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Part B
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# Multivariable and Complex Analysis Midterm

SHOW ALL WORK TO RECEIVE FULL CREDIT

1. (15) Show that the form under the integral sign is exact in the path independent plane from point $a : (\pi/2, \pi)$ to $b : (\pi, 0)$, then evaluate the integral.

$$\int_a^b \left(\frac{1}{2}\cos\frac{1}{2}x\cos 2y\; dx - 2\sin\frac{1}{2}x\sin 2y\; dy\right)$$

a) Exact if curl $F = 0$. So let's check:

P: $\frac{1}{2}\cos\frac{1}{2}x\cos 2y\, dx$ Q: $-2\sin\frac{1}{2}x\sin 2y\, dy$

$\frac{\partial P}{\partial y} = -\cos\left(\frac{x}{2}\right)\sin(2y)$ $\frac{\partial Q}{\partial x} = -\cos\left(\frac{x}{2}\right)\sin(2y)$

The partials are the same so the integrand is exact. (subtracting them would equal zero)

b) To evaluate, convert to double integral using Green's Theorem:

$$= \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_C (F_1\, dx + F_2\, dy)$$

and we see, provided the domain is simply connected, since Stokes's Theorem is a generalization of Green's and it is said it is path independent, we can apply Stokes's Theorem and "since curl $F = 0$ in the surface integral and hence the line integral are zero." (Kreyszig, pg. 468) The function is perfectly symmetrical over the region.

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2. (10) Find the eigenvalues and corresponding eigenvectors of **A**. *To receive full credit, you must show all work.* Is **A** diagonalizable?

$$A = \begin{bmatrix} 13 & 4 & 2 \\ 2 & 7 & -8 \\ 5 & 4 & 7 \end{bmatrix}$$

$$(A-\lambda I) = \begin{bmatrix} 13-\lambda & 4 & 2 \\ 2 & 7-\lambda & -8 \\ 5 & 4 & 7-\lambda \end{bmatrix} = \text{matrix } B$$

$$\det(B) = -\lambda^3 + 27\lambda^2 - 245\lambda + 783 = 0 \quad \text{(triangle rule)}$$

(calculator): The roots are $\lambda_1 = 12.1$, $\lambda_2 = 7.45 + i(3.035)$, $\lambda_3 = 7.45 - i(3.035)$

Now we find Eigenvectors by substituting back into $(A-\lambda I) = 0$:

For $\lambda_1$: $\left[\begin{array}{ccc|c} 0.9 & 4 & 2 & 0 \\ 2 & -5.1 & -8 & 0 \\ 5 & 4 & -5.1 & 0 \end{array}\right]$ After RREF, $A-\lambda_1 I = 0$ is $\left[\begin{array}{ccc|c} 1 & 0 & -1.732 & 0 \\ 0 & 1 & 0.889 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

So $x_1 = 1.732x_3$, $x_2 = -0.889x_3$, $x_3 = x_3$ ... Let $x_3 = 1$, $\therefore v_1 = \begin{bmatrix} 1.732 \\ -0.889 \\ 1 \end{bmatrix}$

For $\lambda_2$: $\left[\begin{array}{ccc|c} 5.55-i(3.035) & 4 & 2 & 0 \\ 2 & -0.45-i(3.035) & -8 & 0 \\ 5 & 4 & -0.45-i(3.035) & 0 \end{array}\right]$ RREF.

$A-\lambda_2 I = 0$ is $\left[\begin{array}{ccc|c} 1 & 0 & -0.826+i(0.957) & 0 \\ 0 & 1 & 0.921-i(1.955) & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ So, $x_1 = (0.826 - i(0.957))x_3$
$x_2 = (-0.921 + i(1.955))x_3$
$x_3 = x_3$

Let $x_3 = 1$, $\therefore v_2 = \begin{bmatrix} 0.826-i(0.957) \\ -0.921+i(1.955) \\ 1 \end{bmatrix}$

For $\lambda_3$: $\left[\begin{array}{ccc|c} 5.55+i(3.035) & 4 & 2 & 0 \\ 2 & -0.45+i(3.035) & -8 & 0 \\ 5 & 4 & -0.45+i(3.035) & 0 \end{array}\right]$ RREF.

$A-\lambda_3 I = 0$ is $\left[\begin{array}{ccc|c} 1 & 0 & -0.826-i(0.957) & 0 \\ 0 & 1 & 0.921+i(1.955) & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ So, $x_1 = (0.826 + i(0.957))x_3$
$x_2 = (-0.921 - i(1.955))x_3$
$x_3 = x_3$

Let $x_3 = 1$, $\therefore v_3 = \begin{bmatrix} 0.826+i(0.957) \\ -0.921-i(1.955) \\ 1 \end{bmatrix}$

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3. (10) Find and sketch the parametric representation of a circle in the $yz$-plane with center $(4, 0)$ and passing through the point $(0, 3)$.

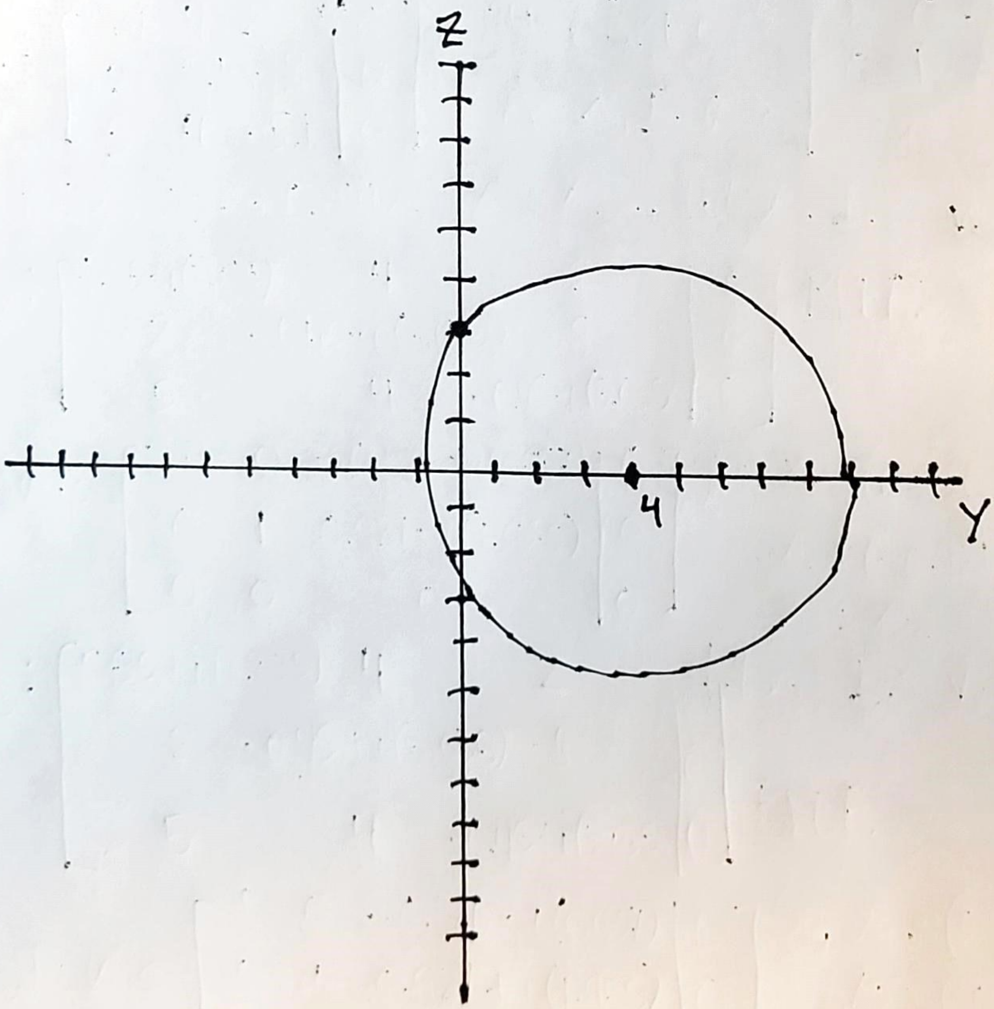
Equation of a circle: $(y-a)^2 + (z-b)^2 = r^2$

Plug in: $(y-4)^2 + z^2 = r^2 = 25$, $r = 5$.

Circle with center $(a, b)$ has parametric representation:

$[a + r\cos\theta, b + r\sin\theta]$ we plug in and get:

$[0, 4 + 5\cos\theta, 5\sin\theta]$ because $x = 0$.



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4. (10) Find basis for null space of **A**, where

$$A = [\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3] = \begin{bmatrix} -1 & 2 & 3 \\ -4 & 1 & 1 \end{bmatrix}.$$

The Null Space of A is all the solutions of $A\bar{x} = 0$. It's the vectors that go to zero. So, to find the basis we solve A augmented with $\bar{0}$ for the free variables:

$$\left[\begin{array}{ccc|c} -1 & 2 & 3 & 0 \\ -4 & 1 & 1 & 0 \end{array}\right] \xrightarrow{R_1 = -R_1} \left[\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ -4 & 1 & 1 & 0 \end{array}\right] R_2 = R_2 + 4R_1$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ 0 & -7 & -11 & 0 \end{array}\right] R_2 = R_2/-7 \left[\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ 0 & 1 & 11/7 & 0 \end{array}\right] R_1 = R_1 + 2R_2$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 1/7 & 0 \\ 0 & 1 & 11/7 & 0 \end{array}\right]$$ So, $$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1/7 x_3 \\ -11/7 x_3 \\ x_3 \end{bmatrix} =$$

$$= x_3 \begin{bmatrix} -1/7 \\ -11/7 \\ 1 \end{bmatrix} = x_3 \bar{v} = \{\bar{v}\} = \{7\bar{v}\} = \left\{\begin{bmatrix} -1 \\ -11 \\ 7 \end{bmatrix}\right\}$$

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5. (15) Evaluate the surface integral $\iint_S (curl\ \mathbf{F}) \cdot \mathbf{n}\, dA$ directly for

$$\mathbf{F} = [-13\sin y, 3\sinh z, x],$$

and $S$ is the rectangle with vertices $(0,0,2)$, $(4,0,2)$, $(4,\pi/2,2)$, $(0,\pi/2,2)$.

First, curl F = $\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -13\sin y & 3\sinh z & x \end{vmatrix}$

$= (0 - 3\cosh z)\bar{i} - (1-0)\bar{j} + (0 + 13\cos y)\bar{k} = [-3\cosh z\,\bar{i}, \bar{j}, 13\cos y\,\bar{k}]$

Bounds of the rectangle:

$0 \le x \le 4$
$0 \le y \le \pi/2$
$z = 2$

$n = \bar{k}$, so $(\text{curl}F)\cdot n = 13\cos y$

Set up integral:

$$\int_{x=0}^{4}\int_{y=0}^{\pi/2} 13\cos y\, dy\, dx \;\; \cancel{dx\,dy} = 13\int_{x=0}^{4} \sin y \Big|_0^{\pi/2} dx = 13\int_{x=0}^{4} 1\, dx$$

$= 13(x\Big|_0^4 = 13 \times 4 = 52$