1. (20 points, 4 points each) Clearly write the word **TRUE** or **FALSE** next to each statement. Give a brief justification for your answer.

   (a) A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A is one-to-one if A has a pivot in every row.

   False. T is one-to-one if A has a pivot in every *column*. T is *onto* if A has a pivot in every row.

   (b) For $n \geq 1$, $\mathbb{R}_n$ is a subspace of $\mathbb{R}^{n+1}$.

   False. $\mathbb{R}_n$ is not a subset of $\mathbb{R}^{n+1}$, so is not a subspace.

   (c) If $Q(x_1, x_2)$ is a quadratic form, then $Q(x_1, x_2) = 1$ is an equation of either an ellipse or a hyperbola.

   False. If Q is negative definite, then no (x_1, x_2) satisfies $Q(x_1, x_2) = 1$. Example: $Q(x_1, x_2) = -x_1^2 - x_2^2$.

   (d) If $\det(A - 3I) \neq 0$, then 3 is an eigenvalue of A.

   False. A number λ is an eigenvalue of A if and only if $A - \lambda I$ is noninvertible. By the Invertible Matrix Theorem, $A - 3I$ is noninvertible if and only if $\det(A - 3I) \neq 0$. It follows that 3 is *not* an eigenvalue of A.

   (e) Fix a vector $\vec{v}$ in $\mathbb{R}^n$. If $\vec{v} \cdot \vec{w} = 0$ for all vectors $\vec{w}$ in $\mathbb{R}^n$, then $\vec{v} = \vec{0}$.

   True. In particular, if we take $\vec{w} = \vec{v}$, then we get $\vec{v} \cdot \vec{v} = 0$. This implies $\vec{v} = \vec{0}$.

2. (20 points, 4 points each) Are the following statements ALWAYS, SOMETIMES, or NEVER true? Clearly write **ALWAYS**, **SOMETIMES**, or **NEVER** next to each, and briefly justify all your answers.

(a) An eigenvalue of A is also an eigenvalue of A^2.

Sometimes. If $A\mathbf{x} = \lambda\mathbf{x}$, then $A^2\mathbf{x} = \lambda^2\mathbf{x}$. It follows that if λ is an eigenvalue of A, then λ^2 is an eigenvalue of A^2. If λ equals 0 or 1, then $\lambda^2 = \lambda$, and λ is an eigenvalue of A^2. Otherwise, λ may not be an eigenvalue of A^2. Example: $A = 2I$, with $\lambda = 2$.

(b) The image of a square of area 2 under the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with standard matrix $A = \begin{bmatrix} 2 & 3 \\ 5 & 2 \end{bmatrix}$ is a parallelogram of area 22.

Always. Since $\det A = -11$, a square of area 2 transforms to a parallelogram of area $2\,|\det A| = 22$.

(c) If A is row equivalent to B, then $\text{Col } A = \text{Col } B$.

Sometimes. True if $A = B$; false if $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

(d) For any invertible matrix A, $\det A^{-1} = \det A$.

Sometimes. In general, $\det A^{-1} = (\det A)^{-1}$. If $\det A$ equals 1 or -1, then $\det A^{-1} = \det A$.

(e) If A is a square matrix, then $|\det A|$ is the product of the singular values of A.

Always. Write A as $A = U\Sigma V$. Then $|\det(A)| =| \det(U\Sigma V)| = |\det A|\cdot|\det(\Sigma)|\cdot|\det(V)|$ by properties of determinant and absolute value. Since U and V are orthogonal, their determinant is ± 1 and the absolute value of their determinant is then 1. Simplifying the expression then gives $|\det(A)| = |\det(\Sigma)|$. Since Σ is a diagonal matrix, its determinant is the product of the diagonal entries. The diagonal entries are the singular values, with non

3. (6 points) Solve the system of linear equations below. Express the solution set in parametric vector form.

$$\begin{array}{rcrcrcrcr} -x_1 & - & x_2 & & & + & 2x_4 & = & 3 \\ 2x_1 & + & 2x_2 & + & x_3 & - & 2x_4 & = & -1 \\ -x_1 & - & x_2 & + & 2x_3 & + & 6x_4 & = & 13 \end{array}$$

Solution

Find the reduced echelon form of the augmented matrix of the linear system:

$$\begin{bmatrix} -1 & -1 & 0 & 2 & 3 \\ 2 & 2 & 1 & -2 & -1 \\ -1 & -1 & 2 & 6 & 13 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & -2 & -3 \\ 2 & 2 & 1 & -2 & -1 \\ -1 & -1 & 2 & 6 & 13 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & -2 & -3 \\ 0 & 0 & 1 & 2 & 5 \\ 0 & 0 & 2 & 4 & 10 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & -2 & -3 \\ 0 & 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We see that x_2 and x_4 are free, and the solutions are:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_2 + 2x_4 - 3 \\ x_2 \\ -2x_4 + 5 \\ x_4 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ 0 \\ 5 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

4. (6 points total) Let $A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 1 & 1 & 3 & 3 \\ 2 & 1 & 5 & 6 \end{bmatrix}$.

(a) (4 points) Find a basis of Col A.

Solution

The pivot columns of A form a basis of Col A. Row reduce A to find its pivot entries:

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 1 & 1 & 3 & 3 \\ 2 & 1 & 5 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We see that columns 1 and 2 are the pivot columns. A basis of Col A is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

(b) (2 points) What is dim Nul A?

Solution

By the Rank Theorem,

$$\text{dim Col } A + \text{dim Nul } A = \text{\# of columns of } A$$

We saw that dim Col $A = 2$, and A has 4 columns, so it follows that dim Nul $A = 2$.

5. (4 points) Find the determinant of $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 3 & 0 & 3 & 0 \\ -1 & 2 & 2 & 1 \end{bmatrix}$.

Solution

The easiest method is to use expansion across the second row, since the second row only has one nonzero entry:

$$\det A = 2 \cdot (-1)^{2+3} \begin{vmatrix} 1 & 1 & 1 \\ 3 & 0 & 0 \\ -1 & 2 & 1 \end{vmatrix}$$

To evaluate the determinant of the 3×3 matrix above, again expand across the second row:

$$\begin{vmatrix} 1 & 1 & 1 \\ 3 & 0 & 0 \\ -1 & 2 & 1 \end{vmatrix} = 3 \cdot (-1)^{2+1} \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 3$$

It follows that $\det A = -6$.

6. (6 points) Suppose A is a positive definite matrix and $A = U\Sigma V^T$ is a singular value decomposition of A. Show that $U = V$ and Σ consists of the eigenvalues of A

Solution

First show that Σ consists of the eigenvalues of A: Since A is given to be positive definite, A is a symmetric matrix (and thus square) with positive eigenvalues. Since A is symmetric, it is orthogonally diagonalizable as $A = PDP^T$, where P is orthogonal and D is diagonal consisting of the eigenvalues λ_i of A. P consists of the unit orthonormal eigenvectors of A. Looking also at the singular value decomposition, the singular values are the square roots of the eigenvalues of $A^TA = A^2 = PD^2P^T$. The diagonal matrix D^2 consists of λ_i^2. Then $\sigma_i = \sqrt{\lambda_i^2} = |\lambda_i| = \lambda_i$ since A is positive definite and thus $\lambda_i > 0$. The diagonal matrix D is then equal to Σ.

Now show that $U = V$: The unit eigenvectors of $AA^T = A^TA = A^2$, which are also the unit eigenvectors of A. This implies the right singular vectors (columns of V) which are the unit eigenvectors of A^TA are equal to the left singular vectors (columns of U), which are the unit eigenvectors of AA^T.

Note that the columns of P, which are also unit eigenvectors are either equal to the vectors in U and V or -1 times those vectors.

Note also that if A is positive *semi*-definite, with the possibility that and entry in Σ could be zero, then but there is no guarantee that $U = V$. Indeed the columns of U and V corresponding to the zero eigenvalues could be any orthonormal decomposition of the null space of A, with sign flips allowed independently on U and V.

7. (8 points total) For each matrix below, either find a diagonalization (that is, find P and D such that $A = PDP^{-1}$), or show that the matrix cannot be diagonalized.

   (a) (4 points) $A = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 3 & 0 \\ 2 & 1 & -4 \end{bmatrix}$

   Solution

   The matrix A is lower triangular, so its eigenvalues are its diagonal entries: $\lambda = -1, 3, -4$. Since the eigenvalues are distinct, A is diagonalizable. The columns of P are eigenvectors of A. For each eigenvalue λ, we find a corresponding eigenvector by row reducing the augmented matrix of the system $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

   $$\underline{\lambda_1 = -1}: \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 2 & 1 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -12/7 & 0 \\ 0 & 1 & 3/7 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{v}_1 = \begin{bmatrix} 12 \\ -3 \\ 7 \end{bmatrix}$$

   $$\underline{\lambda_2 = 3}: \begin{bmatrix} -4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 1 & -7 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -7 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 7 \\ 1 \end{bmatrix}$$

   $$\underline{\lambda_3 = -4}: \begin{bmatrix} 3 & 0 & 0 & 0 \\ 1 & 7 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

   It follows that $A = PDP^{-1}$, where $P = \begin{bmatrix} 12 & 0 & 0 \\ -3 & 7 & 0 \\ 7 & 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -4 \end{bmatrix}$.

   (b) (4 points) $A = \begin{bmatrix} -1 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & -4 \end{bmatrix}$

   Solution

   Since A is upper triangular, its eigenvalues are its diagonal entries: $\lambda = -1$ (mult 2), -4. But we see that A only has a one-dimensional space of eigenvectors corresponding to $\lambda = -1$, so A is not diagonalizable:

   $$\underline{\lambda_1 = -1}: \begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \text{Eigenspace} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

8. (6 points total)

   (a) (4 points) Find the orthogonal projection of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ onto Span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

   Solution

   Let $\mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, and $\mathbf{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$. Since $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal, the orthogonal projection is:

$$\begin{aligned}
\hat{\mathbf{y}} &= \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 \\
&= \frac{2}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 2/3 \\ 1/3 \\ 4/3 \end{bmatrix}
\end{aligned}$$

   (b) (2 points) Find the least-squares solution of $\begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

   Solution

   The least-squares solution is the solution of the system $\begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 2/3 \\ 1/3 \\ 4/3 \end{bmatrix}$. The components of $\mathbf{x}$ are the weights of the linear combination $\hat{\mathbf{y}} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2$ found in part (a):

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1/3 \end{bmatrix}$$

9. (5 points) Find an orthonormal basis of the subspace of $\mathbb{R}^4$ spanned by $\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -6 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ 18 \\ -4 \end{bmatrix} \right\}$.

Solution

Let $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} -1 \\ 2 \\ -6 \\ 2 \end{bmatrix}$, and $\mathbf{x}_3 = \begin{bmatrix} 5 \\ 2 \\ 18 \\ -4 \end{bmatrix}$. We'll use Gram-Schmidt to find an orthogonal basis of $\text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$.

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix}$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \\ -6 \\ 2 \end{bmatrix} - \frac{-9}{9}\begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ -4 \\ 2 \end{bmatrix}$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = \begin{bmatrix} 5 \\ 2 \\ 18 \\ -4 \end{bmatrix} - \frac{45}{9}\begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix} - \frac{-72}{36}\begin{bmatrix} 0 \\ 4 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Since $\mathbf{v}_3 = \mathbf{0}$, we throw it out. An orthogonal basis of $\text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ -4 \\ 2 \end{bmatrix} \right\}$.

Divide each vector by its length to find an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2\}$.

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \\ 0 \end{bmatrix}$$

$$\mathbf{u}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{1}{6}\begin{bmatrix} 0 \\ 4 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$$

10. (6 points total)

(a) (3 points) Show that if a matrix A is diagonalizable and invertible, then A^{-1} is diagonalizable.

Solution

If $A = PDP^{-1}$, then

$$\begin{aligned} A^{-1} &= (PDP^{-1})^{-1} \\ &= (P^{-1})^{-1}D^{-1}P^{-1} \\ &= PD^{-1}P^{-1} \end{aligned}$$

We know the diagonal matrix D is invertible because its diagonal entries are the eigenvalues of A, and 0 is not an eigenvalue of an invertible matrix. We conclude that $A^{-1} = PD^{-1}P^{-1}$ is a diagonalization of A.

(b) (3 points) Show that for any vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$, the equation below (the Polar Identity) holds. (In this formula, $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y}$.)

$$\langle \mathbf{x}, \mathbf{y} \rangle = \frac{1}{4}\|\mathbf{x}+\mathbf{y}\|^2 - \frac{1}{4}\|\mathbf{x}-\mathbf{y}\|^2$$

Solution

Note that

$$\|\mathbf{x}+\mathbf{y}\|^2 = \langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{x} \rangle + 2\langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle$$

Similarly,

$$\|\mathbf{x}-\mathbf{y}\|^2 = \langle \mathbf{x}-\mathbf{y}, \mathbf{x}-\mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{x} \rangle - 2\langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle$$

Therefore,

$$\frac{1}{4}\|\mathbf{x}+\mathbf{y}\|^2 - \frac{1}{4}\|\mathbf{x}-\mathbf{y}\|^2 = \frac{1}{4}(\langle \mathbf{x}, \mathbf{x} \rangle + 2\langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle) - \frac{1}{4}(\langle \mathbf{x}, \mathbf{x} \rangle - 2\langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle)$$

$$= \langle \mathbf{x}, \mathbf{y} \rangle$$

11. (7 points total) Let $Q(x_1, x_2) = x_1^2 + 6x_1x_2 + x_2^2$.

(a) (2 points) Find the maximum value of $Q(\mathbf{x})$, where $\mathbf{x}$ is a unit vector.

Solution

The maximum value is the largest eigenvalue of the symmetric matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ of Q. The characteristic polynomial of A is $\lambda^2 - 2\lambda - 8$, from which it follows that the eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = -2$. The maximum value is 4.

(b) (3 points) Make a change of variable $\mathbf{x} = P\mathbf{y}$ to eliminate the cross-product term in Q.

Solution

The columns of P are unit eigenvectors of A. It is easily seen that eigenvectors $\mathbf{v}_1$ (for $\lambda_1 = 4$) and $\mathbf{v}_2$ (for $\lambda_2 = -2$) are:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Normalizing these vectors, we see that we can take

$$P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

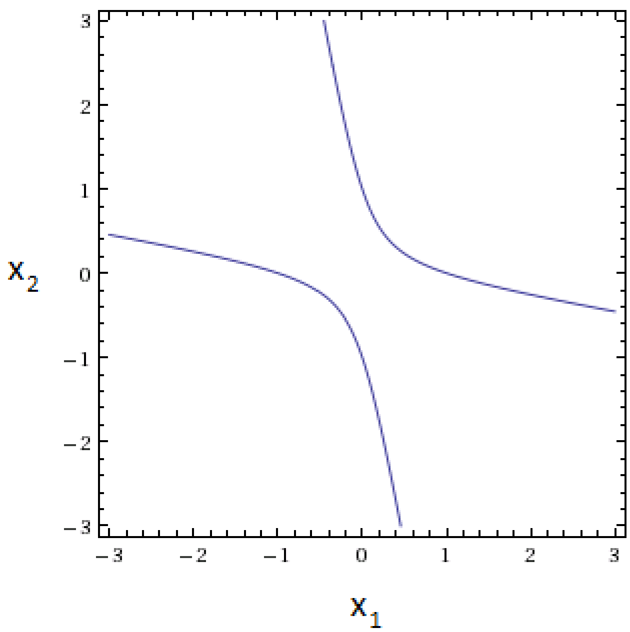
In terms of $\mathbf{y}$, the quadratic form is

$$Q(y_1, y_2) = 4y_1^2 - 2y_2^2$$

(c) (2 points) Make a rough sketch of the curve $Q(\mathbf{x}) = 1$ in the x_1x_2-plane.

Solution

The quadratic form Q is indefinite, so $Q(\mathbf{x}) = 1$ is a hyperbola. It opens in the directions of the eigenvectors corresponding to the positive eigenvalue 4.



12. (6 points) Find a singular value decomposition of $A = \begin{bmatrix} 1 & 1 \\ 2 & -2 \\ 0 & 0 \end{bmatrix}$.

Solution

First find eigenvalues and unit eigenvectors of A^TA.

$$A^TA = \begin{bmatrix} 5 & -3 \\ -3 & 5 \end{bmatrix} \qquad \lambda^2 - 10\lambda + 16 = 0 \quad \Rightarrow \quad \lambda = 2, 8$$

The singular values of A are the square roots of these eigenvalues, and they appear as diagonal elements of the matrix Σ in decreasing order:

$$\Sigma = \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}$$

An eigenvector for $\lambda = 8$ is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, which normalizes to $\mathbf{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$.

An eigenvector for $\lambda = 2$ is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, which normalizes to $\mathbf{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$.

Therefore, we may take $V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$.

Since both singular values are nonzero, we normalize both $A\mathbf{v}_1$ and $A\mathbf{v}_2$ to find the first two columns of U:

$$A\mathbf{v}_1 = \begin{bmatrix} 0 \\ 2\sqrt{2} \\ 0 \end{bmatrix} \qquad \Rightarrow \qquad \mathbf{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$A\mathbf{v}_2 = \begin{bmatrix} \sqrt{2} \\ 0 \\ 0 \end{bmatrix} \qquad \Rightarrow \qquad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

The third column $\mathbf{u}_3$ of U is any unit vector that is orthogonal to both $\mathbf{u}_1$ and $\mathbf{u}_2$. The vector $\mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ works. Therefore,

$$A = U\Sigma V^T = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$