Image Denoising

Suyash P. Awate

Bayesian Image Denoising

- Optimal noiseless image is the one that maximizes the posterior PDF

$$P(\text{NoiselessImage}|\text{NoisyImage}) = \frac{P(\text{NoisyImage}|\text{NoiselessImage})P(\text{NoiselessImage})}{P(\text{NoisyImage})}$$

  - **Likelihood** PDF = **noise** model = probability of generating the data given the noiseless image
  - **Prior** PDF = our prior beliefs about the noiseless image **before** observing the data
  - **Posterior** PDF: product of likelihood and prior
    - What we get “post” / after observing the data

Optimization for Denoising

- Noiseless image X = x
  - X is a MRF
- Observed image data Y = y
- Noise model for intensities given noiseless intensities (i.i.d) $P(Y \mid X) := \Pi_i \; P(Y_i \mid X_i)$
  - e.g., If noise is additive i.i.d. zero-mean Gaussian, $P(Y_i \mid X_i) = G(y_i \mid x_i, \sigma^2)$
- Let θ = parameters underlying noise model and MRF model (in general)

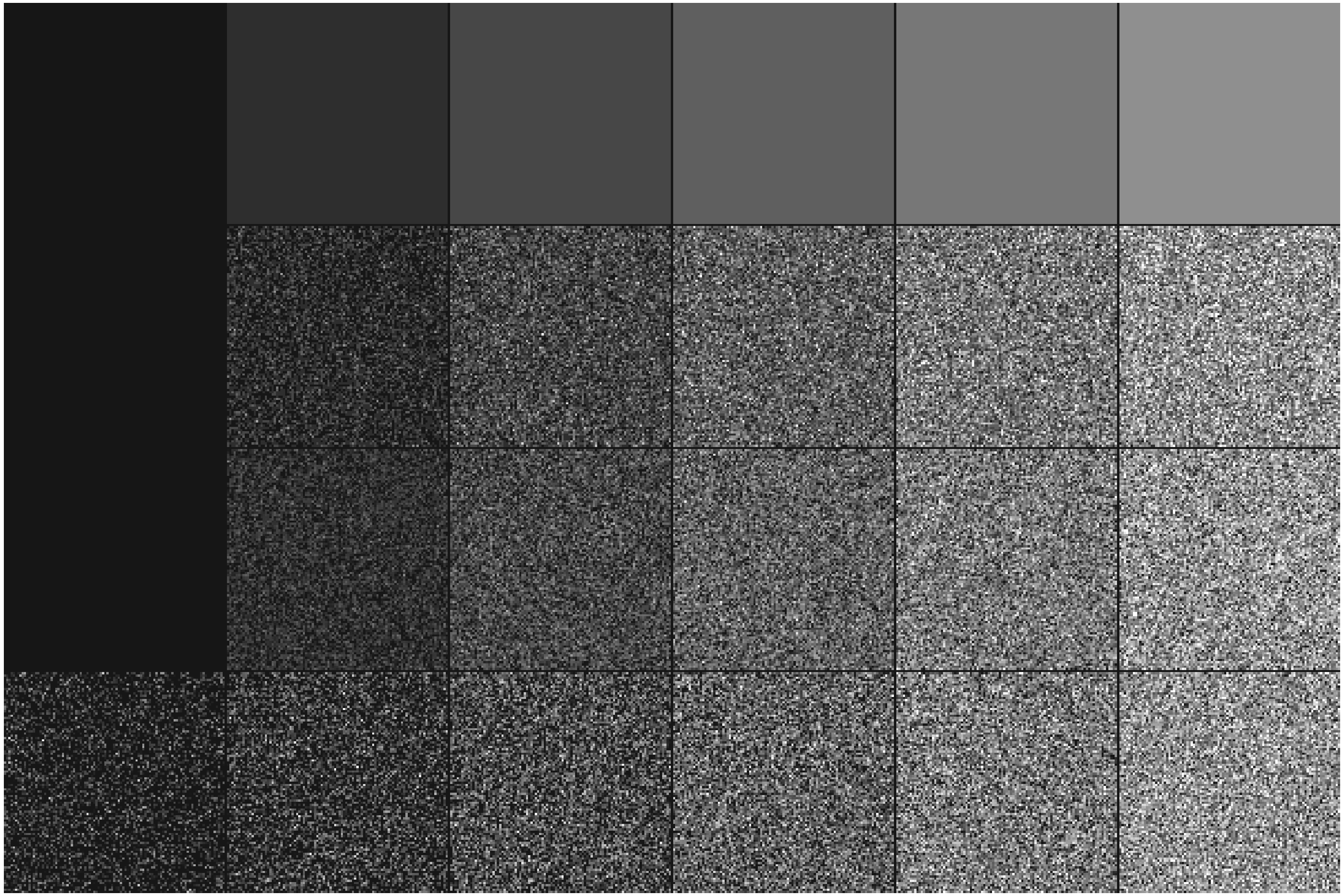
Noise Models

- Which of the noise models are "additive" in nature ?
  - Gaussian ?
  - Poisson ?
  - Rician ?

  - Gaussian noise is independent of pixel intensity
  - Poisson, Rician noise depends on pixel intensity

Noise Models

- Which of the noise models are “additive” in nature ?
  - What if approximate
    Poisson (mu) = Gaussian (mean = mu, variance = mu) ?
    - Row 1 true
      - Mu values
    - Row 2 Poisson
    - Row 3
      Gauss. approx.
    - Row 4
      Gauss. Approx.
      with
      Fixed Variance (for all mu)



Noise Models

- Variance stabilization
  - Assume variance of X depends on its mean
  - Search for transformation Y = f(X), such that variance of Y doesn’t depend on mean of Y
  - If X is Poisson: f(X) = \sqrt (X) makes variance nearly constant
    - Anscombe transform: f(X) = 2 \sqrt (X + 3/8) transforms
      - Poisson data ‘x’ with mean = ‘m’ to →
        Approx. Gaussian data with
        variance = 1,
        mean = 2 \sqrt (m + 3/8) – 1 / (4 \sqrt(m))

Optimization for Denoising

- Optimization Problem and Strategy

  (1) Assume: MRF parameters are user controlled

  - No need to optimize

  (2) Assume: noise level is already known

  - e.g., using the ML estimate in the background region, where signal is known to be zero

  (3) Get MAP estimate for noiseless image 'x' :

  $\max_x P(x \mid y, \theta)$

Optimization for Denoising

- Lets see **what happens at voxel i ?**
- Rewrite the objective function
  - P (X | y, theta)

$= P(X_i, X_{\sim i}|y,\theta)$

$= P(X_i|X_{\sim i},y,\theta)P(X_{\sim i}|y,\theta)$ Conditional Probability

$= P(X_i|X_{N_i},y,\theta)P(X_{\sim i}|y,\theta)$ Markov assumption on X

$= P(X_i|X_{N_i},y_i,\theta)P(X_{\sim i}|y,\theta)$ Conditional independence assumption in noise model

Optimization for Denoising

- Optimization Algorithm 1
  - Iterated Conditional Mode (ICM)
  - **Consider optimization over a single voxel i**
    - Perform $\max_{x_i} P(X \mid y, \theta)$

$$= \max_{x_i} P(X_i|X_{N_i}, y_i, \theta) P(X_{\sim i}|y, \theta)$$

$$= \max_{x_i} P(X_i|X_{N_i}, y_i, \theta) \text{ Second term doesn't depend on } x_i$$

$$= \max_{x_i} \frac{P(y_i|X_i, X_{N_i}, \theta) P(X_i|X_{N_i}, \theta)}{P(y_i|X_{N_i}, \theta)} \text{ Bayes Rule}$$

$$= \max_{x_i} P(y_i|X_i, X_{N_i}, \theta) P(X_i|X_{N_i}, \theta) \text{ Denominator doesn't depend on } x_i$$

$$= \max_{x_i} P(y_i|X_i, \theta) P(X_i|X_{N_i}, \theta) \text{ Conditional independence assumption in noise model}$$

Optimization for Denoising

- Optimization Algorithm 1
  - Iterated Conditional Mode (ICM)
  - $\max_{x_i} P(y_i|X_i, \theta) P(X_i|X_{N_i}, \theta)$
  - 1st term P (yi | Xi , θ) = likelihood function
    - Noise model
  - 2nd term P (Xi | X_{Ni} , θ) = local / conditional prior on noiseless image
    - Image-regularity / smoothness model
  - ICM seeks mode of local / conditional posterior

Optimization for Denoising

- Various Optimization Algorithms
  - Order of Intensity Updates:
    - We want every update to increase the posterior P (x|y, θ)

      (1) Sequentially: Column by column, and then row by row
      - May lead to artifacts

      (2) Sequentially: Randomized order each iteration
      - Need to generate random sequence each iteration
      - Are artifacts eliminated ?

      (3) In Parallel: If seeking <u>mode</u>, doesn't guarantee increase in posterior probability (INVALID) unless …

      (4) In Parallel: Go towards the mode <u>and</u> monitor objective function
      - Gradient ascent : Dynamic step size + Objective-function monitoring
      - Guarantees increase in posterior probability

Optimization for Denoising

- Note on gradient ascent
  - Dynamic step sizing at each iteration :
    - Increase step size by (say) 10% when initial step size increases probability
      - Prevents very slow convergence when far away from optimum
    - Decrease step size by (say) 50% when initial step size decreases probability
      - Adapts step size as you get close to optimum
    - This dual strategy also prevents over-sensitivity to initial step size
  - Termination criteria :
    - Allowable step size becomes very small, e.g., 1e-8
    - Step doesn't increase posterior image probability by much, e.g., 0.01% of probability at current solution

Optimization for Denoising

- Various Optimization Algorithms
  - Gradient ascent needs :

(1) Derivative of local conditional PDF w.r.t. xi

$$P(X_i|X_{N_i},\theta) = \frac{1}{Z_i}\exp\left(-\sum_{a\in A} V_a(x_a)\right)$$

where A is the set of cliques that contain site i

(2) Derivative of noise model w.r.t. xi

MRI (Complex) (Noise: Gaussian)

- Noise Model
  - Circularly-symmetric univariate Gaussian (complex)

$$P(y|x) = \Pi_i P(y_i|x_i) = \Pi_i G_{\mathbb{C}}(y_i|x_i, \sigma^2) = \Pi_i \frac{1}{\sigma^2 \pi} \exp\left(-\frac{|y_i - x_i|^2}{\sigma^2} \right)$$

MRI (Complex) (Noise: Gaussian)

- For ICM optimization, at chosen voxel i, perform

$$\begin{aligned}
\max_{x_i} P(x|y,\theta) &= \max_{x_i} P(y_i|x_i,\theta) P(x_i|x_{N_i},\theta) \\
&= \max_{x_i} \Big(\log P(y_i|x_i,\theta) + \log P(x_i|x_{N_i},\theta) \Big) \\
&= \max_{x_i} \left(\frac{-|y_i - x_i|^2}{\sigma^2} + \sum_{a \in A_i} (-V_a(x_a)) \right) \\
&= \min_{x_i} \left(\frac{|y_i - x_i|^2}{\sigma^2} + \sum_{a \in A_i} V_a(x_a) \right)
\end{aligned}$$

  - 1st term = Fidelity term = penalizes deviation (infidelity) of estimate x from data y
  - 2nd term = Regularity term: penalizes roughness of x

MRI (Complex) (Noise: Gaussian)

- For gradient-descent optimization, at voxel i, derivative is:

$$\frac{\partial P(x|y,\theta)}{\partial x_i} = \frac{1}{\sigma^2} 2(x_i - y_i) + \frac{\partial}{\partial x_i} \sum_{a \in A_i} V_a(x_a)$$

- For entire image x, gradient (column vector) is:

$$g_2(x) := \left(\cdots, \frac{\partial P(x|y,\theta)}{\partial x_i}, \cdots \right)$$

- Current solution x^n at iteration n. Stepsize τ . Updated solution is: $x^{n+1} = x^n - \tau g(x)$

MRI (Magnitude) (Noise: Rician)

- Observed noisy (magnitude-MR) image data y is real
- Noiseless (magnitude-MR) image x is real
- Prior PDF remains same
- Likelihood PDF is :

$$P(y|x) := \Pi_i P(y_i|x_i) := \Pi_i \frac{y_i}{\sigma^2} \exp\left(-\frac{y_i^2 + x_i^2}{2\sigma^2}\right) I_0\left(\frac{y_i x_i}{\sigma^2}\right)$$

where I0 (z) = modified Bessel function of 1st kind, order 0

MRI (Magnitude) (Noise: Rician)

- For ICM optimization, at a chosen voxel i, perform

$$\max_{x_i} P(x|y,\theta) = \min_{x_i} \left(\frac{y_i^2 + x_i^2}{2\sigma^2} - \log I_0 \left(\frac{y_i x_i}{\sigma^2} \right) + \sum_{a \in A_i} V_a(x_a) \right)$$

$$P(y|x) := \Pi_i P(y_i|x_i) := \Pi_i \frac{y_i}{\sigma^2} \exp \left(-\frac{y_i^2 + x_i^2}{2\sigma^2} \right) I_0 \left(\frac{y_i x_i}{\sigma^2} \right)$$

MRI (Magnitude) (Noise: Rician)

- For gradient-descent optimization, at chosen voxel i, the derivative is :

$$\frac{\partial P(x|y,\theta)}{\partial x_i} = \frac{x_i}{\sigma^2} - \frac{I_1\left(\frac{y_i x_i}{\sigma^2}\right)}{I_0\left(\frac{y_i x_i}{\sigma^2}\right)} \frac{y_i}{\sigma^2} + \frac{\partial}{\partial x_i} \sum_{a \in A_i} V_a(x_a)$$

where I1 (z) = modified Bessel function of 1^{st} kind, order 1

Ultrasound Magn. (Noise: Speckle)

- Observed noisy image data y is real
- Noiseless image x is real
- Prior PDF remains same
- Speckle-Noise model is : $Y = X + \sqrt{X}\, Z$
  where $P(Z) := G(0, \sigma^2)$
- What is $P(Y \mid X)$?
  - We need this because this is the likelihood PDF

Ultrasound Magn. (Noise: Speckle)

- What is P (Y | X) ?
  - Use transformation of random variables

RV $Z, P(z) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{z^2}{2\sigma^2}\right)$

RV $Y_1 := \sqrt{x}Z, P(y_1) = \frac{1}{\sqrt{x}\sigma\sqrt{2\pi}} \exp\left(-\frac{y_1^2}{2x\sigma^2}\right)$

Transformation maintains PDF as Gaussian and scales std. dev. by $\sqrt{x}$

RV $Y_2 := Y_1 + x, P(y_2) = \frac{1}{\sqrt{x}\sigma\sqrt{2\pi}} \exp\left(-\frac{(y_2 - x)^2}{2x\sigma^2}\right)$

Transformation maintains PDF as Gaussian and translates mean by x

  - So, likelihood PDF is :

$$P(y|x) := \Pi_i P(y_i|x_i) := \Pi_i G(y_i|x_i, x_i\sigma^2) = \frac{1}{\sqrt{x}\sigma\sqrt{2\pi}} \exp\left(-\frac{(y_i - x_i)^2}{2x_i\sigma^2}\right)$$

Ultrasound Magn. (Noise: Speckle)

- For ICM optimization, at a chosen voxel i, perform

$$\max_{x_i} P(x|y,\theta) = \min_{x_i} \left(\frac{\log(x_i)}{2} + \frac{(y_i - x_i)^2}{2x_i\sigma^2} + \sum_{a \in A_i} V_a(x_a) \right)$$

$$P(y|x) := \Pi_i P(y_i|x_i) := \Pi_i G(y_i|x_i, x_i\sigma^2) = \frac{1}{\sqrt{x}\sigma\sqrt{2\pi}} \exp\left(-\frac{(y_i - x_i)^2}{2x_i\sigma^2} \right)$$

Ultrasound Magn. (Noise: Speckle)

- For gradient-descent optimization, at voxel i, the derivative is :

$$\frac{\partial P(x|y,\theta)}{\partial x_i} = \frac{1}{2x_i} + \frac{1}{2\sigma^2}\left(\frac{x_i^2 - y_i^2}{x_i^2}\right) + \frac{\partial}{\partial x_i}\sum_{a \in A_i} V_a(x_a)$$

Weighting Likelihood & MRF Prior

- Motivation

  (1) We want to control the strength of the prior model based on certain criteria (e.g., noise level)

  (2) We want to balance the enforcement of fidelity and regularity based on certain criteria (e.g., noise level)

- Thumb Rules

  (1) Very high noise levels ⇒ data is highly corrupted ⇒ use a strong prior

  (2) Very low noise levels ⇒ data is high quality ⇒ use a weak prior

Weighting Likelihood & MRF Prior

- How to do it ?
  - Introduce a user-controlled parameter $\beta \in [0, 1]$ that specifies the balance between :

    (1) enforcing / strength of the prior model and

    (2) enforcing / strength of the likelihood model

  - A more straightforward way of thinking about β
    - If we don't know the true prior model,
      then β could be a parameter of the potential function itself.
      This parameter is unknown, so we need to tune it.

Weighting Likelihood & MRF Prior

- Weighting MRF Prior
  - In the **prior PDF**, introduce a parameter **β** ∈ [0, 1] s.t.

$$P(x) := \frac{1}{Z(\beta)} \exp\left(-\beta \frac{1}{T} U(x)\right) \text{ where}$$

$$U(x) := \sum_{c \in C} V_c(x_c) \text{ where}$$

$$Z(\beta) := \sum_{x} \exp\left(-\beta \frac{1}{T} U(x)\right)$$

  - This changes the local conditional prior to

$$P(x_i | x_{N_i}, \theta) = \frac{1}{Z_i(\beta)} \exp\left(-\beta \frac{1}{T} \sum_{a \in A} V_a(x_a)\right)$$

  - Introducing β is similar to changing temperature T

Weighting Likelihood & MRF Prior

- Weighting Likelihood (Complex-Gaussian Noise)
  - Introduce a parameter $\alpha \in [0, 1]$, where **$\alpha := 1 - \beta$**

$$P(y|x) := \Pi_i P(y_i|x_i) := \Pi_i G_\alpha(y_i|x_i, \sigma^2)$$

$$G_\alpha(y_i|x_i, \sigma^2) := \frac{1}{Z(\sigma, \alpha)} \exp\left(-\alpha \frac{|y_i - x_i|^2}{\sigma^2}\right),$$

    where $Z(\sigma, \alpha) = 1 / ((\sigma/\alpha)^2 \pi)$

  - Interpretation
    - Introducing α is similar to changing the “specified” noise level / standard deviation σ
    - This is the Complex-Gaussian PDF with parameters (x_i , σ^2/α)

Weighting Likelihood & MRF Prior

- Modified Optimization Problem (Complex-Gaussian Noise)
  - For ICM optimization, at a chosen voxel i, perform :

$$\max_{x_i} P(x|y,\theta) = \min_{x_i} \left((1-\beta)\frac{|y_i - x_i|^2}{\sigma^2} + \beta \sum_{a \in A_i} V_a(x_a) \right)$$

  - For gradient-descent optimization, at a chosen voxel i, the derivative is :

$$\frac{\partial P(x|y,\theta)}{\partial x_i} = (1-\beta)\frac{1}{\sigma^2} 2(x_i - y_i) + \beta \frac{\partial}{\partial x_i} \sum_{a \in A_i} V_a(x_a)$$

  - β=0 (T → ∞) ignores the prior; we get the ML estimate
  - β=1 (α = 0; σ → ∞) makes the likelihood a uniform PDF

Weighting Likelihood & MRF Prior

- Weighting Likelihood (Rician Noise)
  - Introduce a parameter α ∈ [0, 1], where α := 1 − β

$$P(y|x) := \Pi_i P(y_i|x_i) := \Pi_i \frac{\alpha y_i}{\sigma^2} \exp\left(-\alpha \frac{y_i^2 + x_i^2}{2\sigma^2}\right) I_0\left(\alpha \frac{y_i x_i}{\sigma^2}\right)$$

  - Interpretation
    - Introducing α is similar to changing noise level σ
    - This is the Rician PDF with parameters (xi , σ / sqrt(α))

Weighting Likelihood & MRF Prior

- Modified Optimization Problem (Rician Noise)
  - For ICM optimization, at a chosen voxel i, perform

$$\max_{x_i} P(x|y,\theta) = \min_{x_i} \left((1-\beta)\frac{y_i^2 + x_i^2}{2\sigma^2} - \log I_0\left((1-\beta)\frac{y_i x_i}{\sigma^2}\right) + \beta \sum_{a \in A_i} V_a(x_a) \right)$$

  - For gradient-descent optimization, at a chosen voxel i, the derivative is

$$\frac{\partial P(x|y,\theta)}{\partial x_i} = (1-\beta)\frac{x_i}{\sigma^2} - \frac{I_1\left((1-\beta)\frac{y_i x_i}{\sigma^2}\right)}{I_0\left((1-\beta)\frac{y_i x_i}{\sigma^2}\right)}(1-\beta)\frac{y_i}{\sigma^2} + \beta\frac{\partial}{\partial x_i}\sum_{a \in A_i} V_a(x_a)$$

  - $\beta=0$ ($T \to \infty$) ignores the prior; we get the ML estimate
  - $\beta=1$ ($\alpha = 0$; $\sigma \to \infty$) makes the likelihood a uniform PDF

Weighting Likelihood & MRF Prior

- Weighting Likelihood (Speckle Noise)
  - Introduce a parameter $\alpha \in [0, 1]$, where $\alpha := 1 - \beta$

$$P(y|x) := \Pi_i P(y_i|x_i) := \Pi_i G(y_i|x_i, x_i\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}\sqrt{x_i^\alpha/\alpha}} \exp\left(-\frac{(y_i - x_i)^2}{2(x_i^\alpha/\alpha)\sigma^2}\right)$$

  - Interpretation
    - Introducing α is similar to changing the noise level / standard deviation σ
    - This is the Gaussian PDF with parameters (x_i , σ^2 (x_i^α /α))

Weighting Likelihood & MRF Prior

- Modified Optimization Problem (Speckle Noise)
  - For ICM optimization, at a chosen voxel i, perform

$$\max_{x_i} P(x|y,\theta) = \min_{x_i} \left(\alpha \frac{1}{2} \log(x_i) + \alpha \frac{1}{2\sigma^2} \frac{(y_i - x_i)^2}{x_i^\alpha} + \beta \sum_{a \in A_i} V_a(x_a) \right)$$

  - For gradient-descent optimization, at a chosen voxel i, the derivative is

$$\frac{\partial P(x|y,\theta)}{\partial x_i} = \alpha \frac{1}{2x_i} + \alpha \frac{1}{2\sigma^2} \frac{\partial}{\partial x_i} \frac{(y_i - x_i)^2}{x_i^\alpha} + \beta \frac{\partial}{\partial x_i} \sum_{a \in A_i} V_a(x_a)$$

  - $\beta=0$ ($T \rightarrow \infty$) ignores the prior; we get the ML estimate
  - $\beta=1$ ($\alpha = 0$; $\sigma \rightarrow \infty$) makes the likelihood a uniform PDF