

Linear Algebra  
Practice Problems for Unit 1

Definitions

You should know the definitions of the following vocabulary terms from the text, although you will not be asked to give any of these definitions specifically on the exam.

1. a linear equation, a linear system, a solution of a system, the solution set of a system, equivalent systems
2. consistent and inconsistent systems
3.  $m \times n$  matrix
4. coefficient matrix and augmented matrix of a linear system
5. elementary row operations, row equivalent matrices
6. echelon and reduced echelon forms of a matrix
7. pivot position
8. free variable, basic variable
9. vector, scalar, scalar multiple, linear combination
10.  $\text{Span}\{\vec{a}_1, \dots, \vec{a}_p\}$
11. Matrix-vector product  $A\vec{x}$
12. Vectors  $\{\vec{a}_1, \dots, \vec{a}_p\}$  span  $\mathbb{R}^m$
13. Homogeneous, nonhomogeneous linear systems
14. Trivial, nontrivial solutions
15. General solution of a linear system in parametric vector form
16. Linearly independent, linearly dependent sets of vectors
17. Linear transformation, domain, codomain, image, range
18. Standard matrix for a linear transformation
19. One-to-one, onto linear transformations

TRUE or FALSE

Expect roughly five questions like these to appear on the exam. You will be asked to justify all your answers!

1. If  $A$  is a matrix in reduced echelon form, then at least one entry in every column of  $A$  must be 1.

2. The matrix

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is in reduced echelon form.

3. No set of two vectors in  $\mathbb{R}^3$  can span  $\mathbb{R}^3$ .
4. If two  $m \times n$  matrices are row equivalent, they have the same number of pivot positions.
5. If two  $m \times n$  matrices have the same pivot positions, then they must be row equivalent.
6. There exist three vectors in  $\mathbb{R}^3$ , no two scalar multiples of each other, whose span is a line through the origin.
7. There exist three vectors in  $\mathbb{R}^3$ , no two scalar multiples of each other, whose span is a plane through the origin.
8. If  $A$  is an  $m \times n$  matrix whose columns span  $\mathbb{R}^m$ , then the equation  $A\vec{x} = \vec{b}$  is consistent for every  $\vec{b} \in \mathbb{R}^m$ .
9. If  $S$  and  $T$  are both linear transformations from  $\mathbb{R}^2$  to  $\mathbb{R}^2$ , then  $S \circ T$  is also a linear transformation from  $\mathbb{R}^2$  to  $\mathbb{R}^2$ .
10. If  $S$  and  $T$  are functions from  $\mathbb{R}^2$  to  $\mathbb{R}^2$  that are both nonlinear, then the composition  $S \circ T$  must also be nonlinear.
11. There is a linear transformation  $T : \mathbb{R} \rightarrow \mathbb{R}$  that sends the interval  $[-1, 1]$  to the interval  $[1, 3]$ .
12. There is a set of 2015 vectors in  $\mathbb{R}^{2011}$  that is linearly independent.
13. The transformation

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+1 \\ y+2 \end{bmatrix}$$

is linear.

14. There is a linear transformation  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^1$  that is one-to-one but *not* onto.

ALWAYS, SOMETIMES, or NEVER

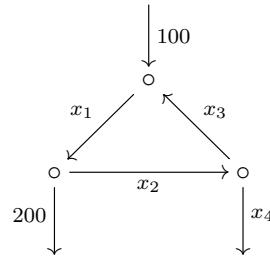
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1. A linear system of  $n$  equations in  $m$  unknowns has an augmented matrix of size  $m \times (n + 1)$ .
2. If  $\vec{b}$  is a vector in  $\text{Span}\{\vec{a}_1, \dots, \vec{a}_p\}$ , then the vector equation  $\vec{b} = x_1\vec{a}_1 + \dots + x_p\vec{a}_p$  has a unique solution.
3. A linear transformation  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  can be represented as a matrix transformation.
4. The function  $A(x) := ax + b$  is a linear transformation from  $\mathbb{R}^1$  to  $\mathbb{R}^1$ .
5. Suppose  $A = [\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3]$  and we know that  $A \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \vec{b}$ . Then there is a solution of the equation
 
$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{b} \end{bmatrix} \vec{x} = \vec{a}_3.$$
6. Suppose  $T : \mathbb{R}^4 \rightarrow \mathbb{R}^4$  is a *one-to-one* linear transformation and  $A$  is its standard matrix. Then every row in  $A$  contains a pivot.
7. If a linear transformation  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  sends some nonzero vector (in  $\mathbb{R}^n$ ) to the zero vector (in  $\mathbb{R}^m$ ), then its standard matrix contains a column of all zeros.
8. The span of a linearly dependent set of vectors  $\{\vec{a}, \vec{b}, \vec{c}\}$  in  $\mathbb{R}^3$  contains two linearly independent vectors.
9. A linear transformation  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  with  $m \times n$  standard matrix  $A$  sends some linearly independent set of vectors to a linearly dependent set of vectors, but the equation  $A\vec{x} = \vec{0}$  has only one solution (namely  $\vec{x} = \vec{0}$ ).
10. A linearly dependent set of vectors  $\{\vec{v}_1, \vec{v}_2\}$  in  $\mathbb{R}^7$  can be made into a linearly independent set  $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$  by adding another (carefully chosen) vector  $\vec{v}_3 \in \mathbb{R}^7$ .

## Short Answer

Expect roughly five questions like these to appear on the exam. Be sure to read each question carefully.

- Suppose we are studying the flow of traffic along a number of one-way streets, as illustrated in the diagram below, where each arrow gives the direction of travel and each label gives the flow along that section of street in vehicles per hour. Find the general solution for the unknown flows  $x_1, x_2, x_3$ , and  $x_4$ , and give any necessary conditions on any free variables that appear to guarantee that all the flows will be positive.



- Find all values of  $h$  and  $k$  such that the vectors

$$\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ h \\ k \end{bmatrix}$$

- (a) span  $\mathbb{R}^3$ ;  
 (b) are linearly independent.
- Suppose  $A$  is the matrix  $\begin{bmatrix} 1 & 3 & 4 \\ 4 & 1 & 3 \\ 2 & 8 & s \end{bmatrix}$  and  $\vec{b}$  is the vector  $\begin{bmatrix} t \\ 3 \\ 2 \end{bmatrix}$ .  
 (a) For which values of  $s, t$  will the system  $A\vec{x} = \vec{b}$  be inconsistent?  
 (b) For which values of  $s, t$  will there be *exactly one* solution to  $A\vec{x} = \vec{b}$ ?
- Suppose  $A$  is a matrix for which it is known that

$$A \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \text{ and } A \begin{bmatrix} 5 \\ 0 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

Find a solution to the equation

$$A\vec{x} = \begin{bmatrix} 5 \\ 2 \\ 7 \end{bmatrix}$$

- Determine by inspection whether or not the given sets of vectors are linearly independent. Justify each answer.

(a)  $\left\{ \begin{bmatrix} 2 \\ -6 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 9 \\ 6 \\ -3 \end{bmatrix} \right\}$

(b)  $\left\{ \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix} \right\}$

(c)  $\left\{ \begin{bmatrix} 4 \\ -1 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -5 \end{bmatrix}, \begin{bmatrix} -1 \\ 7 \\ 5 \end{bmatrix} \right\}$

(d)  $\left\{ \begin{bmatrix} -15 \\ 10 \\ 20 \end{bmatrix}, \begin{bmatrix} 9 \\ -6 \\ 12 \end{bmatrix} \right\}$

6. Suppose  $A$  is an  $m \times n$  matrix. Explain why  $A$  has a pivot in every column if and only if the equation  $A\vec{x} = \vec{0}$  has only the trivial solution.
7. Suppose  $S : \mathbb{R}^5 \rightarrow \mathbb{R}^2$  and  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^5$  are linear transformations. The linear transformation  $(T \circ S) : \mathbb{R}^5 \rightarrow \mathbb{R}^5$  is defined by  $(T \circ S)\vec{x} := T(S\vec{x})$ .
  - (a) Explain how you know that  $S$  is not one-to-one.
  - (b) Explain how you know that  $(T \circ S)$  is not one-to-one.
  - (c) Suppose  $A$  is the standard matrix of the linear transformation  $(T \circ S)$ . How many rows and columns are there in  $A$ ?
  - (d) Can every column of  $A$  contain a pivot?
  - (e) Can every row of  $A$  contain a pivot?
  - (f) Explain how you know that the columns of  $A$  don't span  $\mathbb{R}^5$ .
8. Given  $a, b, c \in \mathbb{R}$ , define a function

$$f_{a,b,c}(t) := ae^t + b \cos t + c \sin t.$$

- (a) Find formulas (in terms of  $a, b, c$ ) for  $p, q, r \in \mathbb{R}$  so that the derivative of  $f_{a,b,c}$  is
$$f'_{a,b,c}(t) = pe^t + q \cos t + r \sin t$$
- (b) Show, using the formulas for  $p, q, r$  you found above, that the transformation  $D$  given by
$$D \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

is linear.

- (c) Find the standard matrix of  $D$ .

9. (a) Suppose  $T : \mathbb{R}^n \rightarrow \mathbb{R}^3$  is a linear transformation and you know

$$T\vec{v} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad T\vec{w} = \begin{bmatrix} 0 \\ 4 \\ 9 \end{bmatrix}.$$

Determine  $T(3\vec{v} - 2\vec{w})$ .

(b) Explain how you know that there is no linear transformation  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that

$$T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad T \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad T \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

10. Find all values (if any) of  $s$  and  $t$  so that, for the matrix,

$$A = \begin{bmatrix} 2 & 0 & 4 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & s & t \end{bmatrix}$$

- (a) the matrix transformation determined by  $A$  is *onto*.
- (b) the matrix transformation determined by  $A$  is *one-to-one*.
- (c) there is exactly one free variable in the matrix equation  $A\vec{x} = \vec{0}$ .