Linear Algebra
Solutions to Practice Problems for Unit 2

TRUE or FALSE

Expect roughly five questions like these to appear on the exam. You will be asked to justify all your answers!

1. If A is a 5×5 matrix whose columns span $\mathbb{R}^5$, then the columns of A^5 also span $\mathbb{R}^5$

   Solution: True.
   Since the columns of A span $\mathbb{R}^5$, A is invertible by the Invertible MAtrix Theorem. That means that A^5 is also invertible (with inverse $(A^{-1})^5$). Now the Invertible Matrix Theorem applied to A^5 says that the columns of A^5 have to be linearly independent.

2. If A is a 7×7 matrix and the columns of A^{10} form a basis for $\mathbb{R}^7$, then the columns for A itself also form a basis for $\mathbb{R}^7$.

   Solution: True.
   If the columns of A^{10} form a basis for $\mathbb{R}^7$, A^{10} is invertible. This follows since basis vectors are linearly independent and the Invertible Matrix Theorem. That means A is invertible since if A were singular, $\det(A)$ would equal 0. Then $\det(A^{10}) = (\det(A))^{10} = 0^{10} = 0$. But we know A^{10} is invertible so its determinant can not be 0. By the Invertible Matrix Theorem, the columns of A are linearly independent and span $\mathbb{R}^7$, so they are a basis for $\mathbb{R}^7$.

3. If A is an $n \times n$ matrix for which the equation $A\vec{x} = \vec{b}$ has at least one solutions for every $\vec{b} \in \mathbb{R}^n$, then the equation $A^3\vec{x} = \vec{0}$ has only the trivial solution.

   Solution: True.
   Since $A\vec{x} = \vec{b}$ has a solution for every $\vec{b} \in \mathbb{R}^n$, A is invertible (IMT). So A^3 is invertible (with inverse $(A^{-1})^3$) and so by the IMT again, $(A^3)\vec{x} = \vec{0}$ has only the trivial solution.

4. If X is the set of vectors with only integer entries:

$$X = \left\{ \begin{bmatrix} m \\ n \end{bmatrix} \in \mathbb{R}^2 \,|\, m, n \in \mathbb{Z} \right\},$$

   then X is a subspace of $\mathbb{R}^2$.
   Solution: False.
   X is not closed under scalar multiplication since $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in X$ but $\frac{1}{2} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \notin X$.

5. If A is the following matrix, then the columns of A^{25} are linearly independent.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 1 \\ 2 & 3 & 1 & 5 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

   Solution: True.
   Since A^{25} is a 4×4 matrix, the IMT says that its columns are linearly independent if and only if it is invertible.

Since $\det(A^{25}) = (\det(A))^{25}$, A^{25} is invertible if and only if A is invertible (remember: a square matrix is invertible if and only if its determinant is not equal to zero). So we want to know if A has a pivot in every row, or really any of the other equivalent conditions in the IMT. To see the pivots of a matrix, we have to row reduce:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 1 \\ 2 & 3 & 1 & 5 \\ 1 & 1 & 1 & 1 \end{bmatrix} \overset{R3-2R1}{\longrightarrow} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & -5 & -3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \overset{R4-R1}{\longrightarrow} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & -5 & -3 \\ 0 & -1 & -2 & -3 \end{bmatrix} \overset{R3+R2}{\longrightarrow} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -5 & -2 \\ 0 & -1 & -2 & -3 \end{bmatrix} \overset{R4+R2}{\longrightarrow}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -5 & -2 \\ 0 & 0 & -2 & -2 \end{bmatrix} \overset{-\frac{1}{5}R3}{\longrightarrow} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2/5 \\ 0 & 0 & -2 & -2 \end{bmatrix} \overset{R4+2R3}{\longrightarrow} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2/5 \\ 0 & 0 & 0 & -6/5 \end{bmatrix}$$

There is a pivot in every row for 4 pivots total.

6. For all $n \times n$ matrices A, $\det(AA^T) \geq 0$.

   Solution: True.
   $\det(AA^T) = \det(A) \cdot \det(A^T) = \det(A) \cdot \det(A) = \det(A)^2 \geq 0$.

7. There exists a 2×3 matrix A and a 3×2 matrix B such that the product AB is invertible.
   Solution: True.
   If $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ then the product is $\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$ which has determinant 2, so is invertible.

8. Every 2×2 matrix has at least one real eigenvalue.

   Solution: False.
   If $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, then $\det(A - \lambda I_2) = \lambda^2 + 1$. Setting this to zero to find the eigenvalues gives $\lambda = \pm i \notin \mathbb{R}$.

9. Every symmetric (that is, $A = A^T$) 2×2 matrix always has two real eigenvalues, counting multiplicites.

   Solution: True.
   Consider a symmetric matrix $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ for some $a, b, c \in \mathbb{R}$. Then $\det(A - \lambda I_2) = (a - \lambda)(c - \lambda) - b^2 = \lambda^2 + \lambda(-a - c) + (ac - b^2)$. Setting this equal to zero and using the quadratic formula gives:

   $$\lambda = \frac{(a + c) \pm \sqrt{(-a - b)^2 - 4(ac - b^2)}}{2}$$

   Simplifying, we get

   $$\lambda = \frac{(a + c) \pm \sqrt{(a^2 - 2ac + c^2) + 4b^2}}{2}$$

   The discriminant, or number under the square root, is $(a - c)^2 + 4b^2$ which is always non-negative, so λ is always real. Note that a symmetric matrix of any size will always have real eigenvalues, but this is much harder to show for $n > 2$.

10. If A is similar to λI for some scalar λ, then $A = \lambda I$.

    Solution: True.
    If A is similar to λI_n, then there is some invertible matrix P such that $A = P(\lambda I_n)P^{-1}$. Since scalars can move in an out of a matrix product, $A = \lambda(PI_nP^{-1}) = \lambda I_n$.

11. If A is a 3×3 matrix with eigenvalues 1,2 and 3, then $\det(A) = 6$.

    Solution: True.
    Since A is a 3×3 matrix with 3 sidtinct real eigenvalues, A is diagonalizable. That means that we can find an invertible matrix P such that $A = P^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} P$. Then $\det(A) = \det(P)\det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \det(P^{-1}) = 6$.

12. If $A^n = 0$ and λ is an eigenvalue of A, then $\lambda = 0$.

 Solution: True.
 Since λ is an eigenvalue for A, there is a nonzero vector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$. Multiply both sides by A^{n-1} to get:

$$\begin{aligned} A^n\vec{x} &= A^{n-1}\lambda\vec{x} \\ &= \lambda A^{n-1}\vec{x} \\ &= \lambda^n\vec{x} \end{aligned}$$

 Since $A^n = 0$, we get $0 = \lambda^n\vec{x}$. Since $\vec{x}$ is nonzero, λ must be zero. So the only eigenvalue of A is 0.

13. If λ is an eigenvalue of A and μ is an eigenvalue of B, then the product $\lambda\mu$ must be an eigenvalue of AB.

 Solution: False.
 Consider the case with $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 0 \\ 0 & 7 \end{pmatrix}$. Then $AB = \begin{pmatrix} 10 & 0 \\ 0 & 21 \end{pmatrix}$. The eigenvalues of AB are the diagonal entries, $\lambda = 10$ and $\lambda = 21$. On the other hand, 2 is an eigenvalue of A and 7 is an eigenvalue for B, but $2 \cdot 7 = 14$ is not an eigenvalue for AB.

14. The identiy matrix is not similar to any other matrix except itself.
 Solution: True.
 This is question 10 above with $\lambda = 1$.

15. If A and B are nonzero matrices but $AB = 0$, then 0 must be an eigenvalue of both A and B.
 Solution: False.
 Rmember, we should not assume that A and B are square unless we are told to. Consider the counterexample:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

 Since neither A nor B are square, they do not have eigenvalues. Note however that if A and B are both assumed to be square and of the same size then this statement is true.

16. For all square matrices A, $\det(\mathrm{A}) = \det(A^T)$.
 Solution: True.
 This is a Theorem in Chapter 3. The proof is by induction on the size of A.

17. If λ is an eigenvalue, then the dimension of the corresponding eigenspace E_λ must be less than or equal to the algebraic multiplicity of λ.
 Solution: True.
 This is a Theorem in Chapter 5.3. Remember, the algebraic multiplicity of λ is the number of times the eigenvalue appears as a root of the characteristic polynomial. The dimension of the corresponding eigenspace E_λ is sometimes called the *geometric multiplicity*. This theorem is sometimes seen as:

$$\text{alg. mult } \lambda \geq \dim E_\lambda$$

18. If A is a 2×2 real matrix with complex eigenvalues $\lambda = 1 \pm i$, then $A^4 = I_2$.
 Solution: False.
 Since A has two complex eigenvalues, it rotates and scales vectors. Multiplication by A scales vectors by a factor of $|\lambda| = \sqrt{1^2 + 1^2} = \sqrt{2}$. A rotates vectors by Arg $\lambda = \tan^{-1}(1) = \frac{\pi}{4}$. (Notice since λ is in the first quadrant, Arg λ is given by $\tan^{-1}(y/x)$). So multiplication by A four times scales vectors by $\sqrt{2}^4 = 4$ and rotates by $4 \cdot \frac{\pi}{4} = \pi$. This is clearly not the identity matrix.

19. Every 3×3 matrix with real entries will have at least one real eigenvalue.
 Solution: True.
 Since the degree of the characteristic polynomial is equal to the size of the matrix, the characteristic polynomial will have degree 3. It is a fact from Calculus that any polynomial of degree 3 has at least one real root (follows from the Intermediate Value Theorem), so since the roots of the characteristic polynomial are the eigenvalues, this statement is true.

ALWAYS, SOMETIMES, or NEVER

Expect roughly five questions like these to appear on the exam. You will be asked to justify all your answers!

1. If A and B are 5×5 matrices, then rank(AB) =rank$(A)\cdot$ rank(B).
   Solution: Sometimes
   True if $A = B = 0_5$ (the 5×5 zero matrix). Then $A = B = AB$ and rank(A) =rank(B) =rank$(AB) = 0$.
   False if $A = B = I_5$ (the 5×5 identity matrix.) Then $A = B = AB$ and rank(A) =rank(B) =rank$(AB) = 5$.
   In general, it is true that rank$(AB) \leq \min(\text{rank}(A), \text{rank}(B)$

2. If $\{\vec{a}_1, \ldots, \vec{a}_p\}$ is a linearly dependent set of vectors, then some subset of these vectors forms a basis for Span$\{\vec{a_1}, \ldots, \vec{a}_p\}$.

   Solution: Always
   This question is equivalent to saying that every subspace of $\mathbb{R}^n$ has as basis, which is true. To find a basis, we just need a set of linearly independent vectors from the given set of linearly dependent vectors. We can create a matrix A whose columns are the vectors $\vec{a}_1, \ldots, \vec{a}_p$. Then the Span$\{\vec{a_1}, \ldots, \vec{a}_p\}$. is the same as the column space of A, or Col(A). Row reduce the matrix to see which columns contain pivots. Then those columns in A will be a linearly independent set so will for a basis for Col(A)=Span$\{\vec{a_1}, \ldots, \vec{a}_p\}$.

3. Suppose H is a four-dimensional subspace of $\mathbb{R}^7$. Then any set of four linearly independent vectors in H will span H.

   Solution: Always
   By the Basis Theorem, a set of four linearly independent vectors in H form a basis for H. By definition, a basis $\mathcal{B}$ of H must span all of H, which is equivalent to saying Span $\mathcal{B} = H$.

4. If A and B are 5×5 matrices and dim(Nul$(B)) = 4$, then Col(BA) is either a point or a line in $\mathbb{R}^5$.

   Solution: Always
   Since dim(Nul(B)) = 4, by the Rank Theorem, rank$(B) = 1$. Asking if Col(AB) is either a point or a line is the same as asking if rank$(AB) \leq 1$. Since AB is 5×5, by the Rank theorem, to show rank$(AB) \leq 1$ is the same as showing that rank$(AB) = 5$ - dim Nul$(AB) \leq 1$. This is equivalent to showing that dim Nul$(AB) \geq 4$. Since we know that Multiplying by B sends four dimensional space to the zero vector (dim(Nul$(B)) = 4$). Then Multiplying by A sends $\vec{0}$ to $\vec{0}$, so A must send everything in the Null Space of B to zero as well. So the composition AB must have a nullspace of dimension at least 4 since Nul(B)

5. If $A\vec{x} = \lambda\vec{x}$, then $\vec{x}$ is an eigenvector of A.

   Solution: Sometimes
   The definition of an eigenvector requires $\vec{x}$ to be a *nonzero* vector $\vec{x}$ such that $(A - \lambda I) = \vec{0}$ has a nontrivial solution.

6. If A and B are similar matrices, and $\vec{x}$ is an eigenvector for A, then $\vec{x}$ is also an eigenvector for B.

   Solution: Sometimes
   This is true of course is A is equal to B. This is false if for example $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and
   $B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = P^{-1}AP = \begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix}$. Here A is similar to B, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector for A but it is not an eigenvector for B.

7. If $A^2 = A$, then 2 is not an eigenvalue of A.

   Solution: Always
   Let $\vec{v}$ be a nonzero vector such that $A\vec{v} = \lambda\vec{v}$. Then multiplying both sides by A gives:

$$\begin{aligned} A^2\vec{v} &= A\lambda\vec{v} \\ &= \lambda A\vec{v} \\ &= \lambda^2\vec{v} \end{aligned}$$

Since $A^2\vec{v} = A\vec{v} = \lambda\vec{v}$, we get $\lambda\vec{v} = \lambda^2\vec{v}$, or, since $\vec{v} \neq \vec{0}$, $\lambda^2 = \lambda$. λ can not equal 2. (It is in fact equal to 0 or 1).

8. If A is invertible and $\vec{x}$ is an eigenvector for A, then $\vec{x}$ is also an eigenvector for A^{-1}.

   Solution: Always
   If A is invertible, 0 is not an eigenvalue. If λ is an eigenvalue and $\vec{v}$ the corresponding eigenvector, then $A\vec{v} = \lambda\vec{v}$. Multiply both sides by A^{-1} to give $\vec{v} = \lambda A^{-1}\vec{v}$.. Since $\lambda \neq 0$, divide by λ to get $A^{-1}\vec{v} = \frac{1}{\lambda}\vec{v}$. This means $\vec{v}$ is an eigenvector for A^{-1} with eigenvalue $\frac{1}{\lambda}$.

9. If A and B are $n \times n$ matrices, then $\det(AB) = \det(BA)$. ***Solution:*** Always
   $\det(AB) = \det(A)\det(B) = \det(B)\det(A) = \det(BA)$. Here we used the fact that multiplication of real numbers is commutative.

10. If A and B are both $n \times n$ and $\det(A) = \det(B)$, then A and B are similar. ***Solution:*** Sometimes
    True if $A = B$, false if $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Here, B is not diagonalizable, so is not similar to A which is diagonalizable.

11. If A and B are both $n \times n$, then $\det(A + B) = \det(A) + \det(B)$ ***Solution:*** Sometimes
    This is true if A and B are the zero matrix. False if $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

Short Answers

Expect roughly five questions like these to appear on the exam. Be sure to read each question carefully.

1. For all invertible matrices A, what is $\det(A^{-1})$ and what is $\det(\text{Adj}A)$ in terms of $\det(A)$?
   Solution:
   Taking the determinant of both sides of the equation $AA^{-1} = I$, we get $\det(A)\det(A^{-1}) = \det(I) = 1$. Solving for $\det(A^{-1})$ gives

   $$\det(A^{-1}) = \frac{1}{\det(A)}.$$

   For the second part of the question, use the equation:

   $$A^{-1} = \frac{\text{Adj } A}{\det\ A}$$

   Let's set $\det\ A = \text{k}$ to make the notaion easier. Rearranging and taking the determinant of the above formula gives:

   $$\det\ \text{Adj}\ A = \det\ kA^{-1} = k^n \det(A^{-1}) = k^n(\det)^{-1} = k^n(k^{-1}) = k^{n-1}$$

2. Use determinants to final all values of x making the vectors linearly **dependent**.

   $$\left\{ \begin{bmatrix} x \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ x \\ 1 \end{bmatrix} \right\}$$

   Solution:
   The vectors

   $$\left\{ \begin{pmatrix} x \\ 4 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ x \\ 1 \end{pmatrix} \right\}$$

are linearly independent if and only if $\det \begin{pmatrix} x & 1 & 0 \\ 4 & -1 & x \\ -1 & -2 & 1 \end{pmatrix} \neq 0$. Calculating the determinant we get the equation $\det = 2(x-1)(x-2)$. So this set is linearly dependent if and only if $x = -1$ or $x = 2$.

3. Suppose we know $\det(A)$=9, where A is the matrix given below.

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \quad B = \begin{bmatrix} a+6g & b+6h & c+6i \\ d-4g & e-4h & f-4i \\ g & h & i \end{bmatrix} \quad C = \begin{bmatrix} 7g & 7h & 7i \\ d & e & f \\ a & b & c \end{bmatrix}$$

   (a) Find $\det(B)$.
   Solution:
   Since B is formed by adding a multiple of one row to another starting from A, which is an elementary row operation that does not change the determinant, we must have that $\det(B) = \det(A) = 9$.

   (b) Find $\det(C)$.
   Solution:
   The matrix C is formed by switching two rows of A and multiplying one row by 7. Switching a row causes a sign change in the determinant, and scaling a row scales the determinant. $\det(C)$ = -7*$\det(A)$ = -63.

   (c) Find $\det(BC)$.
   Solution:
   $\det(BC) = \det(B)\det(C) = 9 \cdot -63 = -567$.

   (d) Find $\det(ABC)$.
   Solution:
   $\det(ABC) = \det(A)\det(B)\det(C) = 9^2 \cdot -63 = -5103$

4. Suppose that

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 0 \\ 4 & 3 & 6 & 5 \end{bmatrix}$$

   Find a nonzero vector $\vec{x}$ such that $A^{1000}\vec{x} = \vec{0}$. Justify your answer.
   Solution:
   We need a nonzero solution to $A^{1000}\vec{x} = \vec{0}$. Clearly we do not actually want to calculate A^{1000}. Instead, let's notice that if there is a nontrivial solution to $A^{1000}\vec{x} = \vec{0}$, then A^{1000} is *not* invertible. Then A is not invertible either (othewise $A^{1000}(A^{-1})^{1000} = I_n$ would have made A^{1000} invertible. So we know by the Invertible Matrix Theorem that $A\vec{x} = \vec{0}$ has a nontrivial solution. If we can find a nontrivial solution $\vec{v}$ to the easier equation $A\vec{x} = \vec{0}$, then we know that it is also a solution to the equation $A^{1000}\vec{v} = A^{999}(A\vec{v}) = A^{999}\vec{0} = \vec{0}$. So we really just need to find a nontrivial solution to $A\vec{x} = \vec{0}$. Afterrowreducing A we see the solution set to $A\vec{x} = \vec{0}$ is Span $\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$. In particular, $\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$ is a solution to $A^{1000}\vec{x} = \vec{0}$.

5. Let

$$A = \begin{bmatrix} 3 & -2 \\ 4 & -3 \end{bmatrix}, \vec{b} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}.$$

   Solve the equation $A^{2011}\vec{x} = \vec{b}$
   Solution:
   We want to solve $A^{2011}\vec{x} = \vec{b}$. Since $\vec{b} \neq \vec{0}$, the trick we used in the the previous problem will not work. We need to actually calculate A^{2011}. The only easy way we know to raise a matrix to a higher power is to diagonalize it, if it is in fact diagonalizable. Let's try this approach. An easy calculation shows that the eigenvalues of A are

± 1. The eigenspace for $\lambda = -1$ is then the span of $\left\{ \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \right\}$. The eigenspace for $\lambda = 1$ is the span of $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$. Define $P = \begin{pmatrix} 1/2 & 1 \\ 1 & 1 \end{pmatrix}$ so $P^{-1} = \begin{pmatrix} -2 & 2 \\ 2 & -1 \end{pmatrix}$. Then

$$A = P \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} P^{-1}$$

So

$$\begin{aligned} A^{2011} &= P \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}^{2011} P^{-1} \\ &= P \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} P^{-1} \\ &= A \end{aligned}$$

Solving $A^{2011}\vec{x} = \vec{b}$ is the same as solving $A\vec{x} = \vec{b}$ since we just showed $A^{2011} = A$. To solve $A\vec{x} = \vec{b}$, row reduce to get that $x_1 = 17$ and $x_2 = 23$. So the vector $\vec{x} = \begin{pmatrix} 17 \\ 23 \end{pmatrix}$ is a solution to $A^{2011}\vec{x} = \vec{b}$.

6. Suppose that A is a nonzero matrix, but that $A^2 = 0$. Show that A can not be diagonalizable.
   Solution:
   Assume that A were diagonalizable. That is, assume there is some invertible P and diagonal D such that $A = PDP^{-1}$. Squaring both sides gives $A^2 = PD^2P^{-1} = 0$, since $A^2 = 0$. Since P is invertible and not 0,$D^2 = 0$, which then gives that $D = 0$. Since $A = PDP^{-1} = P0P^{-1} = 0$, A must be zero. This contradicts the fact that A was given as a nonzero matrix. So our initial assumption that A was diagonalizable must be wrong.

7. Suppose that A and B are both $n \times n$ and there is some $n \times n$ invertible matrix P such that $P^{-1}AP$ is a diagonal matrix and $P^{-1}BP$ is also a diagonal matrix. Show that $AB = BA$.
   Solution:
   If A and B are both $n \times n$ matrices are there is some $n \times n$ matrix P such that $P^{-1}AP$ is diagonal and $P^{-1}BP$ is diagonal, then:

$$\begin{aligned} AB &= A(PP^{-1})B \\ &= (PP^{-1})A(PP^{-1})B(PP^{-1}) \\ &= P(P^{-1}AP)(P^{-1}BP)P^{-1} \\ &= P(P^{-1}BP)(P^{-1}AP)P^{-1} \text{ (since diagonal matrices commute)} \\ &= (PP^{-1})B(PP^{-1})A(PP^{-1}) \\ &= BA \end{aligned}$$

8. Suppose that H and K are two subspaces of R^{10}, and that

$$\mathcal{B} = \{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4, \vec{u}_5\}$$

   is a basis of H, and

$$\mathcal{C} = \{\vec{v}_1, \vec{v}_2\}$$

   is a basis of K. Suppose that $T : \mathbb{R}^{10} \to \mathbb{R}^{10}$ is a linear transformation and we know that for all vectors $\vec{x} \in H$, the image $T(\vec{x}) \in K$.

   (a) Let $A = [\vec{u}_1\, \vec{u}_2\, \vec{u}_3\, \vec{u}_4\, \vec{u}_5]$. Explain why, for all vectors $\vec{v} \in \mathbb{R}^5$, $A\vec{v} \in H$.
   (b) Explain why, for all vectors $\vec{v} \in R^5, \vec{v} = [A\vec{v}]_{\mathcal{B}}$.
   (c) For $\vec{y} \in K$, let $[\vec{y}]_{\mathcal{C}}$ be the vector of the $\mathcal{C}$-coordinate of $\vec{y}$. Define a linear transformation $S : \mathbb{R}^5 \to \mathbb{R}^2$ by letting

$$S(\vec{v}) = [T(A\vec{v})]_{\mathcal{C}}.$$

   Show that if $\vec{v} \in \mathrm{Nul}(S)$, then $A\vec{v} \in \mathrm{Nul}(T)$.

Solution:

(a) Let A be the matrix whose columns consist of the vectors of $\mathcal{B}$. If $\vec{v} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \in \mathbb{R}^5$, then $A\vec{v} = [\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4, \vec{u}_5] \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \sum_{i=1}^{5} x_i \vec{u}_i$, which is a linear combination of the vectors $\vec{u}_1, \ldots, \vec{u}_5$. Since $H = \text{Span}\{\vec{u}_1, \ldots, \vec{u}_5\}$ and $A\vec{v}$ is a linear combination of the $\vec{u}_i$'s, $A\vec{v} \in H$.

(b) Remember that the coordinate vector of $A\vec{v} = [\sum_{i=1}^{5} x_i \vec{u}_i]_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$. So $[A\vec{v}]_{\mathcal{B}} = [\sum_{i=1}^{5} x_i \vec{u}_i]_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \vec{v}$.

(c) Given $\vec{v} \in \mathbb{R}^5$, we first apply A to get $A\vec{v} \in H$. Then apply T to $A\vec{v}$ to get a vector $T(A\vec{v}) \in K$. Now find the vector $[T(A\vec{v})]_{\mathcal{C}}$ and for simplicity, let's call it $S\vec{v}$. Suppose that $\vec{v} \in \text{Nul }(S)$. Then $S\vec{v} = \vec{0}$. Then $[T(A\vec{v})]_{\mathcal{C}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. This is the same as saying $T(A\vec{v}) = 0 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2$. Then since $T(A\vec{v}) = \vec{0}$, $A\vec{v}$ is in the Null space of T.

9. Suppose A is the following (invertible) matrix:

$$A = \begin{bmatrix} 1 & 3 & 5 & 7 & 1 \\ 2 & 4 & 2 & 1 & 2 \\ 3 & 1 & 1 & 4 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 5 & 4 & 3 & 2 & 1 \end{bmatrix}.$$

(a) Which of the following vectors is a solution to $A^{-1} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 2 \\ 3 \end{pmatrix}$? (circle your answer)

$$\vec{u}_1 = \begin{pmatrix} 17 \\ 12 \\ 4 \\ 11 \\ 27 \end{pmatrix}, \vec{u}_2 = \begin{pmatrix} 31 \\ 24 \\ 23 \\ 8 \\ 33 \end{pmatrix}, \vec{u}_3 = \begin{pmatrix} 6 \\ 7 \\ 9 \\ 1 \\ 2 \end{pmatrix}, \vec{u}_4 = \begin{pmatrix} 28 \\ 24 \\ 13 \\ 8 \\ 33 \end{pmatrix}, \vec{u}_5 = \begin{pmatrix} 17 \\ 35 \\ 27 \\ 29 \\ 21 \end{pmatrix}$$

(b) Find the solution set to the equation $A^{-1} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 2 \\ 3 \end{pmatrix}$.

Solution:

(a) One option to find a solution to $A^{-1}\vec{x} = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 2 \\ 3 \end{pmatrix}$ is to first find A^{-1}, then augment it with $\begin{pmatrix} 3 \\ 2 \\ 1 \\ 2 \\ 3 \end{pmatrix}$, and then row reduce to find the solution. This is too long! Let's try a better way.

If $A^{-1}\vec{x} = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 2 \\ 3 \end{pmatrix}$, then multiplying both sides by A gives $\vec{x} = A \begin{pmatrix} 3 \\ 2 \\ 1 \\ 2 \\ 3 \end{pmatrix}$. A simple calculation gives the answer $\vec{x} = \begin{pmatrix} 31 \\ 24 \\ 23 \\ 8 \\ 33 \end{pmatrix}$

(b) Since A^{-1} is invertible, the equation $A^{-1}\vec{x} = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 2 \\ 3 \end{pmatrix}$ has a unique solution. The solution set us just $\begin{pmatrix} 31 \\ 24 \\ 23 \\ 8 \\ 33 \end{pmatrix}$.

10. Compute the determinant of the following matrix two ways, first using cofactor expansion and then using row reduction.

$$\begin{bmatrix} 4 & -1 & 3 \\ 2 & 5 & -1 \\ -8 & 2 & -6 \end{bmatrix}$$

Remember, both methods should give you the same answer. Here, we omit the details since the book has lots of worked examples, but whichever method you use, the determinant of the above matrix is 0. You should know how to compute using both methods for the exam.

11. Suppose A is the matrix

$$A = \begin{bmatrix} 3 & 7 \\ -1 & -2 \end{bmatrix}.$$

(a) Find a rotation matrix

$$C = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

so that $A = PCP^{-1}$ for some invertible matrix P. You do not have to find P!

Solution:

First, we find the eigenvalues to be the complex conjugates $\lambda = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$. Notice that for either value of λ, the length $|\lambda| = \sqrt{(1/2)^2 + (\sqrt{3}/2)^2} = 1$. Arg $\lambda = \tan^{-1}((\frac{\sqrt{3}}{2})/\frac{1}{2}) = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$. Notice that we could use arctan since we were in the first quadrant of the complex plane.

So our rotation matrix is:

$$C = \begin{bmatrix} \cos(\frac{\pi}{3}) & -\sin(\frac{\pi}{3}) \\ \sin(\frac{\pi}{3}) & \cos(\frac{\pi}{3}) \end{bmatrix}$$

(b) Explain, without doing any matrix multiplications, why $A^3 = -I_2$.

Solution:

From part a), we saw that multiplication by A scales by $|\lambda| = 1$ and rotates by $\frac{\pi}{3}$ Then A^3 multiplies by $1^3 = 1$ and rotates by $3 \cdot \frac{\pi}{3} = \pi$ radians. This is a rotation from $\vec{x}$ to $-\vec{x}$ so $A^3 = -I_2$.