More generally, we will need to describe the movement of charge from one place to another. This is captured by a quantity known as the *current density* $\mathbf{J}(\mathbf{x},t)$, defined as follows: for every surface S, the integral

$$I = \int_S \mathbf{J}\cdot d\mathbf{S}$$

counts the charge per unit time passing through S. (Here $d\mathbf{S}$ is the unit normal to S). The quantity I is called the *current*. In this sense, the current density is the current-per-unit-area.

The above is a rather indirect definition of the current density. To get a more intuitive picture, consider a continuous charge distribution in which the velocity of a small volume, at point $\mathbf{x}$, is given by $\mathbf{v}(\mathbf{x},t)$. Then, neglecting relativistic effects, the current density is

$$\mathbf{J} = \rho\mathbf{v}$$

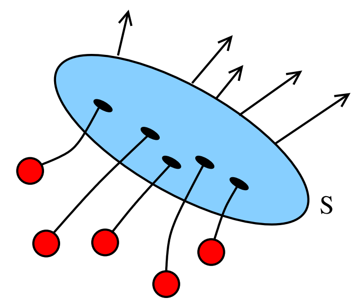

Figure 1: Current flux

In particular, if a single particle is moving with velocity $\mathbf{v} = \dot{\mathbf{r}}(t)$, the current density will be $\mathbf{J} = q\mathbf{v}\delta^3(\mathbf{x}-\mathbf{r}(t))$. This is illustrated in the figure, where the underlying charged particles are shown as red balls, moving through the blue surface S.

As a simple example, consider electrons moving along a wire. We model the wire as a long cylinder of cross-sectional area A as shown below. The electrons move with velocity $\mathbf{v}$, parallel to the axis of the wire. (In reality, the electrons will have some distribution of speeds; we take $\mathbf{v}$ to be their average velocity). If there are n electrons per unit volume, each with charge q, then the charge density is $\rho = nq$ and the current density is $\mathbf{J} = nq\mathbf{v}$. The current itself is $I = |\mathbf{J}|A$.

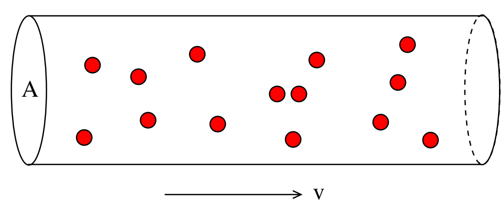

Figure 2: The wire

Throughout this course, the current density $\mathbf{J}$ plays a much more prominent role than the current I. For this reason, we will often refer to $\mathbf{J}$ simply as the "current" although we'll be more careful with the terminology when there is any possibility for confusion.

1.1.1 The Conservation Law

The most important property of electric charge is that it's conserved. This, of course, means that the total charge in a system can't change. But it means much more than that because electric charge is conserved *locally*. An electric charge can't just vanish from one part of the Universe and turn up somewhere else. It can only leave one point in space by moving to a neighbouring point.

The property of local conservation means that ρ can change in time only if there is a compensating current flowing into or out of that region. We express this in the *continuity equation*,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \tag{1.1}$$

This is an important equation. It arises in any situation where there is some quantity that is locally conserved.

To see why the continuity equation captures the right physics, it's best to consider the change in the total charge Q contained in some region V.

$$\frac{dQ}{dt} = \int_V d^3x\ \frac{\partial \rho}{\partial t} = -\int_V d^3x\ \nabla \cdot \mathbf{J} = -\int_S \mathbf{J} \cdot d\mathbf{S}$$

From our previous discussion, $\int_S \mathbf{J} \cdot d\mathbf{S}$ is the total current flowing out through the boundary S of the region V. (It is the total charge flowing *out*, rather than in, because $d\mathbf{S}$ is the outward normal to the region V). The minus sign is there to ensure that if the net flow of current is outwards, then the total charge decreases.

If there is no current flowing out of the region, then $dQ/dt = 0$. This is the statement of (global) conservation of charge. In many applications we will take V to be all of space, $\mathbf{R}^3$, with both charges and currents localised in some compact region. This ensures that the total charge remains constant.

1.2 Forces and Fields

Any particle that carries electric charge experiences the force of electromagnetism. But the force does not act directly between particles. Instead, Nature chose to introduce intermediaries. These are *fields*.

In physics, a "field" is a dynamical quantity which takes a value at every point in space and time. To describe the force of electromagnetism, we need to introduce two fields, each of which is a three-dimensional vector. They are called the *electric field* $\mathbf{E}$ and the *magnetic field* $\mathbf{B}$,

$$\mathbf{E}(\mathbf{x},t) \quad \text{and} \quad \mathbf{B}(\mathbf{x},t)$$

When we talk about a "force" in modern physics, we really mean an intricate interplay between particles and fields. There are two aspects to this. First, the charged particles create both electric and magnetic fields. Second, the electric and magnetic fields guide the charged particles, telling them how to move. This motion, in turn, changes the fields that the particles create. We're left with a beautiful dance with the particles and fields as two partners, each dictating the moves of the other.

This dance between particles and fields provides a paradigm which all other forces in Nature follow. It feels like there should be a deep reason that Nature chose to introduce fields associated to all the forces. And, indeed, this approach does provide one overriding advantage: all interactions are local. Any object — whether particle or field — affects things only in its immediate neighbourhood. This influence can then propagate through the field to reach another point in space, but it does not do so instantaneously. It takes time for a particle in one part of space to influence a particle elsewhere. This lack of instantaneous interaction allows us to introduce forces which are compatible with the theory of special relativity, something that we will explore in more detail in Section 5

The purpose of this course is to provide a mathematical description of the interplay between particles and electromagnetic fields. In fact, you've already met one side of this dance: the position $\mathbf{r}(t)$ of a particle of charge q is dictated by the electric and magnetic fields through the Lorentz force law,

$$\mathbf{F} = q(\mathbf{E} + \dot{\mathbf{r}} \times \mathbf{B}) \tag{1.2}$$

The motion of the particle can then be determined through Newton's equation $\mathbf{F} = m\ddot{\mathbf{r}}$. We explored various solutions to this in the *Dynamics and Relativity* course. Roughly speaking, an electric field accelerates a particle in the direction $\mathbf{E}$, while a magnetic field causes a particle to move in circles in the plane perpendicular to $\mathbf{B}$.

We can also write the Lorentz force law in terms of the charge distribution $\rho(\mathbf{x},t)$ and the current density $\mathbf{J}(\mathbf{x},t)$. Now we talk in terms of the *force density* $\mathbf{f}(\mathbf{x},t)$, which is the force acting on a small volume at point $\mathbf{x}$. Now the Lorentz force law reads

$$\mathbf{f} = \rho\mathbf{E} + \mathbf{J} \times \mathbf{B} \tag{1.3}$$