Linear Algebra
Practice Problems for Chapters 6,7

Definitions

In addition to the vocabulary from chapters 1, 2, 3, 4, and 5 you should know the definitions of the following vocabulary terms from chapters 6 and 7, although you will not be asked to give any of these definitions specifically on the exam.

1. dot product of two vectors
2. length (or norm) of a vector
3. unit vector
4. normalizing a vector
5. distance between two vectors
6. orthogonal vectors, orthogonal set, orthogonal basis
7. orthogonal complement
8. orthogonal projection of a vector $\vec{v}$ onto a subspace W, $\text{proj}_W\vec{v}$
9. orthonormal set, orthonormal basis
10. orthogonal matrix
11. best approximation
12. Gram-Schmidt process
13. QR factorization
14. least-squares solution, least squares line
15. normal equations for $A\vec{x} = \vec{b}$
16. Cauchy-Schwarz inequality
17. triangle inequality
18. symmetric matrix
19. orthogonally diagonalizable matrix
20. spectral theorem
21. spectral decomposition
22. quadratic form
23. matrix of a quadratic form
24. principal axes of a quadratic form
25. positive definite, negative definite, positive semidefinite, negative semidefinite, indefinite
26. singular value decomposition

TRUE or FALSE

Expect three questions like these to appear on the exam. You will be asked to justify all your answers!

1. In $\mathbb{R}^n$, any two linearly independent vectors must be orthogonal.
2. In $\mathbb{R}^n$, any two orthogonal vectors must be linearly independent.
3. The zero vector, $\vec{0}$, is the only vector of length 0.
4. Let $\vec{v} \in \mathbb{R}^3$ and suppose $\|\vec{v}\| = 2$. Let W denote the orthogonal complement of $\operatorname{Span}\{\vec{v}\}$. There exists some $\vec{w} \in W$ such that $\|\vec{v} + \vec{w}\|^2 < 4 + \|\vec{w}\|^2$.
5. Let $V = \mathbb{R}^n$ with a subspace W. For all nonzero vectors $\vec{v} \in V$, $\|\text{proj}_W \vec{v}\| < \|\vec{v}\|$.
6. Suppose W is a subspace of $\mathbb{R}^n$. If $\vec{w} \in W$, then the orthogonal projection of $\vec{w}$ onto W is equal to $\vec{w}$.
7. If A is symmetric, and B is similar to A, then B must also be symmetric.
8. There exists a quadratic form $Q(\vec{x}) = \vec{x}^\top A\vec{x}$ such that the quadratic form Q is positive semidefinite and the matrix A is negative definite.

ALWAYS, SOMETIMES, or NEVER

Expect three questions like these to appear on the exam. You will be asked to justify all your answers!

1. Suppose $A^\top = A^{-1}$, and $\{\vec{v_1}, \vec{v_2}, ..., \vec{v_n}\}$ is an orthonormal basis of V. Then $\{A\vec{v_1}, A\vec{v_2}, ..., A\vec{v_n}\}$ is also an orthonormal basis of V.
2. Suppose V is any subspace containing the vectors $\vec{v}$ and $\vec{0}$. Then $\vec{v}$ and $\vec{0}$ are orthogonal.
3. Suppose V is a subspace containing vectors $\vec{v_1}$, $\vec{v_2}$, and $\vec{v_3}$, and let $W = \text{Span}\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$. Then $W^\perp$ is a subspace of $V^\perp$.
4. Suppose the vectors $\vec{v}$ and $\vec{w}$ are in the same subspace. Then $|\vec{v} \cdot \vec{w}| = \|\vec{v}\| \cdot \|\vec{w}\|$.
5. Applying the Gram-Schmidt process to the linearly independent sets $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ and $\{\vec{v_2}, \vec{v_3}, \vec{v_1}\}$ will yield identical (possibly permuted) orthonormal sets.
6. Applying the Gram-Schmidt process to the sets $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ and $\{\vec{v_2}, \vec{v_3}, \vec{v_1}\}$ will yield two possibly distinct bases of the same space.
7. Suppose A is an $m \times n$ matrix whose columns form a basis of an n-dimensional subspace of $\mathbb{R}^m$. Then the rows of $A^\top A$ are linearly independent.
8. Let $A = \begin{bmatrix} -1 & 3 & 0 & 4 \\ 3 & 2 & 1 & 7 \\ 0 & 1 & -6 & 5 \\ 4 & 7 & 5 & 3 \end{bmatrix}$. Suppose $\lambda_1 \neq \lambda_2$ with $A\vec{v_1} = \lambda_1\vec{v_1}$ and $A\vec{v_2} = \lambda_2\vec{v_2}$. Then $\vec{v_1} \cdot \vec{v_2} = 1$.

1. Let $W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \right\}$. Find an orthonormal basis for $W^{\perp}$.

2. Apply the Gram-Schmidt process to the following set in $\mathbb{R}^4$. $\left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\}$

3. Find the QR-factorization of the matrix $A = \begin{bmatrix} \sqrt{3} & 3\sqrt{3} \\ 3 & 5 \\ 0 & 1 \end{bmatrix}$.

4. Find the spectral decomposition of the matrix $A = \begin{bmatrix} -3 & 2\sqrt{3} \\ 2\sqrt{3} & 1 \end{bmatrix}$.

5. Consider the matrix equation $\begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.
   (a) Show this equation is inconsistent.
   (b) Find the least-squares solution of this equation.

6. Write the matrix of the quadratic form on $\mathbb{R}^2$ defined by $Q(\vec{x}) = 2x_1^2 - 4x_1x_2 + 4x_2^2$. Classify this quadratic form as positive definite, negative definite, positive semidefinite, negative semidefinite, or indefinite.