Linear Algebra
Solutions to Practice Problems for Unit 1

TRUE or FALSE

Expect roughly five questions like these to appear on the exam. You will be asked to justify all your answers!

1. If A is a matrix in reduced echelon form, then at least one entry in every column of A must be 1.

   Solution: False.
   For example:
   $$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

2. The matrix
   $$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
   is in reduced echelon form.

   Solution: True.
   Since there are no nonzero rows, there are no rows with leading entries to worry about, and all conditions of the definition are satisfied.

3. No set of two vectors in $\mathbb{R}^3$ can span $\mathbb{R}^3$.

   Solution: True.
   If $\vec{u}, \vec{v} \in \mathbb{R}^3$, then the matrix $A = \begin{bmatrix} \vec{u} & \vec{v} \end{bmatrix}$ has at most one pivot in each column, so no more than two pivots in all. Since there are three rows, at least one does <u>not</u> contain a pivot, so the columns don't span $\mathbb{R}^3$.

4. If two $m \times n$ matrices are row equivalent, then they have the same number of pivot positions.
   Solution: True.
   Since they are row equivalent, they have the same reduced echelon form.

5. If two $m \times n$ matrices have the same pivot positions, then they must be row equivalent.

   Solution: False.
   Suppose $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}$. Both are already in reduced echelon form, and have the same pivot positions. However, any sequence of row operations applied to A will still leave it with a third column of zeros, so A cannot be transformed into B via elementary row operations, and hence A is not row equivalent to B.

6. There exist three vectors in $\mathbb{R}^3$, no two scalar multiples of each other, whose span is a line through the origin.

   Solution: False.
   If $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$, and no two of these vectors are scalar multiples of each other, then in particular the set $\{\vec{u}, \vec{v}\}$ is linearly independent. The span of two linearly independent vectors in $\mathbb{R}^3$ is a plane through the origin, and the plane Span $\{\vec{u}, \vec{v}\}$ is a subset of Span $\{\vec{u}, \vec{v}, \vec{w}\}$.

7. There exist three vectors in $\mathbb{R}^3$, no two scalar multiples of each other, whose span is a plane through the origin.

   Solution: True.
   $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ span the xy-plane in $\mathbb{R}^3$, but no two are scalar multiples of each other.

8. If A is a $m \times n$ matrix whose columns span $\mathbb{R}^m$, then the equation $\mathrm{A}\vec{x} = \vec{b}$ is consistent for every $\vec{b} \in \mathbb{R}^m$.

   Solution: True.
   Let $\vec{a}_1, ..., \vec{a}_n$ be the columns of A, and let $\vec{b}$ be an arbitrary vector in $\mathbb{R}^m$. Since Span $\{\vec{a}_1, ..., \vec{a}_n\} = \mathbb{R}^m$, there exist scalars $c_1, ..., c_n$ such that $c_1\vec{a}_1 + ... + c_n\vec{a}_n = \vec{b}$. It follows that $A\vec{x} = \vec{b}$ has a solution $\vec{x} = (c_1, ..., c_n)$.

9. If S and T are both linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$, then $S \circ T$ is also a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$.

   Solution: True.
   The transformation $(S \circ T)$: $\mathbb{R}^2 \to \mathbb{R}^2$ is defined by $(S \circ T)\vec{x} = S(T\vec{x})$.

   (a) $(S \circ T)(\vec{x} + \vec{y})$
   $= S(T(\vec{x} + \vec{y}))$
   $= S(T(\vec{x}) + T(\vec{y}))$ *(Since T is linear)*
   $= S(T\vec{x}) + S(T\vec{y}))$ *(Since S is linear)*
   $= (S \circ T)\vec{x} + (S \circ T)\vec{y}$

   (b) $(S \circ T)(c\vec{x})$
   $= (S)(T(c\vec{x}))$
   $= (S)(c \cdot T\vec{x})$ *(Since T is linear)*
   $= (cS)(T(\vec{x}))$ *(Since S is linear)*
   $= c(S \circ T)(\vec{x})$

10. If S and T are functions from $\mathbb{R}^2 \to \mathbb{R}^2$ that are both nonlinear, then $S \circ T$ must also be nonlinear.

    Solution: False.
    If $S, T : \mathbb{R} \to \mathbb{R}$ are defined by $S(x) = x + 1$ and $T(x) = x - 1$, then neither S nor T are linear ($S(0) \neq 0$ and $T(0) \neq 0$). However, $(S \circ T)(x) = S(T(x)) = S(x - 1) = (x - 1) + 1 = x$, so that $S \circ T$ is the identity transformation, which *is* linear.

11. There is a linear transformation T: $\mathbb{R} \to \mathbb{R}$ that sends the interval $[-1, 1]$ to the interval $[1, 3]$

    Solution: False.
    A linear transformation $T : \mathbb{R} \to \mathbb{R}$ always has the form $T(x) = ax$, for some scalar a. It follows that T takes the interval $[-1, 1]$ to the interval $[-a, a]$.

12. There is a set of 2015 vectors in $\mathbb{R}^{2011}$ that is linearly independent.

    Solution: False.
    A 2011×2015 matrix cannot have a pivot in each column. If there are more vectors in a set than there are entries in each vector, then the set is linearly dependent.

13. The transformation

    $$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 1 \\ y + 1 \end{bmatrix}$$

    is linear.

    Solution: False.
    The transformation doesn't send $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

14. There is a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^1$ that is one-to-one but not onto.

    Solution: False.
    The standard matrix A of T is a 1×2 matrix, so at least one column is <u>not</u> a pivot column. That means T can't be one-to-one. If $A \neq 0$, then A has a pivot in its row, so in that case T is onto.

ALWAYS, SOMETIMES, or NEVER

Expect roughly five questions like these to appear on the exam. You will be asked to justify all your answers!

1. A linear system of n equations in m unknowns has an augmented matrix of size $m \times (n+1)$.
   Solution: Sometimes
   \# equations = # of rows in coefficient or augmented matrix
   \# unknowns = # columns in coefficient matrix = one less than # columns in augmented matrix
   So the augmented matrix is $\underline{n \times (m+1)}$.
   The question asked if it is $\underline{m \times (n+1)}$, which is true if and only if $m = n$.

2. If $\vec{b}$ is a vector in Span $\{\vec{a}_1, \ldots, \vec{a}_p\}$, then the vector equation $\vec{b} = x_1\vec{a_1} + \ldots + x_p\vec{a_p}$ has a unique solution.

   Solution: Sometimes
   True if $\vec{a}_1, \ldots, \vec{a}_p$ are linearly independent; false otherwise.

3. A linear transformation $T \colon \mathbb{R}^n \to \mathbb{R}^m$ can be represented as a matrix transformation.

   Solution: Always
   Use its standard matrix, for example. The i^{th} column of the standard matrix is $T(\vec{e}_i)$.

4. The function $T(x) = ax + b$ is a linear transformation from $\mathbb{R}^1$ to $\mathbb{R}^1$.

   Solution: Sometimes
   True if $b = 0$. In that case, $T(x) = ax$, and we see that

   (a) $T(x+y) = a(x+y) = ax + ay = T(x) + T(y)$
   (b) $T(cx) = a(cx) = cT(x)$

5. Suppose $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix}$ and we know that $A \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \vec{b}$. Then there is a solution of the equation $\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{b} \end{bmatrix} \vec{x} = \vec{a}_3$.

   Solution: Always
   $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix}$, so $A \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \vec{b}$ means $\vec{a}_1 + 2\vec{a}_2 + 3\vec{a}_3 = \vec{b}$. Equivalently, $\vec{a}_1 + 2\vec{a}_2 - \vec{b} = -3\vec{a}_3$. Therefore, $-\frac{1}{3}\vec{a}_1 - \frac{2}{3}\vec{a}_2 + \frac{1}{3}\vec{b} = \vec{a}_3$, so $\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{b} \end{bmatrix} \begin{bmatrix} -1/3 \\ -2/3 \\ 1/3 \end{bmatrix} = \vec{a}_3$.

6. Suppose $T : \mathbb{R}^4 \to \mathbb{R}^4$ is a *one-to-one* linear transformation and A is its standard matrix. Then every row in A contains a pivot.

   Solution: Always
   Since T is one-to-one, every column of A contains a pivot. Since A is 4×4, this means that every row of A has a pivot as well.

7. If a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ sends some nonzero vector (in $\mathbb{R}^n$) to the zero vector (in $\mathbb{R}^m$), then its standard matrix contains a column of all zeros.

   Solution: Sometimes

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}}_{\text{no columns of all zeros}} \times \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{\text{nonzero vector}} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\text{zero vector}}$$

$$\underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\text{columns of all zeros}} \times \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{\text{nonzero vector}} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\text{zero vector}}$$

8. The span of a linearly dependent set of vectors $\{\vec{a},\vec{b},\vec{c}\}$ in $\mathbb{R}^3$ contains two linearly independent vectors.

   Solution: Sometimes

   $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a linearly dependent set containing the linearly independent vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

   On the other hand, Span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \right\}$ doesn't contain two linearly independent vectors.

9. A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ with $m \times n$ standard matrix A sends some linearly independent set of vectors to a linearly dependent set of vectors, but the equation $A\vec{x} = \vec{0}$ has only one solution (namely $\vec{x} = \vec{0}$).

   Solution: Never

   Suppose $\{\vec{u}_1, \cdots, \vec{u}_k\}$ is a linearly independent set, but $\{A\vec{u}_1, \cdots, A\vec{u}_k\}$ is linearly dependent. Then there is a choice of $c_1, \ldots, c_k \in \mathbb{R}$ (not all equal to zero) such that $c_1 A\vec{u}_1 + \ldots + c_k A\vec{u}_k = \vec{0}$ (this is from the definition of linear dependence). But multiplication by A is linear, so $A(c_1\vec{u}_1 + \cdots c_k\vec{u}_k) = c_1 A\vec{u}_1 + \cdots + c_k A\vec{u}_k = \vec{0}$, and since $\{\vec{u}_1, \cdots, \vec{u}_k\}$ is linearly independent (and at least one of $c_1, \ldots, c_k$ is nonzero), we know that $(c_1\vec{u}_1 + \cdots c_k\vec{u}_k) \neq \vec{0}$. Thus we see that the equation $A\vec{x} = \vec{0}$ has a nontrivial solution $\vec{x} = c_1\vec{u}_1 + \cdots + c_k\vec{u}_k$.

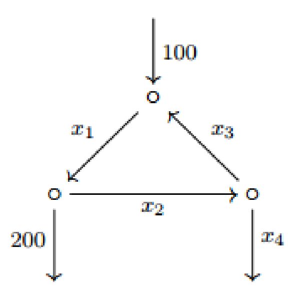
10. A linearly dependent set of vectors $\{\vec{v}_1, \vec{v}_2\}$ in $\mathbb{R}^7$ can be made into a linearly independent set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ by adding another (carefully chosen) vector $\vec{v}_3 \in \mathbb{R}^7$.

    Solution: Never

    Since $\{\vec{v}_1, \vec{v}_2\}$ are linearly dependent, we can find $c_1, c_2 \in \mathbb{R}$ (not both zero) such that $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$. No matter what $\vec{v}_3$ is, we can write $\vec{0}$ as a nontrivial linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$ by writing $c_1\vec{v}_1 + c_2\vec{v}_2 + 0\vec{v}_3 = \vec{0}$.

Short Answers

1. Suppose we are studying the flow of traffic along a number of one-way streets, as illustrated in the diagram below, where each arrow gives the direction of travel and each label gives the flow along that section of street in vehicles per hour. Find the general solution for the unknown flows x1, x2, x3, and x4, and give any necessary conditions on any free variables that appear to guarantee that all the flows will be positive.



Solution: Each node (vertex) in the diagram gives us a linear equation.

	No. in	No. out
A:	$x_3 + 100$	$= x_1$
B:	x_1	$= x_2 + 200$
C:	x_2	$= x_3 + x_4$

Also, the total flow into the network equals the total flow out ($100 = 200 + x_4$), so we want to solve the linear system

$$\begin{array}{rcrcrcrcr} x_1 & & & - & x_3 & & & = & 100 \\ x_1 & - & x_2 & & & & & = & 200 \\ & & x_2 & - & x_3 & - & x_4 & = & 0 \\ & & & & & & x_4 & = & -100 \end{array}$$

Row reduce the augmented matrix of the system:

$$\left[\begin{array}{rrrr|r} 1 & 0 & -1 & 0 & 100 \\ 1 & -1 & 0 & 0 & 200 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -100 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{rrrr|r} 1 & 0 & -1 & 0 & 100 \\ 0 & -1 & 1 & 0 & 100 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -100 \end{array}\right] \xrightarrow{-R_2}$$

$$\left[\begin{array}{rrrr|r} 1 & 0 & -1 & 0 & 100 \\ 0 & 1 & -1 & 0 & -100 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -100 \end{array}\right] \xrightarrow{R_3 - R_2} \left[\begin{array}{rrrr|r} 1 & 0 & -1 & 0 & 100 \\ 0 & 1 & -1 & 0 & -100 \\ 0 & 0 & 0 & -1 & 100 \\ 0 & 0 & 0 & 1 & -100 \end{array}\right] \xrightarrow{R_4 + R_3}$$

$$\left[\begin{array}{rrrr|r} 1 & 0 & -1 & 0 & 100 \\ 0 & 1 & -1 & 0 & -100 \\ 0 & 0 & 0 & -1 & 100 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-R_3} \left[\begin{array}{rrrr|r} 1 & 0 & -1 & 0 & 100 \\ 0 & 1 & -1 & 0 & -100 \\ 0 & 0 & 0 & 1 & -100 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Therefore,

$$\begin{aligned} x_1 &= 100 + x_3 \\ x_2 &= -100 + x_3 \\ x_3 &= \text{ free} \\ x_4 &= -100 \end{aligned}$$

Unfortunately, x_4 is <u>always</u> negative, so there are no choices for x_3 that make all flows positive.

2. Find all values of h and k such that the vectors

$$\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ h \\ k \end{bmatrix}$$

(a) span $\mathbb{R}^3$;

(b) are linearly independent.

Solution: The columns of a 3×3 matrix A span $\mathbb{R}^3$ if and only if there is a pivot in every row of A. The columns are linearly independent if and only if there is a pivot in every column. Row reducing A gives

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & 1 & h \\ 1 & 1 & k \end{bmatrix} \xrightarrow{-R_1} \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & h \\ 1 & 1 & k \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & h+1 \\ 1 & 1 & k \end{bmatrix} \xrightarrow{R_3 - R_1}$$

$$\begin{bmatrix}1 & -1 & -1\\ 0 & 2 & h+1\\ 0 & 2 & k+1\end{bmatrix} \xrightarrow{R_3-R_2} \begin{bmatrix}1 & -1 & -1\\ 0 & 2 & h+1\\ 0 & 0 & k-h\end{bmatrix}$$

which is in echelon form. The diagonal numbers 1 and 2 are pivots, while $k-h$ is a pivot if and only if $k \neq h$.

(a) The vectors span $\mathbb{R}^3$ if and only if $\underline{k \neq h}$ (so every row contains a pivot).

(b) The vectors are linearly independent if and only if $\underline{k \neq h}$ (so every column contains a pivot).

3. Suppose A is the matrix $\begin{bmatrix}1 & 3 & 4\\ 4 & 1 & 3\\ 2 & 8 & s\end{bmatrix}$ and $\vec{b}$ is the vector $\begin{bmatrix}t\\ 3\\ 2\end{bmatrix}$

   (a) For which values of s, t will the system $A\vec{x} = \vec{b}$ be inconsistent?

   (b) For which values of s, t will there be exactly one solution to $A\vec{x} = \vec{b}$?

Solution:

(a) The equation $A\vec{x} = b$ is inconsistent if and only if the rightmost column of the augmented matrix $(A \,|\, \vec{b})$ is not a pivot column. If $s \neq 114/11$, then row reduction of the augmented matrix goes like this:

$$\left[\begin{array}{ccc|c}1 & 3 & 4 & t\\ 4 & 1 & 3 & 3\\ 2 & 8 & s & 2\end{array}\right] \xrightarrow{\text{swap R1 and R2}} \left[\begin{array}{ccc|c}4 & 1 & 3 & 3\\ 1 & 3 & 4 & t\\ 2 & 8 & s & 2\end{array}\right] \xrightarrow{R_2-\frac{1}{4}R_1}$$

$$\left[\begin{array}{ccc|c}4 & 1 & 3 & 3\\ 0 & 11/4 & 13/4 & -3/4+t\\ 2 & 8 & s & 2\end{array}\right] \xrightarrow{4\cdot R_2} \left[\begin{array}{ccc|c}4 & 1 & 3 & 3\\ 0 & 11 & 13 & -3+4t\\ 2 & 8 & s & 2\end{array}\right] \xrightarrow{R3-\frac{1}{2}\cdot R1}$$

$$\left[\begin{array}{ccc|c}4 & 1 & 3 & 3\\ 0 & 11 & 13 & -3+4t\\ 0 & 15/2 & -3/2+s & 1/2\end{array}\right] \xrightarrow{2\cdot R3} \left[\begin{array}{ccc|c}4 & 1 & 3 & 3\\ 0 & 11 & 13 & -3+4t\\ 0 & 15 & -3+2s & 1\end{array}\right] \xrightarrow{\text{swap } R2 \text{ and } R3}$$

$$\left[\begin{array}{ccc|c}4 & 1 & 3 & 3\\ 0 & 15 & -3+2s & 1\\ 0 & 11 & 13 & -3+4t\end{array}\right] \xrightarrow{R3-\frac{11}{15}R2} \left[\begin{array}{ccc|c}4 & 1 & 3 & 3\\ 0 & 15 & -3+2s & 1\\ 0 & 0 & 76/5-22s/15 & -56/15+4t\end{array}\right] \xrightarrow{\frac{15}{2}\cdot R3}$$

$$\left[\begin{array}{ccc|c}4 & 1 & 3 & 3\\ 0 & 15 & -3+2s & 1\\ 0 & 0 & 114-11s & -28+30t\end{array}\right] \xrightarrow{-\frac{-3+2s}{114-11s}R3+R2} \left[\begin{array}{ccc|c}4 & 1 & 3 & 3\\ 0 & 15 & 0 & \frac{30+45s+90t-60st}{114-11s}\\ 0 & 0 & 114-11s & -28+30t\end{array}\right] \xrightarrow{R1-\frac{-3}{114-11s}R3}$$

$$\left[\begin{array}{ccc|c}4 & 1 & 0 & 3+\frac{84-90t}{114-11s}\\ 0 & 15 & 0 & \frac{30+45s+90t-60st}{114-11s}\\ 0 & 0 & 114-11s & -28+30t\end{array}\right] \xrightarrow{R1-\frac{1}{15}\cdot R2} \left[\begin{array}{ccc|c}4 & 0 & 0 & \frac{-424-4s(-9+t)+96t}{-114+11s}\\ 0 & 15 & 0 & \frac{30+45s+90t-60st}{114-11s}\\ 0 & 0 & 114-11s & -28+30t\end{array}\right] \xrightarrow{\frac{1}{4}\cdot R1}$$

$$\left[\begin{array}{ccc|c}1 & 0 & 0 & \frac{106+s(-9+t)-24t}{114-11s}\\ 0 & 15 & 0 & \frac{30+45s+90t-60st}{114-11s}\\ 0 & 0 & 114-11s & -28+30t\end{array}\right] \xrightarrow{\frac{1}{15}\cdot R2} \left[\begin{array}{ccc|c}1 & 0 & 0 & \frac{106+s(-9+t)-24t}{114-11s}\\ 0 & 1 & 0 & \frac{2+3s+6t-4st}{114-11s}\\ 0 & 0 & 114-11s & -28+30t\end{array}\right] \xrightarrow{1/(114-11s)\cdot R3}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{106+s(-9+t)-24t}{114-11s} \\ 0 & 1 & 0 & \frac{2+3s+6t-4st}{114-11s} \\ 0 & 0 & 1 & \frac{-28+30t}{114-11s} \end{array}\right]$$

If $s \neq 114/11$, then the final column of the augmented matrix has no pivot, so the system $A\vec{x} = \vec{b}$ is consistent. If $s = 114/11$, then we stop the row reduction at this step:

$$\left[\begin{array}{ccc|c} 4 & 1 & 3 & 3 \\ 0 & 15 & -3+2s & 1 \\ 0 & 0 & 114-11s & -28+30t \end{array}\right] = \left[\begin{array}{ccc|c} 4 & 1 & 3 & 3 \\ 0 & 15 & -3+2s & 1 \\ 0 & 0 & 0 & -28+30t \end{array}\right]$$

In this case, we see that $A\vec{x} = \vec{b}$ is consistent if and only if $t = 14/15$.

(b) There is exactly one solution to $A\vec{x} = \vec{b}$ if the system is consistent <u>and</u> every column of A contains a pivot. This happens if and only if $s \neq 114/11$.

4. Suppose A is a matrix for which it is known that

$$A \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ and } A \begin{bmatrix} 5 \\ 0 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

Find a solution to the equation

$$A\vec{x} = \begin{bmatrix} 5 \\ 2 \\ 7 \end{bmatrix}$$

Solution: We know that matrix transformations are linear. Let's see if we can find $c_1, c_2 \in \mathbb{R}$ so that

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 7 \end{bmatrix}$$

If we can do this then:

$$\begin{bmatrix} 5 \\ 2 \\ 7 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

$$= c_1 A \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \end{bmatrix} + c_2 A \begin{bmatrix} 5 \\ 0 \\ 4 \\ 0 \end{bmatrix}$$

$$= A \underbrace{\left(c_1 \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 0 \\ 4 \\ 0 \end{bmatrix} \right)}_{\text{solution to } A\vec{x} = \begin{bmatrix} 5 \\ 2 \\ 7 \end{bmatrix}}$$

We find:

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 2 & 2 \\ 1 & 2 & 7 \end{bmatrix} \xrightarrow{R_3 - R_1} \begin{bmatrix} 1 & 0 & 5 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 0 & 5 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{array}{l} c_1 = 5 \\ c_2 = 1 \end{array}$$

Substituting these values:

$$5 \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 5 \\ 0 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 20 \\ 10 \\ 9 \\ 5 \end{bmatrix}$$

so

$$A \begin{bmatrix} 20 \\ 10 \\ 9 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 7 \end{bmatrix}$$

5. Determine by inspection whether or not the given sets of vectors are linearly independent. Justify each answer.

(a) $\left\{ \begin{bmatrix} 2 \\ -6 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 9 \\ 6 \\ -3 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix} \right\}$

(c) $\left\{ \begin{bmatrix} 4 \\ -1 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} -1 \\ 7 \\ 5 \end{bmatrix} \right\}$

(d) $\left\{ \begin{bmatrix} -15 \\ 10 \\ 20 \end{bmatrix}, \begin{bmatrix} 9 \\ -6 \\ 12 \end{bmatrix} \right\}$

Solution:

(a) Linearly dependent, because $\vec{v}_2 = -\frac{3}{2}\vec{v}_1$.

(b) Linearly dependent, because any set containing the zero vector is linearly dependent.

(c) Linearly dependent, because any set of four or more vectors in $\mathbb{R}^3$ is linearly dependent.

(d) Linearly independent, because the two vectors are not multiples of one another. (This reasoning only works for sets of two vectors.)

6. Suppose A is a $m \times n$ matrix. Explain why A has a pivot in every column if and only if the equation $A\vec{x} = \vec{0}$ has only the trivial solution.

   Solution: Let A be an $m \times n$ matrix. If every column of A contains a pivot, then the pivot columns of the $m \times (n+1)$ augmented matrix $(A \,|\, \vec{0})$ are the first n columns. That means there are no free variables, so the only solution to the equation

$$A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \vec{0}$$

   is the zero vector.

   On the other hand, if the only solution to $A\vec{x} = \vec{0}$ is the trivial solution, then there cannot be any free variables in the linear system corresponding to $(A|\vec{0})$. That means every column in the coefficient matrix A must contain a pivot.

7. Suppose $S : \mathbb{R}^5 \to \mathbb{R}^2$ and $T : \mathbb{R}^2 \to \mathbb{R}^5$ are linear transformations. The linear transformation $(T \circ S) : \mathbb{R}^5 \to \mathbb{R}^5$ is defined by $(T \circ S)\vec{x} := T(S\vec{x})$.

   (a) Explain how you know that S is not one-to-one.

   (b) Explain how you know that $(T \circ S)$ is not one-to-one.

   (c) Suppose A is the standard matrix of the linear transformation $(T \circ S)$. How many rows and columns are there in A?

   (d) Can every column of A contain a pivot?

   (e) Can every row of A contain a pivot?

   (f) Explain how you know that the columns of A don't span $\mathbb{R}^5$.

   Solution:

   $S : \mathbb{R}^5 \to \mathbb{R}^2$ is linear. Call its standard matrix B.
   $T \circ S : \mathbb{R}^5 \to \mathbb{R}^5$ is also linear. Call its standard matrix A.

   (a) B is a 2×5 matrix. Each row contains at most one pivot, so it has at most 2 pivots. There are 5 columns, so not every column contains a pivot. This means the equation $B\vec{x} = \vec{0}$ has at least one free variable (so it has <u>infinitely many</u> solutions). That means there are infinitely many vectors that all get sent to the same vector (namely $\vec{0}$) by T.

   (b) From (a) we know that S is not one-to-one, so there is a nonzero vector $\vec{v}$ such that $S\vec{v} = \vec{0}$. Since T is linear, it sends the zero vector in $\mathbb{R}^2$ to the zero vector in $\mathbb{R}^5$. Thus,

$$(T \circ S)\vec{v} = T(S\vec{v}) = T(\vec{0}) = \vec{0}$$

   Since $T \circ S$ sends a nonzero vector to $\vec{0}$, it isn't one-to-one.

   (c) A is a 5×5 matrix.

   (d) $T \circ S$ is not one-to-one, so its standard matrix A has columns that do not contain pivots.

   (e) Since A is a 5×5 matrix and not every column contains a pivot, there are no more than 4 pivots. There are 5 rows, so not every row contains a pivot.

   (f) Not every row of A contains a pivot, so $T \circ S$ is not onto.

8. Given $a, b, c \in \mathbb{R}$, define a function

$$f_{a,b,c}(t) := ae^t + b\cos t + c\sin t$$

   (a) Find formulas (in terms of a, b, c) for $p, q, r \in \mathbb{R}$ so that the derivative of $f_{a,b,c}$ is

$$f'_{a,b,c}(t) := pe^t + q\cos t + r\sin t$$

(b) Show, using the formulas for p, q, r you found above, that the transformation D given by

$$D\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

is linear.

(c) Find the standard matrix of D.

Solution: $f_{a,b,c}(t) = a \cdot e^t + b \cdot \cos(t) + c \cdot \sin(t)$

(a) The derivative is

$$\begin{aligned} f'_{a,b,c}(t) &= ae^t - b\sin t + c\cos t \\ &= ae^t + c\cos t - b\sin t \end{aligned}$$

Therefore, $p = a$, $q = c$, and $r = -b$.

(b) $D\left(\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}\right) = D\begin{bmatrix} a_1 + a_2 \\ b_1 + b_2 \\ c_1 + c_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 \\ c_1 + c_2 \\ -b_1 - b_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ c_1 \\ -b_1 \end{bmatrix} + \begin{bmatrix} a_2 \\ c_2 \\ -b_2 \end{bmatrix} = D\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} + D\begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$

$$D\left(k\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = D\begin{bmatrix} ka \\ kb \\ kc \end{bmatrix} = \begin{bmatrix} ka \\ kc \\ -kb \end{bmatrix} = k\begin{bmatrix} a \\ c \\ -b \end{bmatrix} = kD\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

(c) We see that

$$D\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad D\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \qquad D\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

so the standard matrix of D is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$.

9. (a) Suppose $T : \mathbb{R}^n \to \mathbb{R}^3$ is a linear transformation and you know

$$T\vec{v} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad T\vec{w} = \begin{bmatrix} 0 \\ 4 \\ 9 \end{bmatrix}$$

Determine $T(3\vec{v} - 2\vec{w})$.

(b) Explain how you know that there is no linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$T\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad T\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad T\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Solution:

(a) Since T is linear,

$$T(3\vec{v} - 2\vec{w}) = 3T(\vec{v}) - 2T(\vec{w}) = 3\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 9 \\ -5 \\ -18 \end{bmatrix}$$

(b) If $T : \mathbb{R}^3 \to \mathbb{R}^3$ is linear and

$$T\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}2\\1\\2\end{bmatrix}, \quad T\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}3\\1\\1\end{bmatrix}$$

then

$$T\begin{bmatrix}1\\1\\1\end{bmatrix} = T\left(\begin{bmatrix}1\\0\\0\end{bmatrix} + \begin{bmatrix}0\\1\\1\end{bmatrix}\right) = T\begin{bmatrix}1\\0\\0\end{bmatrix} + T\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}2\\1\\2\end{bmatrix} + \begin{bmatrix}3\\1\\1\end{bmatrix} = \begin{bmatrix}5\\2\\3\end{bmatrix}$$

Therefore, we cannot have $T\begin{bmatrix}1\\1\\1\end{bmatrix} \neq \begin{bmatrix}1\\0\\0\end{bmatrix}$.

10. Find all values (if any) of s and t such that, for the matrix

$$A = \begin{bmatrix}2 & 0 & 4 & 0\\0 & 1 & 1 & 0\\1 & 1 & s & t\end{bmatrix}$$

(a) the matrix transformation determined by A is *onto*.

(b) the matrix transformation determined by A is *one-to-one*.

(c) there is exactly one free variable in the matrix equation $A\vec{x} = \vec{0}$.

Solution:

(a) The transformation is onto iff every row of A contains a pivot.

$$\begin{bmatrix}2 & 0 & 4 & 0\\0 & 1 & 1 & 0\\1 & 1 & s & t\end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix}1 & 0 & 2 & 0\\0 & 1 & 1 & 0\\1 & 1 & s & t\end{bmatrix} \xrightarrow{R_3 - R_1}$$

$$\begin{bmatrix}2 & 0 & 4 & 0\\0 & 1 & 1 & 0\\0 & 1 & s-2 & t\end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix}1 & 0 & 2 & 0\\0 & 1 & 1 & 0\\0 & 0 & s-3 & t\end{bmatrix}$$

The diagonal elements 1 and 1 are pivots, while one of $(s-3)$ or t may be a pivot, depending on the values of s and t. Every row contains a pivot if and only if either $\boxed{s \neq 3\,(t \text{ may take any value})}$ or $\boxed{s = 3, t \neq 0}$.

(b) The transformation is one-to-one if and only if every column of A contains a pivot. Because A has four columns but at most three pivots, this never happens.

(c) This happens if and only if A has three pivots, so the answer is the same as in part (a): either $\boxed{s \neq 3\,(t \text{ may take any value})}$ or $\boxed{s = 3, t \neq 0}$