Directions

This is the Practice Final, to be used as practice for the graded final exam. On the final exam, you will have 180 minutes (3 hours) to complete the exam and will not be permitted to use a calculator, notes, your textbook, or other resources. It is highly recommended that you take this practice final under the same conditions. Clearly explain your reasoning using relevant linear algebra techniques so a reader would not have to guess what you mean. When done, check your work against the solutions provided in the course.

1. (20 points, 4 points each) Clearly write the word **TRUE** or **FALSE** next to each statement. Give a brief justification for your answer.

   (a) A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A is one-to-one if A has a pivot in every row.

   (b) For $n \geq 1$, $\mathbb{R}_n$ is a subspace of $\mathbb{R}^{n+1}$.

   (c) If $Q(x_1, x_2)$ is a quadratic form, then $Q(x_1, x_2) = 1$ is an equation of either an ellipse or a hyperbola.

   (d) If $\det(A - 3I) \neq 0$, then 3 is an eigenvalue of A.

   (e) Fix a vector $\vec{v}$ in $\mathbb{R}^n$. If $\vec{v} \cdot \vec{w} = 0$ for all vectors $\vec{w}$ in $\mathbb{R}^n$, then $\vec{v} = 0$.

2. (20 points, 4 points each) Are the following statements ALWAYS, SOMETIMES, or NEVER true? Clearly write **ALWAYS**, **SOMETIMES**, or **NEVER** next to each, and briefly justify all your answers.

   (a) An eigenvalue of A is also an eigenvalue of A^2.

   (b) The image of a square of area 2 under the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with standard matrix $A = \begin{bmatrix} 2 & 3 \\ 5 & 2 \end{bmatrix}$ is a parallelogram of area 22.

   (c) If A is row equivalent to B, then Col A = Col B.

   (d) For any invertible matrix A, $\det A^{-1} = \det A$.

   (e) If A is a square matrix, then $|\det A|$ is the product of the singular values of A.

3. (6 points) Solve the system of linear equations below. Express the solution set in parametric vector form.

$$\begin{array}{rcrcrcrcr} -x_1 & - & x_2 & & & + & 2x_4 & = & 3 \\ 2x_1 & + & 2x_2 & + & x_3 & - & 2x_4 & = & -1 \\ -x_1 & - & x_2 & + & 2x_3 & + & 6x_4 & = & 13 \end{array}$$

4. (6 points total) Let $A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 1 & 1 & 3 & 3 \\ 2 & 1 & 5 & 6 \end{bmatrix}$.

(a) (4 points) Find a basis of Col A.

(b) (2 points) What is dim Nul A?

5. (4 points) Find the determinant of $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 3 & 0 & 3 & 0 \\ -1 & 2 & 2 & 1 \end{bmatrix}$.

6. (6 points) Suppose A is a positive definite matrix and $A = U\Sigma V^T$ is a singular value decomposition of A. Show that $U = V$ and Σ consists of the eigenvalues of A.

7. (8 points total) For each matrix below, either find a diagonalization (that is, find P and D such that $A = PDP^{-1}$), or show that the matrix cannot be diagonalized.

   (a) (4 points) $A = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 3 & 0 \\ 2 & 1 & -4 \end{bmatrix}$

   (b) (4 points) $A = \begin{bmatrix} -1 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & -4 \end{bmatrix}$

8. (6 points total)

(a) (4 points) Find the orthogonal projection of $\begin{bmatrix}1\\1\\1\end{bmatrix}$ onto Span $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\1\end{bmatrix}\right\}$.

(b) (2 points) Find the least-squares solution of $\begin{bmatrix}1 & -1\\0 & 1\\1 & 1\end{bmatrix}\mathbf{x} = \begin{bmatrix}1\\1\\1\end{bmatrix}$.

9. (5 points) Find an orthonormal basis of the subspace of $\mathbb{R}^4$ spanned by $\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -6 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ 18 \\ -4 \end{bmatrix} \right\}$.

10. (6 points total)

(a) (3 points) Show that if a matrix A is diagonalizable and invertible, then A^{-1} is diagonalizable.

(b) (3 points) Show that for any vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$, the equation below (the Polar Identity) holds. (In this formula, $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y}$.)

$$\langle \mathbf{x}, \mathbf{y} \rangle = \frac{1}{4}\|\mathbf{x} + \mathbf{y}\|^2 - \frac{1}{4}\|\mathbf{x} - \mathbf{y}\|^2$$

11. (7 points total) Let $Q(x_1, x_2) = x_1^2 + 6x_1x_2 + x_2^2$.

   (a) (2 points) Find the maximum value of $Q(\mathbf{x})$, where $\mathbf{x}$ is a unit vector.

   (b) (3 points) Make a change of variable $\mathbf{x} = P\mathbf{y}$ to eliminate the cross-product term in Q.

   (c) (2 points) Make a rough sketch of the curve $Q(\mathbf{x}) = 1$ in the x_1x_2-plane.

12. (6 points) Find a singular value decomposition of $A = \begin{bmatrix} 1 & 1 \\ 2 & -2 \\ 0 & 0 \end{bmatrix}$.