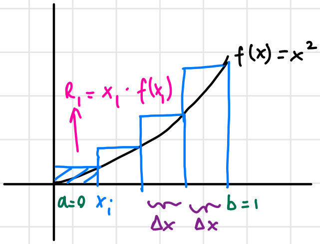
5.1 Areas Under Curves



$A \approx R_n = R_1 + R_2 + \cdots + R_n$ $\qquad R_i = \Delta x \cdot f(x_i)$

$x_i = a + i\,\Delta x$ $\qquad$ a: starting point

i: current point

Δx: distance between two consecutive points

$\Delta x = \dfrac{b-a}{n}$ $\quad$ where n: # of rectangles $\quad$ x_i : x coordinate of current point

Riemann sum

The area of region under graph $f(x)$ continuous on $x \in [a,b]$ with $\Delta x = \dfrac{b-a}{n}$

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} (R_1 + R_2 + \cdots + R_n) = \lim_{n\to\infty} \left(\Delta x \cdot f(x_1) + \Delta x \cdot f(x_2) + \cdots + \Delta x \cdot f(x_n) \right)$$

$$= \lim_{n\to\infty} \Delta x \left[f(x_1) + f(x_2) + \cdots + f(x_n) \right] = \lim_{n\to\infty} \Delta x \sum_{i=1}^{n} f(x_i)$$

$$\boxed{A = \lim_{n\to\infty} \Delta x \sum_{i=1}^{n} f(x_i)}$$

x_i : sample point $(x_i = a + i\Delta x)$

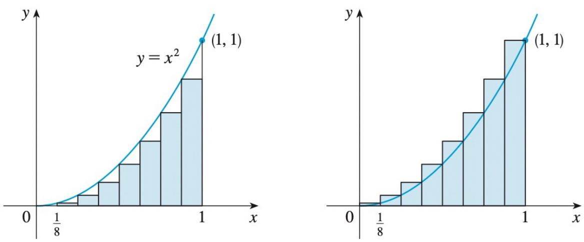
Δx: distance between two consecutive points, base of the rectangles

$f(x_i)$: height

Some useful formulas:

- $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
- $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

Left endpoint on curve, right endpoint on curve, midpoint



(a) Using left endpoints $\qquad$ (b) Using right endpoints

$a=0$, $b=1$, $\Delta x = \frac{1}{8}$, $n=8$

(b) $A = \Delta x \left[f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_8) \right]$

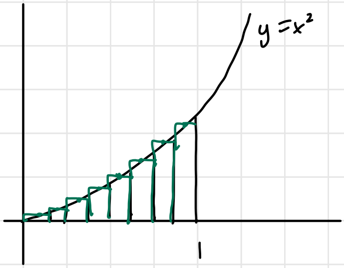
$= \frac{1}{8} \left[f\left(\frac{1}{8}\right) + f\left(\frac{2}{8}\right) + f\left(\frac{3}{8}\right) + \cdots + f(1) \right]$

$= \frac{1}{8} \left[\left(\frac{1}{8}\right)^2 + \left(\frac{2}{8}\right)^2 + \left(\frac{3}{8}\right)^2 + \cdots + 1^2 \right]$

(a) $A = \Delta x \left[f(0) + f\left(\frac{1}{8}\right) + f\left(\frac{2}{8}\right) + \cdots + f\left(\frac{7}{8}\right) \right]$

$= \frac{1}{8} \left[0 + \left(\frac{1}{8}\right)^2 + \cdots + \left(\frac{7}{8}\right)^2 \right]$

Using midpoints:



$x_1 = \frac{1}{2}\left(0 + \frac{1}{8}\right) = \frac{1}{16}$

$x_2 = \frac{1}{2}\left(\frac{1}{8} + \frac{2}{8}\right) = \frac{3}{16}$

$x_3 = \frac{5}{16}$

$\vdots$

$x_8 = \frac{15}{16}$

$A = \frac{1}{8}\left[f\left(\frac{1}{16}\right) + f\left(\frac{3}{16}\right) + \cdots + f\left(\frac{15}{16}\right)\right]$

underestimate, overestimate, or neither

2 Definition of a Definite Integral If f is a function defined for $a \leqslant x \leqslant b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0 \, (= a), x_1, x_2, \ldots, x_n \, (= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any **sample points** in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the **definite integral of f from a to b** is

$$\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x}$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is **integrable** on $[a, b]$.

5.2 Properties of the Definite Integral

1. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

2. $\int_a^a f(x)\,dx = 0$

3. $\int_a^b c\,dx = c(b-a)$

4. $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

5. $\int_a^b c \cdot f(x)\,dx = c\int_a^b f(x)\,dx$

6. $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$

7. if $f(x) \geq g(x)$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

Fundamental Theorem of Calculus

Definition An antiderivative of $f(x)$ is function $F(x)$ such that $F'(x) = f(x)$.

Part 1

If $f(x)$ is a smooth function, $a \in \text{Domain}(f)$, then $A(t) := \int_a^t f(x)\,dx$ is an antiderivative of $f(x)$.

$$\frac{d}{dt}\int_a^t f(x)\,dx = f(t)$$

Part 2

If $f(x)$ is a smooth function on $[a,b]$ and $F(x)$ is any antiderivative of $f(x)$

then $\int_a^b f(x)\,dx = F(b) - F(a)$

$$\int_a^b f(x)\,dx = \int_a^b \frac{d}{dx}[F(x)]\,dx = \frac{d}{dx}\int_a^b F(x)\,dx = F(a) - F(b)$$

Ex State an antiderivative of $f(x) = e^{x^2}$ with $A(2) = 0$

$A(t) := \int_2^t e^{x^2}\,dx$ (we pick 2 as the lower boundary to satisfy the condition that when $t = 2$, $A(2) = \int_2^2 e^{x^2}\,dx = 0$ (follows from a property))

Important Ex. from class

$$\lim_{a\to 0} \frac{1}{a} \cdot \int_0^a e^{-x^2}\,dx = \lim_{a\to 0} \frac{\int_0^a e^{-x^2}\,dx}{a}$$

Apply L'Hopital's Rule: with respect to a

$$\lim_{a\to 0} \frac{\frac{d}{da}\int_0^a e^{-x^2}\,dx}{\frac{d}{da}(a)}$$

Fundamental Theorem of Calculus : e^{-a^2}

$$= \lim_{a\to 0} \frac{e^{-a^2}}{1} = e^0 = 1$$

CAREFUL!

Ex $\frac{d}{dx}\left(\int_2^{1/x} \arctan(t)\,dt\right) = \arctan\left(\frac{1}{x}\right) \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}\arctan(1/x)$

Indefinite Integrals

1. $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$
2. $\int x^{-1} dx = \int \frac{1}{x} dx = \ln|x| + C$
3. $\int \sin x \, dx = -\cos x + C$
4. $\int \cos x \, dx = \sin x + C$
5. $\int a^x \, dx = \frac{a^x}{\ln a} + C$
6. $\int \sec^2 x \, dx = \tan x + C$
7. $\int \sec x \tan x = \sec x + C$
8. $\int \csc^2 x \, dx = -\cot x + C$
9. $\int \csc x \cdot \cot x \, dx = -\csc x + C$
10. $\int \frac{1}{x^2+1} dx = \tan^{-1} x + C$
11. $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$

Ex $\int \frac{1+\sqrt{x}+x}{x} dx = \int \frac{1}{x} dx + \int \frac{\sqrt{x}}{x} dx + \int \frac{x}{x} dx = \int \frac{1}{x} dx + \int x^{-1/2} dx + \int 1 \, dx$

$= \ln|x| + 2x^{1/2} + x + C$

Substitution Rule

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du$$

Ex $\int 5^t \sin(5^t)\, dt \xrightarrow{u\text{-sub}} \int \sin(u) \cdot \frac{1}{\ln 5} du = \frac{1}{\ln 5} \int \sin(u)\, du = \frac{1}{\ln 5}(-\cos(u)) + C$

$u = 5^t \quad du = 5^t \ln 5 \, dt \longrightarrow 5^t dt = \frac{1}{\ln 5} du$

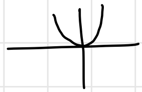
$= \boxed{\frac{-\cos(5^t)}{\ln 5} + C}$

Ex $\int \frac{a+bx^2}{\sqrt{3ax+bx^3}} dx \xrightarrow{u\text{-sub}} \int \frac{1}{\sqrt{u}} \frac{1}{3} du = \frac{1}{3} \cdot 2u^{1/2} + C = \boxed{\frac{2}{3}\sqrt{3ax+bx^3} + C}$

$u = 3ax + bx^3 \quad du = 3a + 3bx^2 \, dx \longrightarrow \frac{1}{3} du = a + bx^2 \, dx$

5.5 Definite Integrals of Symmetric Functions

Even function: $f(-x) = f(x)$



symmetric about the y-axis

Odd function: $f(-x) = -f(x)$

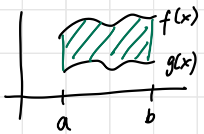
A symmetric boundary: $[-a, a]$

- If f is continuous on $[-a, a]$:

  if $f(x)$ is even: $\int_{-a}^{a} f(x)dx = 2 \cdot \int_{0}^{a} f(x)\, dx$

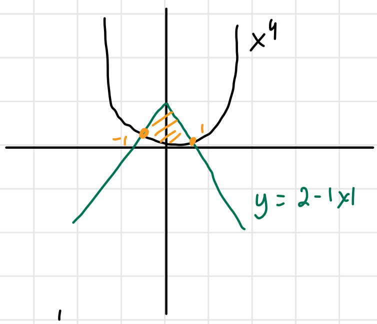
  if $f(x)$ is odd: $\int_{-a}^{a} f(x)\, dx = 0$

CHAPTER 6



$A = \int_{a}^{b} f(x) - g(x)\, dx$

Ex) Find the area of the region enclosed by $y = x^4$ and $y = 2 - |x|$



Find intersecting points:

$x^4 = 2 - |x|$

$x^4 + |x| = 2$

$x^4 + x = 2$

$x(x^3 + 1) = 2$

$1 \cdot 2$

$x = 1$

$x^4 - x = 2$

$x(x^3 - 1) = 2$

$-1 \cdot -2$

↳ check: $(-1)\,(-1 - 1) = (-1)(-2) = 2$ ✓

$x = -1$

Area $= \int_{-1}^{1} (2 - |x| - x^4)\, dx$

BE CAREFUL! For $x < 0$, $|x| = -x$ and for $x > 0$, $|x| = x$, so we set up the integral as

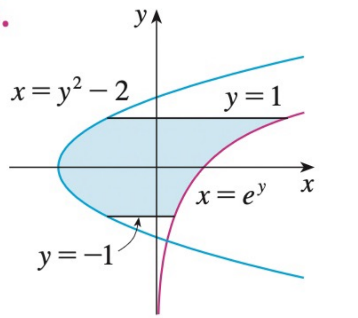
$$\int_{-1}^{1} (2 - |x| - x^4)\, dx = \int_{-1}^{0} (2 + x - x^4)\, dx + \int_{0}^{1} (2 - x - x^4)\, dx$$

$$A = \int_{-1}^{1} (2 - |x| - x^4)\, dx = \left[2x + \frac{x^2}{2} - \frac{x^5}{5}\right]_{-1}^{0} + \left[2x - \frac{x^2}{2} - \frac{x^5}{5}\right]_{0}^{1}$$

$$= \left[-\left(-2 + \frac{1}{2} + \frac{1}{5}\right)\right] + \left[2 - \frac{1}{2} - \frac{1}{5}\right] = 2 - \frac{1}{2} - \frac{1}{5} + 2 - \frac{1}{2} - \frac{1}{5}$$

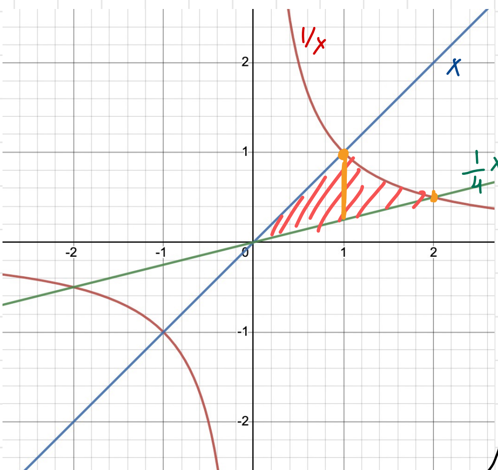
$$= 4 - 1 - \frac{2}{5} = \frac{13}{5}$$

Ex **3.**



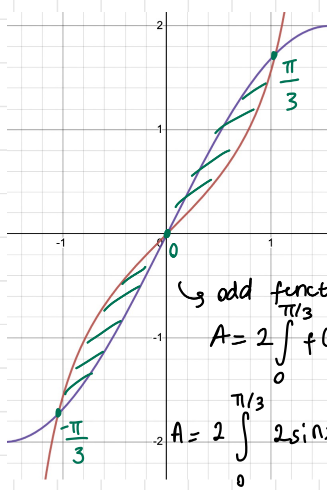
$$A = \int_{-1}^{1} e^y - (y^2 - 2)\, dy$$

Ex $y = 1/x$, $y = x$, $y = \frac{1}{4}x$, $x > 0$ Find the area enclosed by the given curves.



$$A = \int_0^1 x - \frac{1}{4}x\, dx + \int_1^2 \frac{1}{x} - \frac{1}{4}x\, dx$$

Ex $y = \tan x$, $y = 2\sin x$, $-\frac{\pi}{3} \le x \le \frac{\pi}{3}$



↳ odd function!

$$A = 2\int_0^{\pi/3} f(x)\, dx$$

$$A = 2\int_0^{\pi/3} 2\sin x - \tan x\, dx$$

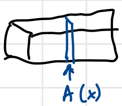
Check intersection points: $\tan x = 2\sin x$

$\frac{\sin x}{\cos x} = 2\sin x$ $\quad 2\sin x\cos x - \sin x = 0$

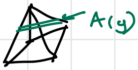
$\sin x\,(2\cos x - 1) = 0$

$\sin x = 0$ $\quad x = 0$ $\quad \cos x = \frac{1}{2}$ $\quad x = \frac{\pi}{3}, \frac{-\pi}{3}$

6.2 Volumes Using Cross Sections



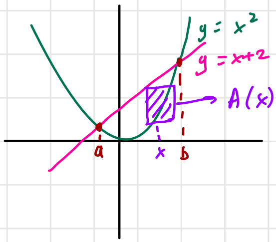
$$V = \int_a^b A(x)\,dx \leftarrow \text{vertical cross sections}$$



$$V = \int_c^d A(y)\,dy \leftarrow \text{horizontal cross sections}$$

(Ex) Find vol of solid whose base is region bounded by $y=x^2$, $y=x+2$ s.t. slices parallel to y-axis are perfect squares.

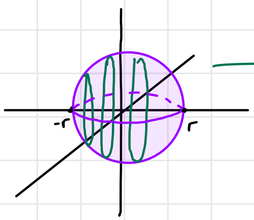
$a,b:\ x^2 = x+2 \rightarrow x=-1,2$



$\rightarrow A(x) = (\text{side})^2 = (x+2-x^2)^2$

$$V = \int_{-1}^{2} (x+2-x^2)\,dx$$

(Ex) Volume of a sphere



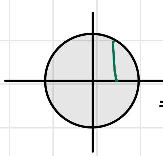
$\rightarrow A(x) = \text{area of a circle}$

$= \pi r^2$ (r being the height)

$= \pi y^2$

$= \pi(r^2-x^2)$

Cross-section:



$x^2+y^2=r^2$

$y=\sqrt{r^2-x^2}$

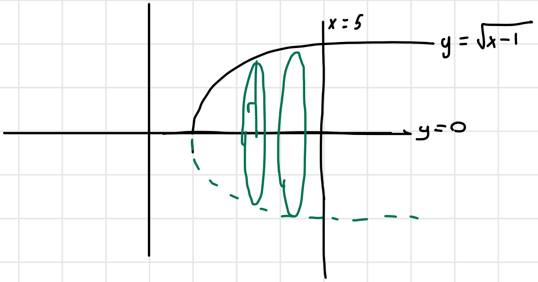
= cross-section radius

$$V = \int_{-r}^{r} A(x)\,dx = \int_{-r}^{r} \pi(r^2-x^2)\,dx$$

- When the cross-section is a disk, find the radius of the disk (in terms of x or y) and use

$$A = \pi(\text{radius})^2$$

(Ex) Find the volume obtained by rotating the region bounded by $y=\sqrt{x-1}$, $y=0$, $x=5$ about the x-axis.



radius $= \sqrt{x-1}$

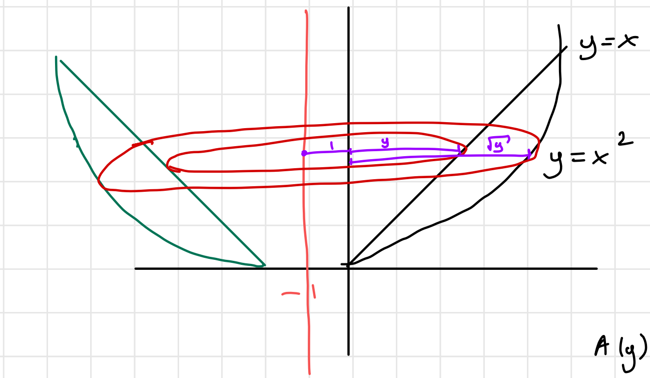
Area of disk $= \pi r^2 = \pi \sqrt{x-1}$

$$V = \int_1^5 \pi \sqrt{x-1}\, dx$$

• When the cross-section is a washer, find the inner and outer radii, compute the area of the washer as

$$A = \pi\,(\text{outer radius})^2 - \pi\,(\text{inner radius})^2$$

Ex Find the volume of the solid obtained by rotating the region enclosed by $y=x$ and $y=x^2$ rotated about the line $x=-1$.



washer

horizontal cross-sections → need area in y, $A(y)$, and radius in y

$y = x \rightarrow x = y$

$y = x^2 \rightarrow x = \sqrt{y}$

outer radius $= 1 + \sqrt{y}$

inner radius $= 1 + y$

$$A(y) = \pi\,(1+\sqrt{y})^2 - \pi\,(1+y)^2$$

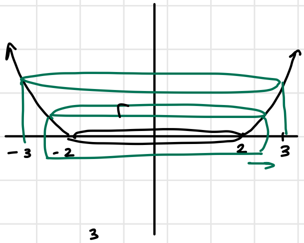
determining bounds: $y = 0$

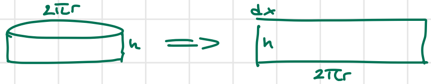
intersection: $y = \sqrt{y}$ @ $y = 1$.

$$V = \int_0^1 A(y)\,dy = \int_0^1 \pi\,(1+\sqrt{y})^2 - \pi\,(1+y)^2\,dy$$

6.3 Volumes by Cylindrical shells

$y = x^2 - 4$, x-axis, $x = 2$, $x = 3$, about y-axis



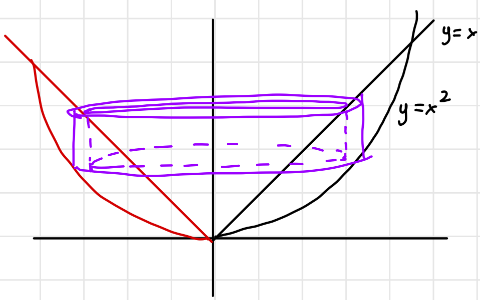


$$V = \int_2^3 \text{Area of cylindrical shell } dx = \int_2^3 2\pi \underbrace{r}_{\text{changes with } x} \cdot \underbrace{h}_{\text{changes with } y} \, dx$$

$$= \int_2^3 2\pi x (x^2 - 4) \, dx$$

$$\int_a^b \underbrace{(2\pi x)}_{\text{circumference}} \overbrace{[f(x)]}^{\text{height}} \underbrace{dx}_{\text{thickness}}$$

Ex Find the volume of the solid by rotating about the y-axis the region between $y = x$ and $y = x^2$.



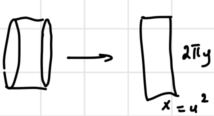
shell height = distance between curves
$= x - x^2$

$$V = \int_0^1 2\pi x (x - x^2) \, dx$$

The Shell Method - Integrating "dy"

$$V = \int_c^d 2\pi y \, h(y) \, dy$$

Radius is the y-value, $h(y)$ is the x-value which is $x = y^2$



6.4 Work

F constant : $W = F \cdot \Delta D = F \cdot \Delta x$

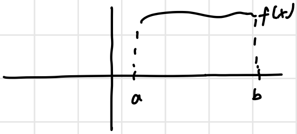
F changes with x, $F = F(x)$ between a and b : $W = \int_a^b F(x)dx$

Hooke's Law : $F(x) = kx$ k : spring constant

d : displacement from rest

6.5 Average Value of a Function

$$f_{ave} = \frac{1}{b-a} \int_a^b f(x)\,dx$$



7.1 Integration by Parts

Formula

$$\int u\,dv = uv - \int v\,du$$

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

There is generally an order with which you pick your u and dv:

L I A T E $\longrightarrow$ whichever comes first should be u

- L: logarithmic
- I: inverse trig
- A: algebraic
- T: trig
- E: exponential

(Ex) $\int_1^{\sqrt{3}} \arctan\left(\frac{1}{x}\right) dx$ (hidden dv!) TRICKY

$$\int_1^3 \arctan\left(\frac{1}{x}\right)\cdot 1\,dx$$

$u = \arctan\left(\frac{1}{x}\right)$ $\quad du = \frac{1}{1+\left(\frac{1}{x}\right)^2}\cdot\left(\frac{-1}{x^2}\right)dx$

$v = x$ $\quad dv = dx$

$$= \arctan\left(\frac{1}{x}\right)x - \int x\left(\frac{-1}{x^2}\right)\frac{1}{1+\left(\frac{1}{x}\right)^2}dx = \arctan\left(\frac{1}{x}\right)\cdot x + \int \frac{1}{x}\,\frac{x^2}{x^2+1}$$

$$= \arctan\left(\frac{1}{x}\right)\cdot x + \int\frac{x}{x^2+1}dx = \left[\arctan\left(\frac{1}{x}\right)\cdot x + \frac{1}{2}\ln|x^2+1|\right]_1^3$$

$u = x^2+1$ $\quad du = 2x\,dx$

$$\int \frac{1}{u}\frac{du}{2} = \frac{1}{2}\ln|u| = \frac{1}{2}\ln|x^2+1|$$

Another tricky example (repeating!)

$$\int e^x\cos x\,dx = e^x\sin x - \int \sin x\, e^x\,dx = e^x\sin x + e^x\cos x - \int e^x\cos x\,dx$$

$u = e^x$ $\quad du = e^x\,dx$

$v = \sin x$ $\quad dv = \cos x\,dx$

$u = e^x$ $\quad du = e^x\,dx$

$v = -\cos x$ $\quad dv = \sin x\,dx$

$$2\int e^x\cos x\,dx = e^x\sin x + e^x\cos x \qquad \int e^x\cos x = \frac{e^x\sin x + e^x\cos x}{2} + C$$

7.2 Trigonometric Integrals: Powers of Sine and Cosine

Solving $\int \sin^m x \cos^n x \, dx$

① either m, n is odd

(1.1) n is odd: factor out $\cos x \longrightarrow \int \sin^m x \cos x \cos^{n-1} x \, dx$

use identity $\sin^2 x + \cos^2 x = 1$

u-sub: $u = \sin x$ (implied $du = \cos x \, dx$)

(1.2) m is odd: factor out $\sin x \longrightarrow \int \sin x \sin^{m-1} x \cos^n x \, dx$

use $\cos^2 x + \sin^2 x = 1$

u-sub: $u = \cos x$ (implies $du = -\sin x \, dx$)

② m and n are <u>both</u> even

Use half-angle identities:

$$\sin^2 x = \frac{1 - \cos(2x)}{2} \qquad \cos^2 x = \frac{1 + \cos(2x)}{2}$$

7.2 Trigonometric Integrals: Powers of Tangent and Secant

Solving $\int \tan^m x \sec^n x \, dx$

① if n is even and $n \geq 2$: 1) factor out $\sec^2 x$ (because $\frac{d}{dx} \tan x = \sec^2 x$)

$$\downarrow$$

$$\int \tan^m x \sec^{n-1} x \sec^2 x \, dx$$

2) use $\sec^2 x = 1 + \tan^2 x$

3) $u = \tan x$ $du = \sec^2 x \, dx$

② if m is odd, $m \geq 3$: 1) factor out $\sec x \tan x \, dx$ (because $\frac{d}{dx} \sec x = \sec x \tan x$)

$$\downarrow$$

$$\int \tan^{m-1} x \sec^{n-1} x (\sec x \tan x) \, dx$$

2) use $\tan^2 x = \sec^2 x - 1$

3) u-sub: $u = \sec x$ ($du = \sec x \tan x \, dx$)

<u>Careful:</u> $\int \sec^2 x \, dx = \tan x$

NOT intuitive

$$\int \sec x \, dx = \int \sec x \cdot \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx$$

$u = \sec x + \tan x$

$du = \sec x \tan x + \sec^2 x \, dx$

$$= \int \frac{1}{u} \, du = \ln |u| = \boxed{\ln |\sec x + \tan x| + C}$$

7.2 Trigonometric Integrals: Product Identities for Sines and Cosines

Solving $\int \sin(mx)\cos(nx)\,dx$, $\int \sin(mx)\sin(nx)\,dx$, $\int \cos(mx)\cos(nx)\,dx$

Trig identities to know: $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$

$$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

$$\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$$

7.3 Trigonometric Substitution

For	Substitute in	Implies	Use identity
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$dx = a\cos\theta\,d\theta$	$1-\sin^2\theta = \cos^2\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$	$dx = a\sec^2\theta\,d\theta$	$1+\tan^2\theta = \sec^2\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$dx = a\sec\theta\tan\theta\,d\theta$	$\sec^2\theta - 1 = \tan^2\theta$

Know:

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + C$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$$

$$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}|x| + C$$

Ex $\int \tan x \sec^3 x\,dx$ (doesn't fit the cases)

$$\int \tan x \sec x \cdot \sec^2 x\,dx \overset{u\text{-sub}}{=} \int u^2\,du = \frac{u^3}{3} + C$$

$u = \sec x$ $du = \sec x \tan x\,dx$

Ex $\int \sqrt{\cos\theta}\,\sin^3\theta\,d\theta = \int \sqrt{\cos\theta}\,\sin\theta\,\sin^2\theta\,d\theta = \int \sqrt{\cos\theta}\,\sin\theta\,(1-\cos^2\theta)\,d\theta$

$u = \cos\theta$ $du = -\sin\theta\,d\theta$

$$= -\int \sqrt{u}\,(1-u^2)\,du = -\int u^{1/2} - u^{5/2}\,du = -\frac{2u^{3/2}}{3} + \frac{2u^{7/2}}{7} + C$$

$$= -\frac{2\cos^{3/2}\theta}{3} + \frac{2\cos^{7/2}\theta}{7} + C$$

Ex TRICKY

$$\int \frac{dx}{\sqrt{x^2+2x+5}} = \int \frac{1}{\sqrt{(x+1)^2+4}}\,dx \qquad \sqrt{(x+1)^2+2^2} \longrightarrow x+1 = 2\tan\theta \qquad x = 2\tan\theta - 1$$

$$dx = 2\sec^2\theta\, d\theta$$

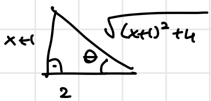
$$= \int \frac{1}{\sqrt{(2\tan\theta)^2+4}}\, 2\sec^2\theta\, d\theta = \int \frac{1}{\sqrt{4\tan^2\theta+4}}\, 2\sec^2\theta\, d\theta = \int \frac{2\sec^2\theta}{2\sec\theta}\, d\theta = \int \sec\theta\, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

Construct the triangle:

$x+1 = 2\tan\theta$

$\tan\theta = \frac{x+1}{2}$



$\sec\theta = \frac{\sqrt{(x+1)^2+4}}{2}$

$$\int = \ln\left|\frac{\sqrt{(x+1)^2+4}}{2} + \frac{x+1}{2}\right| + C = \ln\left|\sqrt{x^2+2x+5} + x+1\right| - \ln|2| + C$$

Partial Fractions

Partial Fraction Decomposition Summary

Case I:
$$\frac{P(x)}{(a_1x+b_1)(a_2x+b_2)\cdots(a_nx+b_n)} = \frac{A}{a_1x+b_1} + \frac{B}{a_2x+b_2} + \cdots + \frac{Z}{a_nx+b_n}$$

Case II:
$$\frac{P(x)}{(a_1x+b_1)^{r_1}(a_2x+b_2)^{r_2}} = \frac{A}{(a_1x+b_1)} + \frac{B}{(a_1x+b_1)^2} + \cdots + \frac{R}{(a_1x+b_1)^{r_1}} + \frac{S}{(a_2x+b_2)} + \frac{T}{(a_2x+b_2)^2} + \cdots$$

Case III: "distinct irred quadratics"
$$\frac{x}{(x-2)^2(x^2+1)(x^2+4)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{Cx+D}{x^2+1} + \frac{Ex+F}{x^2+4}$$

Case IV: "repeated irred quad"
$$\frac{1}{x(x-1)(x^2+x+1)(x^2+1)^3} = \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+x+1} + \frac{Ex+F}{x^2+1} + \frac{Gx+H}{(x^2+1)^2} + \frac{Ix+J}{(x^2+1)^3}$$

Careful !

$$\frac{5x^3-3x^2+2x-1}{x^4+x^2} = \frac{5x^3-3x^2+2x-1}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$$

other example: $\frac{2x+3}{(x^3+2)\,x} = \frac{A}{x} + \frac{Bx^2+Cx+D}{x^3+2}$

Ex $\int \frac{x^4}{x-1}\,dx$ LONG DIVISION! TRICKY

$$\begin{array}{l} x^4 \\ \underline{-\,x^4 - x^3} \\ x^3 \\ \underline{-\,x^3 - x^2} \\ x^2 \\ \underline{-\,x^2 - x} \\ x \\ \underline{-\,x - 1} \\ 1 \end{array} \qquad \begin{array}{|l} x-1 \\ \hline x^3+x^2+x+1 \end{array}$$

$$\int x^3+x^2+x+1+\frac{1}{x-1}\,dx$$

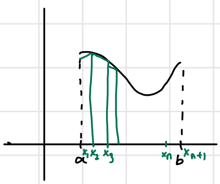
Tables of Integration

We can actually integrate a very small portion of functions, we can approximate some

Using Trapezoids to Estimate Areas

Trapezoids $\Rightarrow$ less error than rectangles

$A = \text{base} \cdot \frac{1}{2}(\text{heights})$



$h_1 = \frac{1}{2}[f(x_1)+f(x_2)]$

$h_2 = \frac{1}{2}[f(x_2)+f(x_3)]$

$\vdots$

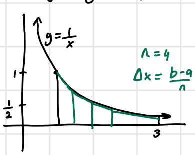
base $= \Delta x = \frac{b-a}{n}$

$$A \approx \frac{b-a}{n} \cdot \frac{1}{2}\left[f(x_1)+f(x_2)+f(x_2)+f(x_3)+f(x_3)+\cdots+f(x_n)+f(x_{n+1})\right]$$

Thm $A \approx \frac{b-a}{2N}\left[f(x_1)+2f(x_2)+2f(x_3)+\cdots+2f(x_n)+f(x_{n+1})\right]$

Ex $\int_1^3 \frac{1}{x}\,dx = \ln|x|\Big]_1^3 = \ln 3 - \cancel{\ln 1}^{\,0} = \ln 3$

Solving by trapezoid estimation:



$n=4$

$\Delta x = \frac{b-a}{n} = \frac{3-1}{4} = \frac{1}{2}$

$$A \approx \frac{1}{2}\cdot\frac{1}{2}\left[f(1)+2f\left(\tfrac{3}{2}\right)+2f(2)+2f\left(\tfrac{5}{2}\right)+f(3)\right]$$

7.8 Improper Integrals of Type I

Type I: Horizontal Asymptotic

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

$$\int_{-b}^{\infty} f(x)dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{t\to-\infty} \int_t^0 f(x)\,dx + \lim_{t\to\infty} \int_0^t f(x)\,dx$$

- $\lim_{t\to\infty} \tan^{-1}(t) = \frac{\pi}{2}$ and $\lim_{t\to-\infty} \tan^{-1}(t) = -\frac{\pi}{2}$

7.8 Improper Integrals of Type II

When your function is not continuous within the given bounds, i.e. when the function is undefined at any point within the boundary

Ex from class

$$\int_1^4 \frac{1}{(x-2)^2}\,dx$$

Function $\frac{1}{(x-2)^2}$ is undefined / discontinuous / has an asymptote at $x=2$, which is w/in bounds

$$\int_1^4 \frac{1}{(x-2)^2}\,dx = \lim_{t\to 2^-} \int_1^t \frac{1}{(x-2)^2}\,dx + \lim_{t\to 2^+} \int_t^4 \frac{1}{(x-2)^2}\,dx$$

7.8 Determining Convergence or Divergence of an Improper Integral

Thm (Comparison)

Let f, g cts, $f(x) \geq g(x) \geq 0$ for $x \geq a$

- if $\int_a^\infty f(x)\,dx < \infty$ (converges), then $\int_a^\infty g(x) < \infty$
- if $\int_a^\infty g(x)\,dx = \infty$ (diverges), then $\int_a^\infty f(x) = \infty$

a. $\displaystyle\int_0^\infty \frac{\arctan x}{2+e^x}\,dx$

$0 \le \arctan x \le \pi/2$

$e^x \le 2 + e^x$

$$\frac{\arctan x}{2+e^x} \le \frac{\pi/2}{e^x}$$

$$\int_0^\infty \frac{\pi/2}{e^x}\,dx = \frac{\pi}{2}\int_0^\infty \frac{1}{e^x}\,dx = \frac{\pi}{2}\int_0^\infty e^{-x}\,dx = \lim_{b\to\infty}\left(\frac{-\pi}{2}e^{-x}\right)\Big]_0^b = \lim_{b\to\infty}\frac{-\pi}{2}\left(\frac{1}{e^b}-\frac{1}{1}\right) = \frac{\pi}{2} \quad \text{(convergent)}$$

- Original function must be convergent