambda+1$

$= -(\lambda-1)(\lambda^2+1) = 0$

✗ $\lambda = 1$, There are not enough linear factors, which means there are not enough eigenvalues.

d) $\begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{bmatrix}$. $\begin{vmatrix} 3-\lambda & 1 & 1 \\ 2 & 4-\lambda & 2 \\ -1 & -1 & 1-\lambda \end{vmatrix} = (3-\lambda)(\lambda^2-5\lambda+6) - 1(-2\lambda+4) + 1(2-(\lambda-4))$

$= -\lambda^3+8\lambda^2-20\lambda+16 = -(\lambda-4)(\lambda-2)^2 = \emptyset$

✓ $\lambda_1 = 4$, $\lambda_2 = \lambda_3 = 2$, There are enough λ_n.

$\begin{bmatrix} 3-4 & 1 & 1 \\ 2 & 4-4 & 2 \\ -1 & -1 & 1-4 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 \\ 2 & 0 & 2 \\ -1 & -1 & -3 \end{bmatrix} = \cancel{\left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 2 & 4 & 0 \\ -1 & -1 & -3 & 0 \end{array}\right]}$ RREF $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

calculator used

$x_1 = -x_3$, $x_2 = -2x_3$, $x_3 = x_3$ $\quad \bar{V}_1 = x_3\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$ $\quad \bar{V}_2 = x_3\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 3-2 & 1 & 1 \\ 2 & 4-2 & 2 \\ -1 & -1 & 1-2 \end{bmatrix} = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 0 \\ -1 & -1 & -1 & 0 \end{array}\right]$ RREF calc used. $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $x_1 = -x_2 - x_3$, $x_2 = x_2$, $x_3 = x_3$ $\quad \bar{V}_3 = x_2\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

✓ There are enough eigenvectors.

✓ $A = PDP^{-1}$ is:

$$\begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 & 3/2 \\ -1 & 0 & -1 \\ -1/2 & -1/2 & -1/2 \end{bmatrix}.$$

Note: D is always the matrix of the eigenvalues on the diagonal with zeros otherwise, P is always the matrix of eigenvectors placed in the same order.

2) To find the general expression for A^n of matrix $A = \begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix}$, we put it in the form of $A^n = PD^nP^{-1}$:

$$\begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix}^n = \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^n\begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}$$

This diagonalization was found previously in part 1) b).

3) As I just stated, the diagonalization of the standard matrix $\begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix}$ was found in part 1) b). We can use this to find a Basis B of $\mathbb{R}^2$ such that this matrix relative to B is diagonal:

$$D_B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad B = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$ ... the eigenvectors.

4) We are given $A = \begin{bmatrix} 1 & -1 \\ 9 & 1 \end{bmatrix}$ and asked to find an invertible matrix P and a rotation matrix C, such that $A = PCP^{-1}$

$$\begin{vmatrix} 1-\lambda & -1 \\ 9 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 10 \rightarrow x = \frac{2 \pm \sqrt{-36}}{2} = \frac{2 \pm 6i}{2} = 1 \pm 3i$$

$$\lambda_1 = 1+3i, \quad \lambda_2 = 1-3i$$

$$\lambda_1: \left[\begin{array}{cc|c} -3i & -1 & 0 \\ 9 & -3i & 0 \end{array}\right] \text{RREF}_{\text{calc used.}} = \left[\begin{array}{cc|c} 1 & -i/3 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \begin{matrix} x_1 = i/3\, x_2 \\ x_2 = x_2 \end{matrix} \quad \vec{v}_1 = x_2 \begin{bmatrix} i/3 \\ 1 \end{bmatrix}, \rightarrow \begin{bmatrix} i \\ 3 \end{bmatrix}$$

$$\lambda_2: \left[\begin{array}{cc|c} 3i & -1 & 0 \\ 9 & 3i & 0 \end{array}\right] \text{RREF}_{\text{calc used.}} = \left[\begin{array}{cc|c} 1 & i/3 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \begin{matrix} x_1 = -i/3\, x_2 \\ x_2 = x_2 \end{matrix} \quad \vec{v}_2 = x_2 \begin{bmatrix} -i/3 \\ 1 \end{bmatrix}, \rightarrow \begin{bmatrix} -i \\ 3 \end{bmatrix}$$

continued →

P is of the form: $[\operatorname{Re}\bar{v}, \operatorname{Im}\bar{v}]$ and
C is $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ where a is Real and b is Imaginary:
(parts of an λ)

$A = PCP^{-1}$ is:

$$\begin{bmatrix} 1 & -1 \\ 9 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1/6 & 1/6 \\ 1/2 & -1/2 \end{bmatrix}$$

C is a complex rotation on a circle made out of A which is a rotation on an orbit. C and A are similar.