# Methods for estimation of clipped signals

Chinmay Talegaonkar | 15D070046

November 26, 2018

**Abstract**

It is often observed in the process of signal acquisition that the signal is clipped due to sensor saturation. This study aims at exploring methods to recover the original signal from the clipped measurements. The original signal is known to be sparse in a given basis. In particular, the focus is on grid-less estimation from saturated signals, with the original signal assumed to be a sparse combination of sinusoids. The theory is backed up by simulations for some of the methods mentioned in the paper [ESJ17] for recovery from noiseless measurements. This report also presents a comparison of the atomic norm minimization based methods with the compressed sensing formulation for this estimation problem, highlighting the similarities and differences between the two.

## 1 Introduction and related work

In most practical measurement setups the signal saturates at either the minimum or maximum (or both) thresholds of the sensor. The approaches to recover the underlying true signal from the saturated measurements typically treat the saturated samples to be missing. There have been attempts to reconstruct the signals using interpolation assuming an underlying distribution or signal model [LMV86]. Lately, several reconstruction techniques like compressed sensing also use the underlying sparsity of the signal in a given sparsifying basis like DFT or DCT. In[DMH$^+$13], a compressed sensing formulation was used by the authors to develop a method to enhance the perceived audio signal quality. The comparison in this report focuses only on the basic compressed formulation, not the model specific to auditory signals since the underlying signal structure assumed for this project is different.

This project focuses on the recovery of signals that assume a prior underlying structure, that the signal can be well modeled as a sum of decaying sinusoidals. The algorithm studied in this project (will be referred as Q1 in this report hence forth) is based on formulating an estimator of the unknown parameters detailing the assumed signal structure [ESJ17], taking into account both the available and the saturated samples. On top of it, leveraging the signal sparsity enables the algorithm to exploit the information available in the saturated samples, while still being robust to the presence of additive noise.

## 2 Recovery methods

Throughout this report, it is assumed that the signal to be recovered is of the form -:

$$\mathbf{y} = \mathbf{A}\mathbf{d} + \mathbf{e}$$

$\mathbf{d}$ is a $K \times 1$ amplitude vector, $\mathbf{e}$ is the additive noise (which is not being considered for our simulation). $\mathbf{A} = [\mathbf{a_1}, \mathbf{a_2} \dots, \mathbf{a_K}]$ is the dictionary of columns $a_i = [sin(2\pi f_i t_1), sin(2\pi f_i t_2) \dots, sin(2\pi f_i t_N)]$, where $f_i$ is the $i^{th}$ frequency. Let $\Omega^-$, $\Omega^+$, and $\Omega$ be the sets of indices of $\mathbf{y}$ which are clipped from below, clipped from above and not clipped respectively. For successful recovery of the signal, estimation of $\mathbf{d}$ (amplitudes), $f_i$ (frequencies) and $K$ (the number of independent components) is required. The subsections ahead briefly describe the approaches studied under this project for estimation of the signal $\mathbf{y}$.

## 2.1 Atomic norm minimization

Dictionary learning techniques using a predefined grid suffer in performance when the true parameters of the grid are not known. A similar situation is encountered in our case, as the columns of the matrix $\mathbf{A}$ whose values are governed by the frequency values $f_i$, are unknown to us. To solve this problem, an atomic norm minimization can be used, which has been exploited successfully in the past to develop estimators using off grid components [TBSR13]. One of the formulations in the paper [ESJ17] (referred as Q0 in this report) uses an atomic norm formulation to exploit the structure of the assumed signal, but throws away the information at the saturation indices. Q1 builds over this formulation to also take into account the information from the saturated samples.

Defining an atom set $A_0 = \{\mathbf{a}(f) : f \in [0,1]\}$ where $[\mathbf{a}(f)]_t = sin(2\pi f t)$, $\mathbf{y}$ can be expressed as a sum of $K$ sinusoids -:

$$\mathbf{y} = \sum_{k=1}^{K} d_k \mathbf{a}(f_k)$$

. The atomic norm over $A_0$ is defined as -:

$$||y||_{A_0} = \inf \left\{ \sum_k d_k : \mathbf{y} = \sum_{k=1}^{K} d_k \mathbf{a}(f_k), d_k \geq 0, f_k \in [0,1] \right\}$$

. The atomic norm can be equivalently written as an SDP as shown in [TBSR13]. This SDP formulation of the atomic norm is referred as **Q0** -:

$$\begin{aligned}
\underset{x,\mathbf{z},\mathbf{u}}{\text{minimize}} \quad & x + u_1 + \frac{1}{2}\|\mathbf{y}_\Omega - \mathbf{z}_\Omega\|_2^2 \\
\text{subject to} \quad & \begin{bmatrix} x & \mathbf{z}^H \\ \mathbf{z} & \mathbf{T}(\mathbf{u}) \end{bmatrix} \geq 0 \\
& \mathbf{T}(\mathbf{u}) \in \mathbb{T}^{N \times N}
\end{aligned}$$

$\mathbf{T}(\mathbf{u})$ is the toeplitz matrix with $\mathbf{u}$ as its first column. $\mathbf{z}$ is the signal to be estimated. $\mathbf{z}_{\boldsymbol{\Omega}}$ and $\mathbf{y}_{\boldsymbol{\Omega}}$ denote the signal measurements over non-clipped samples. Since this is a convex problem, it can be solved using an SDP solver like CVX (which was used for simulations in this project). Solving this optimization problem yields the complete estimated signal $\mathbf{z}$, and the matrix $\mathbf{T}(\mathbf{u})$, the vandermonde decomposition of which gives the frequency and amplitude estimates of the original signal ($y$). [YXS16] $x$ is s scalar corresponding to the sum of the absolute magnitude estimates. This effectiveness of this approach has been observed in recovering the signal as well the frequency estimates, but it suffers from the drawback of throwing away the information from the saturated indices of the signal $\mathbf{y}$, i.e. $\mathbf{y}_{\boldsymbol{\Omega}^+}$ and $\mathbf{y}_{\boldsymbol{\Omega}^-}$. Hence, modifying the formulation Q0 to incorporate the information given by the saturated samples gives the formulation **Q1** - :

$$\begin{aligned}
\underset{x,\mathbf{z},\boldsymbol{\epsilon},\mathbf{u}}{\text{minimize}} \quad & \mu\,(x + u_1) + \lambda\,\|\boldsymbol{\epsilon}\|_1 + \frac{1}{2}\|\mathbf{y}_\Omega - \mathbf{z}_\Omega\|_2^2 \\
\text{subject to} \quad & \begin{bmatrix} x & \mathbf{z}^H \\ \mathbf{z} & \mathbf{T}(\mathbf{u}) \end{bmatrix} \geq 0 \\
& \mathbf{T}(\mathbf{u}) \in \mathbb{T}^{N \times N} \\
& \mathbf{z}_{\Omega^+} + \boldsymbol{\epsilon}^+ \geq \gamma \\
& \mathbf{z}_{\Omega^-} + \boldsymbol{\epsilon}^- \leq -\gamma
\end{aligned}$$

where

$$\boldsymbol{\epsilon} = \begin{bmatrix} \boldsymbol{\epsilon}^+ & \boldsymbol{\epsilon}^- \end{bmatrix}$$

This formulation can be explained in the following way Since the thresholds for minimum and maximum level for saturation ($\gamma$ and -$\gamma$) are known along with the indices where clipping occurs, one can incorporate this information as 2 linear constraints on $\mathbf{z}_{\mathbf{\Omega}^+}$ and $\mathbf{z}_{\mathbf{\Omega}^-}$ as shown in Q1. The constraints also take into account the fact that the noise can push the sample over (or under) the saturation limit. To incorporate this effect, the variables $\epsilon^+$ and $\epsilon^-$ are introduced [ESJ17]. They capture the discrepancy between the observed and the true signal waveform for the samples saturated due to the additive noise. Since this project deals with noiseless measurements only, $[\epsilon^+ \ \epsilon^-] = [0 \ 0]$. As mentioned for Q0, the frequency estimates for Q1 can also be found using the Vandermonde decomposition of $\mathbf{T}(\mathbf{u})$. This method performs much better than Q0, which will be shown in the experiments and results section. To improve the amplitude estimates further, one can solve a modified version of Q1 described in [ESJ17] which explicitly estimates the amplitudes $d_k$ from the precomputed frequency estimates from $\mathbf{T}(\mathbf{u})$, to get better estimates of the amplitudes as compared to Q1. The performance of this method is marginally better than that of Q1 for very noisy signals. The next subsection describes the compressed sensing formulation for this problem in a noiseless setting.

## 2.2 Compressed sensing formulation

Compressed sensing leverages the sparsity of the underlying signal $y \in R^N$ in some basis dictionary $\Psi$ to recover $y$ from a reduced number of measurements $x \in R^M$, where $M < N$ [Don06]. The signal acquisition in compressed sensing is assumed to be of the following form $x = \Phi y$, i.e. a measurement matrix of $\Phi$ of dimensions $M \times N$ is applied on the vector to be measured, i.e. $y$. Therefore, the equation for acquiring compressed measurements can be written as

$$x = \Phi \Psi s$$

where $s$ is a sparse vector in the basis $\Psi$. Hence compressed sensing is different from the compression process on a fundamental level. Compression involves measuring the full signal and then compressing it by removing tail coefficients in some basis, while compressed sensing takes compressed measurements and guarantees reconstruction of the original signal under certain conditions. One of such sufficient conditions which is the most widely used criterion for proving recovery guarantees and performance bounds of CS algorithms is the *Restricted Isometric property* or the RIP condition [BMW15]. It states that if for the sensing matrix $A$ defined above, there exists a $\delta \in (0, 1)$ such that

$$(1 - \delta_k)||s||_2 \leq ||As||_2 \leq (1 + \delta_k)||s||_2$$

is valid for all $k$ sparse vectors $s$ then $A$ is said to be satisfying RIP of the order $k$. In general, if $A$ satisfies RIP of the order $2k$ with $\delta_{2k} \in (0, 0.5)$ then this is a sufficient condition for a variety of algorithms to successfully recover any $k$ sparse vector s, hence the corresponding measurement $y$. Provided $A$ satisfies RIP, exact recovery can be obtained by solving the following constrained optimization problem -

$$\begin{aligned} z^* = \ & \underset{z}{\text{minimize}} && ||z||_1 \\ & \text{subject to} && Az = x \end{aligned}$$

The underlying signal can then be recovered as $\hat{y} = \Psi z^*$. For recovery from saturated measurements, one can simply discard the measurements and hence the measurement matrix $\Phi$ will be a sub matrix of the identity matrix, entirely defined by the clipping pattern, i.e. the rows corresponding to the clipped indices are removed, [DMH+13]. Just like Q1 incorporated the information from the clipped signals, the same linear constraints can be used to get a the following compressed sensing formulation **Q2** for a noiseless case - :

$$\begin{aligned} z_m^* = \ & \underset{z}{\text{minimize}} && ||z||_1 \\ & \text{subject to} && A_m z = x, \\ & && C_m^+ z \geq \gamma, \\ & && C_m^- z \leq -\gamma \end{aligned}$$

Here $C_m^+$ and $C_m^-$ are sub- matrices of an identity matrix composed of the rows corresponding to positively and negatively clipped samples. Note that this formulation works for the noiseless case. When the signal is noisy, the objective function needs to be changed to take the additive noise into account as mentioned in [DMH+13]

# 3 Experiments and Inferences

The experiments and simulations primarily focus on comparing the 3 approaches Q0, Q1 and Q2. This report presents the comparisons of the reconstruction errors obtained of the three formulations for varying levels of saturation in the signal. The original signal $y$ (a $100 \times 1$ vector, i.e. using 100 time samples) was generated as mentioned in section 2 as a combination of $K = 2$ sinusoids. For a fixed $y$, the positive and negative saturation levels ($\gamma$) were varied, as to get the saturated signals of the corresponding variable fraction of non-clipped values. For the compressed sensing formulation, $\Psi$ was chosen to be the DCT basis. The plot below shows the RMSE errors for the complete signals as opposed to the amplitude and frequency estimate errors reported separately in the original paper.

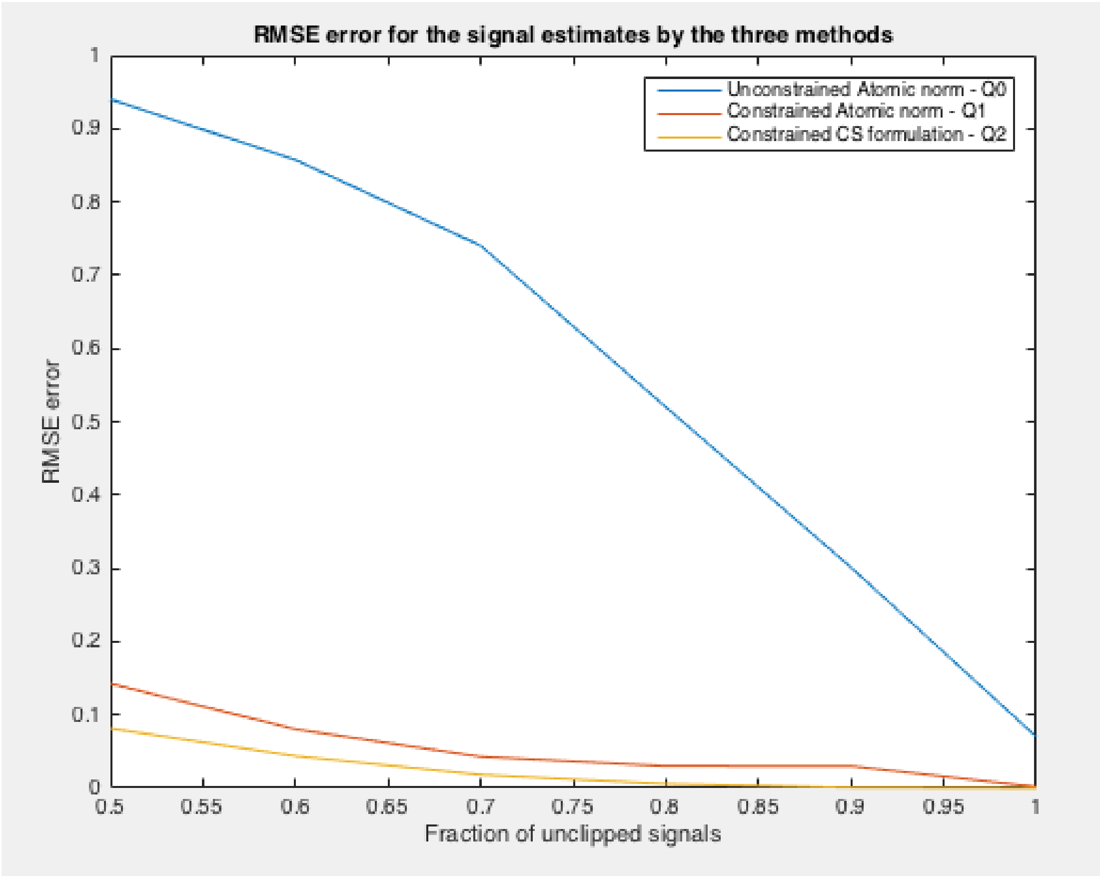


Figure 1: RMSE errors for the complete signal estimates using unconstrained atomic norm, constrained atomic norm and constrained Compressed sensing formulation.

Hence from the plots it can be inferred that the constrained Compressed sensing formulation performs the best of the studied techniques. While CS and atomic norm minimization both take the advantage of the sparsity of the underlying signal. The fact that the DCT (discrete cosine transform) basis was used as a sparsifying basis in our experiments enhances the performance of CS as compared to atomic norm minimization since the chosen signal $y$ is indeed sparse in the DCT basis. Moreover the fact that the sensing matrix for this case obeys RIP further allows for the stricter constraints to be applied which gives better result as compared to the relatively loose semi-definiteness constraint in the atomic norm minimization formulation. The plots in the appendix show reconstructions of the original signal $y$ using constrained and unconstrained versions of the compressed sensing and atomic norm formulation, all at a saturation fraction of 27%.

# 4 Conclusion and Future Work

This project explored various ways to recover the original signals from clipped measurements, and provided the comparison between 3 of these methods for the noiseless case. It was observed that incorporating information from the saturated indices in the form of linear constraints gives better results in both atomic norm and CS based approach. The reconstructions obtained using CS are better than the ones using constrained atomic norm formulation, since the positive semi-definiteness constraint used in the atomic norm formulation is looser as compared to the equality constraint in the CS formulation. The fact that the sensing matrix obeys RIP allows for solutions to the CS problem which has stricter constraints hence better results. But one needs to take note of an important caveat here that the perfect recovery guarantee only holds under the assumption of random positions of the clipped samples, which is not always expected for the signals and thresholding scheme chosen for this experiment. Moreover, it is likely that RIP won't hold for some dictionaries and higher sparsity values of the signal.

For the noisy case, the extension is quite straight forward for Q1, with slight modifications to the linear constraints in the form of slack variables to handle the effect of noise on saturation. The simulations for noisy measurements as done in the paper keep the saturation index fixed and vary the SNR levels to compare the performance. In context of compressed sensing, adding noise destroys the sparsity of the signal, but the signal is still compressible (depending on the value of the sparse coefficients of the original signal) and hence the objective function of the CS formulation can be augmented with an $l_2$ norm regularizer term to handle noise. Further extensions to this problem can be handling saturation of signals for non-additive noise models like Poisson noise. Also, the modification to the Q1 formulation needs to be studied thoroughly, with respect to quantifying analytically that how does it reduce the bias in amplitude estimates. Finding Vandermonde decompositions requires using specific libraries, one of which has been provided in the code repository for the project which can be accessed through this link.

## References


[BMW15] A. Barg, A. Mazumdar, and R. Wang. Restricted isometry property of random subdictionaries. *IEEE Transactions on Information Theory*, 61(8):4440–4450, Aug 2015.

[DMH+13] B. Defraene, N. Mansour, S. De Hertogh, T. van Waterschoot, M. Diehl, and M. Moonen. Declipping of audio signals using perceptual compressed sensing. *IEEE Transactions on Audio, Speech, and Language Processing*, 21(12):2627–2637, Dec 2013.

[Don06] D. L. Donoho. Compressed sensing. *IEEE Transactions on Information Theory*, 52(4):1289–1306, April 2006.

[ESJ17] F. Elvander, J. Swärd, and A. Jakobsson. Grid-less estimation of saturated signals. 15(3):372–376, Oct 2017.

[LMV86] A. Lepschy, G. Mian, and U. Viaro. Comments, with reply, on "arma spectral estimation of time series with missing observations" by b. porat and b. friedlander. *IEEE Transactions on Information Theory*, 32(4):601–602, July 1986.

[TBSR13] G. Tang, B. N. Bhaskar, P. Shah, and B. Recht. Compressed sensing off the grid. *IEEE Transactions on Information Theory*, 59(11):7465–7490, Nov 2013.

[YXS16] Z. Yang, L. Xie, and P. Stoica. Vandermonde decomposition of multilevel toeplitz matrices with application to multidimensional super-resolution. *IEEE Transactions on Information Theory*, 62(6):3685–3701, June 2016.

# 5 Appendix

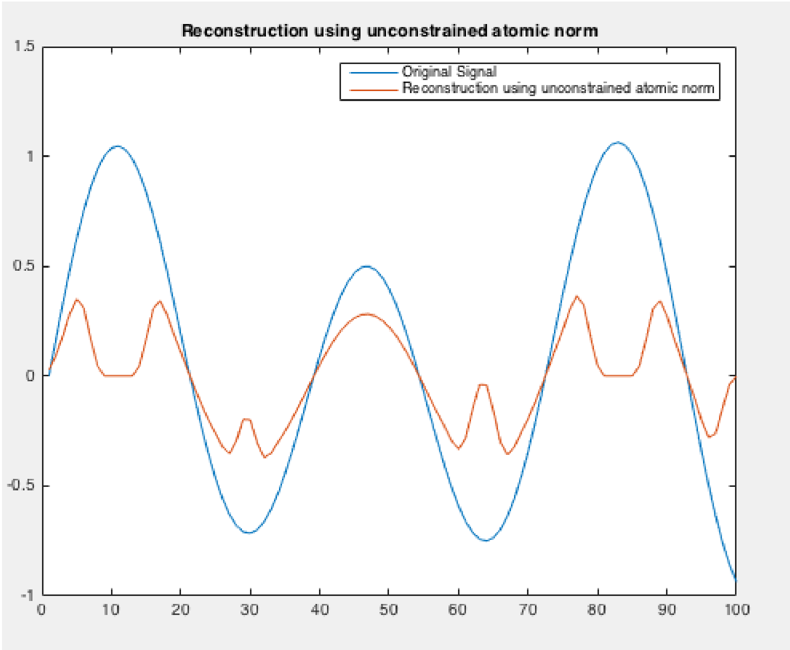


Figure 2: Unconstrained atomic norm

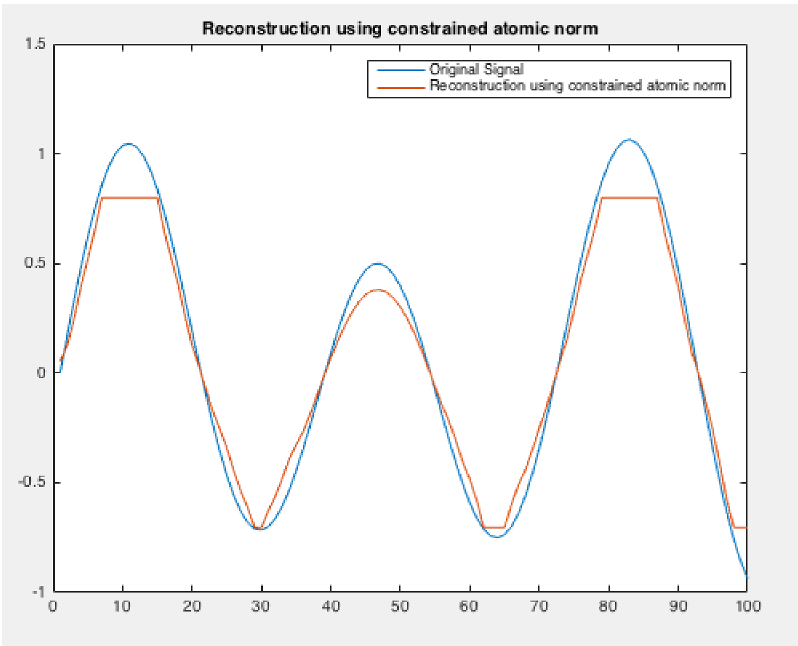


Figure 3: Constrained atomic norm

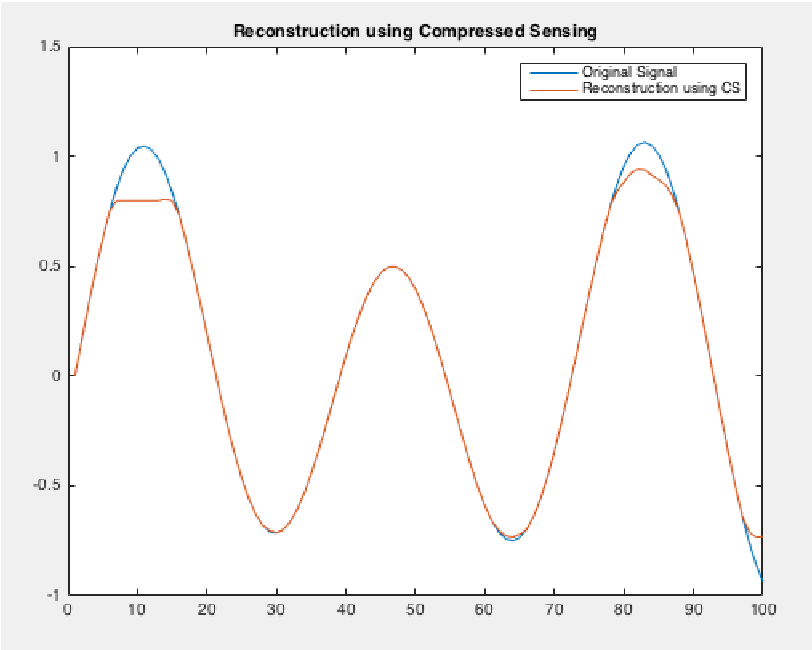


Figure 4: Constrained Compressed Sensing

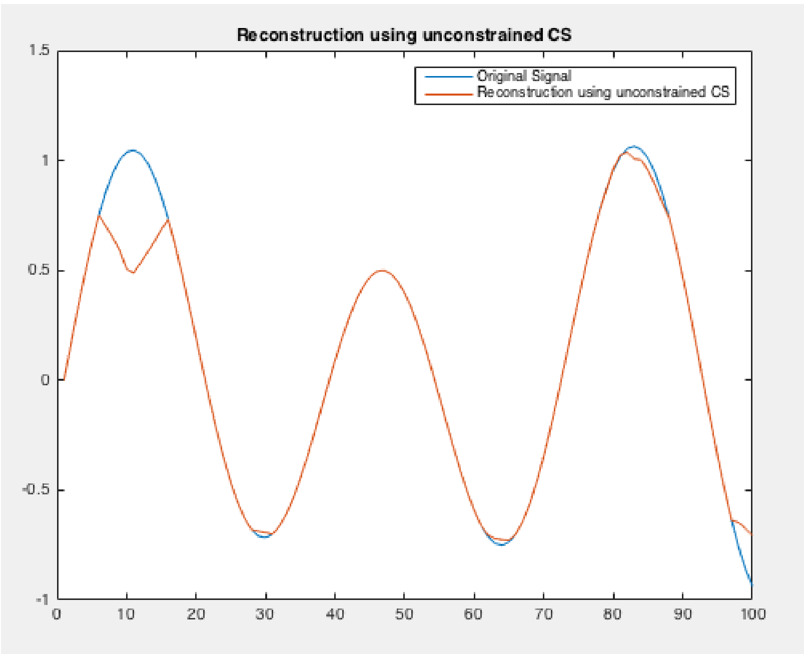


Figure 5: Unconstrained Compressed Sensing