Linear Algebra
Practice Problems **Solutions** for Chapters 6,7

TRUE or FALSE

Expect three questions like these to appear on the exam. You will be asked to justify all your answers!

1. In $\mathbb{R}^n$, any two linearly independent vectors must be orthogonal.

   Solution: False.
   For example in $\mathbb{R}^2$ with $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\0\end{bmatrix}$. The vectors are linearly independent but their dot product is not 0, so they are not orthogonal.

2. In $\mathbb{R}^n$, any two orthogonal vectors must be linearly independent.

   Solution: False.
   For example: The $\vec{0}$ vector is orthogonal to every vector and also linearly dependent with every vector. Note that any two *nonzero* orthogonal vectors are linearly independent.

3. The zero vector, $\vec{0}$, is the only vector of length 0.

   Solution: True.
   This is by the definition of the length as the square of the dot product of the vector with itself.

4. Let $\vec{v} \in \mathbb{R}^3$ and suppose $\|\vec{v}\| = 2$. Let W denote the orthogonal complement of $\text{Span}\{\vec{v}\}$. There exists some $\vec{w} \in W$ such that $\|\vec{v}+\vec{w}\|^2 < 4 + \|\vec{w}\|^2$.

   Solution: False.
   Since $\vec{v}$ and $\vec{w}$ are orthogonal, the Pythagorean Theorem states that $\|\vec{v}+\vec{w}\|^2 = \|\vec{v}\|^2+\|\vec{w}\|^2 = 4 + \|\vec{w}\|^2$.

5. Let $V = \mathbb{R}^n$ with a subspace W. For all nonzero vectors $\vec{v} \in V$, $\|\text{proj}_W\vec{v}\| < \|\vec{v}\|$.

   Solution: False.
   If $\vec{v} \in W$, then $\text{proj}_W\vec{v} = \vec{v}$, so $\|\text{proj}_W\vec{v}\| = \|\vec{v}\|$.

6. Suppose W is a subspace of $\mathbb{R}^n$. If $\vec{w} \in W$, then the orthogonal projection of $\vec{w}$ onto W is equal to $\vec{w}$.

   Solution: True.
   The orthogonal projection $\text{proj}_W\vec{w}$ gives the vector in W that is closest to $\vec{w}$. If $\vec{w} \in W$, then that closest vector is $\vec{w}$ itself.

7. If A is symmetric, and B is similar to A, then B must also be symmetric.

   Solution: False.
   To show that this is false, it suffices to exhibit a single counterexample. For this counterexample, let $B = \begin{bmatrix}1 & 2\\0 & 2\end{bmatrix}$. The eigenvalues of B are 1 and 2, so B is similar to the diagonal matrix $A = \begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}$. This A is symmetric, but B is not symmetric. Thus the claim is false.

8. There exists a quadratic form $Q(\vec{x}) = \vec{x}^\top A\vec{x}$ such that the quadratic form Q is positive semidefinite and the matrix A is negative definite.

   Solution: False.
   By definition, if this quadratic form is positive semidefinite, then so is the corresponding matrix.

ALWAYS, SOMETIMES, or NEVER

Expect three questions like these to appear on the exam. You will be asked to justify all your answers!

1. Suppose $A^\top = A^{-1}$, and $\{\vec{v_1}, \vec{v_2}, ..., \vec{v_n}\}$ is an orthonormal basis of V. Then $\{A\vec{v_1}, A\vec{v_2}, ..., A\vec{v_n}\}$ is also an orthonormal basis of V.

   Solution: Always.
   Since $A^\top = A^{-1}$, we know A is an orthogonal matrix. Thus $\|A\vec{v_k}\| = \|\vec{v_k}\| = 1$ for all k. Likewise,
   $$\langle A\vec{v_k}, A\vec{v_j}\rangle = \langle \vec{v_k}, \vec{v_j}\rangle = \begin{cases} 1, & k = j; \\ 0, & k \neq j. \end{cases}$$
   Thus the set $\{A\vec{v_1}, A\vec{v_2}, ..., A\vec{v_n}\}$ is also an orthonormal basis of V.

2. Suppose V is any subspace containing the vectors $\vec{v}$ and $\vec{0}$. Then $\vec{v}$ and $\vec{0}$ are orthogonal.

   Solution: Always.
   The zero vector dot any vector is always zero and by definition, the vectors are orthogonal.

3. Suppose V is a subspace containing vectors $\vec{v_1}$, $\vec{v_2}$, and $\vec{v_3}$, and let $W = \text{Span}\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$. Then $W^\perp$ is a subspace of $V^\perp$.

   Solution: Sometimes.
   This is true if say $V = W$, but this is false if $V = \mathbb{R}^3$ and W is a line through the origin. Then $W^\perp$ is a plane orthogonal to the line through the origin and $V^\perp = \{\vec{0}\}$ the trivial subspace. In this case, $W^\perp$ is not a subspace of $V^\perp$.

4. Suppose the vectors $\vec{v}$ and $\vec{w}$ are in the same subspace. Then $|\vec{v} \cdot \vec{w}| = \|\vec{v}\| \cdot \|\vec{w}\|$.

   Solution: Sometimes.
   The Cauchy-Schwarz inequality is $|\vec{v} \cdot \vec{w}| \leq \|\vec{v}\|\|\vec{w}\|$. Equality occurs if and only if $\vec{v}$ and $\vec{w}$ are parallel.

5. Applying the Gram-Schmidt process to the linearly independent sets $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ and $\{\vec{v_2}, \vec{v_3}, \vec{v_1}\}$ will yield identical (possibly permuted) orthonormal sets.

   Solution: Sometimes.
   This is true if the initial sets are already orthogonal. Otherwise, it may not be true. For example, suppose $\vec{v_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{v_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Then applying the Gram-Schmidt process to the set $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ yields the standard basis of $\mathbb{R}^3$, whereas applying this process to the set $\{\vec{v_2}, \vec{v_3}, \vec{v_1}\}$ yields the orthonormal set $\left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \right\}$.

6. Applying the Gram-Schmidt process to the sets $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ and $\{\vec{v_2}, \vec{v_3}, \vec{v_1}\}$ will yield two possibly distinct bases of the same space.

   Solution: Always.
   In both cases the Gram-Schmidt process will yield an orthogonal basis of the subspace $\text{Span}\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$.

7. Suppose A is an $m \times n$ matrix whose columns form a basis of an n-dimensional subspace of $\mathbb{R}^m$. Then the rows of $A^\top A$ are linearly independent.

   Solution: Always.
   The matrix $A^\top A$ is invertible if and only if the columns of A are linearly independent. Since the columns of A form a basis, they are linearly independent. Thus, $A^\top A$ is invertible, so the rows of $A^\top A$ are linearly independent.

8. Let $A = \begin{bmatrix} -1 & 3 & 0 & 4 \\ 3 & 2 & 1 & 7 \\ 0 & 1 & -6 & 5 \\ 4 & 7 & 5 & 3 \end{bmatrix}$. Suppose $\lambda_1 \neq \lambda_2$ with $A\vec{v_1} = \lambda_1 \vec{v_1}$ and $A\vec{v_2} = \lambda_2 \vec{v_2}$. Then $\vec{v_1} \cdot \vec{v_2} = 1$.

   Solution: Never.
   Note that the matrix A is symmetric. Thus, eigenvectors corresponding to different eigenvalues are orthogonal. This means $\vec{v_1} \cdot \vec{v_2} = 0$.

Short Answer

1. Let $W = \text{Span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\-2\end{bmatrix}\right\}$. Find an orthonormal basis for $W^\perp$.

   Solution: We first find a basis for $W^\perp$. This will be the same as a basis for the null space of $\begin{bmatrix}1 & 1 & 1\\1 & 0 & -2\end{bmatrix} \sim \begin{bmatrix}1 & 0 & -2\\0 & 1 & 3\end{bmatrix}$. Thus, a basis of $W^\perp$ is given by $\begin{bmatrix}2\\-3\\1\end{bmatrix}$, so it suffices to normalize this vector.

$$\left\|\begin{bmatrix}2\\-3\\1\end{bmatrix}\right\|^2 = 4+9+1$$
$$\left\|\begin{bmatrix}2\\-3\\1\end{bmatrix}\right\| = \sqrt{14}$$

   Thus, an orthonormal basis of $W^\perp$ is given by $\left\{\frac{1}{\sqrt{14}}\begin{bmatrix}2\\-3\\1\end{bmatrix}\right\}$.

2. Apply the Gram-Schmidt process to the following set in $\mathbb{R}^4$. $\left\{\begin{bmatrix}2\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\2\\3\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\3\end{bmatrix}\right\}$

   Solution: To begin, we normalize the first vector and obtain $\vec{v_1} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$.

   Next we make the second vector $\vec{x_2}$ orthogonal to $\vec{v_1}$ to obtain $\begin{bmatrix}0\\2\\3\\0\end{bmatrix}$. We normalize this to find $\vec{v_2} = \begin{bmatrix}0\\\frac{2}{\sqrt{13}}\\\frac{3}{\sqrt{13}}\\0\end{bmatrix}$.

   For the third vector, $\vec{x_3} = \begin{bmatrix}0\\0\\1\\3\end{bmatrix}$, we note that $\vec{x_3}$ is already orthogonal to $\vec{v_1}$, so we just need to subtract off the component in the direction of $\vec{v_2}$.

$$\vec{x_3} - \langle \vec{x_3}, \vec{v_2} \rangle \vec{v_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix} - \frac{3}{\sqrt{13}} \begin{bmatrix} 0 \\ \frac{2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \\ 0 \end{bmatrix}$$

$$\vec{x_3} - \langle \vec{x_3}, \vec{v_2} \rangle \vec{v_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 0 \\ \frac{6}{13} \\ \frac{9}{13} \\ 0 \end{bmatrix}$$

$$\vec{x_3} - \langle \vec{x_3}, \vec{v_2} \rangle \vec{v_2} = \begin{bmatrix} 0 \\ -\frac{6}{13} \\ \frac{4}{13} \\ 3 \end{bmatrix}$$

We normalize this to find $\vec{v_3}$.

$$\left\| \begin{bmatrix} 0 \\ -\frac{6}{13} \\ \frac{4}{13} \\ 3 \end{bmatrix} \right\|^2 = \frac{36+16}{13^2} + 9$$

$$\left\| \begin{bmatrix} 0 \\ -\frac{6}{13} \\ \frac{4}{13} \\ 3 \end{bmatrix} \right\|^2 = \frac{4}{13} + \frac{117}{13}$$

$$\left\| \begin{bmatrix} 0 \\ -\frac{6}{13} \\ \frac{4}{13} \\ 3 \end{bmatrix} \right\| = \sqrt{\frac{121}{13}}$$

$$\vec{v_3} = \sqrt{\frac{13}{121}} \begin{bmatrix} 0 \\ -\frac{6}{13} \\ \frac{4}{13} \\ 3 \end{bmatrix}$$

Putting these pieces together, we have obtained the orthonormal set

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \\ 0 \end{bmatrix}, \sqrt{\frac{13}{121}} \begin{bmatrix} 0 \\ -\frac{6}{13} \\ \frac{4}{13} \\ 3 \end{bmatrix} \right\}$$

3. Find the QR-factorization of the matrix $A = \begin{bmatrix} \sqrt{3} & 3\sqrt{3} \\ 3 & 5 \\ 0 & 1 \end{bmatrix}$.

Solution: We begin by applying the Gram-schmidt process to the columns of A. We normalize the first column as follows.

$$\left\| \begin{bmatrix} \sqrt{3} \\ 3 \\ 0 \end{bmatrix} \right\|^2 = 3 + 9$$

$$\left\| \begin{bmatrix} \sqrt{3} \\ 3 \\ 0 \end{bmatrix} \right\| = 2\sqrt{3}$$

$$\vec{v_1} = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \end{bmatrix}$$

This $\vec{v_1}$ will be the first column of the matrix Q, and the first column of the matrix R is $\begin{bmatrix} 2\sqrt{3} \\ 0 \end{bmatrix}$. Next we subtract off the component of the second column of A in the direction of $\vec{v_1}$. We will call the resulting vector $\vec{y_2}$, and when we normalize $\vec{y_2}$ we will obtain $\vec{v_2}$, the second column of Q.

$$\begin{aligned}
\vec{y_2} &= \vec{x_2} - \langle \vec{x_2}, \vec{v_1} \rangle \vec{v_1} \\
\vec{y_2} &= \begin{bmatrix} 3\sqrt{3} \\ 5 \\ 1 \end{bmatrix} - \left\langle \begin{bmatrix} 3\sqrt{3} \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \end{bmatrix} \right\rangle \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \end{bmatrix} \\
\vec{y_2} &= \begin{bmatrix} 3\sqrt{3} \\ 5 \\ 1 \end{bmatrix} - \left(\frac{3\sqrt{3}}{2} + \frac{5\sqrt{3}}{2} \right) \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \end{bmatrix} \\
\vec{y_2} &= \begin{bmatrix} 3\sqrt{3} \\ 5 \\ 1 \end{bmatrix} - \begin{bmatrix} 2\sqrt{3} \\ 6 \\ 0 \end{bmatrix} \\
\vec{y_2} &= \begin{bmatrix} \sqrt{3} \\ -1 \\ 1 \end{bmatrix} \\
\|\vec{y_2}\|^2 &= 5 \\
\vec{v_2} &= \begin{bmatrix} \sqrt{\frac{3}{5}} \\ \frac{-1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}
\end{aligned}$$

Thus the QR-factorization of A is given by

$$\begin{bmatrix} \sqrt{3} & 3\sqrt{3} \\ 3 & 5 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \sqrt{\frac{3}{5}} \\ \frac{\sqrt{3}}{2} & \frac{-1}{\sqrt{5}} \\ 0 & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 2\sqrt{3} & 4\sqrt{3} \\ 0 & \sqrt{5} \end{bmatrix}$$

4. Find the spectral decomposition of the matrix $A = \begin{bmatrix} -3 & 2\sqrt{3} \\ 2\sqrt{3} & 1 \end{bmatrix}$.

Solution: We begin by finding the eigenvalues and eigenvectors of A.

$$
\begin{aligned}
\det(A - \lambda I) &= 0 \\
\det \begin{bmatrix} -3-\lambda & 2\sqrt{3} \\ 2\sqrt{3} & 1-\lambda \end{bmatrix} &= (-3-\lambda)(1-\lambda) - (2\sqrt{3})^2 \\
\det \begin{bmatrix} -3-\lambda & 2\sqrt{3} \\ 2\sqrt{3} & 1-\lambda \end{bmatrix} &= \lambda^2 + 2\lambda - 15 \\
\det \begin{bmatrix} -3-\lambda & 2\sqrt{3} \\ 2\sqrt{3} & 1-\lambda \end{bmatrix} &= (\lambda+5)(\lambda-3) \\
\lambda &= -5, 3
\end{aligned}
$$

When $\lambda = -5$, we find

$$
\begin{aligned}
A - \lambda I &= A + 5I \\
A - \lambda I &= \begin{bmatrix} 2 & 2\sqrt{3} \\ 2\sqrt{3} & 6 \end{bmatrix} \\
\begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} 2 & 2\sqrt{3} \\ 2\sqrt{3} & 6 \end{bmatrix} \begin{bmatrix} \sqrt{3} \\ -1 \end{bmatrix} \\
\left\| \begin{bmatrix} \sqrt{3} \\ -1 \end{bmatrix} \right\|^2 &= 3 + 1 \\
\left\| \begin{bmatrix} \sqrt{3} \\ -1 \end{bmatrix} \right\| &= 2
\end{aligned}
$$

Thus, a unit eigenvector for $\lambda = -5$ is $\begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{-1}{2} \end{bmatrix}$.

When $\lambda = 3$, we find

$$
\begin{aligned}
A - \lambda I &= A - 3I \\
A - \lambda I &= \begin{bmatrix} -6 & 2\sqrt{3} \\ 2\sqrt{3} & -2 \end{bmatrix} \\
\begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} -6 & 2\sqrt{3} \\ 2\sqrt{3} & -2 \end{bmatrix} \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} \\
\left\| \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} \right\|^2 &= 1 + 3 \\
\left\| \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} \right\| &= 2
\end{aligned}
$$

Thus, a unit eigenvector for $\lambda = 3$ is $\begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$. Then the spectral decomposition of A is

$$
\begin{bmatrix} -3 & 2\sqrt{3} \\ 2\sqrt{3} & 1 \end{bmatrix} = -5 \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{-1}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \end{bmatrix} + 3 \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}
$$

$$
\begin{bmatrix} -3 & 2\sqrt{3} \\ 2\sqrt{3} & 1 \end{bmatrix} = -5 \begin{bmatrix} \frac{3}{4} & -\frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix} + 3 \begin{bmatrix} \frac{1}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{3}{4} \end{bmatrix}
$$

5. Consider the matrix equation $\begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

   (a) Show this equation is inconsistent.

   Solution: Let $A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$. The columns of A are clearly linearly independent, since they are not scalar multiples of each other. Thus, the system is inconsistent if and only if the augmented matrix is invertible.

$$\det \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 3 \end{bmatrix} = \det \begin{bmatrix} 0 & 1 & -5 \\ 0 & 1 & 2 \\ 1 & 0 & 3 \end{bmatrix}$$
$$\det \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 3 \end{bmatrix} = 7$$

   Thus the augmented matrix is invertible, so the system is inconsistent.

   (b) Find the least-squares solution of this equation.

   Solution:

$$\hat{x} = (A^\top A)^{-1} A^\top \vec{b}$$
$$\hat{x} = \left[\begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \right]^{-1} \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$
$$\hat{x} = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$
$$\hat{x} = \frac{1}{6} \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$
$$\hat{x} = \frac{1}{6} \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$
$$\hat{x} = \begin{bmatrix} \frac{2}{3} \\ \frac{5}{6} \end{bmatrix}$$

6. Write the matrix of the quadratic form on $\mathbb{R}^2$ defined by $Q(\vec{x}) = 2x_1^2 - 4x_1x_2 + 4x_2^2$. Classify this quadratic form as positive definite, negative definite, positive semidefinite, negative semidefinite, or indefinite.

   Solution: We must make the matrix symmetric, so we split the cross terms into two equal pieces. The matrix is $A = \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix}$. In order to classify this quadratic form, we find the eigenvalues of the corresponding matrix.

$$
\begin{aligned}
\det(A - \lambda I) &= \det \begin{bmatrix} 2-\lambda & -2 \\ -2 & 4-\lambda \end{bmatrix} \\
\det(A - \lambda I) &= (2-\lambda)(4-\lambda) - 4 \\
\det(A - \lambda I) &= \lambda^2 - 6\lambda + 8 - 4 \\
\lambda^2 - 6\lambda + 4 &= 0 \\
\lambda &= \frac{6 \pm \sqrt{36-16}}{2} \\
\lambda &= 3 \pm \sqrt{5}
\end{aligned}
$$

All of the eigenvalues of A are strictly positive, so the matrix and the quadratic form are positive definite.