CHAPTER 8

Quantum Optics

8.1 Introduction: The Classical and the Quantum

Optics is all about electromagnetic fields characterized by spatial variations over extremely small distances (∼500 nm), where the field is a dynamical system that can be treated as a *classical* one in explaining a large body of optical phenomena. The simplest classical systems are ones with a single degree of freedom, while the basic theoretical principles applying to such systems can be generalized in a straightforward manner to systems with a larger number of degrees of freedom as well. The electromagnetic field, on the other hand, is a system with a non-denumerably infinite number of degrees of freedom but this too can be described within the framework of the classical theory, with the Maxwell equations accounting for the space-time variations of the field vectors, the latter being the basic dynamical variables relevant to the system.

A more fundamental description of the electromagnetic field, however, is provided by quantum principles, where the quantum theory of a system differs from the classical one in several basic aspects.

In Section 8.2 I recall very briefly a number of basic principles of classical mechanics, where the mathematical formulation of the notions of states and observables (or 'dynamical variables') of a system is outlined, along with the equations describing its dynamics. This is followed by Section 8.3, where I outline the basic principles of the *quantum theoretic* description of systems. In the remaining sections of this chapter I will present an elementary outline of quantum optics ie, the quantum theory of optical fields.

8.2 The Classical Description of Systems

8.2.1 The Phase Space: Pure States and Observables

One basic difference between the classical and the quantum modes of description relates to the way the dynamical *state* of a system is specified. For a classical system, such as one with a single degree of freedom, the state is completely specified by means of the values of the position variable (q) and the momentum variable (p), these being two basic *observable*

quantities for the system. This constitutes a complete specification of the state in that the value for any other observable quantity is unambiguously determined in terms of q and p.

The observable quantities ('observables' in brief) are the measurable properties of the system under consideration and are defined as real functions of the basic variables (q and p for a system with a single degree of freedom) since all results of measurements are ultimately expressed as real-valued quantities (which can be combined into complex-valued ones if necessary).

An observable quantity or *dynamical variable* of crucial importance is the *Hamiltonian* of the system, which in numerous situations of interest represents its energy, and which determines the time evolution of the state, as we see below in Eq. (8.2). As an example, I refer to the Hamiltonian of the *harmonic oscillator*:

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2, \tag{8.1}$$

where m represents the mass and ω the angular frequency of the oscillator.

The instantaneous state of a system with a single degree of freedom can be represented geometrically by a point in a two-dimensional (2D) *phase space* as shown in Fig. 8.1A. The time evolution of the state, described by *Hamilton's equations of motion*,

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}, \tag{8.2}$$

is represented by a curve in the phase space, referred to as its 'trajectory,' an example of which is shown in Fig. 8.1A. Possible motions of the system, resulting from various different *initial conditions* correspond to a *family* of trajectories in the phase space, a number of trajectories belonging to such a family for a harmonic oscillator being shown in Fig. 8.1B. On each of these trajectories the energy of the oscillator, which is given by the value of the Hamiltonian function (Eq. 8.1) for any given values of q and p, remains constant.

Generalizations to systems with more than one degree of freedom are straightforward. Thus for a system with coordinates $q_1,\ q_2, \ldots, q_n$, and momenta $p_1,\ p_2, \ldots, p_n$, the phase space is $2n$-dimensional, and a point in this phase space, corresponding to specified values of these variables, constitutes a complete description of the instantaneous state of the system. The Hamiltonian $H(\{q\}, \{p\})$ (I adopt a convenient abbreviation in notation) determines the time evolution of the state described by Hamilton's equations—a set of $2n$ first-order differential equations in time:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad (i = 1, 2, \ldots, n). \tag{8.3}$$

As one other instance of an observable quantity, or a 'dynamical variable,' I mention the angular momentum of a system of particles which actually is a vector function of the position

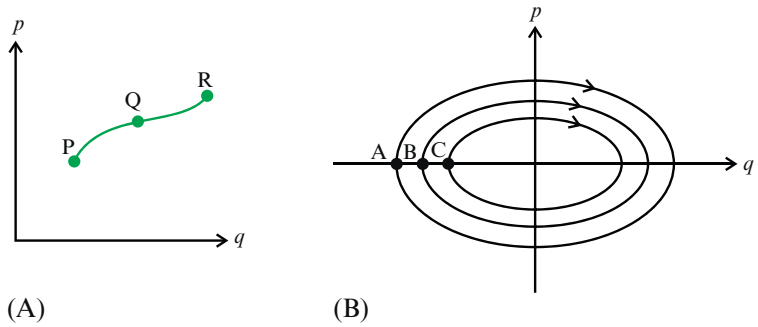

Fig. 8.1
Geometrical representation of the state of a classical system and that of the time evolution of a state. (A) A system with a single degree of freedom and, correspondingly, a two-dimensional phase space made up of the coordinate q and the momentum p. A state is represented by a point such as P, while the time evolution of the state is represented by a curve in the phase space such as PQR, termed its *trajectory*. (B) A family of trajectories for a one-dimensional harmonic oscillator. The points A, B, and C represent instantaneous states at any given instant of time. Given these as initial conditions, the states evolve in time, giving rise to elliptic trajectories, on each of which the energy of the oscillator, represented by the Hamiltonian function, has a constant value.

and momentum coordinates, and is made up of three distinct dynamical variables. Once the state of the system is known, corresponding to a point in the phase space at any given instant of time, the value of a dynamical variable is precisely determined in terms of the values of the position and momentum variables for that point.

> The value of a dynamical variable corresponding to any given point in the phase space can be confirmed by experimental observations to within the limits set by the experimental errors.

However, this description of a state in terms of a single point in the phase space is *maximal* in the sense that it is the most complete and precise description possible. Such a description specifies what is referred to as a *pure state* of the system under consideration. More generally, a system can be said to be in a *mixed state* (see Section 8.2.2 below), where a pure state can be regarded as a limiting case of a mixed state.

8.2.2 Mixed Classical States: Distribution Functions

The classical description of the state of a system in terms of the position and momentum coordinates corresponding to a point in the phase space is the most precise one possible in that it gives one all the information that one can conceivably have about the system at any given instant of time. It is also an *idealized* description since it is only rarely that one can pin the state down to a precisely defined point in the phase space. For instance, if you try to determine

the position and momentum of the bob of a simple pendulum at any given instant of time, you can do so only within certain ranges of possible errors determined by the experimental conditions.

What this means is that, in practice, one can identify a *set of points* in the phase space such that the instantaneous state may correspond to any one of these points, and one can also specify the *probabilities* of the states represented by the various different points belonging to the set. In other words, the state is then defined in terms of a *probability distribution* in the phase space.

For instance, if the state is equally likely to correspond to all the points in a discrete set of N points in the phase space, then the probability distribution is uniform over the set (ie, the probability for each point is $\frac{1}{N}$).

A mixed state is, more generally, described not in terms of a probability distribution over a discrete set of points but of one over a continuously distributed set or, in a broader sense, over the *entire* phase space. In such a description one specifies a *probability density* function (also referred to as the 'probability distribution' or the 'distribution function') $\rho(q,p)$ over the phase space such that the probability of the phase point, representing the instantaneous state, being located within a small range q to $q+dq$, p to $p+dp$ is $\rho(q,p)dqdp$. Evidently, $\rho(q,p)$ has to satisfy the normalization condition (we consider a single degree of freedom for simplicity)

$$\int dqdp\rho(q,p) = 1, \tag{8.4}$$

where the integration is over the entire phase space, since the probability of the phase point being located anywhere in the phase space without any restriction has to be unity.

1. A word on notation. It is common to adopt a short-hand notation for systems with more than one degree of freedom, described by position and momentum coordinates, say, $q_1, q_2, \ldots, q_n; p_1, p_2, \ldots, p_n$, where $\rho(q,p)$ actually means $\rho(q_1, q_2, \ldots, q_n; p_1, p_2, \ldots, p_n)$, q to $q+dq$ actually means q_1 to $q_1+dq_1, \ldots, q_n$ to q_n+dq_n, and so on. At times a symbol such as $\{q\}$ is used to denote a set $\{q_1, q_2, \ldots, q_n\}$.
2. Symbols such as dq or dp are used in this book in the 'physicists' sense,' mostly to denote small intervals; strictly speaking, these symbols are meaningful only under the integral sign.
3. The position and momentum coordinates $\{q\}$ and $\{p\}$ do not always stand for the Cartesian components of the position and momentum vectors of the particles making up the system under consideration, but are in the nature of *generalized* position and momentum variables, some of which may be, for instance, angular coordinates and angular momenta. We will assume that all the generalized coordinates are independent ones (ie, they are not subject to any equations of constraint between them). Corresponding to each generalized coordinate q, one can define a canonically conjugate momentum such that all the conjugate pairs so defined make up the sets $\{q\}$ and $\{p\}$.

For instance, suppose that the only thing known about the instantaneous state of a harmonic oscillator is that its energy lies between E and $E + \delta E$, states (ie, points in the phase space) being equally likely within this range of their energies. This corresponds to a probability density

$$\rho(q,p) = \frac{\omega}{2\pi\delta E} \tag{8.5}$$

at all points in an annulus of elliptic shape (see Fig. 8.2) in the 2D phase space, having an area $\frac{2\pi}{\omega}\delta E$, with the probability density being zero for all points lying outside this area.

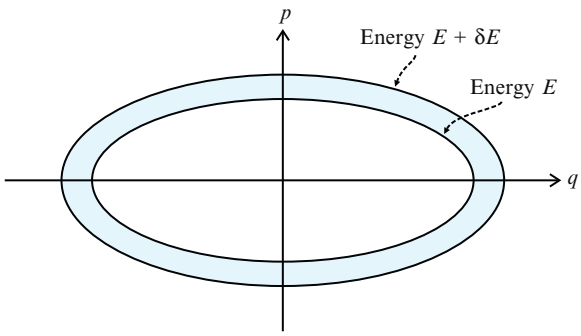

Fig. 8.2
Annular area of elliptic shape in the two-dimensional phase space of a harmonic oscillator, corresponding to the energy lying within a small range from E to $E + \delta E$. The semi-axes of the inner (or the outer) ellipse are $\sqrt{\frac{2E}{m\omega^2}}$ and $\sqrt{2mE}$. For a mixed state with uniform probability distribution within the annular region, the probability density is given by Eq. (8.5) within the annulus and is zero for points outside the annulus.

Formally speaking, even a pure state can be characterized by a probability distribution, which tells us that a pure state is a special (or limiting) instance of a mixed state. Thus a pure state represented by the phase point (q_0, p_0) in a 2D phase space corresponds to a probability density that is concentrated at a single point in the phase space and is given by

$$\rho(q,p) = \delta(q - q_0)\delta(p - p_0), \tag{8.6}$$

where the symbol δ denotes the Dirac delta function.

Since a pure state at (q,p) corresponds to precise values of the position and momentum coordinates, an observable $A(q,p)$ also has a precisely defined value in a pure state. In a mixed state, on the other hand, the observable A does not, in general, have a precise value since the

mixed state corresponds to various possible pure states, with a probability distribution over the latter. The result of a measurement of A in the mixed state characterized by the probability distribution $\rho(q,p)$ in the phase space thus corresponds to a random variable, of which the mean or *expectation value* is given by

$$\langle A\rangle = \int dqdpA(q,p)\rho(q,p) \tag{8.7}$$

(reason this out). In the special case of a pure state (q_0, p_0), the use of Eq. (8.6) gives the value $A(q_0, p_0)$ for $\langle A\rangle$, which is consistent with the fact that A has the precise value $A(q_0, p_0)$ in the pure state.

A mixed state evolves in time as does a pure state. This time evolution is described by the *Liouville equation*—a generalization of the Hamiltonian equation of motion for a pure state—which reads

$$\frac{\partial\rho}{\partial t} = \{H,\rho\}, \tag{8.8a}$$

where $\{H,\rho\}$ stands for the *Poisson bracket* of the functions $H(q,p)$ and $\rho(q,p)$, defined as

$$\{H,\rho\} = \frac{\partial H}{\partial q}\frac{\partial\rho}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial\rho}{\partial q}. \tag{8.8b}$$

8.2.3 Composite Systems and Reduced States

At times one needs to consider a *composite system* made up of two or more subsystems. For simplicity, let us consider systems A and B, each of a single degree of freedom, making up the composite system C. Let the position and momentum coordinates of A be q and p and those of B be Q and P. The phase space of the composite system C is then a four-dimensional (4D) one, depicted symbolically in Fig. 8.3, where the horizontal axis represents the phase space of A and the vertical axis represents the phase space of B. The point P then represents a pure state in the 4D phase space, with coordinates, say, q_0, Q_0, p_0, P_0; it corresponds to a pure state of A at (q_0, p_0) in its own 2D phase space (depicted by the horizontal axis in Fig. 8.3) because these values are unambiguously implied by the point P in the 4D phase space and, likewise, it corresponds to a pure state of B at (Q_0, P_0).

In other words, given a pure state of C, both A and B can be said to be in pure states of their own (as we will see in Section 8.3.12, this is by no means true for a composite *quantum* system). The latter are termed *reduced states* of the subsystems corresponding to the given (pure) state of C. In the case of a *mixed* state of C, one can again define reduced states of the subsystems as indicated below, where these reduced states are, in general, mixed ones.

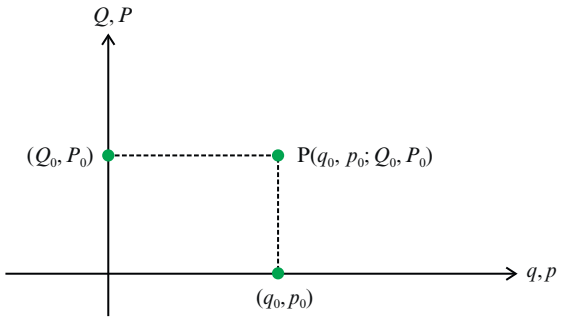

Fig. 8.3
Depicting symbolically the four-dimensional phase space of a composite system (C) made up two subsystems A and B, each with a single degree of freedom. The horizontal axis symbolically depicts the two-dimensional phase space of A, described in terms of coordinates q, p, while the vertical axis similarly depicts the phase space of B, with coordinates Q, P. The point P represents a pure state of C which corresponds to pure states of A and B at (q_0, p_0) and (Q_0, P_0), respectively.

Imagine an observable $A(q,p)$ that pertains to the subsystem A. When looked at in the context of C, it constitutes an observable of a special kind since it depends on q, p alone, and not on Q, P. According to Eq. (8.7), the expectation value of this observable in the mixed state $\rho(q,Q,p,P)$ of C is

$$\langle A\rangle = \int dqdpdQdPA(q,p)\rho(q,Q,p,P) = \int dqdpA(q,p)\tilde{\rho}(q,p), \tag{8.9a}$$

where $\tilde{\rho}(q,p)$ is defined by

$$\tilde{\rho}(q,p) = \int dQdP\rho(q,Q,p,P). \tag{8.9b}$$

This tells us that the statistics of measurement of the observable A in the mixed state ρ of C is the same as that in the mixed state of A corresponding to the probability density $\tilde{\rho}$. One then says that $\tilde{\rho}$ defines the *reduced state* of A corresponding to the state ρ of C, the reduced state of B being similarly defined.

A state of C corresponding to a probability density of the special form

$$\rho(q,Q,p,P) = \rho^{\mathrm{A}}(q,p)\rho^{\mathrm{B}}(Q,P) \tag{8.10}$$

is said to be a *product* state, in which the factors ρ^{A} and ρ^{B} (each assumed to satisfy the normalization condition separately) are precisely the reduced states of A and B, respectively. In this case the subsystems A and B are uncorrelated with each other. On the other hand, a mixed state of C of the form

$$\rho(q,Q,p,P) = \rho_1^{\mathrm{A}}(q,p)\rho_1^{\mathrm{B}}(Q,P) + \rho_2^{\mathrm{A}}(q,p)\rho_2^{\mathrm{B}}(Q,P) \tag{8.11}$$

involves a correlation between A and B. The Fourier transformation theorem tells us that *any* mixed state $\rho(q, Q, p, P)$ of C can be expressed as a sum (or, more generally, an integral) of products of the above form. As we will see in Section 8.3.12, a corresponding statement does not hold in the case of a composite *quantum* system. As a result, a composite quantum system involves a correlation between its subsystems that is quite distinct from the classical correlation that may exist between these subsystems. This exclusively quantum component of the correlation is referred to as *entanglement*, and does not have a classical counterpart.

8.3 The Quantum Description

8.3.1 State Vectors and the State Space

8.3.1.1 The representation of states by vectors: The Hilbert space

The quantum description of systems is of a different nature as compared with the classical description, being subtler and requiring an appropriate *interpretation* so as to relate it to our everyday experience. For a system with one position and one momentum variable (q and p, respectively), a complete specification of the state at any given instant is equivalent to specifying a complex-valued square-integrable function ψ of the position variable (q), of unit norm, known as the *wave function* of the system.

There is a more abstract (and, in a sense, more convenient and flexible) mathematical description of the state in quantum theory—namely, one in terms of an element in an *infinite-dimensional linear vector space*, associated with the field of *complex numbers* as scalars. The vectors (at times referred to as *state vectors*) belonging to the vector space can be *represented* in more ways than one. The wave function constitutes one such representation of the state vector of a quantum system.

The vector space whose elements correspond to the states of a quantum system is in the nature of a *Hilbert space* (we denote it by the symbol $\mathcal{H}$; the term 'state space' is also commonly used) and is characterized by a number of features that can be stated in technical terms. However, I will steer clear of the technicalities in this book and will speak of operations involving the vectors in terms analogous to those relating to a finite-dimensional vector space.

> There exist more general states of a quantum system that can be described in terms of *density operators* in the Hilbert space and these are termed *mixed* states (see Section 8.3.10). By contrast, the states described by vectors in the Hilbert space are termed *pure states*, which occur as limiting instances of mixed states. This is analogous to the distinction between pure and mixed states in the classical description.

A vector in the state space of a system is denoted by means of the symbol $|\cdots\rangle$ so that, for instance, $|u\rangle$ and $|\psi\rangle$ are two vectors where u and ψ are labels for identifying specific states. One can choose a set of vectors $|e_1\rangle$, $|e_2\rangle, \ldots, |e_n\rangle, \ldots$ constituting a *basis* in the Hilbert

space so that any vector $|u\rangle$ can be expressed as a *linear combination* of the basis vectors in the form

$$|u\rangle = u_1|e_1\rangle + u_2|e_2\rangle + \cdots + u_n|e_n\rangle + \cdots = \sum_{n=1}^{\infty} u_n|e_n\rangle, \tag{8.12}$$

where u_i $(i = 1, 2, \ldots)$ stand for an infinite number of scalars depending on the vector $|u\rangle$ and on the basis chosen. Since the Hilbert space is infinite-dimensional, a basis necessarily contains an infinite number of vectors, and the right-hand side of Eq. (8.12) contains, in general, an infinite number of terms. Though an infinite sum of vectors brings in questions of a mathematical nature, such as the one of convergence, one can assume that the infinite sums can be handled in a manner analogous to finite ones, still arriving at meaningful results.

The infinite set of basis vectors in the Hilbert space can be chosen in more ways than one (actually, in an *infinite* number of ways), which means that there exist alternative expansions analogous to the right-hand side of Eq. (8.12) for any chosen vector $|u\rangle$. For any one of the various possible expansions of $|u\rangle$, such as the one in Eq. (8.12), the set of scalars occurring in the expansion may be thought to constitute an infinite column $(u_1\ u_2\ \ldots\ u_n\ \ldots)^{\mathrm{T}}$, which is then said to *represent* the vector $|u\rangle$ in the basis chosen (ie, the one made up of the vectors $|e_1\rangle,\ |e_2\rangle, \ldots, |e_n\rangle, \ldots$ in the present instance).

8.3.1.2 The inner product and the norm

The Hilbert space admits of the operation of the *inner product* (or *scalar* product), whereby any two vectors $|u\rangle$ and $|v\rangle$ produce, by means of this operation, a scalar (ie, a complex number). In this context the concept of *dual* vectors is of relevance. The dual of any chosen vector $|u\rangle$ is denoted by the symbol $\langle u|$ (while $|u\rangle$ is termed a *ket vector*, its dual $\langle u|$ is referred to as a *bra vector*; the names are derived from the splitting of the word 'bracket' into two halves; the bra-ket notation for vectors and their duals is, famously, due to Dirac). Technically speaking, the dual $\langle u|$ resides in a different vector space, namely, the dual of $\mathcal{H}$. A dual such as $\langle u|$ is defined to be an object that acts on any vector, say, $|v\rangle$, where the result of the operation is a scalar that depends linearly on the operand $|v\rangle$.

> The dual vector corresponding to a linear combination of the form $\alpha_1|u_1\rangle + \alpha_2|u_2\rangle$ is $\langle u_1|\alpha_1^* + \langle u_2|\alpha_2^*$.

This operation of a bra vector $\langle u|$ on a ket vector $|v\rangle$ can be chosen to produce the scalar resulting from an *inner product* between vectors $|u\rangle$ and $|v\rangle$ in $\mathcal{H}$, where the scalar is denoted by the symbol $\langle u|v\rangle$. The inner product is characterized by the following important properties:

1. For any two vectors $|u\rangle$ and $|v\rangle$,

$$\langle u|v\rangle = \langle v|u\rangle^*. \tag{8.13a}$$

2. If α_1 and α_2 are any two scalars and $|u\rangle$, $|v_1\rangle$, and $|v_2\rangle$ are arbitrarily chosen vectors, then

$$\langle u|\alpha_1 v_1 + \alpha_2 v_2\rangle = \alpha_1\langle u|v_1\rangle + \alpha_2\langle u|v_2\rangle, \tag{8.13b}$$

   where the symbol $|\alpha_1 v_1 + \alpha_2 v_2\rangle$ stands for the vector represented by the linear combination $\alpha_1|v_1\rangle + \alpha_2|v_2\rangle$.
3. For any and every vector $|u\rangle$,

$$\langle u|u\rangle \geq 0, \tag{8.13c}$$

   where the equality holds if and only if $|u\rangle$ is the null vector.

Two vectors $|u\rangle$ and $|v\rangle$ for which $\langle u|v\rangle = 0$ are said to be *orthogonal* to each other—a relation analogous to the orthogonality of a pair of directed line segments, the latter being elements of a real three-dimensional (3D) vector space.

Given a vector $|u\rangle$, the nonnegative real number $\sqrt{\langle u|u\rangle}$ is referred to as its *norm* (where the positive square root is to be taken) and is denoted by the symbol $\|u\|$. If $|u\rangle$ is not the null vector (the null vector being one of norm zero), then $|v\rangle = \frac{1}{\|u\|}|u\rangle$ is *a unit vector* (ie, a vector of norm unity).

The basic rules of quantum theory imply that a vector $|u\rangle$ and a vector $\alpha|u\rangle$ obtained from it by multiplication with a scalar α correspond to the *same* state of the quantum system under consideration. Among the various different choices of the vector $\alpha|u\rangle$ corresponding to different possible values of the multiplier α, it is convenient to choose one for which the norm is unity. Accordingly, when one speaks of a 'state vector' $|u\rangle$, one commonly refers to a vector of unit norm.

> There still remains an ambiguity in specifying the vector corresponding to a state since if the vector $|u\rangle$ is of unit norm, then so is the vector $e^{i\phi}|u\rangle$, where ϕ is any arbitrarily chosen phase. The phase is of no relevance when one refers to the state under consideration without reference to other possible states of the system.

While a basis can be chosen in a Hilbert space $\mathcal{H}$ in an infinite number of ways, it is of considerable convenience to choose an *orthonormal* one, for which the basis vectors (say, $|e_1\rangle$, $|e_2\rangle, \ldots$) are each of unit norm and are pairwise orthogonal—that is,

$$\langle e_i|e_j\rangle = \delta_{ij} \quad (i,j = 1,2,\ldots), \tag{8.14}$$

where δ_{ij} denotes the Kronecker delta symbol.

If now one considers a state vector $|u\rangle$ of unit norm and expresses it in terms of an orthonormal basis made up of vectors $|e_1\rangle$, $|e_2\rangle, \ldots$, as in Eq. (8.12), then the squared moduli of the coefficients u_i $(i = 1,2,\ldots)$ add up to unity:

$$\sum_i |u_i|^2 = 1. \tag{8.15}$$

These coefficients in the expansion of the state vector $|u\rangle$ as a linear combination of vectors making up an orthonormal basis can be expressed as inner products of $|u\rangle$ with the respective basis vectors:

$$u_i = \langle e_i|u\rangle \tag{8.16a}$$

(check this out)—that is,

$$|u\rangle = \sum_i \langle e_i|u\rangle|e_i\rangle. \tag{8.16b}$$

Finally, here is the formula for the inner product of two vectors $|u\rangle$ and $|v\rangle$ in terms of their representative columns $(u_1, u_2, \ldots)^{\mathrm{T}}$, $(v_1, v_2, \ldots)^{\mathrm{T}}$, referred to the orthonormal basis $\{|e_i\rangle\}$ $(i = 1, 2, \ldots)$:

$$\langle u|v\rangle = \sum_i u_i^* v_i. \tag{8.17}$$

8.3.1.3 The wave function representing a state vector

While a set of basis vectors like $\{|e_1\rangle, |e_2\rangle, \ldots\}$ is a *countable* one, it may sometimes be necessary to consider, in the infinite-dimensional state space $\mathcal{H}$, an *uncountable* set of basis vectors as well like $\{|q\rangle\}$ $(-\infty < q < \infty)$, labeled with a continuously varying real number q. Once again, such a set can be chosen to be an orthonormal one, though the orthonormality condition (8.14) looks different in this case. If $|q\rangle$ and $|q'\rangle$ are two vectors, corresponding to real numbers q and q' $(-\infty < q, q' < \infty)$, belonging to such an orthonormal basis, then the orthonormality condition takes the following form (compare this with Eq. 8.14):

$$\langle q|q'\rangle = \delta(q - q'), \tag{8.18}$$

where $\delta(q - q')$ stands for the Dirac delta function with argument $q - q'$.

Given the orthonormal basis $\{|q\rangle\}$, one can expand any given vector $|\psi\rangle$ as a linear combination of the basis vectors as

$$|\psi\rangle = \int_{-\infty}^{\infty} dq \psi(q)|q\rangle, \tag{8.19a}$$

where the continuously varying set of coefficients $\psi(q)$ is obtained as

$$\psi(q) = \langle q|\psi\rangle \quad (-\infty < q < \infty), \tag{8.19b}$$

which implies that the vector ψ can be expressed in the form

$$|\psi\rangle = \int_{-\infty}^{\infty} dq \langle q|\psi\rangle|q\rangle. \tag{8.19c}$$

Formulae (8.19a), (8.19b), and (8.19c) are analogous to Eqs. (8.12), (8.16a), and (8.16b), respectively, with the difference that one now has integrals over the continuous parameter q instead of discrete summations.

Formula (8.19a) defines the sense in which the vector $|\psi\rangle$ can be said to be represented by the *function* ψ, the latter being analogous to the infinite column (with entries u_i $(i = 1, 2, \ldots)$) representing the vector $|u\rangle$ (see formula (8.12)). This function, which represents the state vector $|\psi\rangle$ in the basis $\{|q\rangle\}$ $(-\infty < q < \infty)$, is commonly referred to as the *wave function* representing the state in the chosen basis.

> At times one writes $\psi(q)$ to denote the function ψ, while, more precisely, $\psi(q)$ denotes the *value* of the function ψ at the point with the coordinate q on the real line depicting the entire range of possible values of q.

With reference to a different orthonormal set of basis vectors, say. $\{|p\rangle\}$ $(-\infty < p < \infty)$, now labeled with the real parameter p, one would have a different representation of the *same* vector $|\psi\rangle$ by the function ϕ, where

$$\phi(p) = \langle p|\psi\rangle \quad (-\infty < p < \infty) \tag{8.19d}$$

(see Section 8.3.5.3).

Two particular instances of such sets of basis vectors, commonly denoted as $\{|q\rangle\}$ and $\{|p\rangle\}$, correspond to what are referred to as the *coordinate* and *momentum representations* for a system with one degree of freedom (see Section 8.3.5, where the generalization to more than one degrees of freedom is also included). As will be seen later, the corresponding functions ψ and ϕ are then related to each other by *Fourier transformation*.

> A basis made up of an uncountable set of vectors, with its attendant delta-function normalization and formulae involving integrals instead of summations, brings in mathematical questions to be settled before one can unambiguously set up a correspondence between mathematical formulae and physical quantities. However, we will not be concerned with such mathematical questions here, and will interpret the relevant formulae in a common sense way, ignoring the questions of a technical nature, since such a simplified interpretation still leads to meaningful results and statements.

With reference to the expansion (8.19c) of a vector $|\psi\rangle$ in terms of an uncountable basis $\{|q\rangle\}$ with the basis vectors labeled with the continuously varying parameter q $(-\infty < q < \infty)$, the condition that $|\psi\rangle$ is of unit norm reads (compare this with formula (8.15) for the case of a countable basis)

$$\int_{-\infty}^{\infty} |\psi(q)|^2 dq = 1. \tag{8.20}$$

The square root of the expression on the left-hand side of Eq. (8.20) is referred as the *norm* of the function ψ, where it is understood that the positive square root is to be taken. It is in this sense that the state corresponding to the vector $|\psi\rangle$ (or, in brief, the *state* $|\psi\rangle$) of a quantum

system can be said to be completely specified by a complex-valued wave function ψ of unit norm. Given a vector $|\psi\rangle$ of norm N (say), not necessarily unity, one can convert it to one of unit norm by multiplying it with the scalar $\frac{1}{\sqrt{N}}$ (or, more generally, with $\frac{e^{i\phi}}{\sqrt{N}}$, where ϕ is any arbitrarily chosen phase), while a corresponding statement holds for the wave function representing $|\psi\rangle$ as well.

Finally, analogous to formula (8.17), the formula for the inner product of any two state vectors $|\psi_1\rangle$ and $|\psi_2\rangle$, with wave functions ψ_1 and ψ_2 referred to an orthonormal basis $\{|q\rangle\}$ $(-\infty < q < \infty)$, can be expressed as the integral

$$\langle\psi_1|\psi_2\rangle = \int_{-\infty}^{\infty} \psi_1^*(q)\psi_2(q)dq. \tag{8.21}$$

8.3.1.4 State spaces of finite dimensions

One sometimes finds that the states of a quantum system are described, in some limited context, as belonging to a *finite*-dimensional vector space. For instance, only the vectors belonging to a finite-dimensional factor in an infinite-dimensional *product* space may be relevant in a problem.

Such a situation arises when one considers states relating to some *internal* degrees of freedom of a system, the most familiar example of which is the *spin* of a particle. The wave function then depends on a finite set of indices corresponding to these internal variables, in addition to its arguments $\{q\}$ or $\{p\}$, where the latter may be regarded as external variables relating to the motional states of the system under consideration in space. In some situations the dependence on the internal variables becomes effectively decoupled from the dependence on the external ones. One can then refer to a finite-dimensional space in respect of the internal states of the system, considered all by themselves. For instance, the description of the spin states of an electron or the polarization states of a photon with a given momentum requires a 2D vector space.

More generally, one may be required to consider a finite-dimensional *subspace* of an infinite-dimensional vector space since the dynamics of states described by vectors belonging to this subspace becomes effectively independent of the dynamics in the bigger space. For instance, the 'two-state atom' is a system for which there occur transitions between the two states with the lowest energies (the 'ground state' and the 'first excited state'), while all the other states can be effectively ignored in the description of the dynamics of the atom.

A *complete* description of the states of a quantum system, however, necessarily involves an infinite-dimensional space.

In the following, the space $\mathcal{H}$—whether infinite-dimensional or effectively finite-dimensional—will be referred to as the *state space* of the system under consideration, while the vectors in this space describing the states of the system will be termed *state vectors*.

8.3.2 Linear Operators

A quantum system is characterized by a number of measurable properties or 'observables,' where the measured values of these observables can change as the state of the system changes with time. This is why these are also referred to as 'dynamical variables.' Examples of these dynamical variables are the position coordinates and momentum components of a particle.

In quantum theory, dynamical variables of a system correspond to *linear operators* in its state space $\mathcal{H}$ (more precisely, to a certain class of linear operators, the *Hermitian* ones; see Section 8.3.3), where a linear operator $\hat{A}$ is a mapping from $\mathcal{H}$ into itself that acts on a vector to produce another vector, preserving linearity. In other words, if $|u\rangle$ and $|v\rangle$ are two vectors and α and β are any two complex numbers, then the defining property for $\hat{A}$ can be expressed as

$$\hat{A}(\alpha|u\rangle + \beta|v\rangle) = \alpha(\hat{A}|u\rangle) + \beta(\hat{A}|v\rangle), \tag{8.22}$$

where $\hat{A}|u\rangle$, for instance, stands for the vector produced by the action of $\hat{A}$ on $|u\rangle$.

A linear operator is completely determined once its action on a set of basis vectors has been specified. For an orthonormal basis made up of vectors $|e_1\rangle$, $|e_2\rangle, \ldots, |e_n\rangle, \ldots$, the action of $\hat{A}$ on, say, $|e_i\rangle$ $(i = 1, 2, \ldots)$ can be expressed as a linear combination of the basis vectors, and hence is of the form

$$\hat{A}|e_i\rangle = \sum_j A_{ji}|e_j\rangle, \tag{8.23}$$

where A_{ji} $(i, j = 1, 2, \ldots)$ are complex numbers depending on the operator $\hat{A}$ and on the basis chosen.

Suppose now that $|u\rangle$ is a vector represented by the column $(u_1, u_2, \ldots)^{\mathrm{T}}$ in the above basis. Then the column $(v_1, v_2, \ldots)^{\mathrm{T}}$ of the vector $|v\rangle = \hat{A}|u\rangle$ in the same basis is given by

$$v_i = \sum_j A_{ij}u_j \quad (i = 1, 2, \ldots). \tag{8.24}$$

In other words, the column representing $|v\rangle$ is obtained from that representing $|u\rangle$ by multiplication of the latter from the left with the *matrix* whose ij-element is A_{ij}. One then says that this matrix represents the operator $\hat{A}$ in the basis chosen.

In the case of a nondenumerable orthonormal basis made up of vectors $|q\rangle$, with the parameter q ranging from $-\infty$ to ∞, formula (8.24) appears in the form

$$\phi(q') = \int A(q', q))\psi(q)dq, \tag{8.25a}$$

where

$$A(q', q) = \langle q'|\hat{A}|q\rangle. \tag{8.25b}$$

Here ψ and ϕ are the wave functions corresponding to the vectors $|\psi\rangle$ and $|\phi\rangle = \hat{A}|\psi\rangle$, respectively, and $A(q', q)$ stands for an element of the matrix representing $\hat{A}$, with row index q' and column index q, where these indices are now continuously varying ones. Incidentally, an expression of the form $\langle u|\hat{A}|v\rangle$ stands for the scalar product of the vectors $|u\rangle$ and $\hat{A}|v\rangle$.

8.3.2.1 Eigenvalues and eigenvectors

For a linear operator $\hat{A}$, suppose that there exists a scalar a and a corresponding nonzero vector $|a\rangle$ such that

$$\hat{A}|a\rangle = a|a\rangle. \tag{8.26}$$

In that case, $|a\rangle$ is termed an *eigenvector* of $\hat{A}$ corresponding to (or belonging to) the *eigenvalue* a. Thus the action of $\hat{A}$ on an eigenvector $|a\rangle$ is of a special type: $|a\rangle$ is multiplied by the scalar a. In the context of a quantum system, $|a\rangle$ and $\hat{A}|a\rangle$ represent the same state of the system, though these may not be of unit norm.

For any given linear operator $\hat{A}$, there may be more than one independent eigenvector corresponding to one and the same eigenvalue a. Consider now the set of all distinct eigenvalues of $\hat{a}$ and, for each of these eigenvalues, the set of all independent eigenvectors belonging to it. The collection of all these eigenvectors belonging to the various distinct eigenvalues forms a bigger set that turns out to be linearly independent, but is not necessarily *complete*. In other words, the set of all possible eigenvectors belonging to the distinct eigenvalues of a linear operator need not form a basis. There exist important classes of operators, though, such that, for an operator belonging to any of these, one does obtain a basis in this way, instances of these being the *unitary* and the *Hermitian* ones (see Sections 8.3.2.2 and 8.3.2.3 below).

Incidentally, if $\tilde{A}$ stands for the matrix representing the operator $\hat{A}$ in some chosen basis and $\tilde{a}$ stands for the column representing the eigenvector $|a\rangle$ belonging to the eigenvalue a in the same basis, then one has

$$\tilde{A}\tilde{a} = a\tilde{a}, \tag{8.27}$$

where $\tilde{A}\tilde{a}$ is the column obtained by the rules of matrix multiplication. More generally, there exists a correspondence between relations involving operators and vectors and those involving matrices and columns. At times it is of no harm to gloss over the distinction between the two sets of categories (operators, vectors, and matrices, columns; the bra vectors correspond to *rows* in this scheme of things). For instance, the operator product $\hat{A}\hat{B}$ may be replaced with the matrix product $\tilde{A}\tilde{B}$, or the unit operator ($\hat{I}$) may be replaced with the unit matrix ($\tilde{I}$), provided that objects of one category are not mixed up with those belonging to the other. Even the notation may be abbreviated by omission of the caret symbol for an operator and the tilde for a matrix (or for a column) such as in writing A for the operator $\hat{A}$.

8.3.2.2 Hermitian operators

Among the linear operators in a vector space $\mathcal{H}$ for a quantum system, the *Hermitian* operators are of special relevance. A Hermitian operator $\hat{A}$ is characterized by the property

$$\hat{A}^{\dagger} = \hat{A}, \tag{8.28}$$

where $A^{\dagger}$ stands for the *Hermitian conjugate* of $\hat{A}$.

> The Hermitian conjugate of an operator $\hat{A}$ is an operator $\hat{A}^{\dagger}$ characterized by the property that, for arbitrarily chosen vectors $|u\rangle$ and $|v\rangle$, the equality
>
> $$\langle v|\hat{A}^{\dagger}|u\rangle = \langle u|\hat{A}|v\rangle^{*} \tag{8.29}$$
>
> holds. From a formal point of view, the operator $\hat{A}^{\dagger}$ acts on vectors in the dual space of $\mathcal{H}$.

Hermitian operators are relevant in quantum theory in that, as I have mentioned earlier, observable quantities for a quantum system are described by means of such operators (see Section 8.3.3).

Hermitian operators are special in the sense that the set of independent eigenvectors of a Hermitian operator belonging to all its eigenvalues (each of which is a real number) constitutes a basis that can be made into an orthonormal one by an appropriate choice of the eigenvectors.

8.3.2.3 Change of basis: Unitary transformations

Also of special relevance in quantum theory are the *unitary* operators, where a unitary operator $\hat{U}$ is characterized by the property

$$\hat{U}^{-1} = \hat{U}^{\dagger}. \tag{8.30a}$$

Here $\hat{U}^{-1}$ stands for the inverse of $\hat{U}$. An equivalent characterization of a unitary operator is,

$$\hat{U}\hat{U}^{\dagger} = \hat{U}^{\dagger}\hat{U} = \hat{I}. \tag{8.30b}$$

Analogous to the special property of a Hermitian operator mentioned in Section 8.3.2.2, a unitary operator $\hat{U}$ is characterized by the property that the set of all its independent eigenvectors, belonging to all its distinct eigenvalues, constitutes an orthogonal basis that can be converted into an *orthonormal* one by an appropriate choice of the norms of the eigenvectors.

Given a unitary operator $\hat{U}$, one can define a *transformation* of vectors and operators in $\mathcal{H}$, where any given vector $|u\rangle$ is transformed to

$$|v\rangle = \hat{U}|u\rangle, \tag{8.31a}$$

with $|v\rangle$ being of the same norm as $|u\rangle$—that is,

$$\langle v|v\rangle = \langle u|u\rangle, \tag{8.31b}$$

(check this out)—and similarly, any given operator $\hat{A}$ is transformed to

$$\hat{B} = \hat{U}\hat{A}\hat{U}^{-1}. \tag{8.31c}$$

For any nonsingular operator (ie, one for which the inverse exists) $\hat{U}$, not necessarily unitary, a transformation of vectors and operators defined by Eqs. (8.31a) and (8.31c) is termed a *similarity transformation*. Such a transformation is of special relevance in that any given eigenvector $|a\rangle$ of $\hat{A}$ belonging to an eigenvalue a is transformed to an eigenvector of $\hat{B}$ belonging to the *same* eigenvalue. Such a transformation leaves invariant the set of eigenvalues of any operator $\hat{A}$, known as its *spectrum*. A similarity transformation by means of a unitary operator is termed a 'unitary transformation.'

For any orthonormal basis in $\mathcal{H}$ made up of the set of vectors $\{|e_1\rangle,\ |e_2\rangle, \ldots\}$, the set $\{\hat{U}|e_1\rangle,\ \hat{U}|e_2\rangle, \ldots\}$ resulting from a unitary transformation by means of $\hat{U}$ may also be seen to constitute an orthonormal basis (check this out).

Put differently, if $\{|e_1\rangle,\ |e_2\rangle, \ldots\}$ and $\{|h_1\rangle,\ |h_2\rangle, \ldots\}$ are two sets of orthonormal basis vectors, then there exists a unitary transformation from the former to the latter where the operator $\hat{U}$ effecting the transformation is the one represented by the matrix U with reference to the basis $\{|e_1\rangle,\ |e_2\rangle, \ldots\}$, such that

$$U_{ij} \equiv \langle e_i|\hat{U}|e_j\rangle = \langle e_i|h_j\rangle \quad (i,j = 1,2\ldots). \tag{8.32}$$

Unitary transformations are used in quantum theory to switch from one representation to another, where a representation relates to the use of one among various possible bases, in terms of which operators and vectors are expressed as matrices and columns, and where one among the possible alternative bases may prove to be more useful and convenient than others. This freedom of switching between alternative representations is analogous to the freedom of making use of canonical transformations in classical mechanics.

Additionally, the time evolution of a state of a quantum system may be described as a series of unitary transformations depending parametrically on time. This is again analogous to the situation in classical mechanics where time evolution of a state can be described as a succession of canonical transformations.

8.3.3 Observations in Quantum Theory

As mentioned in Section 8.2.1 in the context of the classical description of a system, any real function of the basic variables q and p (we continue to consider a system with a single degree of freedom for simplicity) can be regarded as an observable quantity, or a 'dynamical variable,' where q and p are real variables having precise and determinate values for any pure state corresponding to a point in the phase space.

In the quantum description, on the other hand, all the observable quantities, including the position and momentum (the two basic observables), correspond to Hermitian linear operators in the state space $\mathcal{H}$. For instance, an observable quantity A corresponds to a Hermitian operator $\hat{A}$, which, in turn, is represented by a matrix that we commonly denote by the symbol A again, with reference to any chosen basis.

> With reference to the linear vector space relevant to the description of internal dynamical variables, the position and momentum no longer appear as the basic observable quantities. For instance, in the case of the vector space associated with the spin of a particle, the spin components are the basic observable quantities, where the vector space is commonly a finite-dimensional one. Of course, the complete description of the states of a system requires the both internal and the external (or 'orbital') dynamical variables, the latter being functions of position and momentum. One then says that the state space for such a system is the *direct product* of the finite-dimensional space for the internal variables and the infinite-dimensional space for the orbital variables.

The entire theory of state vectors and of operators representing observable quantities is oriented toward expressing *results of observations* in terms of these constructs. In general, the result of a measurement aimed at the determination of the value of an observable quantity in a given state of the system manifests itself as a *random variable* and not as a determinate one as in the classical theory, where a measurement leads to a well-defined value of the observable quantity in any given pure state of the system. A quantum measurement, on the other hand, yields results of an inherently *statistical* nature, with a set of *possible* values and a *probability distribution* over this set. The former is nothing but the set of eigenvalues of $\hat{A}$, while the latter depends on the state of the system in which the measurement is made.

If $|\psi\rangle$ denotes the state of the system in which an observable A is measured and if a_i $(i = 1, 2, \ldots)$ are the eigenvalues of $\hat{A}$, with respective normalized eigenvectors $|a_i\rangle$ (we assume for simplicity that each eigenvalue has associated with it only one independent eigenvector), then any of the eigenvalues is a possible result of the measurement, and the probability that the value a_i is obtained when the measurement is performed is given by

$$P(a_i) = |\langle a_i|\psi\rangle|^2 \quad (i = 1, 2, \ldots). \tag{8.33}$$

This implies that the *expectation value*, or mean value of A, obtained in a large number of measurements performed under identical conditions is given by

$$\langle\hat{A}\rangle = \langle\psi|\hat{A}|\psi\rangle. \tag{8.34}$$

Of special relevance in the context of the measurement of A in the given state $|\psi\rangle$ is the standard deviation or *uncertainty* in the measured values, given by

$$\Delta A \equiv \left(\langle(\hat{A} - \langle\hat{A}\rangle)^2\rangle\right)^{1/2} = \left(\langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2\right)^{1/2}, \tag{8.35}$$

where $\hat{A}^2$ stands for an observable whose measured values are squares of the corresponding measured values of $\hat{A}$. The uncertainty tells us how the measured values of A are spread around the mean value $\langle\hat{A}\rangle$.

> A symbol such as A is commonly used to denote an observable (which is simply the name of a measurable physical quantity), while the corresponding operator in $\mathcal{H}$ will be denoted by $\hat{A}$. The matrix representing the observable in some given basis is commonly denoted by the symbol A again. The terms 'observable' (or 'dynamical variable'), 'operator,' and 'matrix' refer to notions quite distinct from one another, among which the latter two have precise mathematical definitions, while the first is of a physical nature, defined in operational terms with reference to some measurement process. However, one need not be too meticulous in maintaining the distinction, provided one does not lose sight of what one is talking about.

8.3.4 Superposed States

Consider any two states of a quantum system represented by linearly independent unit vectors $|a\rangle$ and $|b\rangle$, each of unit norm. Then the linear combination

$$|c\rangle = c_1|a\rangle + c_2|b\rangle \tag{8.36a}$$

is said to represent a *superposition* of the states $|a\rangle$ and $|b\rangle$, where c_1 and c_2 are complex numbers satisfying

$$|c_1|^2 + |c_2|^2 = 1, \tag{8.36b}$$

the latter being the condition for the vector $|c\rangle$ to be of unit norm (assuming, for the sake of concreteness, that $|a\rangle$ and $|b\rangle$ are orthogonal to each other). More generally, one can speak of superpositions of more than two states. The possibility of more than one state of a quantum system giving rise to new states by superposition is a distinct novelty as compared with the situation in classical mechanics where the notion of superposition does not apply. Indeed, the set of all possible states of a classical system makes up a phase space, which has a mathematical structure quite distinct from the set of all possible states in the quantum description, the latter being a Hilbert space, defined by means of the operations of addition of vectors and multiplication by scalars.

A state generated by superposition as in Eq. (8.36a) engenders novel possibilities in respect of measurements. Thus given a dynamical variable $\hat{A}$, its expectation value in the superposed state $|c\rangle$ is

$$\langle c|\hat{A}|c\rangle = |c_1|^2\langle a|\hat{A}|a\rangle| + |c_2|^2\langle b|\hat{A}|b\rangle + 2\mathrm{Re}[c_1c_2^*\langle b|\hat{A}|a\rangle], \tag{8.37}$$

which differs from the weighted sum (with weights $|c_1|^2$ and $|c_2|^2$) of the expectation values in states $|a\rangle$ and $|b\rangle$ by the third term on the right-hand side, the latter being referred to as the *interference term*. In the special case when $|a\rangle$ and $|b\rangle$ are distinct eigenstates of A belonging

to eigenvalues a and b, respectively, a measurement of P in the state $|c\rangle$ yields either of the values a and b with respective weights $|c_1|^2$ and $|c_2|^2$.

8.3.5 The Coordinate and Momentum Representations

8.3.5.1 The two representations

It is of particular interest to consider the representations of vectors and operators in terms of the bases made up of the two sets of vectors denoted by $\{|q\rangle\}$ $(-\infty < q < \infty)$ and $\{|p\rangle\}$ $(-\infty < q < \infty)$, where these are said to constitute the coordinate and the momentum representations, respectively.

> We continue to consider a system with a single degree of freedom defined classically in terms of the position and momentum variables q and p. Commonly, q corresponds to the displacement, from a chosen origin, along a straight line and varies over the range $(-\infty, \infty)$ (this is the case we consider below for the sake of illustration), while it can also correspond to an angular coordinate varying over the range $[0, 2\pi)$. The conjugate momentum p varies over the range $(-\infty, \infty)$ in either case.

These are defined in terms of the eigenvectors of the operators of the position and the momentum observables $\hat{q}$ and $\hat{p}$, respectively, and in each of these, the set of basis vectors made up of the eigenvectors is non-denumerably infinite:

$$\hat{q}|q\rangle = q|q\rangle, \quad \hat{p}|p\rangle = p|p\rangle. \tag{8.38}$$

We are already familiar with the representations of vectors and operators in terms of the orthonormal basis made up of vectors $\{|q\rangle\}$ $(-\infty < q < \infty)$ (see Sections 8.3.1.3 and 8.3.2), while similar considerations apply to the basis made up of the vectors $\{|p\rangle\}$ $(-\infty < p < \infty)$ as well.

8.3.5.2 The fundamental commutation relation and the transformation formula

The transformation from the coordinate representation to the momentum representation is completely defined by the *commutation relation* between the position and momentum operators:

$$[\hat{p}, \hat{q}] = \frac{\hbar}{\mathrm{i}}\hat{I}, \tag{8.39}$$

where $\hbar$ is $\frac{1}{2\pi}$ times the Planck constant h, and $\mathrm{i} \equiv \sqrt{-1}$. This constitutes a fundamental formula in quantum theory, and is analogous to the *Poisson bracket* relation between the momentum and coordinate variables in classical mechanics.

> The commutator between two operators $\hat{A}$ and $\hat{B}$ is defined as $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$, where $\hat{A}\hat{B}$ and $\hat{B}\hat{A}$ stand for operator products taken in two different orders. The fact that two operators

do not, in general, commute with each other is the distinctive feature of quantum theory as compared with the classical theory.

The basic formula relating the two representations is obtained from the fundamental commutation relation as

$$\langle q|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}qp}. \tag{8.40}$$

8.3.5.3 The wave functions and the basic operators

For any vector $|\psi\rangle$, it is represented in the q- and p-bases by columns with parameters taking up nondenumerably infinite values, which actually means that these correspond to *functions* of the variables q and p, as explained in Section 8.3.1.3. If we define these functions (referred to as the *wave functions* in the two representations) as

$$\langle q|\psi\rangle = \psi(q), \quad \langle p|\psi\rangle = \phi(p), \tag{8.41}$$

the transformation relation (8.40) implies

$$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dq\, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}qp} \psi(q), \tag{8.42}$$

which shows that the two wave functions are related to each other by a *Fourier transformation* (refer once again to Section 8.3.1.3).

Finally, the operator $\hat{p}$ acts in the space of wave functions $\psi(q)$ as

$$\langle q|\hat{p}|\psi\rangle = \frac{\hbar}{\mathrm{i}} \frac{d}{dq} \psi(q), \tag{8.43a}$$

where $\psi(q)$ stands for $|\psi\rangle$ in the coordinate representation. In other words, the matrix elements of $\hat{p}$ in the coordinate representation are given by

$$\langle q'|\hat{p}|q\rangle = \frac{\hbar}{\mathrm{i}} \frac{\partial}{\partial q'} \delta(q' - q). \tag{8.43b}$$

Analogous relations hold for the operator $\hat{q}$ in the momentum representation.

8.3.5.4 Generalization

The generalization to a system with more than one degree of freedom is straightforward. For a system with coordinates $\{q_1, \ldots, q_n\}$ and corresponding momenta $\{p_1, \ldots, p_n\}$, the fundamental commutation relations are

$$[q_i, q_j] = 0, \quad [p_i, p_j] = 0, \quad [q_i, p_j] = \mathrm{i}\hbar\delta_{ij} \quad (i,j = 1, \ldots, n). \tag{8.44}$$

For a single particle moving in three dimensions, one has $n = 3$, where q_1, q_2, q_3 may be taken as the coordinates of the particle with reference to any chosen Cartesian coordinate system and p_1, p_2, p_3 may be taken as the corresponding components of the momentum.

8.3.5.5 Probability density

Relation (8.33) for the probability of an observed value assumes that the eigenvalues of the relevant operator form a discrete set. If, on the other hand, the eigenvalues are distributed over a continuous range, then one has to talk of *probability densities* rather than probabilities. For instance, with reference to the position observable and the corresponding operator $\hat{q}$, the probability of finding a value q in the range, say, $q_1 < q < q_2$ is given by

$$P(q|q_1 < q < q_2) = \int_{q_1}^{q_2} |\langle q|\psi\rangle|^2 dq = \int_{q_1}^{q_2} |\psi(q)|^2 dq, \tag{8.45}$$

where $|\psi\rangle$ stands for the state in which the measurement is being made and ψ stands for the corresponding wave function in the coordinate representation (see Eq. 8.41). Here $|\psi(q)|^2$ can be interpreted as the probability density corresponding to the value q, which means that the probability of obtaining any value in an interval $q - \frac{\delta q}{2} < q < q + \frac{\delta q}{2}$ around q can be approximated by the expression $|\psi(q)|^2 \delta q$ for sufficiently small δq.

In a similar manner, if ϕ stands for the wave function in the momentum representation (at times referred to as the 'momentum space wave function'), then $|\phi(p)|^2$ represents the probability density at momentum p for the state $|\psi\rangle$.

Once again the generalization to higher dimensions is straightforward.

8.3.6 Projection Operators and Their Completeness

If $|u\rangle$ denotes any normalized vector in $\mathcal{H}$, then the object $|u\rangle\langle u|$ stands for an operator whose action on any vector $|v\rangle$ is given by

$$(|u\rangle\langle u|)|v\rangle = \langle u|v\rangle|u\rangle. \tag{8.46}$$

This means that the operator $|u\rangle\langle u|$, acting on any vector, produces a vector proportional to $|u\rangle$ or, in other words, a vector belonging to a one-dimensional (1D) *subspace* of $\mathcal{H}$ *spanned* by $|u\rangle$. One then says that $|u\rangle\langle u|$ is a *projection* operator, projecting into this subspace. More generally, one can think of projection operators projecting into higher-dimensional subspaces of $\mathcal{H}$ as well (as an analogy, think of the operation of projecting a 3D vector into a given plane). A projection operator $\hat{P}$ is characterized by the property

$$\hat{P}^2 = \hat{P}, \tag{8.47}$$

which means that if the action of $\hat{P}$ on any given vector $|\phi\rangle$ produces the vector $|\psi\rangle$, then $|\psi\rangle$ will remain unchanged by further actions of P on it. It follows from Eq. (8.47) that the

eigenvalues of a projection operator are 0 and 1, where the number of independent eigenvectors belonging to the eigenvalue 1 is the same as the dimension of the subspace into which the projection effected by $\hat{P}$ occurs. A projection operator is, moreover, a Hermitian one and can be interpreted as an observable quantity for a quantum system, where a 'measurement' of this observable gives a value of 0 or 1, thereby identifying a subspace of $\mathcal{H}$ to which the state of the system belongs after the measurement.

Consider now an orthonormal basis in $\mathcal{H}$ made up of vectors $|e_1\rangle, |e_2\rangle, \ldots$, where each of the basis vectors has associated with it a projection operator (thus $|e_i\rangle\langle e_i|$ corresponds to the basis vector $|e_i\rangle$ $(i = 1, 2, \ldots)$). The completeness of the basis vectors then implies that the sum of all these projection operators is the unit operator:

$$\sum_i |e_i\rangle\langle e_i| = \hat{I} \tag{8.48a}$$

(reason this out).

It is this completeness relation that ensures that the expressions $P(a_i)$ defined in Eq. (8.33) add up to unity, which they have to so as to qualify as probabilities:

$$\sum_i P(a_i) = 1 \tag{8.48b}$$

(check this out). The completeness relations (8.48a) and (8.48b) assume a slightly modified form in the case of continuously distributed eigenvalues, such as the one corresponding to the operator $\hat{q}$:

$$\int |q\rangle\langle q| dq = \hat{I}, \tag{8.49a}$$

$$P(q| -\infty < q < \infty) = \int_{-\infty}^{\infty} |\psi(q)|^2 dq = 1. \tag{8.49b}$$

8.3.7 Simultaneous Measurements

A fact of fundamental importance following from the basic principles of quantum theory is that two or more observable quantities for a quantum system are, in general, not measurable simultaneously since the operators corresponding to these need not commute with each other. This statement needs a bit of explanation.

While it is generally the case that the measurement of an observable A in a state $|\psi\rangle$ can produce any one of the eigenvalues of $\hat{A}$ with a probability given by Eq. (8.33), there exist special states in each of which the uncertainty in the measurement of A reduces to zero, these being precisely the *eigenstates* of $\hat{A}$. For instance, if the system under consideration is in the

state $|a_i\rangle$ ($i = 1, 2, \ldots$), then Eq. (8.33) implies that the measurement yields the value a_i with certainty (ie, with probability 1). In this sense, then, one says that the observable A is *measurable* in any of the eigenstates of $\hat{A}$. More generally, one can choose the state $|\psi\rangle$ such that the uncertainty ΔA (see Eq. 8.35) assumes as small a value as one wishes.

The question that comes up is, can this be done for several observables at a time? For instance, does there exist a state for which the uncertainties for two given observables, say, A and B, both reduce to zero, or both assume arbitrarily small values? The answer, in general, is no. More precisely, if the operators $\hat{A}$ and $\hat{B}$ do not commute, then there does not exist a *complete* set of states such that each of these is simultaneously an eigenstate of $\hat{A}$ and $\hat{B}$ (one or more simultaneous eigenstates may, however, exist without being numerous enough to form a complete set). As follows from formula (8.39), the position and momentum observables are of this kind. These, moreover, make up a special (though not exceptional) pair in that there exists *no* state in which the uncertainties in q and p are simultaneously zero or assume arbitrarily small values. This is expressed by the *Heisenberg formula*

$$\Delta q \Delta p \geq \frac{\hbar}{2}, \tag{8.50}$$

where Δq and Δp stand for the uncertainties in the measurements of q and p (see Eq. 8.35; in the following we will use the symbol $\Delta\hat{A}$ instead of ΔA for the sake of clarity) in any chosen state. Thus if one chooses a state $|\psi\rangle$ for which one of the two uncertainties assumes an arbitrarily chosen small value, then the other has to be correspondingly *large* so as to make the product of the two satisfy (8.50).

The failure of observables to be simultaneously measurable for a quantum system contrasts with the situation for a classical system, where both the position and momentum have well-defined values for any pure state corresponding to a point (q, p) in the phase space, with a correspondingly well-defined value for any and every observable such as the one defined by a function $A(q, p)$.

Incidentally, while speaking of observables for a quantum system, one includes, along with the external or orbital variables defined in terms of the position and momentum components (referring to the general case of one or more particles in three dimensions), the internal variables such as the spin components as well. In the classical case, the internal variables are not defined.

8.3.8 The Hamiltonian Operator and the Energy Representation

Among the operators characterizing the observable quantities of a system, the *Hamiltonian* operator is of outstanding relevance, being the operator associated with the *energy* of the system, where we assume the latter to be a closed one for simplicity (for a system that may not be a closed one, the Hamiltonian need not correspond to its energy; for a system interacting

with other systems, its energy is not a well-defined concept since only the energy of all the interacting systems taken together is meaningful). It is the Hamiltonian that determines how a state of the system *evolves* with time. While for a classical system the time evolution is given by the Hamiltonian equations of motion (see Eqs. 8.2 and 8.3), in the case of a quantum system it is described by the *Schrödinger* equation, as mentioned in Section 8.3.9 below.

Analogously to the classical case, the energy of a closed system is a conserved quantity (see Section 8.3.9.1) and its Hamiltonian does not depend explicitly on time, being a function of the position and momentum operators, along with the operators for the internal dynamical variables, if any. For such a system, the eigenvalues and the eigenvectors of the Hamiltonian are of relevance, the former constituting the possible values of a result of measurement of the energy in any chosen state and the latter (the 'energy eigenstates') corresponding to the *stationary* states—that is, the states that do not change with time (see Eq. 8.54 in Section 8.3.9).

The following situation is of special relevance since it is often encountered in practice: A closed quantum system with Hamiltonian $\hat{H}_0$ is made to interact weakly with a second system, where the latter may be amenable to a classical description. The Hamiltonian of the system is then modified to a time-dependent one by the addition of a new term of the form $\lambda \hat{V}(t)$, where λ is a small parameter characterizing the strength of the interaction. If, then, one starts from an eigenstate $|\psi_1\rangle$ of $\hat{H}_0$ (a stationary state in the limit $\lambda \to 0$), then there occur *transitions* from this state to other stationary states of $\hat{H}_0$. The probability of transition to a stationary state $|\psi_2\rangle$ of $\hat{H}_0$ in any given time interval t can then be calculated by making use of results of *perturbation theory* and can be compared with experimentally observed values. This constitutes a situation where the eigenstates of a Hamiltonian ($\hat{H}_0$) are not stationary, but where the transition probabilities between such eigenstates are of theoretical and experimental relevance.

If we refer again to a closed system with a Hamiltonian $\hat{H}$, the normalized eigenvectors of $\hat{H}$ can be so chosen as to form an orthonormal basis, termed the 'energy basis.' Vectors and operators can be represented with columns and matrices with reference to this basis, constituting the *energy representation*.

As an example of the Hamiltonian of a closed system, let us consider a *harmonic oscillator*, for which the Hamiltonian function in the classical description is given by Eq. (8.1). The Hamiltonian operator in the quantum description of the oscillator is obtained by reference to the classical analogue and is of the form

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{q}^2, \tag{8.51}$$

where $\hat{q}$ and $\hat{p}$ are the operators for the position and momentum observables of the particle. We will have a good look at the quantum mechanics of the harmonic oscillator in Section 8.4 since it is of great relevance in quantum optics.

Not all quantum mechanical systems have a classical analogue. For instance, a system with one or more internal degrees of freedom such as spin is inherently quantum mechanical since these internal degrees cannot be described classically. For such a system the Hamiltonian operator is constructed by indirect considerations. Even for a system with a classical analogue the Hamiltonian is not always obtained unambiguously because of the noncommutativity of the operators $\hat{q}$ and $\hat{p}$.

8.3.9 Pure States and Their Time Evolution: The Schrödinger Equation

The states described in terms of vectors in the Hilbert space $\mathcal{H}$ are referred to as *pure states* and do *not* exhaust the set of all possible states of a quantum system since *mixed* states are also possible (see Section 8.3.10), analogous to the mixed states of a classical system. The pure states occur as limiting instances of mixed states.

In general, the state of a quantum system changes, or evolves, with time as in the case of a classical system, where this time evolution constitutes the *dynamics* of the system. As in the classical case, the dynamics of a closed quantum system is governed by its *Hamiltonian*, where the Hamiltonian is an operator corresponding to its energy, as indicated in Section 8.3.8.

Let us consider a closed system with Hamiltonian $\hat{H}$ whose state at time t, assumed to be a pure one, is described by the vector $|\psi(t)\rangle$ of unit norm. Then the time evolution of the state is governed by the equation

$$\mathrm{i}\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle, \tag{8.52}$$

where $\frac{d}{dt}|\psi(t)\rangle$ gives the rate of change of the state vector at time t. This is a first-order differential equation in the Hilbert space $\mathcal{H}$, which on integration gives the state at any time t in terms of the state at any initial time t_0, and is referred to as the *Schrödinger equation* for the system under consideration.

The time evolution is of a trivial nature if the initial state $|\psi(t_0)\rangle$ is an energy eigenstate of the system—that is, if it satisfies

$$\hat{H}|\psi(t_0)\rangle = E|\psi(t_0)\rangle, \tag{8.53}$$

where E is an eigenvalue of the Hamiltonian. In this case one obtains

$$|\psi(t)\rangle = \mathrm{e}^{-\frac{\mathrm{i}E(t-t_0)}{\hbar}}|\psi(t_0)\rangle, \tag{8.54}$$

which means that the state at time t continues to be same as that at time t_0, while differing from the latter by just a phase factor. In other words, an eigenstate $\psi(t_0)\rangle$ of the Hamiltonian defines a *stationary state* of the system.

More generally, if the Hamiltonian of a system does not depend explicitly on time, and the initial state, when expressed as a linear combination of the energy eigenstates $|E_i\rangle$ corresponding to eigenvalues E_i $(i = 1, 2, \ldots)$, is of the form

$$|\psi(t_0)\rangle = \sum_n c_n |E_n\rangle \qquad \left(\sum_n |c_n|^2 = 1\right), \tag{8.55a}$$

then the state at time t, as obtained from Eq. (8.52), is given by

$$|\psi(t)\rangle = \mathrm{e}^{-\frac{\mathrm{i}(t-t_0)}{\hbar}\hat{H}}|\psi(t_0)\rangle = \sum_i \mathrm{e}^{-\frac{\mathrm{i}E_n(t-t_0)}{\hbar}} c_n |E_n\rangle, \tag{8.55b}$$

which tells us that the evolution of $|\psi(t)\rangle$ is *multiperiodic* in nature. In spite of the time evolution, however, the expectation value of the Hamiltonian remains constant:

$$\langle \hat{H} \rangle = \langle \psi(t)|\hat{H}|\psi(t)\rangle = \sum_n |c_n|^2 E_n. \tag{8.56}$$

8.3.9.1 Conserved dynamical variables

For a dynamical variable $\hat{A}$ of a quantum system, the expectation value relating to a measurement in a state $|\psi(t)\rangle$ will, in general, depend on the time t because, though the dynamical variable itself may not depend on t (eg, it may be a function of the basic coordinate and momentum variables without an explicit time dependence), the state goes on evolving with time. There may, however, exist dynamical variables of the system for which the expectation value in the state $|\psi(t)\rangle$ remains independent of time in spite of the time evolution of the state itself (the Hamiltonian $\hat{H}$ is a trivial example of such an observable; see Eq. 8.56; we continue to assume that $\hat{H}$ does not depend explicitly on time). Such dynamical variables are referred to as *conserved variables* or as *constants of motion*.

The condition for a dynamical variable $\hat{A}$ to be a constant of motion is that it should *commute with the Hamiltonian*—that is,

$$[\hat{A}, \hat{H}] = \hat{A}\hat{H} - \hat{H}\hat{A} = 0. \tag{8.57}$$

This condition is analogous to the one in classical mechanics that states that a dynamical variable of which the *Poisson bracket* with the Hamiltonian function is zero is a constant of motion.

An instance of a conserved quantity is the angular momentum (which is made up of three Cartesian components) of a particle in a spherically symmetric potential.

8.3.10 Mixed States in Quantum Theory

Analogous to the mixed states in the classical description, a state of a system in the quantum description is, in general, a *mixed* one. This means that the information that you may possess

about a system prepared in a certain specified manner may not be sufficient to pin its state down to a precisely defined vector $|\psi\rangle$ but, on the other hand, may tell you that the system can be in any one of a set of states corresponding to vectors, say, $|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_n\rangle$ with some definite probability distribution over these states, say, with probability w_i for the state $|\psi_i\rangle$ $(i = 1, \ldots, n)$. One then says that a mixed state of the system under consideration has been defined as a mixture of the pure states $|\psi_i\rangle$ $(i = 1, \ldots, n)$, and is described by the *density operator*

$$\hat{\rho} = \sum_{i=1}^{n} w_i |\psi_i\rangle\langle\psi_i|. \tag{8.58}$$

Here $|\psi_i\rangle\langle\psi_i|$ is the density operator representation of the pure state $|\psi_i\rangle$ (this constitutes an alternative means of specifying a pure state, where the density operator is actually a projection operator). Indeed, the pure state $|\psi_i\rangle$ constitutes a special case of a mixed state where $w_i = 1$ and all the other probabilities P_j $(j \neq i)$ are zero. In a representation corresponding to some particular choice of basis vectors, the density operator for the pure state $|\psi_i\rangle$ corresponds to the product of the column representing the ket $|\psi_i\rangle$, with the adjoint row representing the bra $\langle\psi_i|$. With reference to Eq. (8.58), an alternative approach to describing the mixed state corresponding to the density operator $\hat{\rho}$ is to call it an *ensemble* of states $\{|\psi_i\rangle\}$ with probabilities $\{w_i\}$.

For a mixed state of the form (8.58), the pure states $|\psi_i\rangle$ $(i = 1, \ldots, n)$ need not be orthogonal to each other. On the other hand, for the same mixed state described by the density operator $\hat{\rho}$, one can find an orthonormal basis made up of vectors $|e_i\rangle$ $(i = 1, 2, \ldots)$ such that

$$\hat{\rho} = \sum_{i} P_i |e_i\rangle\langle e_i|, \tag{8.59}$$

where $\{P_i\}$ is a set of non-negative real numbers that add up to unity. In other words, $\hat{\rho}$ is also a mixture of the basis states $|e_i\rangle$ with probabilities P_i $(i = 1, 2, \ldots)$, which tells us that the representation (8.58) is not a unique one, and the same mixed state can be regarded as a mixture of pure states in more ways than one. What is more, given any arbitrary orthonormal basis made up of vectors $|u_i\rangle$ $(i = 1, 2, \ldots)$, the same density operator $\hat{\rho}$ can be expressed as

$$\hat{\rho} = \sum_{ij} \rho_{ij} |u_i\rangle\langle u_j|. \tag{8.60}$$

In other words, the matrix representing $\hat{\rho}$ in the given basis has ρ_{ij} as the element in its ith row and jth column. This is referred to as the *density matrix* of the mixed state under consideration in the chosen basis (at times the term 'density matrix' is used loosely to denote a density operator).

From the representation (8.59) of the density operator, one infers that the latter is Hermitian and of trace unity. Hence, referring to representation (8.60) by the density matrix made up of the elements ρ_{ij} $(i, j = 1, 2, \ldots)$, one obtains

$$\rho_{ij} = \rho_{ji}^*, \quad \sum_i \rho_{ii} = \mathrm{Tr}\hat{\rho} = 1. \tag{8.61}$$

One other property of the density operator (and hence of the corresponding density matrix as well) is the following:

$$\mathrm{Tr}\hat{\rho}^2 \leq 1 \tag{8.62}$$

(check this out; use Eq. 8.59).

Incidentally, given a density operator $\hat{\rho}$, the condition for it to represent a pure state is

$$\hat{\rho}^2 = \hat{\rho}, \quad \text{ie, } \mathrm{Tr}\hat{\rho}^2 = 1, \tag{8.63a}$$

while, correspondingly, the condition that it describes a mixed state (and not a pure state) is

$$\mathrm{Tr}\hat{\rho}^2 < 1. \tag{8.63b}$$

Given a mixed state corresponding to the density operator $\hat{\rho}$ of a system, the expectation value of the results of measurement of an observable represented by the operator $\hat{A}$ can be obtained from formula (8.59), and is given by

$$\langle \hat{A} \rangle = \mathrm{Tr}(\hat{A}\hat{\rho}). \tag{8.64}$$

This is the fundamental formula relating the density operator—a mathematical construct—to the statistical distribution of the results of an observation, or a measurement, which is of physical relevance in the context of the system under consideration.

Finally, here is the equation describing the time evolution of a mixed state, which is a first-order differential equation specifying the rate of change of the density operator at any time t:

$$\mathrm{i}\hbar \frac{d}{dt}\hat{\rho}(t) = [H, \hat{\rho}]. \tag{8.65}$$

This form of the evolution equation applies to the special case of a pure state expressed in the form of a density operator, and is the generalization of Eq. (8.52) to include the evolution of mixed states. It is the quantum analog of the Liouville equation (8.8a) describing the time evolution of mixed classical states, and is referred to as the 'Neumann-Liouville equation.'

If the Hamiltonian $\hat{H}$ does not depend explicitly on time, then the evolution equation (8.65) has the solution

$$\hat{\rho}(t) = \mathrm{e}^{-\frac{\mathrm{i}(t-t_0)}{\hbar}\hat{H}}\hat{\rho}(t_0)\mathrm{e}^{\frac{\mathrm{i}(t-t_0)}{\hbar}\hat{H}}, \tag{8.66}$$

where $\hat{\rho}(t_0)$ denotes the density operator at any initial time t_0. The unitary operator

$$\hat{U}(t, t_0) = \mathrm{e}^{-\frac{\mathrm{i}(t-t_0)}{\hbar}\hat{H}} \tag{8.67}$$

is referred to as the 'time evolution operator' for the system. In the more general case of a time-dependent Hamiltonian, the evolution operator assumes a more complex form, and satisfies the differential equation

$$i\hbar \frac{d}{dt}\hat{U} = \hat{H}\hat{U}, \tag{8.68a}$$

subject to the boundary condition

$$\hat{U}(t_0, t_0) = \hat{I}. \tag{8.68b}$$

The time evolution of pure and mixed states can be described in terms of the evolution operator as follows:

$$\text{(pure state) } |\psi(t)\rangle = \hat{U}(t, t_0)|\psi(t_0)\rangle, \tag{8.69a}$$

$$\text{(mixed state) } \hat{\rho}(t) = \hat{U}(t, t_0)\hat{\rho}(t_0)U^{\dagger}(t, t_0). \tag{8.69b}$$

8.3.11 The Three Pictures

Eqs. (8.52) and (8.65) (or, equivalently, Eqs. 8.69a and 8.69b) describe the dynamics of a quantum system with a Hamiltonian $\hat{H}$. The general idea in this description is that the state of the system, whether a pure or a mixed one, evolves in time, but the operators describing the observable quantities remain unchanged unless they carry an explicit time dependence.

This is one among several possible *schemes* or *pictures* that can be invoked to describe the dynamics of a quantum system, and is referred to as the *Schrödinger picture*, while the *Heisenberg picture* and the *interaction picture* are two other schemes of quite considerable importance.

In the Heisenberg picture the dynamics is carried by the operators corresponding to the observable quantities rather than by the state vectors, the latter being independent of time, in contrast to the Schrödinger picture. Indeed, if the time-dependent state vectors of the Schrödinger picture are all transformed by the inverse of the evolution operator, one obtains time-independent vectors. For instance, with any given choice of t_0 (the usual choice is $t_0 = 0$), the state $|\psi(t)\rangle$ transforms to

$$\hat{U}^{\dagger}(t, t_0)|\psi(t)\rangle = |\psi(t_0)\rangle, \tag{8.70}$$

which is a stationary vector and represents the state in the Heisenberg picture. Correspondingly, for a dynamical variable represented by the operator $\hat{A}$ in the Schrödinger picture, the operator in the Heisenberg picture is obtained as $U^{\dagger}(t, t_0)\hat{A}U(t, t_0)$. From now on, we distinguish between state vectors and operators (representing observables) in the Schrödinger picture and the corresponding vectors and operators in the Heisenberg picture with help of a superscript 'S' in the former and a superscript 'H' in the latter. Moreover, we will write $\hat{U}$ for $\hat{U}(t, t_0)$ for brevity when there is no scope for confusion.

With this notation we then have the relations

$$|\psi^{\mathrm{H}}\rangle = \hat{U}^{\dagger}|\psi^{\mathrm{S}}(t)\rangle, \quad \hat{A}^{\mathrm{H}}(t) = \hat{U}^{\dagger}\hat{A}^{\mathrm{S}}\hat{U}. \tag{8.71}$$

As a result of this unitary transformation, the following relation is satisfied

$$\langle\psi^{\mathrm{H}}|\hat{A}^{\mathrm{H}}(t)|\psi^{\mathrm{H}}\rangle = \langle\psi^{\mathrm{S}}(t)|\hat{A}^{\mathrm{S}}|\psi^{\mathrm{S}}(t)\rangle, \tag{8.72}$$

which tells us that the expectation values of observables in given states are not modified in the transformation from one picture to another. This is a requirement of the theory since the different pictures are just different ways of describing the dynamics, and should not give different values for the physically relevant quantities, and is guaranteed by the fact that the transformation from one picture to another is a unitary one.

Since the dynamics of the system under consideration in the Heisenberg picture is carried entirely by the operators corresponding to the various observables, these operators satisfy their own evolution equations, where in each case the evolution is governed by the Hamiltonian, which, however, remains the same in the Schrödinger and Heisenberg pictures in the case of a closed system. For an observable represented by the operator $\hat{A}^{\mathrm{H}}(t)$ in the Heisenberg picture, the equation of motion reads

$$\frac{d}{dt}\hat{A}^{\mathrm{H}}(t) = \frac{\mathrm{i}}{\hbar}[\hat{H}, \hat{A}^{\mathrm{H}}(t)]. \tag{8.73a}$$

At times an operator in the Schrödinger picture carries an explicit time dependence that is transferred to the Heisenberg picture as well, when the above equation of motion appears as

$$\frac{d}{dt}\hat{A}^{\mathrm{H}}(t) = \frac{\mathrm{i}}{\hbar}[\hat{H}, \hat{A}^{\mathrm{H}}(t)] + \frac{\partial}{\partial t}\hat{A}^{\mathrm{H}}(t), \tag{8.73b}$$

where the last term on the right-hand side gives the rate of change due to the explicit time dependence.

The *interaction picture* (also referred to, at times, as the *intermediate picture*) is a convenient scheme to invoke for the description of the dynamics of a system when its Hamiltonian can be split into two parts as

$$\hat{H}^{\mathrm{S}} = \hat{H}_0^{\mathrm{S}} + \hat{V}^{\mathrm{S}}, \tag{8.74}$$

where the part $\hat{V}^{\mathrm{S}}$ arises from the interaction (usually assumed to be a weak one, in a sense that can be made precise in any given context) of the system under consideration with some other external system(s), and $\hat{H}_0^{\mathrm{S}}$ is the Hamiltonian governing the dynamics in the absence of the interaction, and where the operators are all written in the Schrödinger picture.

Corresponding to such a splitting of the Hamiltonian, the evolution operator $\hat{U}(t, t_0)$ is conveniently expressed in the form of a product

$$\hat{U}(t,t_0) = \hat{U}_0(t,t_0)\hat{U}_{\rm I}(t,t_0), \tag{8.75}$$

where $\hat{U}_0$ is the evolution operator for the Hamiltonian $\hat{H}_0$ (ie, for an imagined situation where the interaction is absent) and $\hat{U}_{\rm I}$ can be interpreted as carrying the effect of the interaction.

The interaction picture is then defined as one where the operators (corresponding to observables) carry the dynamics described by $\hat{U}_0$, while the density matrices (or the state vectors in the case of pure states) carry the dynamics described by $\hat{U}_{\rm I}$—that is,

$$\hat{A}^{\rm I}(t) = \hat{U}_0^\dagger(t,t_0)\hat{A}^{\rm S}\hat{U}_0(t,t_0), \quad \hat{\rho}^{\rm I}(t) = \hat{U}_{\rm I}(t,t_0)\hat{\rho}^{\rm S}(t_0)\hat{U}_{\rm I}^\dagger(t,t_0), \tag{8.76a}$$

where a superscript 'I' is used to denote operators and states (either vectors or density operators) in the interaction representation. For completeness, one needs the differential equation satisfied by the operator $U_{\rm I}$, which reads

$$\mathrm{i}\hbar\frac{d}{dt}\hat{U}_{\rm I}(t,t_0) = \hat{V}^{\rm I}(t)\hat{U}_{\rm I}(t,t_0), \tag{8.76b}$$

where $\hat{V}^{\rm I}(t)$ stands for the interaction part of the Hamiltonian ($\hat{V}$, or, more precisely, $\hat{V}^{\rm S}$) expressed in the interaction picture—that is,

$$\hat{V}^{\rm I}(t) = \hat{U}_0^\dagger(t,t_0)\hat{V}^{\rm S}\hat{U}_0(t,t_0). \tag{8.76c}$$

Incidentally, one has to replace $\hat{A}^{\rm S}$ and $\hat{V}^{\rm S}$ with $\hat{A}^{\rm S}(t)$ and $\hat{V}^{\rm S}(t)$, respectively, if the Schrödinger picture operators carry an explicit time dependence.

8.3.12 Composite Systems and Reduced States

Consider two quantum systems A and B, with Hilbert spaces $\mathcal{H}^{\rm A}$ and $\mathcal{H}^{\rm B}$, and the *composite system* C made up of A and B, with the latter two possibly interacting with each other. It is a basic premise of quantum theory that the Hilbert space $\mathcal{H}^{\rm C}$ for the composite system C is the *direct product* of $\mathcal{H}^{\rm A}$ and $\mathcal{H}^{\rm B}$:

$$\mathcal{H}^{\rm C} = \mathcal{H}^{\rm A} \otimes \mathcal{H}^{\rm B}. \tag{8.77}$$

If $\{|e_i^{\rm A}\rangle\}$ $(i = 1, 2, \ldots)$ and $\{|e_j^{\rm B}\rangle\}$ $(j = 1, 2, \ldots)$ are orthonormal bases in $\mathcal{H}^{\rm A}$ and $\mathcal{H}^{\rm B}$, then an orthonormal basis in $\mathcal{H}^{\rm C}$ is made up of pairs of the form $|e_i^{\rm A}\rangle \otimes |e_j^{\rm B}\rangle$ $(i, j = 1, 2, \ldots)$, and a typical vector of unit norm in $\mathcal{H}^{\rm C}$ is of the form

$$|u^{\rm C}\rangle = \sum_{ij} u_{ij}|e_i^{\rm A}\rangle \otimes |e_j^{\rm B}\rangle, \tag{8.78a}$$

where u_{ij} are complex numbers satisfying

$$\sum_{ij} |u_{ij}|^2 = 1. \tag{8.78b}$$

This characterization of the product space holds for finite-dimensional spaces as well as for infinite-dimensional ones with appropriate ranges for the indices i and j where, in the former case, the dimension (N^{C}) of the product space is the product of the dimensions ($N^{\mathrm{A}},\ N^{\mathrm{B}}$) of the factor spaces.

A class of special state vectors in the product space is made up of vectors of the form

$$|u^{\mathrm{C}}\rangle = |u^{\mathrm{A}}\rangle \otimes |u^{\mathrm{B}}\rangle, \tag{8.79a}$$

with $|u^{\mathrm{A}}\rangle$ and $|u^{\mathrm{B}}\rangle$ belonging to $\mathcal{H}^{\mathrm{A}}$ and $\mathcal{H}^{\mathrm{B}}$, respectively, in which case the numbers u_{ij} occurring in Eq. (8.78a) are of the form

$$|u^{\mathrm{A}}\rangle \otimes |u^{\mathrm{B}}\rangle :\ u_{ij} = u_i^{\mathrm{A}} u_j^{\mathrm{B}} \quad (i,j = 1,2,\ldots), \tag{8.79b}$$

u_i^{A} and u_j^{B} $(i,j = 1,2,\ldots)$ being the expansion coefficients of $|u^{\mathrm{A}}\rangle$ and $|u^{\mathrm{B}}\rangle$ in terms of the chosen bases in $\mathcal{H}^{\mathrm{A}}$ and $\mathcal{H}^{\mathrm{B}}$.

A unit vector of the form $|u^{\mathrm{A}}\rangle \otimes |u^{\mathrm{B}}\rangle$ in $\mathcal{H}^{\mathrm{C}}$ represents a *pure product state* of the composite system. A more general class of states in $\mathcal{H}^{\mathrm{C}}$ is made up of mixed states of the form

$$\hat{\rho}^{\mathrm{C}} = \hat{\rho}^{\mathrm{A}} \otimes \hat{\rho}^{\mathrm{B}}, \tag{8.80a}$$

where $\hat{\rho}^{\mathrm{C}}$ appears as the direct product of a density matrix in $\mathcal{H}^{\mathrm{A}}$ with one in $\mathcal{H}^{\mathrm{B}}$. Still more generally, one can have

$$\hat{\rho}^{\mathrm{C}} = P_1 \hat{\rho}_1^{\mathrm{A}} \otimes \hat{\rho}_1^{\mathrm{B}} + P_2 \hat{\rho}_2^{\mathrm{A}} \otimes \hat{\rho}_2^{\mathrm{B}} + P_3 \hat{\rho}_3^{\mathrm{A}} \otimes \hat{\rho}_3^{\mathrm{B}} + \cdots, \tag{8.80b}$$

where $P_1,\ P_2, \ldots$ are a set of real nonnegative weights that add up to unity, $\hat{\rho}_1^{\mathrm{A}},\ \hat{\rho}_2^{\mathrm{A}}, \ldots$ are density operators in $\mathcal{H}^{(\mathrm{A})}$, and $\hat{\rho}_1^{\mathrm{B}},\ \hat{\rho}_2^{\mathrm{B}}, \ldots$ are density operators in $\mathcal{H}^{(\mathrm{B})}$, the number of terms on the right-hand side of Eq. (8.80b) being arbitrary.

Evidently, a state of the system C of the form (8.79a) is a special instance of a state of the type (8.80a) (recall that a pure state $|u\rangle$ in a space $\mathcal{H}$ corresponds to a density operator $|u\rangle\langle u|$), while the latter, in turn, is a special instance of a state of the type (8.80b). However, even the states of the form (8.80b) constitute a very special class of states of C. The most general type of state of C is one that *cannot* be expressed in any of the above forms involving direct products of pure states or density matrices. Even a pure state $|u^{\mathrm{C}}\rangle$ in $\mathcal{H}^{\mathrm{C}}$ is, in general, not of any one of the above forms.

The question of statistical features of results of measurements made on the composite system C is one of great relevance. Given an observable $\hat{A}^{\mathrm{C}}$, its expectation value in the state described by the density operator $\hat{\rho}^{\mathrm{C}}$ is given by (see formula (8.64))

$$\langle \hat{A}^{\mathrm{C}} \rangle = \mathrm{Tr}(\hat{A}^{\mathrm{C}} \hat{\rho}^{\mathrm{C}}). \tag{8.81}$$

In general, the observable $\hat{A}^{\mathrm{C}}$ relates to features of *both* A and B. Consider, however, an observable $\hat{A}^{\mathrm{A}}$ that relates to the system A alone. The expectation value of the result of a

measurement of $\hat{A}^{\mathrm{A}}$ in the state $\hat{\rho}^{\mathrm{C}}$ of the composite system is obtained from Eq. (8.81) by replacement of $\hat{A}^{\mathrm{C}}$ on the right-hand side with $\hat{A}^{\mathrm{A}}$. It turns out that this expectation value is the same *as if* the observable $\hat{A}^{\mathrm{A}}$ were measured in a certain state $\tilde{\hat{\rho}}^{\mathrm{A}}$ of A that is obtained by the taking of the *partial trace* of $\hat{\rho}^{\mathrm{C}}$ over B, defined as

$$\tilde{\hat{\rho}}^{\mathrm{A}} = \mathrm{Tr}_{\mathrm{B}}\hat{\rho}^{\mathrm{C}} = \sum_{ij} \tilde{\rho}^{\mathrm{A}}_{ij} |e^{\mathrm{A}}_i\rangle\langle e^{\mathrm{A}}_j|, \tag{8.82a}$$

with

$$\tilde{\rho}^{\mathrm{A}}_{ij} = \sum_k \langle e^{\mathrm{A}}_i e^{\mathrm{B}}_k | \hat{\rho}^{\mathrm{C}} | e^{\mathrm{A}}_j e^{\mathrm{B}}_k\rangle. \tag{8.82b}$$

In expression (8.82b), $|e^{\mathrm{A}}_i e^{\mathrm{B}}_k\rangle$ $(i, k = 1, 2, \dots)$ stands for the direct product $|e^{\mathrm{A}}_i\rangle \otimes |e^{\mathrm{B}}_k\rangle$, which belongs to an orthonormal product basis—that is, the basis we used in writing Eq. (8.78a).

The density operator $\tilde{\hat{\rho}}^{\mathrm{A}}$ arrived at by the operation of a partial trace over B is referred to as the *reduced density operator* (or, in brief, the *reduced state*) of A obtained from $\hat{\rho}^{\mathrm{C}}$. The reduced state of subsystem B can be similarly defined.

The theory of composite systems can be generalized to include one made up of more than two subsystems as well.

8.3.13 Quantum Correlations: Entanglement

A state of a composite system C made up of subsystems A and B that can be expressed in the form (8.80b) is termed a *separable state*, while if the density matrix cannot be so expressed, then it is referred to as an *entangled state*. We will not address here the question as to whether and how it can be determined if any given density matrix of the composite system C corresponds to a separable or an entangled state because it involves quite nontrivial considerations.

Entangled states of composite quantum systems are of great theoretical and practical relevance because they are characterized by *nonclassical* correlations (ie, ones of a specifically quantum nature) between the subsystems. By contrast, separable states involve only classical correlations between the subsystems. The nonclassical correlations of an entangled state can be made use of in numerous ways, and have opened up stupendous possibilities of practical applications, a number of which have already been realized, while many more are to be realized in days to come. In most cases, the entanglement between subsystems involves *nonlocal* features of a composite system that provide the basis for the extraordinary applications engendered by the property of entanglement in composite systems.

From formula (8.80b), the reduced state of A can be worked out by following definition (8.82a), when one finds

$$\tilde{\hat{\rho}}^{\mathrm{A}} = P_1 \hat{\rho}_1^{\mathrm{A}} + P_2 \hat{\rho}_2^{\mathrm{A}} + \cdots ; \tag{8.83}$$

that is, the reduced state *does not depend* on the density operators $\hat{\rho}_1^{\mathrm{B}}$, $\hat{\rho}_2^{\mathrm{B}}, \ldots$ relating to subsystem B. In other words, the statistical distribution of results of measurement of an observable such as $\hat{A}^{\mathrm{A}}$ are determined without regard to subsystem B or to the composite system C. Similar statements apply to the reduced state of B, obtained by taking the partial trace over A. This is one way of interpreting the statement that there are no quantum correlations between A and B in a separable state. On the other hand, the correspondence between the first factor and the second factor in each term occurring in the summation on the right-hand side of Eq. (8.80a) is an indicator of the classical correlation between the two subsystems.

The exclusively quantum character of entanglement is seen from the fact that even a pure state of C of the form $\frac{1}{\sqrt{2}}(|u_1^{\mathrm{A}}\rangle \otimes |u_1^{\mathrm{B}}\rangle + |u_2^{\mathrm{A}}\rangle \otimes |u_2^{\mathrm{B}}\rangle)$ (the factor of $\frac{1}{\sqrt{2}}$ is for normalization) is, in general, an entangled one (ie, the corresponding density operator cannot be expressed as a sum of direct products). For a composite *classical* system, on the other hand, a pure state is not only separable but is of the direct product form (ie, does not involve any correlation between the subsystems).

Quantum optics provides a superbly fertile field for theoretical and practical developments in the area of quantum *information processing* that is exclusively based on the concept of entanglement.

8.3.14 Electromagnetic Field: The Quantum View

With this background, we can now have a first look at the quantum description of the state of an *electromagnetic field* at any given instant of time. In this, it is useful to refer to the sets of observables $\{\mathbf{E}(\mathbf{r})\}$ and $\{\mathbf{B}(\mathbf{r})\}$, i.e., the collection of electric and magnetic field intensities at all points in some region of space, where we recall that these are now simply the *names* of the observables, not to be confused with their *values*. Indeed, if one measures the value of any of these observables, say, the electric field strength at $\mathbf{r}$ (it is not necessary to worry at this stage as to the practical feasibility of such a measurement), one would not obtain a specific value as a result since the latter actually constitutes a *random variable*. A large number of measurements, repeated each time for the *same* state of the field (once again, we do not ask how this is to be achieved in practice), would give various different values with a certain *probability distribution* determined by the state under consideration, and the expectation value, or the mean of all these measured values, would be a well-defined quantity for this state. One could also think of observables of a more complicated nature—for instance, the *products* of field intensities at two different points—and talk of expectation values of such products. These are referred to as *correlations* between the field intensities at the points under consideration.

Before undertaking a more detailed description of certain types of states of the electromagnetic field and observable effects relating to these states, I include below a brief survey of the quantum mechanical harmonic oscillator since in numerous situations of practical interest the electromagnetic field can be described in terms of a collection of harmonic oscillators. Special states of the harmonic oscillator such as the number states, coherent states, and the squeezed states are of considerable relevance in the quantum description of states of the electromagnetic field.

8.4 The Harmonic Oscillator

8.4.1 The Number States: Creation and Annihilation Operators

The classical harmonic oscillator is described by the Hamiltonian function (8.1), while in the quantum description one refers to the Hamiltonian operator (8.51). Classically, if one starts from a point (q, p) in the phase space at an initial instant of time, then subsequently q and p vary sinusoidally with angular frequency ω, and the representative point in the phase space describes an elliptic trajectory. The representative point at any given time instant represents a pure classical state of the oscillator, while a collection of points making up a region of the phase space, with some particular probability density defined over the region, defines a mixed classical state. While both q and p have precisely defined values for a pure state, their values can be given only in statistical terms for a mixed state.

The quantum theory of the harmonic oscillator is crucially relevant to quantum optics, where it provides the basis for describing quantum states of the electromagnetic field. A set of pure states of central importance for the harmonic oscillator are the *eigenstates* of the Hamiltonian, also termed *stationary states*. These eigenstates form a discrete set that can be enumerated with an index, say, n $(= 0, 1, 2, \ldots)$, a typical normalized eigenstate $|n\rangle$ satisfying

$$\hat{H}|n\rangle = E_n|n\rangle, \tag{8.84}$$

where E_n stands for the energy eigenvalue and is given by

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right). \tag{8.85}$$

Thus the minimum possible energy for the quantum harmonic oscillator is not zero but $\frac{1}{2}\hbar\omega$, known as the *zero point energy* of the oscillator.

A convenient and useful way to characterize the eigenstates is in terms of the *annihilation* and *creation* operators $(\hat{a},\ \hat{a}^\dagger)$, defined as

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{q} + \frac{\mathrm{i}\hat{p}}{m\omega}\right), \tag{8.86a}$$

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{q} - \frac{\mathrm{i}\hat{p}}{m\omega}\right). \tag{8.86b}$$

Using the fundamental commutation relation

$$[\hat{q}, \hat{p}] = \mathrm{i}\hbar\hat{\mathrm{I}}, \tag{8.87}$$

where $\hat{\mathrm{I}}$ stands for the identity operator, one finds that the operators $\hat{a}$ and $\hat{a}^\dagger$ satisfy the basic commutation relation

$$[\hat{a}, \hat{a}^\dagger] = \hat{\mathrm{I}}. \tag{8.88}$$

The Hamiltonian of the harmonic oscillator expressed in terms of the creation and annihilation operators reads

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger a + \frac{1}{2}\right), \tag{8.89}$$

where $\frac{1}{2}$ stands for $\frac{1}{2}\hat{\mathrm{I}}$.

The operator

$$\hat{\mathrm{N}} \equiv \hat{a}^\dagger\hat{a} \tag{8.90}$$

is termed the *number operator* since its eigenvalues are the nonnegative integers $n = 0, 1, 2, \ldots$, a result that implies the energy eigenvalues (8.85). Indeed, the eigenstates of the Hamiltonian (commonly referred to as the *number states*) are the *same* as those of the number operator, some of whose features I list below.

Before we proceed, we observe that, in the Heisenberg picture, the annihilation and creation operators evolve in time as

$$\hat{a}(t) = \hat{a}(0)\mathrm{e}^{-\mathrm{i}\omega t}, \tag{8.91a}$$

$$\hat{a}^\dagger(t) = \hat{a}^\dagger(0)\mathrm{e}^{\mathrm{i}\omega t} \tag{8.91b}$$

(check these formulae out; use the evolution equation (8.73a) in the Heisenberg picture). The relations involving the creation and annihilation operators given here and in later sections mostly hold for all times t unless otherwise stated.

The operators $\hat{a}$ and $\hat{a}^\dagger$ are at times referred to as 'lowering and raising operators' since, operating on a number eigenstate $|n\rangle$, they produce number eigenstates again, with the number eigenvalue decreased and increased by unity, respectively:

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle, \quad \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle. \tag{8.92}$$

Strictly speaking, Eqs. (8.92) hold good only at a given instant of time, say, at $t = 0$, and are to be modified by the inclusion of appropriate phase factors for other values of t. In the Schrödinger picture the operators a and $a^\dagger$ are time independent, while the stationary states evolve by acquiring time-dependent phases. In the Heisenberg picture, on the other hand (which we adopt in most of our following analysis), the states $|n\rangle$ are time independent, while the operators a and $a^\dagger$ evolve as in Eqs. (8.91a) and (8.91b).

More commonly, however, equations such as Eq. (8.92) are to be interpreted as relations involving one or more specified operators and one or more specified vectors at specified time instants, regardless of the time evolution that these may happen to undergo. Equations involving time-dependent state vectors or time-dependent observables may then be obtained by insertion of appropriate time-dependent factors in these relations depending on which scheme or, *picture*, one is working in. While I have referred to the Schrödinger and Heisenberg pictures, the *interaction picture* (see Section 8.3.11) is sometimes of greater use.

Relations (8.92) lead to the following *matrix elements* of the creation and annihilation operators between the number states:

$$\langle m|\hat{a}|n\rangle = \sqrt{n}\delta_{m,n-1}, \quad \langle m|\hat{a}^\dagger|n\rangle = \sqrt{n+1}\delta_{m,n+1} \quad (m, n = 0, 1, \ldots). \tag{8.93}$$

The second relation in Eq. (8.92) further tells us that the number state $|n\rangle$ $(n = 0, 1, 2 \ldots)$ can be built up from the *vacuum state* $|0\rangle$ (the ground state of the oscillator) by repeated application of the creation operator:

$$|n\rangle = \left(\frac{1}{\sqrt{n!}}\hat{a}^{\dagger n}\right)|0\rangle. \tag{8.94}$$

In other words, the entire theory of the quantum harmonic oscillator can be built up with the help of the operators $\hat{a}$ and $\hat{a}^\dagger$, and the number states constructed from the ground state $|0\rangle$.

Contact with the classical theory is established by reference to the operators $\hat{q}$ and $\hat{p}$ representing the position and momentum observables, and looking at their measurement statistics in given quantum states of the oscillator. Classically, a pure state of the oscillator at any time instant t with energy E is characterized by a precise value of position $(q(t))$ and *also* a corresponding value of momentum $(p(t))$. Quantum mechanically, on the other hand, the position and momentum operators at time t *do not* possess precisely defined values in a stationary state $|n\rangle$, though the statistics of their measured values are characterized by well-defined probability distributions. In the case of a stationary state, these probability distributions are time independent, with *uncertainties* (or standard deviations) $\Delta\hat{q}$ and $\Delta\hat{p}$ satisfying the fundamental relation (Heisenberg's uncertainty principle)

$$\Delta\hat{q}\Delta\hat{p} \geq \frac{\hbar}{2}, \tag{8.95}$$

where the equality sign holds for the ground state $|0\rangle$ (the uncertainty relation, of course, holds for *any* state, stationary or not). The ground state is thus one member of a family of *minimum uncertainty states*, also known as *coherent states*.

8.4.2 The Coherent State

8.4.2.1 Minimum uncertainty

We begin by considering a state with wave function

$$\psi_{\bar{q}\bar{p}}(q) = \frac{1}{(\pi q_0^2)^{1/4}} \exp\left(-\frac{(q-\bar{q})^2}{2q_0^2} + \frac{\mathrm{i}}{\hbar}\bar{p}q\right), \tag{8.96}$$

where

$$q_0 \equiv \sqrt{\frac{\hbar}{m\omega}}, \tag{8.97a}$$

and $\bar{q}$ and $\bar{p}$ are two real parameters which we can combine into a single complex parameter:

$$\lambda = \frac{1}{\sqrt{2}}\left(\frac{\bar{q}}{q_0} + \mathrm{i}\frac{q_0}{\hbar}\bar{p}\right). \tag{8.97b}$$

Like any other pure (or, for that matter, mixed) quantum state, this state is characterized by fluctuations, or *dispersion*, in measured values of the position and momentum observables, but these fluctuations are *minimal* in the sense that, for this state, one has

$$\Delta\hat{q} = \frac{q_0}{\sqrt{2}}, \quad \Delta\hat{p} = \frac{\hbar}{\sqrt{2}q_0}, \tag{8.98a}$$

and hence

$$\Delta\hat{q}\Delta\hat{p} = \frac{\hbar}{2}. \tag{8.98b}$$

Such a state can thus be considered to correspond closely to a classical state since the variations in the measured values of position and momentum around their mean values are, in a sense, small, these mean values being given by

$$\langle\hat{q}\rangle = \bar{q}, \quad \langle\hat{p}\rangle = \bar{p}. \tag{8.98c}$$

We refer to Eq. (8.96) as a *coherent state*. It corresponds to the classical state represented by the point $(\bar{q},\bar{p})$ in the phase space of the harmonic oscillator. In particular, the ground state of the oscillator is a coherent state with $\bar{q} = 0$ and $\bar{p} = 0$ (and hence $\lambda = 0$), and corresponds to the equilibrium configuration of the classical oscillator, though it is distinguished from the latter by the existence of the zero point fluctuations, where these fluctuations are again minimal in the sense mentioned above.

In the following, a coherent state characterized by the parameter λ (see Eq. 8.97b) will be denoted by the symbol $|\lambda\rangle$.

8.4.2.2 The quadrature operators

In this context, we define the *quadrature operators*

$$\hat{X}_1 = \sqrt{\frac{m\omega}{2\hbar}}\hat{q}, \quad \hat{X}_2 = \sqrt{\frac{1}{2m\omega\hbar}}\hat{p}, \tag{8.99a}$$

which are simply rescaled position and momentum operators, related to a and $a^\dagger$ as

$$\hat{a} = \hat{X}_1 + \mathrm{i}\hat{X}_2, \quad a^\dagger = X_1 - \mathrm{i}X_2. \tag{8.99b}$$

These satisfy the commutation relation

$$[\hat{X}_1, \hat{X}_2] = \frac{\mathrm{i}}{2}, \tag{8.99c}$$

which implies that, for any arbitrarily chosen state, their fluctuations have to satisfy the uncertainty relation

$$\Delta\hat{X}_1 \Delta\hat{X}_2 \geq \frac{1}{4}. \tag{8.99d}$$

However, a coherent state is very special in being characterized by minimal fluctuations and so, for a coherent state $|\lambda\rangle$,

$$\Delta\hat{X}_1 = \frac{1}{2}, \quad \Delta\hat{X}_2 = \frac{1}{2}, \quad \Delta\hat{X}_1 \Delta\hat{X}_2 = \frac{1}{4}, \tag{8.99e}$$

which tells us, moreover, that the two quadratures are characterized by *equal* fluctuations in a coherent state.

Digression: Coherent state for an arbitrary 1D system

Even though we have defined the coherent state with reference to a harmonic oscillator, this reference is not essential and one can regard q_0 in Eq. (8.96) as any real parameter, not necessarily related to the parameters characterizing a harmonic oscillator in accordance with Eq. (8.97a). This gives us a coherent state for an arbitrarily chosen 1D quantum system (even for a harmonic oscillator, the parameter q_0 need not be related to the parameters of *that* oscillator by Eq. 8.97a). This will then constitute a minimum uncertainty state for the system under consideration, but will not possess many of the other characteristics of the harmonic oscillator coherent state, with q_0 defined in terms of the parameters of the oscillator as in Eq. (8.97a).

One can generalize the 1D coherent state introduced here to systems with higher numbers of degrees of freedom. The N-dimensional coherent state ($N = 1, 2, \ldots$) is an object of central importance in establishing the correspondence between quantum and classical theories.

8.4.2.3 Coherent state: Characteristic features

The normalized ket vector corresponding to the wave function (8.96) representing the coherent state can be written as a linear combination of the stationary states $|n\rangle$ $(n = 0, 1, 2, \ldots)$:

$$|\lambda\rangle = \mathrm{e}^{-\frac{|\lambda|^2}{2}} \sum_n \frac{\lambda^n}{\sqrt{n!}} |n\rangle, \tag{8.100}$$

where the complex parameter λ, as defined in Eq. (8.97b), is used to label the coherent state vector instead of the two real parameters ξ and η. One can generate Eq. (8.100) by applying the operator $\mathrm{e}^{\lambda a^\dagger}$ on the vacuum state $|0\rangle$:

$$|\lambda\rangle = \mathrm{e}^{-\frac{|\lambda|^2}{2}} \mathrm{e}^{\lambda \hat{a}^\dagger} |0\rangle. \tag{8.101}$$

An equivalent expression is

$$|\lambda\rangle = \hat{D}(\lambda)|0\rangle, \tag{8.102a}$$

where, for any given λ, the *displacement operator* $\hat{D}(\lambda)$ is defined as

$$\hat{D}(\lambda) = \mathrm{e}^{\lambda \hat{a}^\dagger - \lambda^* \hat{a}}. \tag{8.102b}$$

One can show that Eq. (8.102a) implies Eq. (8.101) by using the result that for two operators $\hat{A}$ and $\hat{B}$, each of which commutes with $[\hat{A}, \hat{B}]$,

$$\mathrm{e}^{\hat{A}+\hat{B}} = \mathrm{e}^{\hat{A}} \mathrm{e}^{\hat{B}} \mathrm{e}^{-1/2[\hat{A},\hat{B}]},$$

and then choosing $\hat{A} = \lambda \hat{a}^\dagger$ and $B = -\lambda^* \hat{a}$.

In this context, note that the coherent state $|\lambda\rangle$ is an *eigenstate of the annihilation operator*:

$$\hat{a}|\lambda\rangle = \lambda|\lambda\rangle, \tag{8.103}$$

the eigenvalue λ being, in general, complex because the annihilation operator a is non-Hermitian.

Using the time development of the annihilation and creation operators (Eqs. 8.91a and 8.91b) or the time evolution of stationary states, one can look at the evolution of the coherent states. In the Schrödinger picture, $\hat{a}$ and $\hat{a}^\dagger$ are time-independent operators, but the stationary states evolve by acquiring phases, which implies that, if the coherent state $|\lambda\rangle$ is given by Eq. (8.100) with complex parameter λ at $t = 0$, then at time t it evolves, *modulo* an inessential phase factor, to a coherent state with complex parameter

$$\lambda(t) = \lambda \mathrm{e}^{-\mathrm{i}\omega t}. \tag{8.104}$$

In the Heisenberg picture, on the other hand, the state itself remains fixed, but using the time-dependence of $\hat{a}$ in Eq. (8.103), one finds that $|\lambda\rangle$ is an eigenvector of $\hat{a}(t)$ with eigenvalue (8.104), which now becomes the new value of the parameter λ.

It is important to note that the *expectation values* of observables do not depend on the picture employed (ie, an expectation value of the form $\langle\psi|\hat{A}|\psi\rangle$ remains the same regardless of whether one uses the Schrödinger picture or the Heisenberg picture). One can, for instance, work out the expectation values of the position coordinate (q) and momentum (p) in a coherent state and find that in either of the two pictures

$$\begin{aligned}\langle\hat{q}\rangle(t) &= \mathrm{Re}[\sqrt{2}q_0\lambda(t)] \\ &= \sqrt{2}q_0|\lambda|\cos(\omega t-\theta),\end{aligned} \tag{8.105a}$$

$$\begin{aligned}\langle\hat{p}\rangle(t) &= \mathrm{Im}\left[\frac{\sqrt{2}\hbar}{q_0}\lambda(t)\right] \\ &= -\frac{\sqrt{2}\hbar}{q_0}|\lambda|\sin(\omega t-\theta),\end{aligned} \tag{8.105b}$$

where θ stands for the initial phase of λ. These resemble the sinusoidal oscillations of the position and momentum in a pure state of the classical harmonic oscillator. Correspondingly, $\langle\hat{X}_1\rangle(t)$ and $\langle\hat{X}_2\rangle(t)$—that is, the expectation values of the quadrature operators—vary as Cartesian components in a 2D space (the q-p phase space rescaled along the two axes) of a rotating phasor of magnitude $|\lambda|$:

$$\langle\hat{X}_1\rangle(t) = |\lambda|\cos(\omega t-\theta), \quad \langle\hat{X}_2\rangle(t) = -|\lambda|\sin(\omega t-\theta); \tag{8.106}$$

that is, the expectation values of the quadrature operators oscillate with equal amplitudes, and are out of phase with each other by $\frac{\pi}{2}$.

At any given instant of time we can represent the fluctuations in $\hat{X}_1$ and $\hat{X}_2$ in a 2D 'phase space' (recall that the quadrature operators are nothing but rescaled position and momentum operators) as in Fig. 8.4A (we choose $t = 0$ for concreteness), where the measured values $(X_1,\ X_2)$ of the two quadrature operators are represented as the real and imaginary parts of a complex-valued random phasor $(X_1 + \mathrm{i}X_2)$ whose mean is the phasor OA, with components

$$\langle\hat{X}_1\rangle = \mathrm{Re}\,\lambda,\ \langle\hat{X}_2\rangle = \mathrm{Im}\,\lambda. \tag{8.107}$$

The endpoints of the complex random phasor are scattered around A, which is depicted by means of the circle (of radius $\frac{1}{2}$; ie, the value of the uncertainty of either of the quadrature operators) drawn with A as the center, referred to as the *noise disk*. The projections of the radii AB and AC of the noise disk on the two axes represent geometrically the uncertainties $\Delta\hat{X}_1$ and $\Delta\hat{X}_2$. When the time dependence of the coherence parameter λ is taken into account, the noise disk along with the mean phasor OA revolves along a circle around the origin O in the direction of the bent arrow with uniform angular velocity ω, but the two projections remain constant. The revolving noise disk then represents the coherent state in the Schrödinger picture.

As I have already mentioned, the coherent state, being a state with the minimum uncertainty product $\Delta\hat{X}_1\Delta\hat{X}_2$, resembles a classical state. The resemblance increases with the value of $|\lambda|$ since this corresponds to a relative decrease in the values of $\Delta\hat{X}_1$ and $\Delta\hat{X}_2$ in comparison with $\langle\hat{X}_1\rangle$ and $\langle\hat{X}_2\rangle$. The phase angle (θ) of the coherence parameter λ at $t = 0$ and the phase uncertainty ($\Delta\Phi$) are shown in Fig. 8.4A, the latter being a measure of the fluctuation in the phase of the random complex phasor $X_1 + iX_2$ at any chosen time t. Notice that as $|\lambda|$ increases for any given θ, the phase uncertainty decreases, making the coherent state resemble more closely a classical oscillator.

Since the vacuum state $|0\rangle$ is a coherent state with $\lambda = 0$ (see Eq. 8.103), its phase space representation is as in Fig. 8.4B. Notice that the noise disk is the same for all the coherent states. However, the vacuum state, with $|\lambda| = 0$, has the least resemblance to a classical state since it is, at the same time, a number state ($n = 0$). The phase uncertainty for this state is maximal since the phase fluctuates between 0 and 2π.

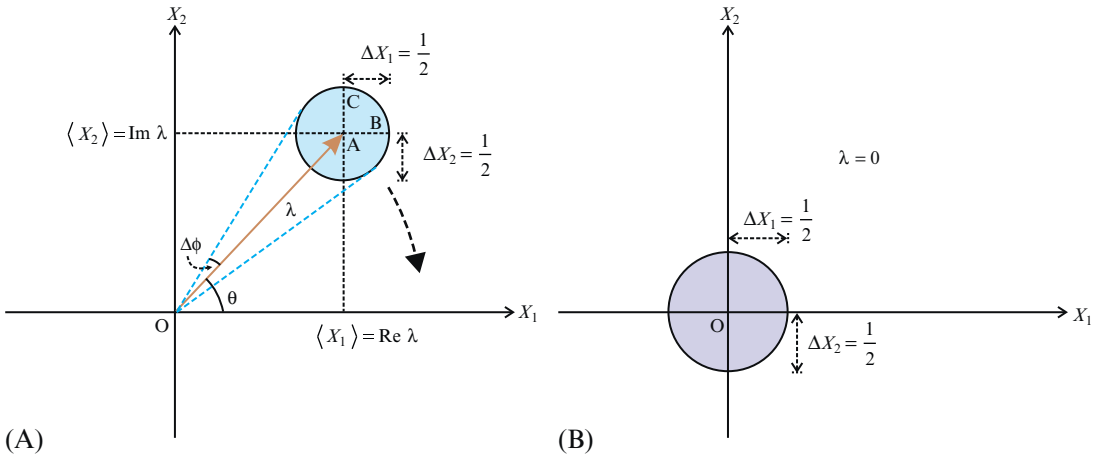

Fig. 8.4

Illustrating the phase space representation of a coherent state. (A) The coherent state with parameter λ is represented by a circular noise disk in a two-dimensional phase space centered around the point with abscissa and ordinate given by $\langle\hat{X}_1\rangle = \text{Re}\,\lambda$ and $\langle\hat{X}_2\rangle = \text{Im}\,\lambda$. The disk has the same radial lengths along the X_1- and X_2-axes corresponding to $\Delta\hat{X}_1 = \Delta\hat{X}_2 = \frac{1}{2}$. The time evolution of the coherent state is indicated with the help of the bent arrow, which corresponds to a uniform rotation of the phasor λ. The initial phase of λ is denoted by θ. The phase uncertainty $\Delta\Phi$ is shown. A coherent state with a comparatively large value of $|\lambda|$ closely resembles a classical state because of reduced phase uncertainty. (B) The vacuum state $|0\rangle$ is represented by a circular noise disk centered at the origin. The vacuum state, being at the same time a number state, is nonclassical. The phase uncertainty is maximal since the phase fluctuates between 0 and 2π.

Incidentally, the displacement operator $\hat{D}(\lambda)$ is so called because its action on the vacuum state displaces the noise disk from its position in Fig. 8.4B to that in Fig. 8.4A.

The time evolution of the coherent state can be described, in summary, by saying that $\langle \hat{X}_1 \rangle$ and $\langle \hat{X}_2 \rangle$ both vary harmonically with frequency ω, while $\Delta \hat{X}_1$ and $\Delta \hat{X}_2$ are both time independent (and equal, with product $\Delta X_1 \Delta X_2 = \frac{1}{4}$).

8.4.2.4 Poisson distribution

Consider the expression (8.100) for the coherent state $|\lambda\rangle$, where one observes that the probability for the number operator $\hat{N}$ having the value n in this state is

$$P(n) = \mathrm{e}^{-|\lambda|^2} \frac{|\lambda|^{2n}}{n!}, \tag{8.108}$$

with mean and variance given by

$$\langle n \rangle = (\Delta n)^2 = |\lambda|^2. \tag{8.109}$$

This is an important result. It tells us that the distribution of the variable n is *Poissonian*, with identical mean and variance. This feature of the coherent state is of relevance in the context of the *photon counting statistics* for an electromagnetic field that can be represented in terms of a coherent state, where the statistics can be described as being *classical* in nature. In contrast, the counting statistics may deviate from the Poissonian type in the case of a field represented by a *squeezed* state described in Section 8.4.3, the latter being then a *nonclassical* state.

8.4.3 Squeezed States

8.4.3.1 Quadrature squeezing: The squeezed coherent state

As already mentioned, the state (8.96) with q_0 defined as in Eq. (8.97a) is a coherent state for a harmonic oscillator with mass m and angular frequency ω, having the following important features:

1. It satisfies the minimum uncertainty relation (8.98b).
2. The uncertainties in the quadrature operators $\hat{X}_1$ and $\hat{X}_2$ are equal to each other, each being the square root of the minimum possible value ($\frac{1}{4}$) of the product.

If, on the other hand, for the harmonic oscillator of mass m and angular frequency ω, we choose q_0 to have some value *other* than Eq. (8.97a), then feature 1 remains unaltered but feature 2 will no longer remain valid (though it will still be valid for some *other* harmonic oscillator of appropriate mass and angular frequency). This feature becomes more pronounced as q_0 is made to deviate further away from Eq. (8.97a), either toward ∞ or toward 0.

The state with wave function (8.96) then constitutes a special instance of a class of states, known as *squeezed states*. It reduces to the coherent state (Eqs. 8.102a and 8.102b) under the choice of q_0, λ given by Eqs. (8.97a) and (8.97b). While it belongs to a family of states depending on the real parameter q_0, the squeezed states will be seen to belong to a larger family characterized by *two* real parameters (or, equivalently, by one complex parameter), in addition to the complex parameter λ.

With reference to a harmonic oscillator of mass m and angular frequency ω and with reference to the squeezed state (8.96) where q_0 is chosen arbitrarily (ie, not necessarily as in Eq. 8.97a), notice that the definition of the quadrature operators (Eq. 8.99a), however, involves the parameters m and ω, and one now has

$$\Delta\hat{X}_1 = \frac{1}{2}\sqrt{\frac{m\omega}{\hbar}}q_0, \quad \Delta\hat{X}_2 = \frac{1}{2}\sqrt{\frac{\hbar}{m\omega}}\frac{1}{q_0}. \tag{8.110}$$

In other words, the fluctuations of the two quadrature operators are, in general, *unequal* in a squeezed state, one being suppressed at the cost of the other (whence the name 'squeezed'). With reference to Fig. 8.4, this corresponds to the circular noise disk (depicting the uncertainties of the two quadrature operators) being compressed in one direction and elongated in the other.

The state (8.96), with q_0 having an arbitrarily chosen value, is referred to as a *squeezed coherent state* since it involves squeezing and, at the same time, is a minimum uncertainty state. The type of squeezing characterizing such a state is termed *quadrature squeezing* since one of the quadrature components is squeezed at the expense of the other. As mentioned above, the term 'squeezed state' refers to a larger class of states that includes the squeezed coherent states.

8.4.3.2 Squeezed states: Definition and construction

I outline below two ways to look at a *squeezed state*, in the more general sense mentioned above, where a squeezed state is of relevance in quantum optics as one corresponding to a *nonclassical* state of the electromagnetic field.

The first approach

Consider the unitary operator $\hat{S}(\xi)$, corresponding to any given complex number ξ, defined as

$$\hat{S}(\xi) = \exp\left(\frac{\xi^*}{2}\hat{a}^2 - \frac{\xi}{2}\hat{a}^{\dagger 2}\right), \tag{8.111a}$$

and the state obtained by the action of $\hat{S}(\xi)$, the so-called *squeeze operator*, on the vacuum state

$$|0;\xi\rangle = \hat{S}(\xi)|0\rangle, \tag{8.111b}$$

where the notation on the left-hand side is meant to convey that this is a special instance of a squeezed state of a more general description, characterized by two complex parameters λ and ξ (see later), Eq. (8.111b) being a state with $\lambda = 0$.

This is referred to as the *squeezed vacuum state*. Expressing ξ in the polar form

$$\xi = re^{2i\phi}, \tag{8.112a}$$

where one can choose $r \geq 0$ by an appropriate choice of ϕ (it is more usual however, to choose ϕ as an acute angle, in which case r can be either positive or negative), one can describe the action of $\hat{S}(\xi)$ on $|0\rangle$ in geometrical terms as follows.

Referring to Fig. 8.4B and looking at the circular noise disk representing the ground state in the X_1-X_2 phase space, imagine a pair of Cartesian axes rotated by an angle ϕ, where these new axes correspond to measured values of operators $\hat{Y}_1$ and $\hat{Y}_2$ related to $\hat{X}_1$ and $\hat{X}_2$ as

$$\hat{Y}_1 = \hat{X}_1 \cos\phi + \hat{X}_2 \sin\phi, \quad \hat{Y}_2 = -\hat{X}_1 \sin\phi + \hat{X}_2 \cos\phi. \tag{8.112b}$$

The new axes, obtained from the old ones by means of a rotation through the angle ϕ, are indicated in Fig. 8.5A. The geometrical significance of Eq. (8.111b) is then seen from its representation by the noise patch shown in Fig. 8.5A. Compared with the circular noise disk representing the vacuum state (see Fig. 8.4B), it is seen to be elliptic, oriented with its principal axes along the new coordinate axes measuring Y_1 and Y_2. The uncertainties in the measured values of $\hat{Y}_1$ and $\hat{Y}_2$ in the state $|0;\xi\rangle$, giving the ratio by which one of the principal axes of the ellipse is squeezed at the expense of the other, are

$$\Delta\hat{Y}_1 = \frac{1}{2}e^{-r}, \quad \Delta\hat{Y}_2 = \frac{1}{2}e^{r}. \tag{8.112c}$$

In other words, the operator $\hat{S}(\xi)$ has the effect of 'rotation and squeeze' in the phase space, whereby the circular disk in Fig. 8.4B representing the vacuum state is transformed to the elliptic patch in Fig. 8.5A representing the squeezed vacuum state.

> While the idea of phase space is, strictly speaking, not an appropriate one in describing quantum states, it can be given a more precise meaning in terms of the so-called *Wigner function*, which can be regarded, with certain qualifications, as a distribution function in the phase space corresponding to a quantum state. The circular disk or elliptic patch in Fig. 8.4 or Fig. 8.5A can also be defined in precise terms by reference to the Wigner function. For our present purpose, we interpret it as a 2D space made up of axes representing possible measured values of the quadrature operators (or of the transformed operators in Eq. 8.112b).

Let us now consider the action of the displacement operator $\hat{D}(\lambda)$ (see Eq. 8.102b) on the squeezed vacuum $|0;\xi\rangle$. As seen from Fig. 8.4, this has the effect of displacing the noise disk along the phasor $\lambda = |\lambda|e^{i\theta}$. In other words, the state

$$|\lambda;\xi\rangle = \hat{D}(\lambda)\hat{S}(\xi)|0\rangle, \tag{8.113}$$

obtained by the application of the squeeze operator $\hat{S}(\xi)$ to the vacuum state $|0\rangle$, followed by the application of the displacement operator $\hat{D}(\lambda)$, is represented in the phase space by the noise disk as shown in Fig. 8.5B, which results from the circular disk in Fig. 8.4B by the operations of *rotation, squeeze*, and *displacement*. This constitutes an instance of a squeezed state of the general type, of which the squeezed vacuum state $|0;\xi\rangle$ is a particular example, corresponding to $\lambda = 0$. The latter means that the vacuum state is squeezed and rotated but not displaced. As another particular instance, recall the properties of the state with wave function (8.96), with an arbitrarily chosen value of q_0. This is a state with quadrature squeezing and displacement but no rotation (ie, with $\phi = 0$). The squeezing factor e^{-r} (see Eq. 8.112c; with $\phi = 0$, $\hat{Y}_1$ and $\hat{Y}_2$ reduce to the old quadrature operators $\hat{X}_1$ and $\hat{X}_2$) is related to q_0 as

$$e^{-r} = \sqrt{\frac{m\omega}{\hbar}}\, q_0. \tag{8.114}$$

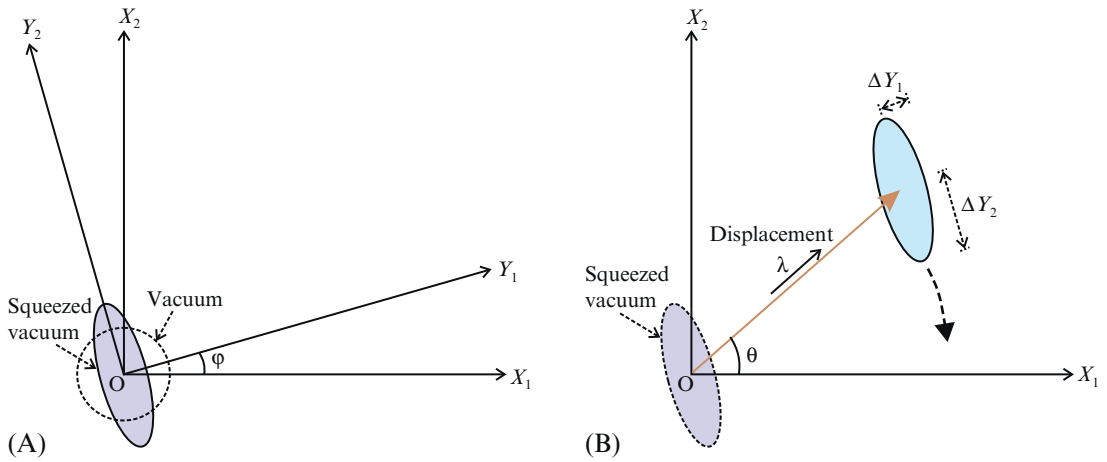

Fig. 8.5

Illustrating the phase space representation of a squeezed state. (A) The squeezed vacuum state. A rotation by ϕ gives the new axes measuring $\hat{Y}_1$ and $\hat{Y}_2$ from the old ones measuring $\hat{X}_1$ and $\hat{X}_2$. The uncertainty patch for the vacuum state (*dotted circle*) is then squeezed and extended along the transformed axes as shown (r has been chosen positive for concreteness), to give a rotated ellipse. (B) The more general squeezed state, obtained from the squeezed vacuum (*dotted ellipse centered at the origin*) by the application of the displacement operator $\hat{D}(\lambda)$. In contrast to the coherent state, where the phasor λ rotates uniformly with its endpoint remaining on a circle, the squeezed state corresponds to the endpoint of the phasor rotating on an ellipse (not shown) with frequency ω, where the length of the phasor varies periodically with frequency 2ω.

In the Schrödinger picture the noise disk representing the squeezed state rotates with an angular velocity ω around the origin along a rotated ellipse (not shown in Fig. 8.5), and the length of the phasor λ varies with time (see Section 8.4.3.3).

The operator $\hat{S}(\xi)$ may be regarded as a *two-photon displacement operator* since it is built up from $\hat{a}^2$ and $\hat{a}^{\dagger 2}$ in a manner similar to the way the displacement operator $D(\lambda)$ is made up from $\hat{a}$ and $\hat{a}^\dagger$.

The second approach

The second approach I will now outline gives an alternative view of the squeezed state. Consider the pair of operators $\hat{b}$ and $\hat{b}^\dagger$ related to the creation and annihilation operators $\hat{a}$ and $\hat{a}^\dagger$ as

$$\hat{b} = \mu\hat{a} + \nu\hat{a}^\dagger, \quad \hat{b}^\dagger = \mu^*\hat{a}^\dagger + \nu^*\hat{a}, \tag{8.115a}$$

where μ and ν are complex numbers satisfying

$$|\mu|^2 - |\nu|^2 = 1. \tag{8.115b}$$

One may regard Eqs. (8.115a) and (8.115b) as a transformation from $\hat{a}$ and $\hat{a}^\dagger$ to new operators $\hat{b}$ and $\hat{b}^\dagger$ by means of a unitary transformation such that $\hat{b}$ and $\hat{b}^\dagger$ can be interpreted as a new pair of annihilation and creation operators, with the same commutation relation as that for the old pair,

$$[\hat{b}, \hat{b}^\dagger] = \hat{I}, \tag{8.116}$$

where the symbol $\hat{I}$ stands for the unit operator.

While $\hat{a}$ and $\hat{a}^\dagger$ are the annihilation and creation operators for the 'quanta' corresponding to the decrease and increase of the eigenvalue of the number operator in the stationary states of the oscillator (the *photons* in the oscillator description of the electromagnetic field; see Section 8.6), the new operators $\hat{b}$ and $\hat{b}^\dagger$ can be interpreted as annihilation and creation operators, but for a different set of excitations of the oscillator (or a different set of excitations of the electromagnetic field in the quantum description) that we term the *generalized number states*. More precisely, these excitations are the eigenstates of the operator

$$\hat{N}_\mathrm{g} = \hat{b}^\dagger\hat{b}, \tag{8.117}$$

whose eigenvalues are the nonnegative integers, analogous to those of $\hat{N} = a^\dagger a$.

> The operators and states defined in terms of $\hat{b}$ and $\hat{b}^\dagger$ will be distinguished by attaching the subscript 'g.' According to this notation, $\hat{b}$ and $\hat{b}^\dagger$ are more appropriately written as $\hat{a}_\mathrm{g}$ and $\hat{a}_\mathrm{g}^\dagger$, though I will stick to $\hat{b}$ and $\hat{b}^\dagger$ instead.

The generalized number states $|n\rangle_{\rm g}$ are built up from the generalized vacuum state $|0\rangle_{\rm g}$ by successive application of $\hat{b}^\dagger$. One can express the generalized vacuum state in terms of the ordinary number states by assuming an expansion of the form $\sum c_n|n\rangle$, and then evaluating the coefficients c_n from the defining relation

$$\hat{b}|0\rangle_{\rm g} = 0. \tag{8.118}$$

From this exercise, one finds that $|0\rangle_{\rm g}$ is made up of ordinary number states with only *even* numbers of quanta (ie, $n = 0, 2, \ldots$) (check this statement out).

One can now define a generalized coherent state $|\beta\rangle_{\rm g}$ characterized by any given complex number β by means of the relation

$$\hat{b}|\beta\rangle_{\rm g} = \beta|\beta\rangle_{\rm g}, \tag{8.119}$$

which is analogous to formula (8.103), and tells us that $|\beta\rangle_{\rm g}$ is an eigenstate of the generalized annihilation operator $\hat{b}$ belonging to the eigenvalue β. In other words, there exists a set of relations in terms of the generalized operators and vectors analogous to those in terms of the old set of operators and vectors. For instance, the generalized coherent state $|\beta\rangle_{\rm g}$ is made up of the generalized number states $|n\rangle_{\rm g}$ in a manner analogous to the way the old coherent state λ is made up of the old number states (formula (8.100)).

One can now identify these generalized coherent states as the squeezed states I introduced above.

1. The generalized coherent states are at times referred to as *two-photon coherent states*. The term is also used in a more restricted sense, to refer to the squeezed vacuum state $\hat{S}(\xi)|0\rangle$ since $\hat{S}(\xi)$ plays the role of a two-photon displacement operator.
2. On the face of it, the state $|\beta\rangle_{\rm g}$ appears to depend on three complex parameters μ, ν, and β. However, on solving for $|\beta\rangle_{\rm g}$ from Eq. (8.119) as a superposition of the ordinary number states, one finds that the superposition coefficients depend on only two complex parameters $\frac{\beta}{\mu}$ and $\frac{\beta}{\nu}$. This is consistent with the fact that a squeezed state also depends, in general, on the two complex parameters λ and ξ. The correspondence between the two sets of parameters λ, ξ and μ, ν, β is given by formulae (8.121a) and (8.121b) below.

However, for this, one has to have the complex parameters λ and ξ corresponding to the eigenvalue β of the generalized annihilation operator, where the latter, in turn, depends on the parameters μ and ν in Eqs. (8.115a) and (8.115b). In other words, given a $|\beta\rangle_{\rm g}$, we have to obtain λ and ξ in terms of μ, ν, and β such that

$$|\lambda; \xi\rangle = |\beta\rangle_{\rm g}. \tag{8.120}$$

This needs a bit of operator algebra, and I will tell you what the end result is. First, the parameters r and ϕ are given by the relations

$$\mu = \cosh r, \quad \nu = \sinh r\, {\rm e}^{2{\rm i}\phi}. \tag{8.121a}$$

We then define ξ in terms of r and ϕ as in Eq. (8.112a). Finally, λ is obtained as

$$\lambda = \mu\beta - \nu\beta^*. \tag{8.121b}$$

With this identification of the parameters, one has the result (8.120). In other words, one can now write

$$|\beta\rangle_{\mathrm{g}} = |\lambda;\xi\rangle = \hat{D}(\lambda)\hat{S}(\xi)|0\rangle. \tag{8.122a}$$

8.4.3.3 Squeezed states: Characteristic features

Time evolution

I include a few words first on the time evolution of a squeezed state. In contrast to the coherent state, for which $\Delta\hat{X}_1$ and $\Delta\hat{X}_2$ are both time independent owing to the time independence of $|\lambda|$, the two uncertainties for the squeezed state both vary periodically with time, with frequency 2ω.

Along with the uncertainties $\Delta\hat{X}_1$ and $\Delta\hat{X}_2$, the product $\Delta\hat{X}_1\Delta\hat{X}_2$ also varies periodically. If the minimum value attained by the product $\Delta\hat{X}_1\Delta\hat{X}_2$ is $\frac{1}{4}$ (the minimum possible value allowed by the uncertainty principle), then the state under consideration is referred to as a *minimum uncertainty squeezed state*. Thus the quadrature-squeezed state given by the wave function (8.96), with an arbitrarily chosen value of q_0, is a minimum uncertainty squeezed state for which the two uncertainties are, moreover, time independent.

The expectation value β of the generalized annihilation operator $\hat{b}$ varies periodically with a fixed magnitude as $\beta(t) = \beta(0)\mathrm{e}^{-\mathrm{i}\omega t}$ (analogously to the expectation value of the ordinary annihilation operator a in the coherent state). This gives the time variation of λ since the latter is related to β by Eq. (8.121b). With progressing time, the phasor λ moves on a rotated ellipse, and $|\lambda|$ varies periodically with frequency 2ω. The time evolution of the squeezed state in the Schrödinger picture can then be described geometrically as a uniform rotation of the elliptic noise disk on this rotated ellipse, as mentioned in Section 8.4.3.2. The phasor ξ, however, does not depend on time.

Number distribution in a squeezed state

The expression for the squeezed state $|\lambda;\xi\rangle$ in terms of the number states $|n\rangle$ does not look as simple as that for the coherent state $|\lambda\rangle$ (Eq. 8.100) but you still need to have a look at it since it constitutes the master formula for whatever special case you may have in mind (eg, one obtains the coherent state $|\lambda\rangle$ from it by putting $\xi = 0$, the squeezed vacuum corresponds to $\lambda = 0$, and so on). It reads

$$|\lambda;\xi\rangle = c_n|n\rangle, \tag{8.123a}$$

where

$$c_n = (n!\cosh r)^{-1/2}\left(\frac{e^{2i\phi}\tanh r}{2}\right)^{n/2}\exp\left(-\frac{1}{2}|\lambda|^2 - \frac{1}{2}\lambda^{*2}e^{2i\phi}\tanh r\right) \times H_n\left(\frac{\lambda e^{-i\phi} + \lambda^* e^{i\phi}\tanh r}{\sqrt{2\tanh r}}\right). \tag{8.123b}$$

In this expression, $H_n(z)$ stands for the *Hermite polynomial* of degree n, with z as argument, the parameters r and ϕ being defined in terms of ξ as in Eq. (8.112a). From this, one can obtain the probability distribution for the number of quanta (n, the eigenvalue of the number operator $\hat{N}$) as

$$P(n) = |c_n|^2 = (n!\cosh r)^{-1}\left(\frac{\tanh r}{2}\right)^n \exp\left(-|\lambda|^2 - \frac{1}{2}(\lambda^2 e^{-2i\phi} + \lambda^{*2}e^{2i\phi})\tanh r\right) \times \left|H_n\left(\frac{\lambda e^{-i\phi} + \lambda^* e^{i\phi}\tanh r}{\sqrt{2\tanh r}}\right)\right|^2. \tag{8.123c}$$

One can show that formula (8.123c) reduces to Eq. (8.108), the number distribution in the coherent state $|\lambda\rangle = |\lambda;\xi = 0\rangle$, in the limit $r \to 0$, by using the fact that, for $z \to \infty$, $H_n(z)$ goes like $(2z)^n$.

Using the distribution (8.123c), one can work out the mean and variance of the distribution and obtain

$$\langle\hat{N}\rangle = \langle\lambda;\xi|\hat{N}|\lambda;\xi\rangle = |\lambda|^2 + \sinh^2 r, \tag{8.124a}$$

$$(\Delta\hat{N})^2 = \langle\lambda;\xi|\hat{N}^2|\lambda;\xi\rangle - \langle\hat{N}\rangle^2 = \left|\lambda\cosh r - \lambda^*\sinh r\, e^{2i\phi}\right|^2 + 2\cosh^2 r\sinh^2 r. \tag{8.124b}$$

As we will see from these expressions, the number distribution $P(n)$ for the general squeezed state $|\lambda;\xi\rangle$ possesses interesting features as compared with the corresponding distribution for the coherent state $|\lambda\rangle$, which relates to the fact that, generally speaking, the squeezed state is *nonclassical* in nature while the coherent state has *classical* features (recall from Section 8.4.2.3 the way a coherent state evolves in time). You will find in Section 8.4.4 a brief explanation of the terms 'classical' and 'nonclassical' as applied to the states of a quantum system.

The nonclassical feature of a squeezed state is relevant in the context of *photon counting statistics* in optics, as we will see in Section 8.17. For instance, the number distribution $P(n)$, given by Eq. (8.123c), can be described as *sub-Poissonian* for certain ranges of values of the parameters μ and ν characterizing a squeezed state. Fig. 8.6 illustrates this idea graphically.

In Fig. 8.6A the distribution $P(n)$ for a coherent state with a given value of λ (graph A) is compared with the distributions (graphs B and C) for squeezed states with the same λ, where graphs B and C both correspond to the same value of r but to different values of the phase ϕ relative to the phase (θ) of λ. One observes that graph B corresponds to a sub-Poissonian distribution, being *narrower* than the Poissonian graph A, while graph C is broader than graph A.

The sub-Poissonian nature of the number distribution for certain ranges of the parameters characterizing a squeezed state is described by the inequality

$$(\Delta\hat{N})^2 < \langle\hat{N}\rangle, \tag{8.125}$$

where, for given values of the parameters, one can check from Eqs. (8.124a) and (8.124b) whether or not the inequality is satisfied. For instance, for $r \geq 0$, $\phi = 0$, and λ real ($\theta = 0$), the graphs in Fig. 8.6B depict the relation between $(\Delta\hat{N})^2$ and $\langle\hat{N}\rangle$. In Fig. 8.6B, graph A again corresponds to coherent states ($r = 0$) with various values of λ (increasing in the direction of the arrow), while graph B represents the variation of $(\Delta\hat{N})^2$ with $\langle\hat{N}\rangle$ for various values of λ with a fixed positive value of r.

One observes that, under the stipulated constraints (fixed $r > 0$, $\phi = 0$, λ real and positive; assumed for concreteness), the number distribution for a squeezed state becomes

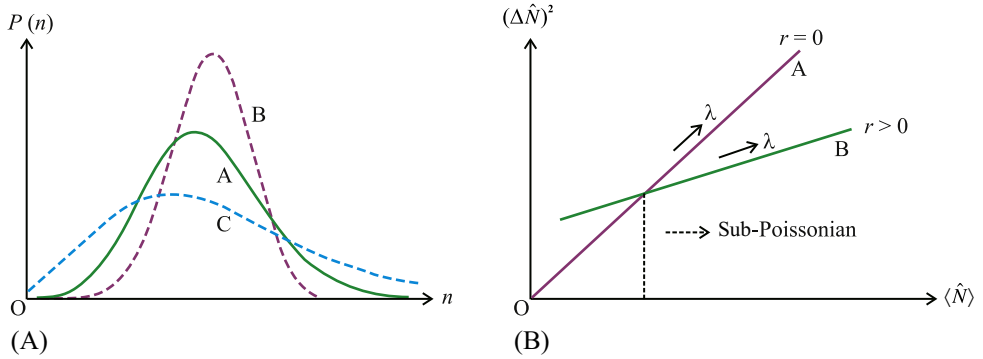

Fig. 8.6

Illustrating how the number distribution in a typical squeezed state differs from the Poisson distribution in a coherent state. (A) Graphs of $P(n)$ against n. Graph A corresponds to a coherent state with a chosen value of λ, while graphs B and C correspond to squeezed states with the same λ. The value of r is the same for the latter two graphs, which differ in the value of ϕ (see Eq. 8.112a) relative to the phase of λ. Graph B corresponds to sub-Poissonian statistics of the number distribution, while graph C is broader than the Poissonian graph (schematic). (B) The relation between $(\Delta\hat{N})^2$ and $\langle\hat{N}\rangle$. Graph A again corresponds to coherent states, while graph B is for a set of squeezed states with a chosen value of $r(> 0)$ and with $\phi = 0$. For each of the two graphs, the arrow represents increasing values of λ assumed real and positive. The squeezed state distribution becomes sub-Poissonian for sufficiently large values of λ. (Adapted from C. Gerry and P. Knight, *Introductory Quantum Optics*, Cambridge University Press, Cambridge, 2005; Fig. 7.9)

sub-Poissonian ($(\Delta\hat{N})^2 < \langle\hat{N}\rangle$) as λ becomes sufficiently large. The number distribution for a squeezed state follows sub-Poissonian statistics for other, more general, conditions on the parameters as well.

8.4.4 Harmonic Oscillator: Classical and Nonclassical States

8.4.4.1 The basic idea

In Section 8.4.3 I described three classes of states of the harmonic oscillator: the number states, the coherent states, and the squeezed states. It is now necessary to link these to another concept—the distinction between *classical* and *nonclassical* states of a quantum system. I will briefly outline the basic idea here with reference to the harmonic oscillator again, the system of prime interest in quantum optics.

Recall that a pure state of a classical system is represented by a single point in the phase space, a space of two dimensions for the harmonic oscillator, made up of the canonical variables q and p. The state evolves in time by way of these two variables executing simple harmonic oscillations. A *mixed* state, on the other hand, corresponds to a *probability distribution* over a set of pure states. For the harmonic oscillator this corresponds to a probability distribution in the 2D phase space, wherein the measured values of q and p in the state appear as random variables.

By contrast, a pure state of a quantum system corresponds to a vector in a linear vector space, and dynamical variables such as the position and the momentum correspond to random variables in such a state, with appropriate probability distributions. Referring to the harmonic oscillator, one might think of the state being described by a probability distribution in the q-p phase space, thereby bringing up the question of comparing a pure quantum state with a mixed classical state. However, a probability distribution in the phase space for a quantum state is ruled out by the fact that q and p are incompatible observables, represented by noncommuting operators (on the other hand, a probability distribution in terms of either q or p is well defined). Moreover, a probability distribution does not, in general, carry the *phase information* inherent in a wave function representing the quantum state.

To meaningfully address the question of how closely a state of a quantum system resembles a classical state, possibly a mixed one, one needs a phase space description in terms of a phase space distribution function for a pure quantum state that carries the phase information of the quantum wave function and, at the same time, resembles as closely as possible a probability distribution function.

It turns out that *such a distribution function can be defined only for a certain class of quantum states*. These are referred to as the 'classical states' of the system under consideration. In the case of the harmonic oscillator, these are precisely the *coherent states*. For other classes of

states, on the other hand, an analogous construction of a phase space distribution function fails in the sense that the distribution function differs from a probability distribution in one or more crucial respects. These are the *nonclassical states*, examples of which are the number states and the squeezed states of the harmonic oscillator.

One approach for the description of quantum states in terms of phase space distribution functions involves the construction and use of the so-called *Wigner distribution*. A more meaningful and appropriate distribution function, especially in the context of the harmonic oscillator, is the *Sudarshan-Glauber P*-representation. However, before I introduce the *P*-representation, I want you to recall two more types of states for quantum systems of relevance in quantum optics.

The first of these is the class of *mixed states* of a quantum system, while the second relates to *composite systems* such as a system made up of a number of independent harmonic oscillators, where these have been explained in Sections 8.3.10, 8.3.12, and 8.3.13. Of the two, composite systems will be seen to be of relevance in the context of multimode configurations of the electromagnetic field. *Entangled* states of composite systems are a class of nonclassical states of especial importance for composite systems. Entangled photonic states are of great relevance in optical *information processing* systems. Mixed quantum states, on the other hand, can occur in single-mode field configurations as well as in multimode ones.

For instance, if a harmonic oscillator is in equilibrium with a thermal reservoir at temperature T, then one cannot specify with certainty as to which number state $|n\rangle$ it is in, but on the other hand, one knows that the probability of finding it in the state $|n\rangle$ is given by the *Boltzmann formula*

$$P(n) = \frac{\exp\left(-\frac{E_n}{k_B T}\right)}{Z} \quad \left(E_n = \hbar\omega\left(n + \frac{1}{2}\right)\right), \tag{8.126a}$$

where k_B stands for the Boltzmann constant, and the *partition function* Z is given by

$$Z = \sum_n \exp\left(-\frac{E_n}{k_B T}\right). \tag{8.126b}$$

Mixed states of this type are relevant in quantum optics in describing *blackbody radiation* (ie, radiation in thermal equilibrium within an enclosure) and thermal equilibrium states involving single or multiple modes in a cavity.

As mentioned in Section 8.3.10, a mixed state of the form

$$\hat{\rho} = \sum_{n=1}^{N} w_n |\psi_n\rangle\langle\psi_n| \tag{8.127}$$

is often referred to as an *ensemble* in statistical physics, the ensemble being made up of the pure states $|\psi_n\rangle$, with respective probabilities w_n $(n = 1, 2, \ldots, N)$.

With the concept of the state of a quantum system broadened to include mixed states, one can now address the question of distinguishing between classical and nonclassical states of the system. Before taking up the question in the context of the electromagnetic field, I will outline the basic idea in Section 8.4.4.2 below by referring to the harmonic oscillator.

8.4.4.2 Sudarshan-Glauber P-representation

As I have already mentioned, a coherent state resembles a state of a classical oscillator, while a squeezed state is a nonclassical one. One manifestation of this distinction relates to the fact the number distribution in the coherent state is Poissonian, while that in a squeezed state may be sub-Poissonian, a distinction that acquires relevance in the context of photon counting statistics in quantum optics.

A convenient criterion for classifying the states of an oscillator into classical and nonclassical states can now be stated by reference to their *P-representations*. For any given state, in general a mixed one, described by the density operator $\hat{\rho}$, its P-representation is defined by reference to the following expression:

$$\hat{\rho} = \int P(\sigma)|\sigma\rangle\langle\sigma| d\sigma_{\mathrm{R}} d\sigma_{\mathrm{I}}, \tag{8.128}$$

where $|\sigma\rangle\langle\sigma|$ is the density operator for the coherent state with parameter σ, and the integration is over all possible values of σ_{R} and σ_{I}, the real and imaginary parts of σ, respectively. This relation defines a real function $P(\sigma)$ of σ_{R} and σ_{I} that resembles a distribution in the phase space made up of the variables σ_{R} and σ_{I}, and is referred to as the 'Sudarshan-Glauber P-representation' of the state under consideration.

1. The term 'phase space' is relevant here since σ_{R} and σ_{I} correspond to the two quadrature operators, and can be regarded as rescaled versions of the position and momentum variables in the classical description.
2. I will briefly explain below the term 'resembles' used above since, strictly speaking, a quantum state cannot be described completely by a probability distribution in the classical phase space. It is the degree of 'resemblance' that determines whether the state under consideration can be described as a classical or a nonclassical one.

While formula (8.128) can be regarded as the definition of the P-function, an explicit, though formal, expression for the latter is given by

$$P(\sigma) = \frac{\mathrm{e}^{|\sigma|^2}}{\pi^2} \int d\alpha_{\mathrm{R}} d\alpha_{\mathrm{I}} \mathrm{e}^{|\alpha|^2} \langle -\alpha|\rho|\alpha\rangle \exp(\alpha^*\sigma - \alpha\sigma^*). \tag{8.129}$$

Here $\alpha = \alpha_{\mathrm{R}} + \mathrm{i}\alpha_{\mathrm{I}}$ is a complex variable for integration over the phase space, and $|\alpha\rangle$ and $|-\alpha\rangle$ stand for coherent states with complex parameters α and $-\alpha$, respectively.

A true probability distribution in the classical phase space is necessarily everywhere positive and, moreover, nonsingular in nature, where the extreme case of a distribution concentrated at

one point (a pure classical state) corresponds to a delta function singularity. Thus a distribution that acquires negative values, or one that possesses singularities stronger than a delta function singularity, cannot correspond to a classical probability distribution. On the other hand, a distribution that is everywhere positive and possesses no singularities stronger than delta function ones can be interpreted as the probability distribution corresponding to a classical state, possibly a mixed one.

This, then, is the criterion for distinguishing 'classical' from nonclassical states of a harmonic oscillator. While this may sound self-contradictory in that the description of the states of a quantum system differs fundamentally from that of a classical one, the basic idea here is to check whether or not any given quantum state can be *operationally* interpreted as a classical one. The practical relevance of the P-representation is that it gives an operational criterion whereby certain states of the electromagnetic field correspond to photocount statistics that may be said to be indistinguishable from the statistics produced by states of the field described classically, while certain others cannot be so characterized. Such a criterion clearly distinguishes those states of the field for which one requires a quantum theoretic formalism for an adequate description of its behavior from the ones for which a classical description suffices.

Looked at from this point of view, the coherent state redeems itself by fulfilling the criterion for a classical state of the harmonic oscillator since one has

$$P(\sigma) = \delta^{(2)}(\lambda - \sigma) \quad \text{for } \hat{\rho} = |\lambda\rangle\langle\lambda|. \tag{8.130}$$

One can derive formula (8.130) by referring to Eq. (8.129) and using the result that for two coherent states $|\alpha\rangle$ and $|\lambda\rangle$,

$$\langle\alpha|\lambda\rangle = \exp\left(\frac{1}{2}(\alpha^*\lambda - \alpha\lambda^*)\right)\exp\left(-\frac{1}{2}|\alpha - \lambda|^2\right). \tag{8.131}$$

By the same criterion, a number state is nonclassical, since it turns out that the P-function for such a state involves derivatives of the delta function, thereby being more singular than the latter. On the other hand, a state of thermal equilibrium at any given temperature (see Eqs. 8.126a and 8.126b) is a classical one since the P-function corresponding to such a state is a Gaussian probability distribution. Finally, the squeezed state is nonclassical, since the P-function for a squeezed state $|\lambda;\xi\rangle$ (with $r \neq 0$; see Eq. 8.112a) is not positive everywhere in phase space.

This last statement can be verified as follows.

Assume, for concreteness, that $r > 0$. Express $\Delta\hat{Y}_1$, the uncertainty in the rotated quadrature operator $\hat{Y}_1$ (see Eq. 8.112b), in terms of $P(\alpha)$ for the state $|\lambda;\xi\rangle$, where α stands for the complex phase space variable. Since $\hat{Y}_1$ can be written as

$$\hat{Y}_1 = \frac{1}{2}(e^{-i\phi}\hat{a} + e^{i\phi}\hat{a}^\dagger) \tag{8.132}$$

(check this out), one has

$$(\Delta\hat{Y}_1)^2 = \frac{1}{4}\langle\Big((\hat{a}-\langle\hat{a}\rangle)e^{-i\phi}+(\hat{a}^\dagger-\langle\hat{a}^\dagger\rangle)e^{i\phi}\Big)^2\rangle$$
$$= \frac{1}{4}+\frac{1}{4}\int d\alpha P(\alpha)\Big((\alpha-\langle\hat{a}\rangle)e^{-i\phi}+(\alpha^*-\langle\hat{a}^\dagger\rangle)e^{i\phi}\Big)^2 \quad (8.133)$$

(check this out with use of Eq. 8.88; the integration is over the 2D phase space). Now, for $r > 0$, we have already seen that $(\Delta\hat{Y}_1)^2 < \frac{1}{4}$ (see Eq. 8.112c; one obtains the squeezed state $|\lambda;\xi\rangle$ from the vacuum state by applying the displacement $\hat{D}(\lambda)$ after the squeeze $\hat{S}(\xi)$, which means that $\Delta\hat{Y}_1$ is the same in $|\lambda;\xi\rangle$ as in the squeezed vacuum). This means that $P(\alpha)$ in Eq. (8.133) cannot be everywhere positive (reason this out).

Even though the P-representation is what is referred to as a *quasi-probability distribution*, it is useful in quantum optics since it directly relates to photocount rate measurements. Other quasi-probability distributions can also be defined in the phase space of the oscillator, or in the composite phase space for a collection of oscillators, of which the Wigner distribution is one. Some of these distributions are relevant in various contexts in quantum optics. One of these, the *Husimi distribution*, is closely related to the P-distribution. It is defined for a state with density operator ρ as

$$Q(\alpha) = \frac{1}{\pi}\langle\alpha|\rho|\alpha\rangle, \quad (8.134)$$

where $|\alpha\rangle$ stands for the coherent state with complex parameter α. The Husimi distribution $Q(\alpha)$ has the desirable feature that it is everywhere nonnegative, though it is still not an exact classical representation of a quantum state (which, in any case, is ruled out in principle), and can be interpreted as a *coarse-grained* classical distribution, where the coarse graining arises from quantum fluctuations.

With this background on the harmonic oscillator, we move on to the description of states of the free electromagnetic field. Interactions of the field with an atom will be taken up next.

8.5 The Free Electromagnetic Field in a Box: Classical Description

8.5.1 Periodic Boundary Condition in a Box: Plane Wave Modes

As an illustration of the classical description of the states of an electromagnetic field, we consider an electromagnetic field in free space (ie, states of the field in regions free of sources and material bodies). Instead of looking at the vectors $\mathbf{E}$ and $\mathbf{H}$ in describing the state of the field, we find it more convenient in this context to refer to the *vector potential* $\mathbf{A}$ in the *Coulomb gauge* (see Section 1.4.2). The latter corresponds to the condition

$$\text{div}\,\mathbf{A} = 0, \quad (8.135)$$

subject to which the electric and magnetic field intensities are given by

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}, \tag{8.136a}$$

$$\mathbf{H} = \frac{1}{\mu_0}\text{curl } \mathbf{A}, \tag{8.136b}$$

where, for the field set up in free space, one can choose the scalar potential $\phi = 0$ in the Coulomb gauge.

Maxwell's equations imply that, subject to Eq. (8.135), the vector potential $\mathbf{A}$ in free space satisfies the wave equation

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = 0, \tag{8.137a}$$

where

$$c = \frac{1}{\epsilon_0 \mu_0}. \tag{8.137b}$$

In the presence of sources, one obtains an inhomogeneous wave equation for $\mathbf{A}$, which involves the *transverse current* on the right-hand side.

The solution to Eq. (8.137a) depends on the *boundary conditions* to be satisfied by $\mathbf{A}(\mathbf{r}, t)$. These boundary conditions depend on the nature and location of sources and material bodies constraining the space- and time dependence of the field components. For the present, we consider the free electromagnetic field with *no* material bodies located anywhere in space, in which case the boundary conditions refer to constraints to be satisfied at infinitely distant points in space. However, for simplicity of mathematical presentation we introduce *periodic boundary conditions on a box*. A further simplification without any essential loss in generality results if the box is taken to cubical, with edge length L, say, and with edges parallel to the coordinate axes of a chosen right-handed Cartesian system. The periodic boundary conditions then correspond to the requirement

$$\mathbf{A}(\mathbf{r}, t) = \mathbf{A}(\mathbf{r} + L\hat{e}, t), \tag{8.138}$$

where $\hat{e}$ stands for a unit vector along any of the three Cartesian coordinate axes.

The periodic boundary conditions allow us to decompose the space- and time-dependent field $\mathbf{A}(\mathbf{r}, t)$ into a discrete spatial Fourier series and then to solve for the time dependence of the Fourier components. Without going through the actual derivation, I present below a particular solution and then the general solution to Eq. (8.137a) subject to Eqs. (8.135) and (8.138). For this we first consider a set of vectors ($\mathbf{k}$) whose Cartesian components are given by

$$k_i = \frac{2\pi n_i}{L} \quad (i = 1, 2, 3;\ n_i = 0, \pm 1, \pm 2, \ldots). \tag{8.139}$$

The set of all such vectors will be referred to as the *allowed set of wave vectors* for the periodic boundary conditions under consideration because these are the ones that arise in the Fourier series expansion referred to above. Further, for any given $\mathbf{k}$ belonging to this allowed set, we introduce a pair of *unit polarization vectors* $\mathbf{e}_{\mathbf{k}s}$ $(s = 1, 2)$, in general complex, satisfying

$$\mathbf{k} \cdot \mathbf{e}_{\mathbf{k}s} = 0, \quad \mathbf{e}^*_{\mathbf{k}s} \cdot \mathbf{e}_{\mathbf{k}s'} = \delta_{ss'} \quad (s, s' = 1, 2), \tag{8.140a}$$

$$\mathbf{e}_{\mathbf{k}1} \times \mathbf{e}_{\mathbf{k}2} = \frac{\mathbf{k}}{|\mathbf{k}|}. \tag{8.140b}$$

Finally, we define

$$\omega_{\mathbf{k}} \equiv c|\mathbf{k}|. \tag{8.141}$$

I will now leave it to you to check that for any allowed wave vector $\mathbf{k}$ and any one of the pair of unit polarization vectors chosen as above, a particular solution to Eq. (8.137a) subject to Eq. (8.135) and to Eq. (8.138) is given by

$$\mathbf{A}(\mathbf{r}, t) = u_{\mathbf{k}s}\mathbf{e}_{\mathbf{k}s}\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)} + u^*_{\mathbf{k}s}\mathbf{e}^*_{\mathbf{k}s}\mathrm{e}^{-\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)}, \tag{8.142}$$

where $u_{\mathbf{k}s}$ $(s = 1, 2)$ is any chosen complex number.

> A pair of real-valued unit polarization vectors corresponds to a linearly polarized plane wave solution, while circularly and elliptically polarized waves are described by complex-valued polarization vectors.

This solution, whose form ensures that $\mathbf{A}(\mathbf{r}, t)$ is real, contains a *positive frequency* component and a *negative frequency* one, these terms being used to designate time variations of the form $\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}t}$ and $\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}t}$, respectively. It represents a monochromatic plane wave with wave vector $\mathbf{k}$, state of polarization characterized by $\mathbf{e}_{\mathbf{k}s}$, and scalar amplitude $|u_{\mathbf{k}s}|$ (in magnitude; the phase of $u_{\mathbf{k}s}$ determines the instantaneous phase of the wave at any given point). We refer to such a particular solution as an 'eigenmode' (or, in brief, a *mode*) of the electromagnetic field under the assumed boundary conditions. Notice that the time dependence of the eigenmode is harmonic, with frequency $\omega_{\mathbf{k}}$.

Defining the complex-valued quantities $A_{\mathbf{k}s}(t)$ as

$$A_{\mathbf{k}s}(t) \equiv u_{\mathbf{k}s}\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}t}, \tag{8.143}$$

the *general solution* to Eq. (8.137a), subject to Eqs. (8.135) and (8.138), can be written as a superposition of eigenmodes corresponding to all possible allowed wave vectors and states of polarization, with arbitrarily chosen amplitudes $u_{\mathbf{k}s}$:

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{\epsilon_0^{\frac{1}{2}} L^{\frac{3}{2}}} \sum_{\mathbf{k},s} [u_{\mathbf{k}s}\mathbf{e}_{\mathbf{k}s} e^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)} + u^*_{\mathbf{k}s}\mathbf{e}^*_{\mathbf{k}s} e^{-\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)}]$$
$$= \frac{1}{\epsilon_0^{\frac{1}{2}} L^{\frac{3}{2}}} \sum_{\mathbf{k},s} [A_{\mathbf{k}s}(t)\mathbf{e}_{\mathbf{k}s} e^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} + A^*_{\mathbf{k}s}(t)\mathbf{e}^*_{\mathbf{k}s} e^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}]. \tag{8.144}$$

Here the summation is over all the allowed wave vectors $\mathbf{k}$ satisfying Eq. (8.139) and over $s = 1, 2$, and the normalization constant $\frac{1}{\epsilon_0^{\frac{1}{2}} L^{\frac{3}{2}}}$ has been introduced for later convenience. Any particular choice of the set of amplitudes $\{u_{\mathbf{k}s}\}$ gives us a particular solution for $\mathbf{A}(\mathbf{r}, t)$. In particular, Eq. (8.142) corresponds to the amplitude of a particular eigenmode being $u_{\mathbf{k}s}$, all other amplitudes being zero.

In the following, we will, for simplicity, replace the mode index represented by a pair such as $\mathbf{k}s$ with one represented by a single index such as α, where the single index will now represent the propagation vector and the polarization index taken together. In some instances the index α may, depending on the context, stand for only one of the two mode indices. Thus ω_α will mean $\omega_{\mathbf{k}}$ since the frequency of any given mode does not depend on the polarization index. At times even the single index α will be suppressed, in which case a summation symbol will imply a sum over all the modes relevant in any given context

Any one of the expressions in Eq. (8.144) can be regarded in one of two related ways: (1) it constitutes the general solution to Maxwell's equations under the periodic boundary conditions (8.138) and, equivalently, (2) for any particular choice of the set of amplitudes $\{u_\alpha\}$, it describes, for any given value of t, the instantaneous *state* of the electromagnetic field, once again under the assumed boundary conditions. Any particular eigenmode then constitutes one special instance of the state of the field.

Using Eqs. (8.136a) and (8.136b), one can express the space-time dependence of the electric and magnetic field intensities as

$$\mathbf{E}(\mathbf{r},t) = \frac{\mathrm{i}}{\epsilon_0^{\frac{1}{2}} L^{\frac{3}{2}}} \sum_{\alpha} \omega_\alpha [A_\alpha(t)\mathbf{e}_\alpha e^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} - A^*_\alpha(t)\mathbf{e}^*_\alpha e^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}], \tag{8.145a}$$

$$\mathbf{H}(\mathbf{r},t) = \frac{\mathrm{i}}{\mu_0\epsilon_0^{\frac{1}{2}} L^{\frac{3}{2}}} \sum_{\alpha} [A_\alpha(t)(\mathbf{k}\times\mathbf{e}_\alpha) e^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} - A^*_\alpha(t)(\mathbf{k}\times\mathbf{e}^*_\alpha) e^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}]. \tag{8.145b}$$

Once again, the choice of all but one of the amplitudes $u_{\mathbf{k}s}$ as zero corresponds to a particular mode of the field with a certain wave vector, amplitude, and state of polarization.

As I have mentioned already, Eqs. (8.145a) and (8.145b), along with definitions (8.143) and (8.141), tell us that either of the fields $\mathbf{E}$ and $\mathbf{H}$, and also the potential $\mathbf{A}$, can be expressed

as a sum of a *positive frequency* part and a *negative frequency* one. For instance, Eq. (8.145a) can be written as

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{2}\left(\mathbf{E}^{(+)}(\mathbf{r},t) + \mathbf{E}^{(-)}(\mathbf{r},t)\right), \tag{8.146a}$$

where

$$\mathbf{E}^{(+)}(\mathbf{r},t) = \frac{2\mathrm{i}}{\epsilon_0^{\frac{1}{2}} L^{\frac{3}{2}}} \sum_\alpha \omega_\alpha A_\alpha(t)\mathbf{e}_\alpha \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}, \quad \mathbf{E}^{(-)}(\mathbf{r},t) = -\frac{2\mathrm{i}}{\epsilon_0^{\frac{1}{2}} L^{\frac{3}{2}}} \sum_\alpha \omega_\alpha A_\alpha^*(t)\mathbf{e}_\alpha^* \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}. \tag{8.146b}$$

Recalling the definition of the analytic signal from Section 7.9, we can relate the positive frequency part $\mathbf{E}^{(+)}$ to the analytic signal $\tilde{\mathbf{E}}(\mathbf{r},t)$ corresponding to the electric field strength at the point $\mathbf{r}$:

$$\mathbf{E}^{(+)}(\mathbf{r},t) = \tilde{\mathbf{E}}(\mathbf{r},t). \tag{8.146c}$$

Note that the state described by Eqs. (8.145a) and (8.145b) for any given value of t is a *pure* one, and this is consistent with the statement that the observables $\{\mathbf{E}(\mathbf{r})\}$ and $\{\mathbf{H}(\mathbf{r})\}$ (ie, the sets of field intensities at all points in space) possess definite values for any classically defined pure state of the electromagnetic field. Examples of other observable quantities that depend on $\{\mathbf{E}(\mathbf{r})\}$ and $\{\mathbf{H}(\mathbf{r})\}$ and possess well-defined values in any given state of the field are the energy-momentum density, angular momentum density, total energy-momentum, and total angular momentum of the field.

For instance, the energy density of the electromagnetic field is given by

$$u(\mathbf{r}) = \frac{1}{2}[\epsilon_0 \mathbf{E}(\mathbf{r})^2 + \mu_0 \mathbf{H}(\mathbf{r})^2]. \tag{8.147}$$

The argument t has not been included in this expression because this refers to an *observable* quantity, where the value of an observable at any given instant will be denoted by inclusion of the argument t. Though this seems like laboring a point, it is of some use to pay attention to the distinction between an observable and its measured value (or the value that would result from a measurement) at any given point of time. In the quantum description the observable corresponds to an operator, while its measured value in a given state corresponds, in general, to a random variable.

In accordance with Eq. (8.147), the observable corresponding to the total energy of the field in the box, represented by its *Hamiltonian*, is given by

$$H = \int_{\text{box}} \frac{1}{2}\left[\epsilon_0 \mathbf{E}(\mathbf{r})^2 + \mu_0 \mathbf{H}(\mathbf{r})^2\right] d^{(3)}\mathbf{r}, \tag{8.148}$$

where the integration is performed over the box, in terms of which the periodic boundary condition was specified to begin with. Using Eqs. (8.145a) and (8.145b), one obtains

$$H = 2\sum_{\alpha} \omega_\alpha^2 |A_\alpha|^2, \tag{8.149}$$

where, once again, the (complex-valued) observable $A_\alpha (\equiv A_{\mathbf{k}s})$ rather than its value $A_\alpha(t)$ has been used since the above equation is meant to be an expression for the observable H.

> The 'observable' A_α, which differs from the commonly encountered ones in being complex valued, is defined as one whose value at any specified time t is given by Eq. (8.143). It occurs in the expression for H through $|A_\alpha|^2$, which is a real-valued observable. In any case, the real and imaginary parts of A_α are observables in the commonly used sense of the term.

It is of interest to note that the expression for the Hamiltonian is, on the face of it, independent of the length L of the box since the factor depending on L is canceled on integration over the volume of the box. However, a dependence on L remains through the set of allowed propagation vectors. Each component of the propagation vector $\mathbf{k}$ assumes a discrete set of allowed values which go over to one with continuously varying values (ranging from $-\infty$ to ∞) in the limit $L \to \infty$. This limit is commonly implied when one is describing an electromagnetic field in free space.

In the following, we make use of the observables like A_α that can all be expressed in terms of $\{\mathbf{A}(\mathbf{r})\}$, $\{\mathbf{E}(\mathbf{r})\}$, and $\{\mathbf{H}(\mathbf{r})\}$, where we do not concern ourselves as to how these can be measured. These are observables insofar as they are defined in terms of the field variables at specified sets of points.

Thus the complex-valued observables A_α introduced above can be expressed in terms of the basic observables (the field variables) as

$$A_\alpha = \frac{\epsilon_0^{\frac{1}{2}}}{2L^{\frac{3}{2}}} \int d^{(3)}\mathbf{r}\, [\mathbf{A}(\mathbf{r}) - \frac{\mathrm{i}}{\omega_\alpha}\mathbf{E}(\mathbf{r})] \cdot \mathbf{e}_\alpha^* \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}, \tag{8.150a}$$

$$A_\alpha^* = \frac{\epsilon_0^{\frac{1}{2}}}{2L^{\frac{3}{2}}} \int d^{(3)}\mathbf{r}\, [\mathbf{A}(\mathbf{r}) + \frac{\mathrm{i}}{\omega_\alpha}\mathbf{E}(\mathbf{r})] \cdot \mathbf{e}_\alpha \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}, \tag{8.150b}$$

where α stands for the pair of indices $\mathbf{k}s$ for any specified propagation vector $\mathbf{k}$ and polarization vector $\mathbf{e}_{\mathbf{k}s}$.

> Formulae (8.150a) and (8.150b) depict relations between observable quantities (I repeat that A_α and A_α^* are not observables in the commonly used sense of the term since they are complex valued; however, their real and imaginary parts correspond to observables) and not between their values in any given state of the field. In the quantum description, these are expressed as relations between the corresponding *operators*, real observables being

represented by *Hermitian* operators. As I have already mentioned, when referring to values of the observables in any given state of the field, we will usually insert the argument t in the relevant expressions.

Using the above set of complex-valued quantities, we introduce, finally, one other set of observables (real valued this time) termed the *normal mode coordinates* and *momenta*, defined as

$$Q_\alpha = A_\alpha + A_\alpha^*, \tag{8.151a}$$

$$P_\alpha = -\mathrm{i}\omega_\alpha[A_\alpha - A_\alpha^*]. \tag{8.151b}$$

The Hamiltonian of the free electromagnetic field (Eq. 8.148) expressed in terms of $\{Q_\alpha\}$ and $\{P_\alpha\}$ reads

$$H = \frac{1}{2}\sum_\alpha [P_\alpha^2 + \omega_\alpha^2 Q_\alpha^2]. \tag{8.152}$$

The observables $\{Q_\alpha\}$ and $\{P_\alpha\}$, which have well-defined values at any given instant of time in the state given by Eq. (8.144), are *canonically conjugate* variables in that their time evolution is given by the classical canonical equations of motion.

Expression (8.152) allows one to interpret the free electromagnetic field in terms of a set of independent *harmonic oscillators*, each of unit mass, where a typical oscillator corresponding to the index α has frequency ω_α. *This interpretation is of central relevance in quantum optics*.

Observe that a pair of the normal mode variables Q_α and P_α is associated with every eigenmode of the field, where these depend on the choice of the unit polarization vector. The set of normal mode coordinates and momenta for all allowed $\mathbf{k}$ and s constitute a complete set of observable quantities for the electromagnetic field in that the observables $\{\mathbf{A}(\mathbf{r})\}$, $\{\mathbf{E}(\mathbf{r})\}$, and $\{\mathbf{H}(\mathbf{r})\}$ can all be expressed in terms of these:

$$\mathbf{A}(\mathbf{r}) = \frac{1}{2\epsilon_0^{\frac{1}{2}} L^{\frac{3}{2}}} \sum_\alpha \left(\left[Q_\alpha + \frac{\mathrm{i}}{\omega_\alpha} P_\alpha \right] \mathbf{e}_\alpha \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} + \text{c.c.} \right), \tag{8.153a}$$

$$\mathbf{E}(\mathbf{r}) = \frac{\mathrm{i}}{2\epsilon_0^{\frac{1}{2}} L^{\frac{3}{2}}} \sum_\alpha \left([\omega_\alpha Q_\alpha + \mathrm{i}P_\alpha] \mathbf{e}_\alpha \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} - \text{c.c.} \right), \tag{8.153b}$$

$$\mathbf{H}(\mathbf{r}) = \frac{\mathrm{i}}{2\mu_0\epsilon_0^{\frac{1}{2}} L^{\frac{3}{2}}} \sum_\alpha \left(\left[Q_\alpha + \frac{\mathrm{i}}{\omega_\alpha} P_\alpha \right] \mathbf{k} \times \mathbf{e}_\alpha \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} - \text{c.c.} \right). \tag{8.153c}$$

The normal mode momenta $\{P_\alpha\}$ are to be distinguished from the momentum density $\mathbf{P}(\mathbf{r})$ or the total field momentum $\mathbf{P}$. The latter can be expressed in terms of the complete set of observables $\{Q_\alpha\}$ and $\{P_\alpha\}$.

8.5.2 The Electromagnetic Field in a Cavity

There are other optical setups where one is led to a consideration of *standing wave modes* of the electromagnetic field. For instance, consider a *cavity*, once again in the shape of a cubical box of edge length L, for which the vector potential and the electric field strength are zero on the walls of the cavity. In this case, a typical eigenmode of the field is of the form

$$\mathbf{A}(\mathbf{r},t) = \frac{2\sqrt{2}}{\epsilon_0^{\frac{1}{2}} L^{\frac{3}{2}}} \sum_{\alpha} \sin(k_1 x)\sin(k_2 y)\sin(k_3 z)(A_\alpha(t)\mathbf{e}_\alpha + \text{c.c.}) \quad (\alpha \equiv \mathbf{k}s), \tag{8.154}$$

where now the allowed wave vectors have components

$$k_i = \frac{\pi n_i}{L} \quad (i = 1,2,3,\ n_i = 1,2,3,\ldots), \tag{8.155}$$

while the unit polarization vectors $\mathbf{e}_\alpha$ are defined as above. This is in the nature of a *standing wave* with nodes on the walls of the cavity, where $\mathbf{e}_\alpha$ describes the state of polarization of the standing wave and $A_\alpha(t)$ represents the time-dependent amplitude. The general solution for the electromagnetic field in the cavity is thus expressed as a superposition of these standing wave eigenmodes (or normal modes; in brief, *modes*) over all the allowed wave vectors and polarization states.

Once again, one can introduce, for each allowed α, the (complex-valued) observable A_α as above, such that its value at time t is $A_\alpha(t)$ given by expression (8.143). Finally, one can express everything in terms of the canonically conjugate sets of observables $\{Q_\alpha\}$ and $\{P_\alpha\}$ defined as in Eqs. (8.151a) and (8.151b), wherein the Hamiltonian assumes the form (8.152). Thus the electromagnetic field in the cavity can also be regarded as being made up of independent normal modes, each of which is equivalent to a harmonic oscillator of unit mass with position and momentum coordinates Q_α and P_α, respectively (for appropriate $\alpha \equiv \mathbf{k}s$), and with angular frequency ω_α.

> The factor of $2\sqrt{2}$ in formula (8.154) owes its origin to the fact that the field is confined within reflecting walls in all three directions.

Another optical setup of considerable importance consists of a pair of parallel mirrors (on which the vector potential and the electric field strength are both zero), say, perpendicular to the z-axis, there being no constraints in the other two directions. However, as before, we can simplify things by assuming periodic boundary conditions in these two directions. We again assume for simplicity that the boundaries constitute the sides of a cubical box of edge length L.

In this case a typical eigenmode is an object of mixed character that can be regarded as a standing wave along the z-axis and traveling waves along the x- and y-axes. While, strictly speaking, the mirrors are to be of infinite extent, the use of finite mirrors in practice leads to a

situation where the electromagnetic field is effectively described as a superposition of standing waves since the modes with $k_1,\ k_2 \neq 0$ can all be made to escape from the cavity. One then has, for the observable $\mathbf{A}$, which now depends only on the coordinate z,

$$\mathbf{A}(\mathbf{r}) = \sqrt{\frac{2}{\epsilon_0 L^3}} \sum_{\alpha} \sin(kz)(A_\alpha \mathbf{e}_\alpha + \text{c.c.}) \quad (\alpha \equiv ks). \tag{8.156}$$

This setup is referred to as the quasi-1D cavity since the field is confined within reflecting walls in one direction while it satisfies the periodic boundary conditions in the other two directions. In contrast to Eq. (8.154), one now has a factor of $\sqrt{2}$ on the right-hand side.

Here k stands for the magnitude of the wave vector, which is now along the z-axis, with allowed values

$$k = \frac{\pi n}{L} \quad (n = 1, 2, 3, \ldots), \tag{8.157}$$

$\mathbf{e}_{ks}$ $(s = 1, 2)$ are a pair of orthogonal unit polarization vectors (in general complex) in the x-y plane, and A_α stands for a complex-valued observable corresponding to the time-dependent amplitude of a standing wave of frequency $\omega_\alpha \equiv ck$.

In writing formula (8.156), we have suppressed the x- and y-dependence of $\mathbf{A}$ since the only relevant modes are the ones with $k_1 = k_2 = 0$.

Corresponding to Eq. (8.156), the expressions for the observables representing the electric and magnetic field vectors are given by

$$\mathbf{E}(\mathbf{r}) = \mathrm{i}\sqrt{\frac{2}{\epsilon_0 L^3}} \sum_{\alpha} \sin(kz)\omega_\alpha (A_\alpha \mathbf{e}_\alpha - \text{c.c.}), \tag{8.158a}$$

$$\mathbf{H}(\mathbf{r}, t) = \frac{1}{\mu_0}\sqrt{\frac{2}{\epsilon_0 L^3}} \sum_{\alpha} k\cos(kz)(A_\alpha(\hat{e}_3 \times \mathbf{e}_\alpha) + \text{c.c.}), \tag{8.158b}$$

where $\hat{e}_3$ stands for the unit vector along the z-direction. As in the case of the free field subject to the periodic boundary conditions and the field in a 3D cavity, the electromagnetic field in this (quasi-)1D cavity can also be regarded as a collection of normal modes, each of which is effectively a harmonic oscillator of unit mass. Each normal mode is characterized by an index α (corresponding to k given by Eq. (8.157) and a polarization index s defining the unit polarization vector $\mathbf{e}_{ks}$), for which a pair of canonical observables Q_α and P_α can be defined. In terms of these variables, the effective Hamiltonian of the field is

$$H = 2\sum_{\alpha} \omega_\alpha^2 |A_\alpha|^2 = \frac{1}{2}\sum_{\alpha}(P_\alpha^2 + \omega_\alpha^2 Q_\alpha^2) \tag{8.159}$$

(check this out).

8.5.3 Summary: Eigenmode Expansion of the Electromagnetic Field

In summary, given an optical setup and the corresponding boundary conditions constraining the electromagnetic field, it is possible to regard the latter as a collection of independent eigenmodes, where each eigenmode is equivalent to a harmonic oscillator of unit mass, described in terms of canonically conjugate coordinates Q and P, so that the total Hamiltonian of the electromagnetic field is given by a sum of terms of the form $\frac{1}{2}(P^2 + \omega^2 Q^2)$, ω being the frequency of the mode under consideration.

For the free field satisfying the periodic boundary conditions, the eigenmodes are monochromatic traveling plane waves, and the expressions for the field variables as sums over the eigenmode contributions are nothing but the corresponding spatial Fourier expansions. For the 3D cubical cavity and the (quasi-)1D cavity, on the other hand, the eigenmodes are standing waves in terms of which the field variables can once again be expressed as sums over eigenmode contributions. More generally, for an appropriate class of boundary conditions, this way of looking at the electromagnetic field can be expected to remain valid, though the eigenmodes will no longer be traveling plane waves and nor will the eigenmode expansion be equivalent to a simple spatial Fourier expansion. For instance, for the electromagnetic field in a spherical cavity, the eigenmodes will be in the nature of standing waves derived from vector spherical waves.

The basic fact of crucial importance here is that the eigenmodes are *independent* of one another, and the Hamiltonian of the field can be expressed as a sum over Hamiltonians corresponding to these eigenmodes. The canonical variables P_α and Q_α for each mode evolve independently of those pertaining to the other ones. This leads one in a straightforward manner to the *quantum* description of the field where each eigenmode can be treated as a quantum harmonic oscillator independently of the other modes. The known results on the quantum harmonic oscillator can then be employed to look at the various possible quantum states of the field and the statistical features relating to the various observable quantities in these states. A useful and effective concept in this quantum description is that of *photons* (see later). It is this concept that provides the basis for Planck's derivation of blackbody radiation, where the latter corresponds to a *mixed state* of the electromagnetic field in a cavity with the Boltzmann distribution characterizing the probabilities of pure states with given photon numbers.

In numerous situations of practical interest, the state of the field involves a sum over a finite number of modes, often only one or two, and not one over *all* possible modes. Correspondingly, one needs to consider a Hamiltonian involving only one or two independent harmonic oscillators each for one mode with some particular wave vector and polarization vector. This simplification carries over to the quantum description as well, where one again considers only a few modes, each of which is effectively a quantum harmonic oscillator of unit mass. The photon description is then based on stationary states of these few oscillators.

In concluding this section, I will write down for your easy reference the expressions for the electric field vector for the 3D box with periodic boundary conditions, an also for the quasi-1D cavity with reflecting walls, for each of which only *a single mode* is assumed to be relevant. In both of these, the propagation vector is assumed to be in the z-direction (ie, $k_1 = k_2 = 0, k_3 = k$). The mode index α and the sum over modes are now no longer necessary. The unit polarization vector (in general, complex) is $\hat{e}$. The formulae below refer to the field quantities regarded as observables:

(3D with periodic boundary conditions)

$$\mathbf{E}(z) = \frac{\mathrm{i}}{\epsilon_0^{\frac{1}{2}} L^{\frac{3}{2}}} \omega (A \mathrm{e}^{\mathrm{i}kz} \hat{e} - A^* \mathrm{e}^{-\mathrm{i}kz} \hat{e}^*) \quad \left(= \frac{1}{2}[E^{(+)} + E^{(-)}] \right). \tag{8.160a}$$

(1D with reflecting walls)

$$\mathbf{E}(z) = \mathrm{i} \frac{\sqrt{2}}{\epsilon_0^{\frac{1}{2}} L^{\frac{3}{2}}} \omega \sin(kz) (A \hat{e} - A^* \hat{e}^*) \quad \left(= \frac{1}{2}[E^{(+)} + E^{(-)}] \right). \tag{8.160b}$$

While much of our discussion below will relate to these simple expressions, two mode fields will also be referred to, in which case one has to consider a sum of two terms in each of the above formulae, one corresponding to each of the two modes.

Note that each of the above two expressions is made up of two parts, of which the first one is the positive frequency part and the second part is the negative frequency one, with a factor of $1/2$ in each case.

> The factor of $1/2$ is included here (see Eq. 8.146a) so as to establish correspondence with the analytic signal relating to the classical field.

The complex-valued observables A and A^* in each of the above formulae are related to the canonically conjugate variables Q and P as

$$Q = A + A^*, \quad P = -\mathrm{i}\omega (A - A^*), \tag{8.160c}$$

and the expression for the Hamiltonian reads

$$H = 2\omega^2 |A|^2 = \frac{1}{2}(P^2 + \omega^2 Q^2). \tag{8.160d}$$

8.6 Quantization of the Electromagnetic Field

With this background on the classical description of the electromagnetic field in terms of normal modes under specified boundary conditions, we now look at the *quantum* description of the free field and at statistical distributions of values of observable quantities in the various possible states of the field.

8.6.1 Mode Expansion of the Field Hamiltonian

Quantization of the electromagnetic field involves a number of technical details which we need not go into. A more convenient approach from our point of view is to refer to the classical eigenmode expansion and to start directly from the Hamiltonian *operator* for the field. A simple prescription is to replace the classical observables P_α and Q_α with *operators* $\hat{P}_\alpha$ and $\hat{Q}_\alpha$, in terms of which the Hamiltonian operator appears as

$$\hat{H} = \sum_\alpha \frac{1}{2}(\hat{P}_\alpha^2 + \omega_\alpha^2 \hat{Q}_\alpha^2), \tag{8.161}$$

where the summation extends to all possible eigenmodes (labeled with the single index α for brevity) of the field subject to specified boundary conditions. This Hamiltonian is simply the sum of an infinite number of equivalent harmonic oscillators, where each mode appears effectively as an oscillator of unit mass and with an appropriate frequency. With reference to the oscillator corresponding to the mode α, $\hat{P}_\alpha$, and $\hat{Q}_\alpha$ are the canonically conjugate operators equivalent to the momentum and position operators considered in Section 8.4.

As mentioned in Section 8.5.1, an eigenmode is characterized by some specific combination of the wave vector $\mathbf{k}$ and the polarization index s, the latter corresponding to the polarization vector $\mathbf{e}_{\mathbf{k}s}$. Other forms of eigenmode expansion such as the one corresponding to boundary conditions on a sphere lead to eigenmodes characterized differently, such as with those relating to vector spherical waves.

8.6.2 Annihilation and Creation Operators: Commutation Relations

While the operators $\hat{P}_\alpha$ and $\hat{Q}_\alpha$ do not appear to have direct physical relevance, they can be invoked to make possible a neat description of the states of the electromagnetic field because of the form of Eq. (8.161) and, at the same time, to obtain expressions for the field operators $\hat{\mathbf{E}}(\mathbf{r}, t)$ and $\hat{\mathbf{B}}(\mathbf{r}, t)$ (the time dependence is included here for the operators in the Heisenberg picture). For instance, in the case of periodic boundary conditions on a cube of side L, the field operators are given by expressions (8.153a)–(8.153c), in which the field variables are to be replaced with the corresponding operators.

Since each of the modes appears as a harmonic oscillator, one can use all the results on the states of a harmonic oscillator and on the probability distributions of values of observables in these states as mentioned in Section 8.4. While these will be of relevance in the context of single-mode fields, multimode field configurations can be dealt with by a straightforward extension to collections of independent harmonic oscillators.

We introduce, for each mode characterized with the index α, annihilation and creation operators in a manner analogous to Eqs. (8.86a) and (8.86b):

$$\hat{Q}_\alpha = \sqrt{\frac{\hbar}{2\omega_\alpha}}(\hat{a}_\alpha + \hat{a}_\alpha^\dagger), \tag{8.162a}$$

$$P_\alpha = -i\sqrt{\frac{\hbar\omega_\alpha}{2}}(\hat{a}_\alpha - \hat{a}_\alpha^\dagger). \tag{8.162b}$$

The operators $\hat{a}_\alpha$ and $\hat{a}_\alpha^\dagger$ are obtained from the classical (complex-valued) observables by means of the following replacements:

$$\hat{A}_\alpha \to \sqrt{\frac{\hbar}{2\omega_\alpha}}\hat{a}_\alpha, \quad \hat{A}_\alpha^* \to \sqrt{\frac{\hbar}{2\omega_\alpha}}\hat{a}_\alpha^\dagger. \tag{8.163}$$

I give below, for easy reference, the expression for the electric field operator for a single-mode field in terms of the creation and annihilation operators (V stands for the volume of the box within which the field is quantized):

(3D with periodic boundary conditions)

$$\hat{\mathbf{E}}(z) = \mathrm{i}\sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}(\mathrm{e}^{\mathrm{i}kz}\hat{e}\hat{a} - \mathrm{e}^{-\mathrm{i}kz}\hat{e}^*\hat{a}^\dagger) \quad \left(= \frac{1}{2}[\hat{\mathbf{E}}^{(+)} + \hat{\mathbf{E}}^{(-)}]\right), \tag{8.164a}$$

(1D with reflecting walls)

$$\hat{\mathbf{E}}(z) = \mathrm{i}\sqrt{\frac{\hbar\omega}{\epsilon_0 V}}\sin(kz)(\hat{e}\hat{a} - \hat{e}^*\hat{a}^\dagger) \quad \left(= \frac{1}{2}[\hat{\mathbf{E}}^{(+)} + \hat{\mathbf{E}}^{(-)}]\right). \tag{8.164b}$$

Recall that the caret symbol over e represents a unit vector (in general, complex), which is to be distinguished from the caret symbol indicating a quantum operator.

One can also express the field operators in terms of the quadrature operators $\{X_{1\alpha}\}$ and $\{X_{2\alpha}\}$ defined as

$$\hat{X}_{1\alpha} = \frac{\hat{a}_\alpha + \hat{a}_\alpha^\dagger}{2}, \quad \hat{X}_{2\alpha} = \frac{\hat{a}_\alpha - \hat{a}_\alpha^\dagger}{2i}. \tag{8.165}$$

In our classical description, the observables such as A_α and A_α^* do not carry any explicit reference to time, while their values are time dependent. This corresponds in the quantum description to the Schrödinger picture (which is the one we have invoked in writing Eqs. 8.164a and 8.164b), while it is more convenient at times to refer to the Heisenberg picture, which constitutes an alternative description scheme in quantum theory. A third scheme, the interaction picture, is useful in describing the field in interaction with any other system.

Invoking the Heisenberg picture, where the annihilation and creation operators are time dependent as in Eqs. (8.91a) and (8.91b), one arrives at the following expressions for the

time-dependent electric field strength operator in the case of a single-mode field, where the vector **k** is assumed to be along the z-axis:

(3D with periodic boundary conditions)

$$\hat{\mathbf{E}}(\mathbf{r},t) = \mathrm{i}\sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}\left(\hat{a}\hat{e}\mathrm{e}^{\mathrm{i}(kz-\omega t)} - \hat{a}^\dagger\hat{e}^*\mathrm{e}^{-\mathrm{i}(kz-\omega t)}\right), \tag{8.166a}$$

(1D with reflecting walls)

$$\hat{\mathbf{E}}(\mathbf{r},t) = \mathrm{i}\sqrt{\frac{\hbar\omega}{\epsilon_0 V}}\sin(kz)\left(\hat{a}\hat{e}\mathrm{e}^{-\mathrm{i}\omega t} - \hat{a}^\dagger\hat{e}^*\mathrm{e}^{\mathrm{i}\omega t}\right). \tag{8.166b}$$

Here $\hat{a}^\dagger$ and $\hat{a}$ stand for the creation and annihilation operators at time $t = 0$.

> Logically speaking, the terms 'single mode' and 'multimode' (see later) apply not to operator expansions in terms of modes but to field *states* (see Section 8.7). Since the physically relevant quantities are the expectation values of the operators in various different states, one needs to keep only a single-mode term in an operator expression when its expectation value is sought to be evaluated in a single-mode state. It is only in this sense that a single-mode expression for an operator such as the one in Eq. (8.166a) or Eq. (8.166b) can be used as an abbreviation for the complete expression involving the sum over all the modes.

In the case of a multimode field, the modes are independent in that the creation and annihilation operators for distinct modes *commute* with one another:

$$[\hat{a}_\alpha, \hat{a}_\beta] = 0, \quad [\hat{a}_\alpha^\dagger, \hat{a}_\beta^\dagger] = 0, \quad [\hat{a}_\alpha^\dagger, \hat{a}_\beta] = 0 \quad (\alpha \neq \beta). \tag{8.167a}$$

On the other hand, the creation and annihilation operators for any given mode satisfy the basic commutation relation (8.88):

$$[\hat{a}_\alpha,\ \hat{a}_\alpha^\dagger] = \hat{I}, \tag{8.167b}$$

where, once again, a_α and $a_\alpha^\dagger$ refer to the operators at any given instant of time (say, $t = 0$) if it is the Heisenberg picture that one is working in. In reality, the equal-time commutation relations (Eqs. 8.167a and 8.167b) themselves are time independent. Commutators between operators at different time instants can be easily evaluated with use of Eqs. (8.91a) and (8.91b).

Using these commutation relations, one can derive commutation relations between field components of specified modes such as $\hat{E}_{\alpha i}(\mathbf{r},t)$ and $\hat{H}_{\alpha i}(\mathbf{r},t)$. It is found that a pair of field components at space-time points $(\mathbf{r},t)$ and $(\mathbf{r}',t')$ for any given mode commute with each other unless the space-time points are related by a *light-like* separation. In other words, if a light signal can connect the two points then a measurement of any one field component at one of these points affects the result of measurement of another component at the other point.

The commutation relations between field components can be used to derive the equations of motion of these field operators. It is found that these equations are nothing but *Maxwell's equations* written with the operators instead of the classical field variables. However, the details of these commutation relations will not concern us here.

8.6.3 Energy Spectrum: Photons

The fact that the Hamiltonian for the free electromagnetic field is simply the sum over independent harmonic oscillator Hamiltonians corresponding to the eigenmodes of the field, leads straightaway to the possible energy eigenvalues of the field (ie, to its *energy spectrum*). In accordance with the rules of quantum theory, *the energy eigenvalues are all possible sums over the energies of the individual modes*, the typical energy eigenvalue of a mode being of the form (8.85), where n stands for the quantum number characterizing the corresponding stationary state.

As indicated in Section 8.3.12, the mathematical formalism in quantum theory for describing a system made up of a number of *independent* components involves the use of the *direct product* of the linear vector spaces for the individual components. For instance, if the system is made up of two independent components whose pure states are described in terms of vectors in vector spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, then the pure states of the composite system will be represented by vectors in the direct product space $\mathcal{H}_1 \otimes \mathcal{H}_2$. A typical pure state in this product space can be expressed as a linear combination of product states of the form $|\alpha\rangle \otimes |\beta\rangle$ or, in brief, $|\alpha\rangle|\beta\rangle$, where $|\alpha\rangle$ and $|\beta\rangle$ denote pure states in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.

Further, if the Hamiltonian of the system is the sum of the Hamiltonians of the two component systems and if $|\alpha\rangle$ and $|\beta\rangle$ are stationary states of the component systems with energies E_α and E_β, respectively, then the product state $|\alpha\rangle|\beta\rangle$ is a stationary state of the composite system, with energy eigenvalue $E_\alpha + E_\beta$. Here we presuppose that the component systems are *distinguishable* from one another, because states of systems made of indistinguishable components have to satisfy certain symmetry properties imposed by the rules of quantum theory.

This distinguishability criterion is satisfied by the electromagnetic field when it is regarded as being composed of independent modes since the modes can certainly be distinguished from one another (through the wave vectors and the polarization indices). Hence a typical stationary state of the field is of the form $\ldots |n_\alpha\rangle \ldots |n_\beta\rangle \ldots$ ($\prod_\alpha |n_\alpha\rangle$ in brief), which is a direct product over all modes (with mode indices $\alpha,\ \beta, \ldots$), the quantum numbers for the modes being, respectively, $n_\alpha,\ n_\beta, \ldots$. The energy eigenvalue corresponding to the stationary state is then

$$\mathcal{E}(\{n_\alpha\}) = \sum_\alpha \hbar\omega_\alpha \left(n_\alpha + \frac{1}{2}\right), \tag{8.168}$$

where the summation is once again over all the modes of the field.

This expression is made up of two parts:

$$\mathcal{E}(\{n_\alpha\}) = E_0 + E(\{n_\alpha\}), \tag{8.169a}$$

where

$$E_0 \equiv \frac{1}{2}\sum_\alpha \hbar\omega_\alpha \tag{8.169b}$$

is a constant part regardless of the quantum numbers $\{n_\alpha\}$ characterizing the stationary states of the modes, and is termed the *zero point energy* of the field. It is the field energy when all the modes are in their respective ground states (ie, none of the modes are *excited*). The other part,

$$E(\{n_\alpha\}) = \sum_\alpha \hbar\omega_\alpha n_\alpha, \tag{8.169c}$$

on the other hand, differs from one stationary state of the field to another. One says that Eq. (8.169c) is the energy of the field when the modes are excited, with the *occupation number* of the mode α being n_α.

Though the zero point energy assumes significance in certain contexts, we may disregard it as an inessential term for the purpose of this brief introduction. Expression (8.169c), by contrast, depends on the degree of excitation (expressed in terms of occupation numbers) of the various modes and leads to the concept of *photons*: one says that the state of the field $\prod_\alpha |n_\alpha\rangle$ is equivalently the state of a collection of noninteracting photons, there being n_α photons of frequency ω_α, each having an energy $\hbar\omega_\alpha$, where the index α ranges over all the possible modes of the field.

While the photons are introduced as energy quanta of the field, they carry particle attributes such as momentum and spin, similarly to other particles such as electrons and protons. A photon of frequency ω corresponding to a traveling wave mode with wave vector $\mathbf{k}$ ($\omega = c|\mathbf{k}|$) carries a momentum $\hbar\mathbf{k}$, and correspondingly the stationary state of the field mentioned above is an eigenstate of the total field momentum operator with eigenvalue $\sum_\alpha \hbar\mathbf{k}_\alpha$. Photons corresponding to standing wave modes of the field, however, carry no momentum and are similar to particles in bound states.

In the previous paragraphs I have given you the expressions for field operators in the form of sums over modes where the mode indices form a *discrete* set because we have used periodic boundary conditions over a finite volume V ($= L^3$). In real-life situations, however, one is often led to consider *continuously* distributed modes corresponding to an infinitely large volume V in which the field is contained. Such a continuous-mode description of the field will be considered in Section 8.9.

8.7 States of the Electromagnetic Field

8.7.1 Single-Mode States

The states of the free electromagnetic field that are conceptually the simplest to work with are the ones where all modes except a single one are in their respective ground states. These single-mode states are met with in numerous situations of practical interest and form the basis of the description of states of a more complex structure.

A simple instance of a single-mode state is a pure state of the form

$$|\psi\rangle = |n\rangle_\alpha \prod_{\alpha'(\neq\alpha)} |0\rangle_{\alpha'}, \tag{8.170}$$

which is a stationary state with the occupation number of the mode α being n, and with all the *other* modes being in their respective ground states. At times reference to these other modes is dropped for convenience and, moreover, the mode index α may also be left understood so that one writes the state as simply $|n\rangle$. This is an eigenstate of the number operator $\hat{N}_\alpha$ for the mode under consideration, which may once again be abbreviated to $\hat{N}$.

In the language of quantum theory, a pure state $|\phi\rangle$ of a system can be described in terms of a *density operator* $|\phi\rangle\langle\phi|$, where, more generally, density operators are used to describe *mixed states*.

Thus the pure state (8.170) can be represented by the density operator $|n\rangle\langle n|$, while a mixed single-mode state is represented by a positive Hermitian operator of unit trace in the state space of the single mode under consideration, where, once again, the fact that the other modes are in their respective ground states is left implied. The density operator $\hat{\rho}$ (say) can be represented in the form of a density *matrix* in any chosen orthonormal basis in the vector space of states of the mode under consideration. One simple choice for the basis vectors is the set of *number states* $\{|n\rangle\}$.

All the considerations of Section 8.4 relating to the quantum harmonic oscillator carry over unchanged to single-mode states and single-mode operators of the free electromagnetic field. In particular, recall that the quadrature operators $\hat{X}_1$ and $\hat{X}_2$ of the mode are defined as (we consistently suppress the mode index α; mode indices will be used later when we talk of more than one mode at a time)

$$\hat{X}_1 = \sqrt{\frac{\omega}{2\hbar}}Q = \frac{1}{2}(\hat{a} + \hat{a}^\dagger), \quad \hat{X}_2 = \sqrt{\frac{1}{2\omega\hbar}}P = \frac{1}{2\mathrm{i}}(\hat{a} - \hat{a}^\dagger), \tag{8.171}$$

where we recall further that each mode is equivalent to a harmonic oscillator of unit mass.

Among the single-mode pure states, particular mention is to be made of the number (or *Fock*) states $|n\rangle$ ($n = 0, 1, 2, \ldots$), introduced in Section 8.4.1, and the coherent states $|\lambda\rangle$ and squeezed states $|\lambda, \xi\rangle$, introduced in Sections 8.4.2 and 8.4.3, where λ and ξ are arbitrary complex numbers. A single-mode mixed state of considerable importance is the *thermal state* described by the density operator (see Section 8.4.4.1)

$$\hat{\rho}_{\text{single-thermal}} = \frac{1}{Z} \sum_n \mathrm{e}^{-\frac{n\hbar\omega}{k_\mathrm{B}T}} |n\rangle\langle n|, \tag{8.172a}$$

where

$$Z = \frac{1}{1 - \mathrm{e}^{-\frac{\hbar\omega}{k_\mathrm{B}T}}}. \tag{8.172b}$$

This is the state that results when a single standing wave mode of the electromagnetic field is made to remain in equilibrium with a thermal bath at temperature T.

Single-mode squeezed light is commonly produced from coherent laser radiation by the nonlinear interaction of the radiation with the constituents of a crystalline medium—for instance, in the processes of *second-harmonic generation*, *optical parametric amplification*, and *parametric down conversion*. You will find a brief introduction to nonlinear optics in Chapter 9. Section 9.8 outlines how a quantum description of such processes can give us an idea of the way squeezed light is generated. The optical parametric oscillator (see Section 9.6.4) is a commonly used device for the production of squeezed light.

8.7.2 Multimode States

While single-mode states are relevant in some setups involving laser sources, multimode states are of more general relevance where several or many of the possible modes of the field are excited. Simple examples of multimode states are direct products of number eigenstates of the form $\prod_\alpha |n_\alpha\rangle$ where more than one n_α is nonzero. Other states of relevance are the multimode coherent states of the form

$$|\{\lambda_\alpha\}\rangle = \prod_\alpha |\lambda_\alpha\rangle, \tag{8.173}$$

and the *thermal* states of the field, a thermal state at temperature T being described by the density operator

$$\hat{\rho}_{\text{thermal}} = \prod_\alpha \sum_\alpha \frac{1}{Z_\alpha} \mathrm{e}^{-\frac{n_\alpha \hbar\omega_\alpha}{k_\mathrm{B}T}} |n_\alpha\rangle\langle n_\alpha|, \tag{8.174a}$$

where

$$Z_\alpha = \frac{1}{1 - \mathrm{e}^{-\frac{\hbar\omega_\alpha}{k_\mathrm{B}T}}}. \tag{8.174b}$$

In Eqs. (8.173), (8.174a), and (8.174b), α is used as the mode index, running over all possible combinations of the wave vector $\mathbf{k}$ and polarization index s, the latter specifying the (complex) unit polarization vector $\mathbf{e}_{\mathbf{k}s}$. Both these multimode states are of a simple structure, being instances of *separable* states (see Section 8.3.13). By contrast, nonseparable or *entangled* states are of a more complex structure, and will be considered later.

While we have considered multimode states here with discretely distributed modes, *continuous-mode* states are often of greater relevance. Section 8.9 will include a brief introduction to the continuous-mode description of the electromagnetic field.

8.8 Statistical Features of Observables

As already mentioned, measured values of observable quantities are represented by random variables even in a pure state such as the single-mode number state $|n\rangle$. Each such random variable is characterized by a probability distribution, which determines its mean or expectation value and also higher moments such as the mean squared value. All the moments taken together constitute a complete characterization of the statistical features of the measurement results. For most purposes, however, moments of relatively low order suffice to describe and analyze the results of experimental observations.

8.8.1 Photon Number Distribution

8.8.1.1 Single-mode number states and coherent states

From the point of view of the photon number distribution, a single-mode number state $|n\rangle$ is trivial in that it contains exactly n photons (ie, the photon number variance is zero).

We consider a single-mode coherent state $|\lambda\rangle$ and look at the probability distribution over the eigenvalues of the number operator $\hat{N}$ in this state. This is referred to as the *photon number distribution* and is obtained straightaway from (see formula (8.108))

$$P(n) = \frac{|\lambda|^{2n}\mathrm{e}^{-|\lambda|^2}}{n!}, \tag{8.175}$$

which one recognizes as a *Poisson distribution* with parameter $|\lambda|^2$. The mean photon number and the variance (square of standard deviation) for this state both work out to $|\lambda|^2$ (see Eq. (8.109)).

8.8.1.2 Single-mode chaotic states

Similarly, the mean and variance of the photon number distribution for the single-mode thermal state can be worked out from Eqs. (8.172a) and (8.172b), and are

$$\langle \hat{N} \rangle = \frac{1}{\mathrm{e}^{\frac{\hbar\omega}{k_B T}} - 1}, \quad (\Delta \hat{N})^2 = \langle \hat{N} \rangle^2 + \langle \hat{N} \rangle, \tag{8.176a}$$

with the photon number distribution given in terms of $\langle \hat{N} \rangle$ by

$$P(n) = \frac{\langle \hat{N} \rangle^n}{(1 + \langle \hat{N} \rangle)^{n+1}} \tag{8.176b}$$

(check this out). Evidently, the distribution is *super-Poissonian* ($(\Delta \hat{N})^2 > \langle \hat{N} \rangle$), though, for ω in the optical range, one has $\langle \hat{N} \rangle \ll 1$, and $\Delta \hat{N}^2 \approx \langle \hat{N} \rangle$ except at very high temperatures.

The thermal state considered here constitutes a special instance of a *chaotic state* of the electromagnetic field (see Section 7.11.3.5 for the classical description of chaotic light, which, however, is, in general, a continuous-mode state).

8.8.1.3 Photon number distribution in a single-mode squeezed state

The photon number distribution in a squeezed state is again obtained directly from the results stated in Section 8.4.3.3 and may be sub-Poissonian or super-Poissonian depending on the parameters λ and ξ defining the state, as shown in Fig. 8.6. The expression for the mean photon number (Eq. 8.124a) is made up of two parts—namely, a coherent contribution ($|\lambda|^2$) and a squeeze contribution ($\sinh^2 r$). The variance, on the other hand, is given by Eq. (8.124b), which can be written as

$$(\Delta \hat{N})^2 = |\lambda|^2 \left(\mathrm{e}^{2r} \sin^2(\theta - \phi) + \mathrm{e}^{-2r} \cos^2(\theta - \phi) \right) + 2 \sinh^2 r \cosh^2 r, \tag{8.177}$$

where θ stands for the phase angle of λ ($\lambda = |\lambda| \mathrm{e}^{\mathrm{i}\theta}$). These expressions for $\langle \hat{N} \rangle$ and $(\Delta \hat{N})^2$ reduce to those for a coherent state in the limit $r \to 0$ and to those for the squeezed vacuum state for $|\lambda| \to 0$, as they should.

In the case of a squeezed state for which the phase angles θ and ϕ satisfy $\theta = \phi$ and the coherent contribution to the mean photon number is large compared with the squeeze contribution the expressions for $\langle \hat{N} \rangle$ and $(\Delta \hat{N})^2$ assume the simple forms

$$\langle \hat{N} \rangle = |\lambda|^2, \quad (\Delta \hat{N})^2 = \langle \hat{N} \rangle \mathrm{e}^{-2r}, \tag{8.178}$$

which means that the photon number distribution is *sub-Poissonian*, a result consistent with Fig. 8.6.

8.8.2 Electric Field Fluctuations

8.8.2.1 The single-mode field operator

We next look at the expectation value and variance of the electric field strength in various single-mode states of the electromagnetic field.

For this we consider a propagating mode and assume that the propagation is along the z-axis, with the field polarized along a given direction in the x-y plane, which means that one can work with a scalar field operator, where in the Heisenberg picture the latter assumes the form

$$\hat{E} = \gamma(\hat{a}\mathrm{e}^{-\mathrm{i}\chi} + \hat{a}^{\dagger}\mathrm{e}^{\mathrm{i}\chi}). \tag{8.179a}$$

In this expression, χ is a time-dependent *phase* given by

$$\chi = \omega t - kz - \frac{\pi}{2}, \tag{8.179b}$$

and the expression for the constant γ is

$$\gamma = \sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}. \tag{8.179c}$$

The electric field operator can alternatively be expressed in terms of the quadrature operators as

$$\hat{E}(\chi) = 2\gamma(\hat{X}_1 \cos\chi + \hat{X}_2 \sin\chi), \tag{8.180a}$$

which means that the random variable (E) representing the possible values of the electric field strength is related to the random variables ($X_1,\ X_2$) representing the values of the quadrature operators in a similar manner:

$$E(\chi) = 2\gamma(X_1 \cos\chi + X_2 \sin\chi). \tag{8.180b}$$

8.8.2.2 Field fluctuations in number states

Using the commutation relation between $\hat{a}$ and $\hat{a}^{\dagger}$, or that between $\hat{X}_1$ and $\hat{X}_2$, one obtains, for a number state $|n\rangle$,

$$\langle\hat{E}(\chi)\rangle = 0 \quad (\Delta E(\chi))^2 = 2\gamma^2\left(n + \frac{1}{2}\right); \tag{8.181}$$

that is, the mean electric field strength in a number state is zero, while the variance is proportional to the energy (recall the classical expression relating the energy density to the squared field strength). Both the mean and the variance are phase independent, and the minimum value of the variance occurs for the vacuum state ($n = 0$), which, in fact, is the lower limit attainable by the variance in any single-mode state.

8.8.2.3 Coherent state: Amplitude and phase fluctuations

In the case of a coherent state $|\lambda\rangle$, the mean and variance of the electric field strength are

$$\langle\hat{E}(\chi)\rangle = \gamma(\lambda\mathrm{e}^{-\mathrm{i}\chi} + \lambda^*\mathrm{e}^{\mathrm{i}\chi}) = 2\gamma|\lambda|\cos(\chi - \theta) \quad (\Delta\hat{E}(\chi))^2 = \gamma^2, \tag{8.182}$$

where θ is the phase angle of λ ($= |\lambda| e^{i\theta}$) (check this out), and where the variance is once again seen to be phase independent. The variance has the minimum attainable value for any single-mode state (recall that the vacuum state is at the same time a coherent state), which is why the coherent state can be regarded as a close approximation to a classical plane wave field (of complex amplitude $\gamma\lambda$).

> In the above formulae, the phasor λ is to be interpreted as a time-independent one—that is, the uniform rotation of $\lambda(t)$ around the origin is not to be considered. Indeed, in the Heisenberg picture, the phasor λ characterizing the coherent state $|\lambda\rangle$ is time independent, while the time dependence appears in the field operator (as in formula (8.179a)). When the expectation values are worked out in the Schrödinger picture, one again obtains the above result, now with λ standing for $\lambda(0)$. The magnitude $|\lambda|$ is time independent for a coherent state.

Fig. 8.7 depicts pictorially the fluctuations of the electric field strength and illustrates the concepts of *amplitude* and *phase* fluctuations. The measured values of the electric field (a real quantity) correspond to a random variable with a mean represented by the segment OP along the horizontal axis, while the analytic signal (a complex random variable) is represented by a random phasor (ie, a 2D vector in a complex plane; the phasor rotates with an angular velocity ω because of the time dependence of χ) whose mean is the phasor OA of magnitude $2\gamma|\lambda|$ and phase angle $(\chi - \theta)$. Since the variance is independent of the phase, the endpoints of the random phasor are depicted as being spread over a circle centered at A, with radius γ. The projection of the circle on the real axis depicts the fluctuation of the real electric field, while the diameter along OA and the one perpendicular to it (along OC) indicate the fluctuations in the *amplitude* and *phase* of the electric field phasor (more precisely, the extent of the phase fluctuations is indicated by the diameter along AD, where OD is tangential to the noise circle). The radius AB ($=\gamma$) denotes the amplitude uncertainty, while the angle AOD represents the phase uncertainty (as briefly explained in Section 8.4.2.3). The phase can be regarded as a concept complementary to the photon number, but a precise definition of a phase operator as one conjugate to the photon number operator is not possible. For relatively large values of $|\lambda|$, the phase and its variance become meaningful in a semiquantitative sense. Thus the phase uncertainty, measured by the angle AOC for large $|\lambda|$, becomes

$$\begin{aligned} \Delta\Phi &= \frac{\gamma}{2\gamma|\lambda|} \\ &= \frac{1}{2(\langle\hat{N}\rangle)^{1/2}} = \frac{1}{2\Delta\hat{N}}, \end{aligned} \tag{8.183}$$

which gives the so-called *number-phase uncertainty relation*

$$\Delta\hat{N}\Delta\Phi = \frac{1}{2}. \tag{8.184}$$

The complementarity between the photon number and the phase is then expressed by saying that there exists an inverse relation between the variances of the two. Similar geometrical considerations apply to the squeezed states, as we will see in Section 8.8.2.4.

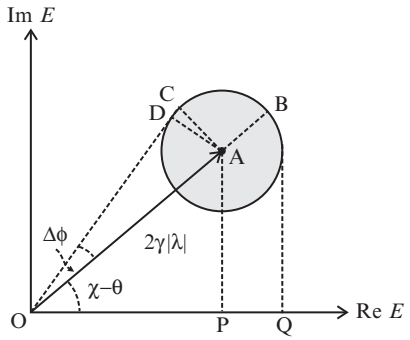

Fig. 8.7

Depicting geometrically the electric field fluctuations in a coherent state where the concepts of amplitude and phase fluctuations of the field are illustrated. The fluctuating electric field is represented by a random complex phasor whose projection on the horizontal axis gives the fluctuations of the real electric field strength. The phasor OA of length $2\gamma|\lambda|$ represents the mean, whose projection OP gives $\langle \hat{E}(\chi)\rangle$ in Eq. (8.182). The random variation of the endpoint of the complex phasor causes it to be scattered over a region, which is depicted by a circle around A since the electric field variance is independent of χ. The uncertainty is represented by the radius of the circle (=PQ); it also equals in the present case the radius AB along OA, which is interpreted as the amplitude uncertainty. The radius AD, which coincides with the radius AC in the direction perpendicular to OA for sufficiently large $|\lambda|$, subtends an angle $\Delta\Phi$ at the origin O, which can be interpreted as the phase uncertainty. Similar geometrical considerations apply to a squeezed state, as shown in Fig. 8.10.

Incidentally, the mean phase is not apparent from the geometry in Fig. 8.7, and is given by

$$\langle \Phi \rangle = \theta, \tag{8.185}$$

the phase angle of the phasor λ. On can establish this by working out the phase probability distribution for the coherent state, and it can be inferred directly from the geometry in Fig. 8.4.

Fig. 8.8 illustrates pictorially the variation in the mean field strength and the field uncertainty with the phase difference $\chi - \theta$ (recall from Eq. 8.179b that χ increases linearly with time). The dotted curve depicts the sinusoidal variation of the mean field strength (refer to $\langle \hat{E}(\chi)\rangle$ in Eq. 8.182) and is the mean of the oscillating ribbon, whose width along the vertical axis for any specified value of χ represents twice the field uncertainty. In the present context of a coherent state, the ribbon is of constant width since the uncertainty is phase independent.

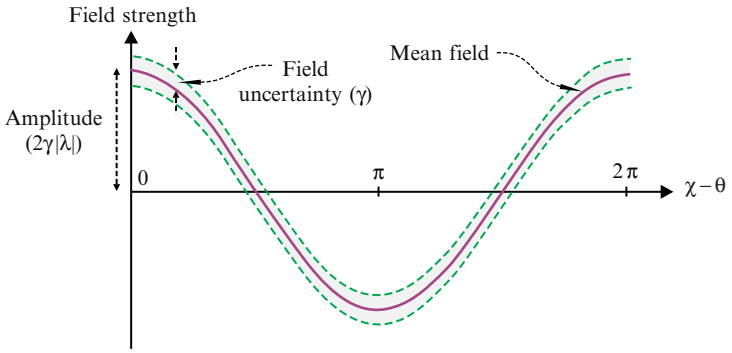

Fig. 8.8
Depicting the variation of the mean electric field strength and the field uncertainty with $\chi - \theta$ for a coherent state. The mean curve of the oscillating ribbon illustrates the sinusoidal variation of the mean field strength, while the vertical width of the ribbon represents twice the field uncertainty; the latter is phase independent in the case of a coherent state. The field uncertainty becomes negligible in a relative sense as $|\lambda|$ is made to attain a large value.

8.8.2.4 *Field fluctuations in a squeezed state*

Considering first the squeezed vacuum state $|0;\xi\rangle$ (see Section 8.4.3.2; recall that all results in respect of the harmonic oscillator translate unchanged to those of single-mode field states) and referring to the field expression (8.179a), one can work out the mean and the variance of the electric field strength for any given phase angle χ, obtaining

$$\mathcal{S} \equiv \langle \hat{E}(\chi) \rangle = 0, \tag{8.186a}$$

$$\mathcal{N} \equiv (\Delta \hat{E}(\chi))^2 = \gamma^2 \left(\mathrm{e}^{2r} \sin^2(\chi - \phi) + \mathrm{e}^{-2r} \cos^2(\chi - \phi) \right) \tag{8.186b}$$

(check these out; for Eq. 8.186a, recall that the squeezed vacuum is a superposition of number states with an even number of photons; formula (8.186b) is obtained from Eq. 8.123a with $\lambda = 0$).

Note that the variance is now phase dependent, in contrast to the results for the number state and the coherent state. In the case of a phase-independent variance, the inequality $(\Delta E(\chi))^2 \geq \gamma^2$ is always satisfied, while in the case of a phase-dependent variance, values in the range $0 \leq (\Delta E(\chi))^2 \leq \gamma^2$ can be achieved for appropriate values of the phase angle χ. The reduced variance, however, is compensated by values larger than γ^2 for phase angles $\chi \pm \frac{\pi}{2}$.

Fig. 8.9 depicts the variation of the field uncertainty (the square root of the variance) as a function of the phase $\chi - \phi$ (recall once again that χ increases linearly with t according to Eq. 8.179b), where the uncertainty for any given value of $\chi - \phi$ is obtained as half the

vertical separation between the upper and lower wavy curves corresponding to $\pm\Delta\hat{E}$, which lie symmetrically with reference to the horizontal axis, the latter corresponding to the mean field ($\langle\hat{E}(\chi)\rangle = 0$). The region between the two wavy lines depicts the possible values of the uncertainty, on either side of the mean, for various possible values of $\chi - \phi$, and is referred to as the *noise band*. The two dotted horizontal lines depict the upper and lower boundaries of the noise band for a coherent state.

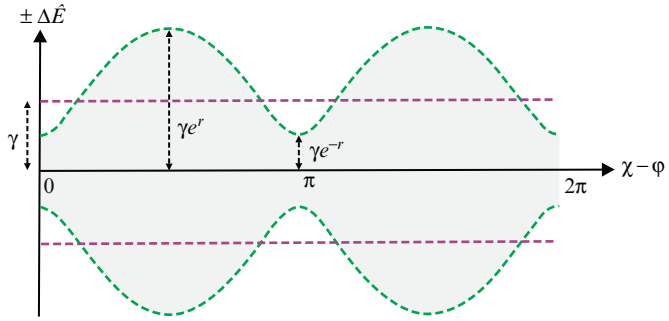

Fig. 8.9

Depicting the noise band for the squeezed vacuum state $|0;\xi\rangle$ in comparison with the same for the coherent state $|\lambda;0\rangle$. The vertical separation between the upper and lower wavy lines gives twice the field uncertainty (square root of the variance, Eq. 8.186b) for any given value of the phase angle $\chi - \phi$. The uncertainty dips below the coherent state value (half separation between the two dotted horizontal lines) for two intervals in each complete cycle of χ. The minimum and maximum values are determined by the squeeze parameter r.

The uncertainty in the squeezed vacuum state attains the minimum (or maximum) value γe^{-r} (or γe^{r}; we assume r to be positive) twice in each complete cycle of χ. The product of the maximum and the minimum uncertainties is phase independent and is the same as the variance (γ^2) for a coherent state.

The variances of the two quadrature operators $\hat{X}_1$ and $\hat{X}_2$ depend on the phase of the squeeze parameter ξ. For instance, with $\phi = 0$, $r > 0$, $\Delta\hat{X}_1$ attains the maximum value of $\frac{1}{2}e^{r}$, while $\Delta\hat{X}_2$ attains the minimum value of $\frac{1}{2}e^{-r}$, the product of the two being $\frac{1}{4}$. These values are interchanged for $\phi = \frac{\pi}{4}$. Thus the uncertainty of one quadrature is squeezed at the expense of that of the other.

We turn our attention now to the squeezed state $|\lambda;\xi\rangle$ ($\lambda \neq 0$), which is a displaced squeezed vacuum. The noise patch is now represented by a rotated and displaced ellipse as explained in Section 8.4.3.2, which changes the mean of the quadrature operators and of the electric field strength (both zero for the squeezed vacuum) to nonzero values:

$$\langle\hat{X}_1\rangle = |\lambda|\cos\theta, \quad \langle\hat{X}_2\rangle = |\lambda|\sin\theta, \quad \mathcal{S} = \langle\hat{E}\rangle = 2\gamma|\lambda|\cos(\chi - \theta), \tag{8.187}$$

these being expressions identical to the corresponding ones for the coherent state (recall that θ is the phase angle of the phasor λ). Contrary to the squeezed vacuum, the *signal* $\mathcal{S}$, as detected in *homodyne detection* (see Section 8.19), is nonzero. On the other hand, the *noise* $\mathcal{N}$ remains the same as in the case of the squeezed vacuum, and is given by expression (8.186b). The *signal-to-noise ratio*, an important characteristic of the field state, is given by

$$\sigma \equiv \frac{\langle \hat{E} \rangle^2}{(\Delta \hat{E})^2} = \frac{4|\lambda|^2 \cos^2(\chi - \theta)}{\mathrm{e}^{2r} \sin^2(\chi - \phi) + \mathrm{e}^{-2r} \cos^2(\chi - \phi)}. \tag{8.188a}$$

As mentioned in connection with the coherent state, one can work out the expectation values in either the Schrödinger picture or the Heisenberg picture. In the Heisenberg picture the time variation of the parameter λ does not appear, and all the time dependence appears in the field operator, as in Eq. (8.179a). In the Schrödinger picture, on the other hand, the time dependence appears through that of the phasor λ, whose length varies with frequency 2ω, as mentioned in Section 8.4.3.3. When this mode of calculation is adopted, the results stated above hold with $|\lambda|$ standing for the length of the phasor at time $t = 0$.

For a given value of the phase χ of the electric field, the signal-to-noise ratio is determined by the two phase angles θ and ϕ, one relating to the displacement phasor λ and the other relating to the squeeze phasor ξ. The maximum possible value of the signal-to-noise ratio is

$$\sigma_{\max} = 4\mathrm{e}^{2r}|\lambda|^2, \tag{8.188b}$$

corresponding to θ and ϕ both being equal to χ. Significantly, the signal strength ($\mathcal{S}$) in the squeezed state is the same as that in the coherent state with the same coherent amplitude λ and, at the same time, its noise is reduced below that of the coherent state for appropriately chosen values of the phase angles.

Fig. 8.10 depicts the electric field fluctuations in the squeezed state, analogously to Fig. 8.7 for the coherent state, for the choice $\theta = \phi$, in which case the minor axis of the noise ellipse (a displaced and rotated one) lies along the mean signal amplitude phasor $2\gamma\lambda$. The semi-minor axis (AB) of the ellipse gives a measure of the amplitude uncertainty of the squeezed state, while the semimajor axis AC, measured in proportion to OA, constitutes an approximate semiquantitative measure of the *phase uncertainty*. For the case shown in Fig. 8.10, the field can be said to be of the *amplitude-squeezed* type, while *phase squeezing* is also possible (eg, with $\theta = \phi + \frac{\pi}{2}$).

Simple expressions for the phase uncertainty and the photon number uncertainty can be worked out in the case of a large coherent signal ($|\lambda| \gg \mathrm{e}^r$), when one obtains

$$\Delta \hat{N} = \langle \hat{N} \rangle^{1/2} \mathrm{e}^{-r}, \quad \Delta\Phi = \frac{\mathrm{e}^r}{2\langle \hat{N} \rangle^{1/2}}. \tag{8.189}$$

Thus the number uncertainty is reduced compared with that for the coherent state, and corresponds to a *sub-Poissonian* distribution. There is a corresponding increase in the phase

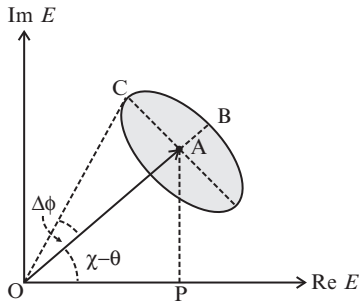

Fig. 8.10

Depicting geometrically the electric field fluctuations in a squeezed state, analogous to Fig. 8.7 for a coherent state. The fluctuating electric field is represented by a random complex phasor whose projection on the horizontal axis gives the fluctuations of the real electric field. The phasor OA of length $2\gamma|\lambda|$ represents the mean, whose projection gives $\langle \hat{E}(\chi) \rangle$ in Eq. (8.187). The random variation of the endpoint of the complex phasor causes it to be scattered over a region, which is depicted by the noise ellipse, whose minor axis is chosen to lie along the signal amplitude phasor OA ($\theta = \phi$). In this case the semiminor axis AB of the ellipse represents the amplitude uncertainty, while the semimajor axis AC determines, up to an approximation, the phase uncertainty.

uncertainty, while the product of the two uncertainties continues to be the same as in Eq. (8.184).

Fig. 8.11 depicts schematically the variation of the mean field strength and the field uncertainty as a function of $\chi - \theta$ (with $\phi = \theta$), where one observes a sinusoidal variation of the mean field and a corresponding variation of the vertical width (twice the field uncertainty) of the oscillating ribbon (the noise band), there being two minima and two maxima in the uncertainty in a complete cycle of the phase. In other words, while the mean field oscillates in time with frequency ω, the uncertainty oscillates with frequency 2ω.

8.8.2.5 Field fluctuations in a single-mode chaotic state

A chaotic state derives from radiation in equilibrium within a cavity at any specified temperature T.

> The term 'chaotic light' describes a classical state of the electromagnetic field (see Section 8.10) and was introduced in general outline in Section 7.11.3.5.

The radiation may be made to come out from the enclosure through a window, and may be made to pass through an appropriate filter when only a single mode passes through it. A single-mode chaotic state may be described in terms of a density operator as in Eqs. (8.172a) and (8.172b), from which one can work out the mean and the variance of the electric field strength by evaluating the expectation values:

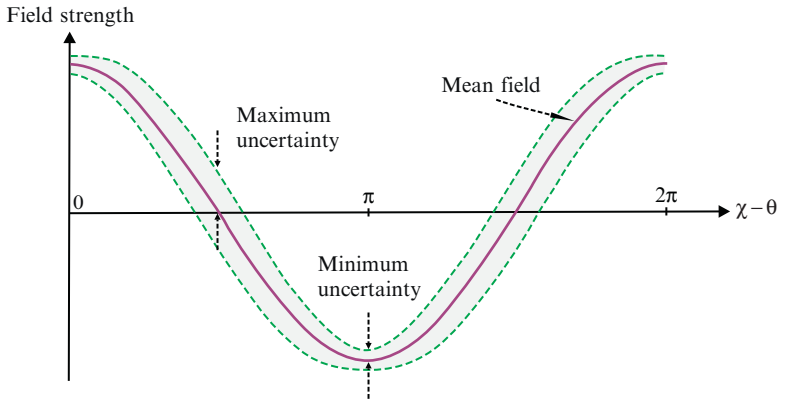

Fig. 8.11
Depicting schematically the variation of the mean electric field strength (*solid curve*) and the field uncertainty (the oscillating *noise band*) with the phase angle $\chi - \theta$ for a squeezed state with $\theta = \phi$, where, at any instant of time t (recall that χ increases linearly with t), the vertical width of the noise band indicates twice the field uncertainty. The uncertainty attains a maximum and a minimum value twice in each complete cycle of the phase angle as in the case of the squeezed vacuum (for which, however, the mean, ie, the signal strength, is zero; see Fig. 8.9). In contrast, the field uncertainty remains constant in a coherent state (see Fig. 8.8).

$$\langle \hat{E} \rangle = \mathrm{Tr}(\hat{E}\hat{\rho}), \quad \langle \hat{E}^2 \rangle = \mathrm{Tr}(\hat{E}^2\hat{\rho}). \tag{8.190a}$$

In these formulae we use the expression for the cavity field operator, assuming a fixed direction of linear polarization in the x-y plane (which allows the use of a scalar field operator), as

$$\hat{E} = \mathrm{i}\gamma(\hat{a}\mathrm{e}^{-\mathrm{i}\omega t} - \hat{a}^\dagger \mathrm{e}^{\mathrm{i}\omega t}) \quad \left(\gamma \equiv \sqrt{\frac{\hbar\omega}{\epsilon_0 V}}\sin(kz)\right) \tag{8.190b}$$

(see Eq. 8.166b). One then obtains, for the single-mode chaotic field,

$$\langle \hat{E} \rangle = 0 \quad (\Delta\hat{E})^2 = 2\gamma^2\left(\langle \hat{N} \rangle + \frac{1}{2}\right), \tag{8.191}$$

where $\langle \hat{N} \rangle$ stands for the mean photon number in the state, given by the first formula in Eq. (8.176a). In other words, the mean electric field strength is zero as in a photon number state, while the expression for the variance is also analogous to that in the case of a number state (see Eq. 8.181), with the mean photon number $\langle \hat{N} \rangle$ appearing for n (check this out). This is because the single-mode chaotic state is a statistical mixture of the number states, the fluctuations in the various number states being independent of one another.

8.8.3 Fluctuations in Multimode States

Multimode field states with discretely distributed modes were considered in Section 8.7.2. Such states are especially relevant in the context of fields within an enclosure, where only a discrete collection of modes can be sustained depending on the shape and size of the enclosure and on the boundary conditions at the enclosure walls. For instance, multimode chaotic light is made up of a number of modes in the enclosure existing in equilibrium at a given temperature, while *blackbody radiation* can be described as a chaotic field state involving *all* the possible modes in an enclosure of large volume V, where in the limit $V \to \infty$, the discrete set of modes goes over to a continuum.

For brevity, I will not mention the results on photon number fluctuations and field fluctuations in discrete multimode states here, since these can be arrived at without new principles being invoked. Instead, I will now consider field states with *continuously* distributed modes (see Section 8.9 below), giving you the basic idea underlying the description of these states. In Sections 8.16 and 8.17 I will state a few results on field correlation functions in discrete multimode states.

8.9 The Continuous-Mode Description

8.9.1 Continuous-Mode Description: The Basics

In a *continuous-mode* description of the field one has to consider all possible directions and magnitudes of the wave vector $\mathbf{k}$ and, for each $\mathbf{k}$, a pair of independent polarization vectors labeled with the index s. For simplicity, however, we consider here a class of field configurations where the possible wave vectors are all directed along a fixed line, which we take as the z-axis of some chosen coordinate system. We will also assume that the relevant field states correspond to a single polarization vector. These assumptions describe field states in the form of superpositions of plane waves along the chosen direction, all in the chosen state of polarization, and allows the use of a scalar field operator $\hat{E}$.

Recalling the procedure of quantization of field modes with periodic boundary conditions referred to a cube of side L, we find the continuous-mode description specified above requires us to go over to the limit $L \to \infty$. However, we consider states of the field that correspond to a parallel polarized beam of cross-sectional area A, which means that the above limit is to be taken only along the chosen axis, while the lengths in the transverse directions are to be kept finite so as to make up a finite area of cross section A.

The modes are now to be labeled with a single index k, the magnitude of the propagation vector, or, equivalently, with the angular frequency ω, where ω can vary continuously over the range 0 to ∞. In reality, the relevant field states correspond to a finite range of frequencies, say, from ω to $\omega + \Delta\omega$. One can then formally assume the range of variation of ω to extend

from $-\infty$ to ∞, since the values of all relevant measurable functions of ω can be set to zero outside the above range. This allows one to make use of Fourier transformations from the frequency domain to the time domain and vice versa ($\omega \to t$ and $t \to \omega$).

I will now skip the mathematical steps required to go from the discrete mode representation to the continuous-mode representation, and will give you a number of basic formulae in the latter, in terms of which one can express measurable quantities.

> You will find these steps outlined in the sixth chapter of
>
> R. Loudon, *The Quantum Theory of Light*, 3rd ed., Oxford University Press, Oxford, 2000.
>
> This is a great text in quantum optics. I have followed it in several sections in the present chapter.

The creation and annihilation operators labeled with the continuously varying parameter ω satisfy the commutation relations (compare with Eqs. 8.167a and 8.167b; in the following, $\hat{a}(\omega)$ is an operator function characterizing the continuous-mode field)

$$[\hat{a}(\omega), \hat{a}(\omega')] = 0, \quad [\hat{a}^\dagger(\omega), \hat{a}^\dagger(\omega')] = 0, \quad [\hat{a}(\omega), \hat{a}^\dagger(\omega')] = \delta(\omega - \omega'). \tag{8.192}$$

In terms of these creation and annihilation operators, the electric field operator, now a scalar one depending on the spatial variable z and time t, is given by

$$\hat{E}(z,t) = \mathrm{i} \int_0^\infty d\omega \left(\frac{\hbar\omega}{4\pi c \epsilon_0 A} \right)^{1/2} \left(\hat{a}(\omega) \mathrm{e}^{-\mathrm{i}\omega(t-\frac{z}{c})} - \hat{a}^\dagger(\omega) \mathrm{e}^{\mathrm{i}\omega(t-\frac{z}{c})} \right), \tag{8.193}$$

where the vector operator is obtained by multiplication with the chosen polarization vector. This expression has a positive frequency part and a negative frequency one ($\hat{E} = \frac{1}{2}(\hat{E}^+ + \hat{E}^-)$), each of which depends on z and t only through the combination $t - \frac{z}{c}$.

The electromagnetic field Hamiltonian is now given by

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \int_0^\infty d\omega \hbar\omega a^\dagger(\omega) a(\omega), \tag{8.194}$$

where $\hat{\mathcal{H}}_0$ stands for the vacuum energy part, which will not be needed for our present purpose.

Instead of the number operator $\hat{N} = a^\dagger a$ for a single discrete mode, one has the number density operator

$$\hat{N}(\omega) = a^\dagger(\omega) a(\omega), \tag{8.195a}$$

and the number operator for the field as a whole is

$$\hat{N} = \int d\omega a^\dagger(\omega) a(\omega). \tag{8.195b}$$

The quantum mechanical operator for the instantaneous intensity, which is obtained from the classical expression by means of normal ordering, is

$$\hat{I}(z,t) = \frac{\hbar}{2\pi A}\int d\omega d\omega' \sqrt{\omega\omega'}\, a^{\dagger}(\omega)a(\omega')\exp\left(\mathrm{i}(\omega-\omega')\left(t-\frac{z}{c}\right)\right). \tag{8.196}$$

It is useful to express this in terms of the time-dependent creation and annihilation operators obtained from $\hat{a}^{\dagger}(\omega)$ and $\hat{a}(\omega)$ by Fourier transformation (recall that it is legitimate to apply a Fourier transform, especially for radiation of narrow bandwidth) as, for instance,

$$\hat{a}(t) = \sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty} d\omega \hat{a}(\omega)\mathrm{e}^{-\mathrm{i}\omega t} \tag{8.197}$$

(note the factor of $\sqrt{\frac{1}{2\pi}}$ at the front, which differs from the way the Fourier transform was defined in Chapters 6 and 7; the present definition gives symmetric-looking formulae in terms of time-dependent and frequency-dependent operators). In terms of the frequency domain and the time domain annihilation operators, the field vacuum is defined as

$$\hat{a}(\omega)|0\rangle = \hat{a}(t)|0\rangle = 0. \tag{8.198}$$

The number operator of Eq. (8.195b) now reads

$$\hat{N} = \int dt a^{\dagger}(t)a(t), \tag{8.199}$$

while the electric field operator appears in the simple form

$$\hat{E}(z,t) = \mathrm{i}\left(\frac{\hbar\omega_0}{2\epsilon_0 cA}\right)^{1/2}\left(\hat{a}\left(t-\frac{z}{c}\right)-\hat{a}^{\dagger}\left(t-\frac{z}{c}\right)\right), \tag{8.200}$$

where we have made the simplifying assumption that the radiation is narrowband radiation, with central frequency ω_0 (this allows one to bring the factor $\left(\frac{\hbar\omega_0}{4\pi c\epsilon_0 A}\right)^{1/2}$ outside the integral in Eq. (8.193) while extending the range of integration to one from $-\infty$ to ∞ at the same time; the integrals that remain are the Fourier transforms of $a(\omega)$ and $a^{\dagger}(\omega)$). This assumption acquires validity in numerous situations of practical interest.

Finally, the intensity operator in Eq. (8.196) also simplifies to

$$\hat{I}(z,t) = \frac{\hbar\omega_0}{A}\hat{a}^{\dagger}\left(t-\frac{z}{c}\right)\hat{a}\left(t-\frac{z}{c}\right), \tag{8.201}$$

with the help of which one can work out intensity fluctuations and intensity correlations in an optical field. The form of this expression suggests that one can interpret

$$\hat{f}(t) = \hat{a}^{\dagger}(t)\hat{a}(t) \tag{8.202}$$

as the *photon flux* operator in an optical field.

An important special case to be considered is that of a stationary light beam, of which the fluctuations at any given point are time independent. For such a beam, the correlation function (i.e., the expectation value of an operator product) in the frequency domain is of the form

$$\langle \hat{a}^\dagger(\omega)\hat{a}(\omega')\rangle = 2\pi f(\omega)\delta(\omega - \omega'), \tag{8.203}$$

where $f(\omega)$ can be interpreted as the mean photon flux per unit frequency interval at frequency ω, since its integral may be seen to be equal to the mean of the photon flux operator in Eq. (8.202), which is time independent in this special case:

$$\int f(\omega)d\omega = \langle \hat{f}(t)\rangle. \tag{8.204}$$

In an actual detection with a photodetector, one measures the photon flux over a period of time, but the photocount rate does not directly give the photon flux because the photodetector responds to the photon flux with only limited efficiency. The theory of photocount distribution will be taken up in Section 8.14.2.

8.9.2 Continuous-Mode Number States

The single-mode states considered in Section 8.7.1 are idealized ones as are the discrete multimode states considered in Section 8.7.2 since real-life setups with a large variety of commonly used sources generally involve a finite frequency bandwidth, with the frequency ranging continuously over some interval. Exceptions to this statement are brought up by laser sources, where the bandwidth is usually much smaller than all other relevant bandwidths. Single-mode fields are also special in their coherence characteristics, as will be seen in Sections 8.16.2 and 8.17.3.

As an instance of the continuous-mode description, a continuous-mode single wavepacket photon number state is characterized by a frequency spectrum defined by some spectral function (also referred to as the 'pulse shape,' since such a state appears in the form of a finite pulse) $\xi(\omega)$, having a certain bandwidth and a mean frequency, where the spectral function depends on the setup for the production and detection of the state. As an example, a Gaussian pulse shape is of the form

$$\xi(\omega) = (2\pi\Delta^2)^{-1/4} \exp\left(\mathrm{i}(\omega - \omega_0)t_0 - \frac{(\omega - \omega_0)^2}{4\Delta^2}\right), \tag{8.205}$$

where ω_0 stands for the mean frequency, Δ for the bandwidth (which we assume is small compared with ω_0), and t_0 for the time at which the peak of the pulse passes the point $z = 0$ on the axis of propagation. Because of the small bandwidth, the narrowband approximation mentioned in Section 8.9.1 applies.

The pulse shape in the time domain is given by the Fourier transform of Eq. (8.205) as

$$\tilde{\xi}(t) = \left(\frac{2\Delta^2}{\pi}\right)^{1/4} \exp\left(-\mathrm{i}\omega_0 t - \Delta^2(t - t_0)^2\right), \tag{8.206}$$

which tells us that the pulse is, in reality, a wave packet, with a slowly decaying envelope.

The photon-wave packet creation operator for the given pulse shape $\xi(\omega)$ is defined as

$$\hat{a}_\xi^\dagger = \int d\omega \xi(\omega)\hat{a}^\dagger(\omega) = \int dt \tilde{\xi}(t)\hat{a}^\dagger(t), \tag{8.207}$$

where the second equality is obtained by Fourier transformation from the frequency domain to the time domain. The suffix 'ξ' is used to indicate that the expression under consideration is defined for the spectral function $\xi(\omega)$.

The annihilation operator $\hat{a}_\xi$, defined by taking the Hermitian conjugate, satisfies

$$[\hat{a}_\xi, \hat{a}_\xi^\dagger] = \hat{I}, \tag{8.208}$$

which is free of singularity, in contrast to the third relation in Eq. (8.192). The photon wave packet number states $|n_\xi\rangle$ ($n_\xi = 1, 2, \ldots$) are then defined in the usual way by successive application of $a_\xi^\dagger$ on the vacuum state $|0\rangle$:

$$|1_\xi\rangle = \hat{a}_\xi^\dagger|0\rangle, \quad |n_\xi\rangle = \frac{1}{\sqrt{n}}\hat{a}_\xi^\dagger|(n-1)_\xi\rangle. \tag{8.209}$$

One also has the important commutation relations

$$[\hat{a}(\omega), \hat{a}_\xi^\dagger] = \xi(\omega)\hat{I}, \quad [\hat{a}(t), \hat{a}_\xi^\dagger] = \tilde{\xi}(t)\hat{I}. \tag{8.210}$$

Using these commutation relations, one finds that the state $|n_\xi\rangle$ is actually the eigenstate of the number operator $\hat{N}$ of Eqs. (8.195b) and (8.199) belonging to the eigenvalue n:

$$\hat{N}|n_\xi\rangle = n|n_\xi\rangle. \tag{8.211}$$

The continuous-mode field operators and their functions, such as $\hat{E}(z,t)$ and $\hat{I}(z,t)$, are defined in terms of $\hat{a}(\omega)$ or $\hat{a}(t)$ as in Eqs. (8.193), (8.196), (8.200), and (8.201). The expectation values of these operators in the wave packet number states can all be determined by use of Eqs. (8.209) and (8.210). Thus, for instance,

$$\langle n_\xi|\hat{E}(z,t)|n_\xi\rangle = 0, \tag{8.212a}$$

which resembles the result for the ordinary number state. The expectation value of the photon flux operator is

$$\langle n_\xi|\hat{f}(t)|n_\xi\rangle = n|\tilde{\xi}(t)|^2. \tag{8.212b}$$

Referring to the case of a narrowband Gaussian photon wave packet, one obtains from this formula the expectation value of the instantaneous intensity as

$$\langle n_\xi|\hat{I}(z,t)|n_\xi\rangle = \sqrt{\frac{2}{\pi}}\left(\frac{\hbar\omega_0\Delta}{A}\right) n \exp\left(-2\Delta^2\left(t - t_0 - \frac{z}{c}\right)^2\right). \tag{8.212c}$$

Evidently, the field fluctuations for a photon wave packet number state are not stationary in nature because of the localized nature of the state in the frequency and time domains.

Among the wave packet number states, the single-photon states are of practical relevance since it is prohibitive to produce a larger number of photons with identical wave packets. The formalism outlined in the present section can be extended to include two-photon wave packet states with the two wave packets in the same continuous-mode field or in two different fields corresponding to distinct beams. Such photon pair states (refer to Section 8.9.5) can be produced by parametric down conversion in a nonlinear crystal, and can be made to give rise to *two-photon interference* (see Section 8.18).

8.9.3 Continuous-Mode Coherent States

A single-mode coherent state is characterized by a complex amplitude λ and is obtained from the vacuum state by the action of the displacement operator in Eq. (8.102b). As mentioned in Section 8.7.2, a multimode coherent state is specified in terms of a complex amplitude defined for each of the discrete modes under consideration. More generally, a continuous-mode coherent state is specified in terms of a complex amplitude *function* $\lambda(\omega)$—that is, a complex amplitude for each value of the frequency belonging to the relevant frequency range—and is obtained by the action of a displacement operator of the form

$$\hat{D}_\lambda = \exp(\hat{a}_\lambda^\dagger - \hat{a}_\lambda) \tag{8.213a}$$

on the vacuum state $|0\rangle$, where $\hat{a}_\lambda$ and $\hat{a}_\lambda^\dagger$ are operators defined for some specified function $\lambda(\omega)$ (which is analogous to the spectral function $\xi(\omega)$ specifying a photon wave packet number state; $\lambda(\omega)$ may similarly be regarded as the spectral function specifying a wave packet coherent state). In analogy to the continuous-mode number state, $\hat{a}_\lambda^\dagger$ is defined as

$$\hat{a}_\lambda^\dagger = \int d\omega \lambda(\omega)\hat{a}^\dagger(\omega) = \int dt \tilde{\lambda}(t)\hat{a}^\dagger(t), \tag{8.213b}$$

where $\tilde{\lambda}(t)$ denotes the Fourier transform of $\lambda(\omega)$. Denoting the continuous-mode coherent state specified by the function $\lambda(\omega)$ by the symbol $|\{\lambda\}\rangle$ (in analogy with discrete multimode coherent states introduced in Section 8.7.2), one has, by definition,

$$|\{\lambda\}\rangle = \hat{D}_\lambda|0\rangle. \tag{8.214}$$

The commutation relations

$$[\hat{a}(\omega), \hat{a}_\lambda^\dagger] = \lambda(\omega)\hat{I}, \quad [\hat{a}(t), \hat{a}_\lambda^\dagger] = \tilde{\lambda}(t)\hat{I}, \tag{8.215}$$

analogous to Eq. (8.210), can be seen to hold, which implies that $|\{\lambda\}\rangle$ is an eigenstate of both $\hat{a}(\omega)$ and $\hat{a}(t)$,

$$\hat{a}(\omega)|\{\lambda\}\rangle = \lambda(\omega)|\{\lambda\}\rangle, \quad \hat{a}(t)|\{\lambda\}\rangle = \tilde{\lambda}(t)|\{\lambda\}\rangle, \tag{8.216}$$

again in analogy to the single-mode and multimode coherent states.

> I want you to pause at this point and recall the context. The object of interest here is a continuous-mode polarized parallel beam of light specified by operator functions $\hat{a}(\omega)$ and $\hat{a}^\dagger(\omega)$ (a different mode, corresponding to a different beam, would correspond to some other functions, say, $\hat{b}(\omega)$ and $\hat{b}^\dagger(\omega)$). The state of the field in the beam may be a photon wave packet number state $|n_\xi\rangle$, corresponding to some specified spectral function $\xi(\omega)$, or a wave packet coherent state specified by the spectral function $\lambda(\omega)$, or one of some other description such as continuous-mode chaotic light or squeezed light, there being a spectral function specifying the state in each case.
>
> Corresponding to the given operator functions $\hat{a}(\omega)$ and $\hat{b}^\dagger(\omega)$, the operators for the electric field strength, photon flux, or the instantaneous intensity are given by expressions as specified in Section 8.9.1, starting from which one can work out their expectation values or higher-order correlations in any one of the different possible types of states specified by the appropriate spectral function.

The mean photon number in the coherent state $|\{\lambda\}\rangle$ (recall the definition of the continuous-mode photon number operator from Eqs. 8.195b and 8.199) is given by

$$\langle \hat{N} \rangle = \int d\omega |\lambda(\omega)|^2 = \int dt |\lambda(t)|^2. \tag{8.217}$$

The continuous-mode coherent state $|\{\lambda\}\rangle$ is an eigenstate of the coherent state wave packet destruction operator as well, belonging to the eigenvalue $n_\lambda (\equiv \langle \hat{N} \rangle)$, the mean photon number,

$$\hat{a}_\lambda |\{\lambda\}\rangle = n_\lambda |\{\lambda\}\rangle, \tag{8.218}$$

while the coherent state wave packet creation and destruction operators satisfy the commutation relation

$$[\hat{a}_\lambda, \hat{a}_\lambda^\dagger] = n_\lambda. \tag{8.219}$$

The mean photon flux in the coherent state $|\{\lambda\}\rangle$—that is, the expectation value of the flux operator $\hat{f}(t)$ in Eq. (8.202)—is given by

$$\langle \hat{f}(t) \rangle = |\tilde{\lambda}(t)|^2, \tag{8.220a}$$

while the mean instantaneous intensity at z is, for a narrowband beam,

$$\langle \hat{I}(z,t) \rangle = \frac{\hbar\omega_0}{A} \left| \tilde{\lambda}\left(t - \frac{z}{c}\right) \right|^2. \tag{8.220b}$$

The wave packet coherent states we have considered in this section are relevant in that they are analogous to completely coherent classical fields, with reference to which any given field state may be diagnosed as one with features of *bunching* or *antibunching*, the latter corresponding to *nonclassical light*. Another related feature characterizing a field indicates whether it has a *super-Poissonian* or a *sub-Poissonian* photon count distribution, where a wave packet coherent

state is characterized by a *Poisson* distribution. A monochromatic beam of laser radiation with a finite cross section may be regarded as a coherent state with a delta function pulse shape in the frequency domain, corresponding to which there is an infinite temporal extension typical of a stationary field. However, strictly speaking, even such a field has a small but finite bandwidth.

8.9.4 Continuous-mode Chaotic States

Continuous-mode descriptions of other types of field states are also possible and are of importance for practical purposes. For instance, a parallel beam of chaotic light can be described in terms of a density operator that is an integral over density operators such as in Eqs. (8.172a) and (8.172b), where once again a spectral function is used. More commonly, however, a stationary beam of chaotic light is described in terms of the function $f(\omega)$ defining the first-order correlation in formula (8.203) since correlations of higher orders can all be expressed in terms of this.

For instance, a beam of chaotic light with a *Lorentzian spectrum* is described by the frequency domain correlation

$$f(\omega) = \frac{\Gamma}{\pi((\omega - \omega_0)^2 + \Gamma^2)} F, \tag{8.221}$$

where ω_0 and Γ are constants corresponding to the mean frequency and the spectral width, and F stands for the total mean flux of radiation given by Eq. (8.204).

The first-order correlation in the time domain is obtained by Fourier transformation as

$$\langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle = F \exp(-\mathrm{i}\omega_0\tau - \Gamma|\tau|). \tag{8.222}$$

As will be indicated in Sections 8.16.2 and 8.17.3 the quantum coherence characteristics of such radiation are the same as those of classical chaotic light (ie, chaotic light has no quantum features distinct from classical ones).

8.9.5 Photon Pair States

Photon pair states are nonclassical ones produced in quantum optical processes such as *cascade emission* and *parametric down conversion*, the former occurring in a gas of three-level atoms and the latter in a nonlinear crystalline environment. The term 'photon pair' here does not refer to a state of the form $|n_\xi\rangle$ introduced in Section 8.9.2 (see Eq. 8.211) with $n = 2$, since the two photons in such a state in a single wave packet described by the spectral function $\xi(\omega)$ are independent of one another and cannot correspond to a photon pair made up of correlated (or *entangled*) photons, where, moreover, the two photons may belong to distinct wave packets.

A photon pair state in a single continuous-mode field is produced by the action of a *photon pair creation operator* on the vacuum state, where the former is defined in terms of the operators $\hat{a}(\omega)$ and $\hat{a}^\dagger(\omega)$ by an expression of the form

$$\hat{a}_\beta^{(2)\dagger} = \frac{1}{\sqrt{2}} \int d\omega d\omega' \beta(\omega, \omega') \hat{a}^\dagger(\omega) \hat{a}^\dagger(\omega') \tag{8.223}$$

and where $\beta(\omega, \omega')$ is the wave packet spectral function, now depending on *two* frequency arguments. We thus have the state

$$|2_\beta^{(aa)}\rangle = \hat{a}_\beta^{(2)\dagger}|0\rangle, \tag{8.224}$$

where the symbol (aa) used as a superscript indicates that the two photons in the pair belong to the mode characterized by the annihilation operator $\hat{a}(\omega)$. The two-photon wave packet spectral function is normalized as

$$\int d\omega d\omega' |\beta(\omega, \omega')|^2 = 1. \tag{8.225}$$

The continuous-mode two-photon spectral function in the time domain, $\tilde{\beta}(t, t')$, is obtained from $\beta(\omega, \omega')$ by Fourier transformation, where the frequency domain and the time domain spectral functions can be chosen so as to satisfy the symmetry relations

$$\beta(\omega, \omega') = \beta(\omega', \omega), \quad \tilde{\beta}(t, t') = \tilde{\beta}(t', t). \tag{8.226}$$

Formula (8.223) is clearly a generalization from the single-photon wave packet creation operator defined in Eq. (8.207), along with which one can work out a generalization of the rest of the formalism as well, using the spectral function $\beta(\omega, \omega')$ and its Fourier transform $\tilde{\beta}(t, t')$. In particular, it is straightforward to verify that Eq. (8.224) is indeed a two-photon eigenstate of $\hat{N}$, the photon number operator $\hat{N}$ in Eqs. (8.195b) and (8.199):

$$\hat{N}|2_\beta^{(aa)}\rangle = 2|2_\beta^{(aa)}\rangle. \tag{8.227}$$

The photon pair state defined above reduces to the uncorrelated two-photon wave packet state $|2_\xi\rangle$ in Section 8.9.2 whenever the spectral function $\beta(\omega, \omega')$ decomposes to a product of the form

$$\beta(\omega, \omega') = \xi(\omega)\xi(\omega'). \tag{8.228}$$

In addition to the photon pair state defined above where the members of the pair belong to a single continuous-mode wave packet, one can also define photon pair states with the members of the pair belonging to two distinct continuous-mode fields corresponding to two separate beams. Such a state is produced by the action of a two-beam photon pair creation operator

$$\hat{a}_\beta^{(1,1)\dagger} = \frac{1}{\sqrt{2}} \int d\omega d\omega' \beta(\omega, \omega') \hat{a}^\dagger(\omega) \hat{b}^\dagger(\omega') \tag{8.229}$$

on the vacuum state $|0\rangle$ as

$$|2_\beta^{(ab)}\rangle = \hat{a}_\beta^{(1,1)\dagger}|0\rangle. \tag{8.230}$$

In the above definition, $\hat{a}^\dagger$ and $\hat{b}^\dagger$ stand for frequency-dependent creation operators for the two continuous-mode fields under consideration, while the normalized spectral function $\beta(\omega, \omega')$ establishes the correlation between the photons belonging to the pair. The creation and annihilation operators corresponding to the two fields commute with one another. The symbol (ab) used as a superscript indicates that the two photons in the pair belong to two distinct continuous-mode fields, characterized by annihilation operators $\hat{a}(\omega)$ and $\hat{b}(\omega)$.

Defining photon number operators $\hat{N}_a$ and $\hat{N}_b$ for the two fields, each as in Eqs. (8.195b) and (8.199), one can easily check that the photon pair state of Eq. (8.230) is an eigenstate of each of these number operators belonging to the eigenvalue unity, and is thus indeed a two-photon state:

$$\hat{N}_a|2_\beta^{(ab)}\rangle = \hat{N}_b|2_\beta^{(ab)}\rangle = |2_\beta^{(ab)}\rangle. \tag{8.231}$$

The single-beam and two-beam photon pair states are both nonclassical ones, as one sees by working out the second-order quantum correlation functions for these states. This will be indicated in Section 8.17.2.

8.9.6 Continuous-Mode Squeezed States

Squeezed states are generated from the vacuum by the action of two-photon continuous-mode displacement operators analogous to the two-photon single-mode displacement operator $\hat{S}(\xi)$ in Section 8.4.3.

Thus consider the single-beam continuous-mode two-photon creation operator $\hat{a}_\beta^{(2)\dagger}$ defined in Eq. (8.223) and the two-photon continuous-mode displacement operator defined as

$$\hat{S}_\beta^{(2)} = \exp(\hat{a}_\beta^{(2)} - \hat{a}_\beta^{(2)\dagger}). \tag{8.232}$$

The single-beam continuous-mode squeezed vacuum state or the continuous-mode *two-photon coherent state* is defined from this as

$$|\{\beta aa\}\rangle = \hat{S}_\beta^{(2)}|0\rangle. \tag{8.233}$$

Such a state, defined in terms of the single-beam continuous-mode spectral function $\beta(\omega, \omega')$, is analogous to the single-mode squeezed vacuum $|0; \xi\rangle$ defined in Section 8.4.3. One can construct from this a state analogous to the single-mode squeezed state $|\lambda; \xi\rangle$, where the resulting state possesses useful properties in the context of optical signal communications.

The function β plays the same role in the operator $\hat{S}_\beta^{(2)}$ as does the squeeze parameter ξ in the operator $\hat{S}(\xi)$.

One can also define in an analogous manner a *two-beam* continuous-mode squeezed vacuum state by the action of a two-photon displacement operator constructed from the two-beam photon pair creation operator defined in Eq. (8.229) as

$$|\{\beta ab\}\rangle = \hat{S}_\beta^{(1,1)}|0\rangle, \quad \hat{S}_\beta^{(1,1)} = \exp(\hat{a}_\beta^{(1,1)} - \hat{a}_\beta^{(1,1)\dagger}). \tag{8.234}$$

8.10 The P-Representation of an Optical Field

8.10.1 P-Representations of Single-Mode Field States

In the classical and quantum descriptions of the electromagnetic field, the latter appears as a dynamical system made up of an infinite number of *modes*, where each mode is equivalent to a harmonic oscillator and where, in a given state of the field, only one or a finite number of modes (or modes distributed continuously over a range of frequencies) may be excited, with the remaining modes being in their unexcited or vacuum states.

The basic system of interest here being a harmonic oscillator, one can go back to the considerations of Section 8.4, where we got introduced to the various types of states of the oscillator, a number of which have been seen in subsequent sections to be relevant in the description of single-mode states of the electromagnetic field. In Section 8.4.4 I outlined the P-representation of harmonic oscillator states. The basic idea of the P-representation applies naturally to single-mode field states, while the extension of the same to multimode states will not be of relevance for our purpose.

> Among the commonly encountered states of the field, the thermal state involves a number of modes or, more generally, all the infinite number of modes, but the density operator decomposes into a product over individual modes, each of which can be considered independently of the others. The P-representation of entangled states will not be considered in this introductory exposition.

Briefly, the P-representation of a state described by the density operator $\hat{\rho}$ corresponds to the real function $P(\sigma)$ of a complex variable σ, as defined by Eq. (8.128), where $P(\sigma)$ can be regarded, in a qualified sense, as a distribution function in a surrogate phase space made up of coordinates σ_R and σ_I. For any given $\hat{\rho}$, the corresponding $P(\sigma)$ can be obtained from formula (8.129), which is, in general, a formal one since the integral occurring in it is not always well defined.

For a certain class of field states, $P(\sigma)$ possesses all the properties of a probability distribution with at most delta function singularities, these being precisely the *classical* states of the field (ie, ones that can, in an *operational sense*, be interpreted as mixed states for which a classical description is sufficient). In general, however, $P(\sigma)$ has deviant characteristics, in which case $\hat{\rho}$ corresponds to a *nonclassical*, or, specifically *quantum* state of the field.

A single-mode field in a coherent state $|\lambda\rangle$ provides an instance of a classical state, where $P(\sigma)$ has a delta function singularity (see Eq. 8.130). On the other hand, a number state $|n\rangle$ ($n \neq 0$) is of the nonclassical type, and so is a squeezed state $|\lambda, \xi\rangle$ ($\xi \neq 0$).

> From the practical point of view, the field of a laser maintained much above its amplification threshold resembles a coherent state, though in an actual setup the phase of the field often undergoes a random drift. The laser field with a randomly drifting phase is also classical in nature, and admits of a P-representation in terms of a probability distribution function, though the latter differs from a 2D delta function.

The radiation from a *thermal source* (ie, a single mode or a number of independent modes in thermal equilibrium at some temperature T) constitutes a commonly encountered field state. The density operator for such a *chaotic* state decomposes into a product of density operators, each corresponding to one of all the possible modes, and being of the form (8.172a). The P-representation for such a single-mode field state is of the classical type, with a Gaussian distribution function.

8.10.2 Optical Equivalence Theorem

Experimentally observed quantities relating to optical setups are all in the form of expectation values of the field variables or their products, or of expressions related to such expectation values, where these field variables are evaluated at given space-time points. Examples of such quantities are the intensity at a point and the intensity correlations relating to specified points. All such experimentally relevant quantities can ultimately be related to expectation values of products of creation and annihilation operators relating to the field where the operators in these products commonly appear in a certain order, referred to as the *normal order*. This is a consequence of the fact that most of the experimentally relevant quantities are defined with reference to *photon absorption* processes.

In a normal ordered product *all the annihilation operators occur to the right of the creation operators*. The ordering among the creation operators or that among the annihilation operators is of no concern here since these commute among themselves. Thus with reference to the annihilation and creation operators for any particular field mode, a product such as $\hat{a}^{\dagger 2}\hat{a}^2$ is in the normal ordered form but the product $\hat{a}\hat{a}^{\dagger 2}\hat{a}$ is not. A product of the creation and annihilation operators with an arbitrary ordering can be converted into a sum of normal ordered products by use of the commutation relations (see Section 8.6.2).

In Section 8.8 we encountered a few simple examples of expectation values of operators in several single-mode field states. What is needed, however, is a systematic procedure for the evaluation of expectation values of normal ordered products of creation and annihilation operators in any given field state. This is provided by the *optical equivalence theorem* formulated by Sudarshan. This theorem expresses the expectation value of a normal ordered product in any specified field state in the form of a *classical* expectation value. What this means is the following.

Confining ourselves to single-mode states for simplicity, we consider a state described by the density operator $\hat{\rho}$, with a density function $P(\sigma)$ as defined in Eq. (8.128). Let, for non-negative integers m and n, $\hat{a}^{\dagger m}\hat{a}^n$ be a normal ordered product of the creation and annihilation operators whose expectation value is to be evaluated in the state $\hat{\rho}$. According to the basic formula (8.64), this is given by

$$\langle \hat{a}^{\dagger m}\hat{a}^n \rangle = \mathrm{Tr}(\hat{a}^{\dagger m}\hat{a}^n\hat{\rho}). \tag{8.235a}$$

We now use the result that, for an operator $\hat{A}$ and a vector $|u\rangle$,

$$\mathrm{Tr}(\hat{A}|u\rangle\langle u|) = \langle u|\hat{A}|u\rangle \tag{8.235b}$$

(check this result out). On using the basic formula (8.103), one arrives at the relation

$$\mathrm{Tr}(\hat{a}^{\dagger m}\hat{a}^n\hat{\rho}) = \int d^{(2)}\sigma P(\sigma)\sigma^{*m}\sigma^n \tag{8.235c}$$

(check *this* out as well), which, in essence, is the optical equivalence theorem. In formula (8.235c), $d^{(2)}\sigma$ stands for the area element $d(\mathrm{Re}\,\sigma)d(\mathrm{Im}\,\sigma)$ in the complex σ-plane, and $\sigma^{*m}\sigma^n$ is the expectation value of $\hat{a}^{\dagger m}\hat{a}^n$ in the coherent state $|\sigma\rangle$. As we saw in Section 8.4.2, the expectation values of the canonical coordinates q and p in a coherent state $|\sigma\rangle$ are directly related to the real and imaginary parts of σ (see formulae (8.99a) and (8.107)), and thus $\mathrm{Re}\,\sigma$ and $\mathrm{Im}\,\sigma$ can be interpreted, in the context of the P-representation, as variables making up a surrogate phase space.

Formula (8.235c) can now be interpreted as stating that the quantum expectation value of $\hat{a}^{\dagger m}\hat{a}^n$ in the state $\hat{\rho}$ can be formally interpreted as the ensemble average of the corresponding phase space function $\sigma^{*m}\sigma^n$ with reference to the probability density $P(\sigma)$ defined over the surrogate phase space. In other words, the quantum expectation value appears formally as a *classical phase space average*.

This, of course, does not mean that the field can be described operationally as a classical system, since the function $P(\sigma)$ is only formally a probability distribution function, often differing in essential ways from an actual one. However, the distinction disappears for the *classical* states of the field, which can indeed be interpreted operationally as mixed states of a classical system. For the *nonclassical states*, though, the interpretation in terms of a classical phase space distribution remains a formal one, while relation (8.235c) continues to be valid.

8.11 Field Transformation by Optical Devices

8.11.1 The Beam Splitter

The beam splitter is an optical device of great importance, effecting a linear *transformation* of fields presented to two *input ports*, so the fields at two *output ports* are related to the input fields in a characteristic manner. It is used in numerous optical setups, such as the

Mach-Zehnder interferometer, various setups of the Hanbury Brown-Twiss type, and the *homodyne* detector (see Section 8.19). It is of crucial relevance in the demonstration of fundamental quantum effects, as in *two-photon* interference experiments.

Fig. 8.12 illustrates the action of a beam splitter in which '1' and '2' indicate two input beams, while the two output beams are indicated by '3' and '4.' What happens in the beam splitter is the partial reflection and refraction of each of the two input beams at the surface S, so that each of the output beams is determined by features of both input beams.

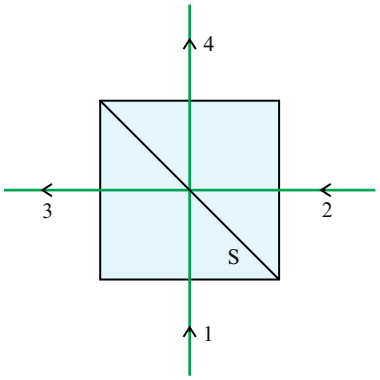

Fig. 8.12
Depicting the input and output arms of a beam splitter. Input beams are directed along the arms marked '1' and '2,' while the arms marked '3' and '4' carry the output beams. Parts of the amplitudes of the two input beams enter each of the output beams by partial reflection and transmission at the surface S. In the classical description the output beam amplitudes are related to the input ones as in Eq. (8.236). In the quantum description the same beam splitter matrix relates the output annihilation operators to those corresponding to the input arms.

In the classical description of the beam splitter action, the complex amplitudes of the two output beams are related linearly to those of the input beams. Assuming all the input and output beams to be in the form of plane waves for which the complex amplitudes of the analytic signals at the surface S are A_i $(i = 1, 2, 3, 4)$, one obtains a relation involving the input and output amplitudes by means of the *beam splitter matrix* as

$$\begin{pmatrix} A_3 \\ A_4 \end{pmatrix} = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, \tag{8.236}$$

where r and t stand for the reflection and transmission coefficients at S for the first beam and r' and t' denote the corresponding coefficients for the second beam (recall the Fresnel formulae in Section 1.14.3, where we saw how the reflected and transmitted amplitudes are related to the incident amplitude at a planar surface; the ratios of the former two to the latter are termed the 'reflection coefficient' and the 'transmission coefficient').

The principle of energy conservation requires that the total power per unit area in the two output beams has to equal the total power per unit area in the input beams (we assume that the beam splitter is lossless), which in turn implies

$$|r|^2 + |t|^2 = |r'|^2 + |t'|^2 = 1, \quad rt'^* + tr'^* = 0 \tag{8.237}$$

(check this out). If the reflection and transmission coefficients are expressed in terms of their magnitudes and phase factors as

$$r = |r|e^{i\phi_1}, \quad t = |t|e^{i\psi_1}, \quad r' = |r'|e^{i\phi_2}, \quad t' = |t'|e^{i\psi_2}, \tag{8.238}$$

then the above formulae can be expressed as

$$|r| = |r'|, \quad |t| = |t'|, \quad |r|^2 + |t|^2 = 1, \quad \phi_1 + \phi_2 - \psi_1 - \psi_2 = \pm\pi. \tag{8.239}$$

These relations can also be arrived at from the results derived in Section 1.14.3, with use of the boundary conditions at the surface S. Additionally, the reflection and transmission coefficients can be made to satisfy a further set of requirements by adjustment of the dielectric constants of the media involved in the reflection and transmission of the two input beams. For instance, one can have the *symmetric* relations

$$\begin{aligned} &\phi_1 = \phi_2 = \phi, \quad \psi_1 = \psi_2 = \psi, \quad \phi - \psi = \pm\frac{\pi}{2}, \\ &r = r' = |r|e^{i\phi}, \quad t = t' = |t|e^{i\psi}. \end{aligned} \tag{8.240}$$

One can, moreover, have, in a *50:50 beam splitter*

$$|r| = |t| = \frac{1}{\sqrt{2}}, \tag{8.241}$$

in which case the 2×2 beam splitter matrix of Eq. (8.236) assumes the form

$$B = \frac{1}{\sqrt{2}} e^{i\phi} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \tag{8.242}$$

while the complex conjugate form B^* is also possible. In general, even without Eqs. (8.240) and (8.241), the beam splitter matrix is a unitary one, which is thus a property implied by the energy conservation in the input and output beams.

In the *quantum* description of the beam splitter action, the relation (8.236) between the input and output field amplitudes is to be replaced with a relation between the input and output field *operators*. Noting that the field amplitudes of the analytic signals correspond to the positive frequency parts of the complex representations of the field functions, we find the required operator relations can be expressed in the matrix form

$$\begin{pmatrix} \hat{a}_3 \\ \hat{a}_4 \end{pmatrix} = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix}, \tag{8.243}$$

where the elements of the 2×2 beam splitter matrix B again satisfy the unitarity relations (8.239). In the special case of a symmetric 50:50 beam splitter, the matrix is again of the simple form (8.242).

In this representation the two beams in the input arms of the beam splitter correspond to two independent modes of the electromagnetic field in the input sides of the surface S, while the beams in the output arms again make up two independent modes on the output sides. Thus $\hat{a}_1$ and $\hat{a}_2$ stand for the annihilation operators of the input modes, while $\hat{a}_3$ and $\hat{a}_4$ represent the annihilation operators of the output modes. The matrix relating the creation operators of the output modes to those of the input modes is simply the complex conjugate of the matrix B.

The states of the field in the two input arms are obtained from the two respective vacuum states by the application of appropriate functions of the relevant creation and annihilation operators. One then obtains the states in the two output arms by expressing the input creation and annihilation operators in terms of the output ones and then applying the resulting operator functions on the output vacuum states.

> The beam splitter is a passive device that transforms input vacuum states to vacuum states in the output arms. In the present section we assume for simplicity that the field states in the input arms are single-mode ones, which implies that the states carried by the two output arms are, in general, two-mode entangled states. More generally, the beam in either of the input arms can be a plane parallel one including a number of modes, all having the same (or nearly the same) direction of the propagation vectors.

Starting from the relations between the input and output operators expressed in Eq. (8.243) and using the unitarity relations (8.239), one can check that the creation and annihilation operators for the various input and output arms satisfy the standard commutation relations for independent harmonic oscillator modes:

$$[\hat{a}_i, \hat{a}_i^\dagger] = \hat{I}, \quad [\hat{a}_i, \hat{a}_j^\dagger] = 0 \ (i \neq j), \quad [\hat{a}_i, \hat{a}_j] = 0, \quad [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0 \quad (i,j = 1,2,3,4). \tag{8.244}$$

In other words, the commutation relations, the unitarity of the beam splitter matrix, and the energy conservation are all equivalent conditions. In the quantum description the last named condition appears in the form

$$\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 = \hat{a}_3^\dagger \hat{a}_3 + \hat{a}_4^\dagger \hat{a}_4. \tag{8.245}$$

A fundamental distinction between the classical and quantum descriptions of the beam splitter action relates to the fact that when no light beam is presented to one of the two input arms, the field in that arm drops out of consideration in determining the fields in the two output arms in the classical description, while in the quantum description the vacuum field in the input arm under consideration plays a crucial role in determining the states of the field in the two output arms.

As a simple example, let us assume that the input '1' carries a single photon state, while the input '2' carries the vacuum state. The composite input state is then

$$|\Psi_{\text{in}}\rangle = (\hat{a}_1^\dagger \otimes \hat{I})(|0\rangle \otimes |0\rangle), \tag{8.246}$$

where, in an obvious notation, the first factor in the direct product of operators or states relates to input arm '1' and the second factor relates to arm '2,' which means that $\hat{I}$ in expression (8.246) is to be interpreted as the unit operator pertaining to arm '2.' A similar notation will apply to the pair of output arms (at times, the relevant unit operators are left implied).

By using the inverse of the matrix B, the output state of the composite system made up of the two output arms is seen to be

$$|\Psi_{\text{out}}\rangle = (r\hat{a}_3^\dagger \otimes \hat{I} + \hat{I} \otimes t\hat{a}_4^\dagger)(|0\rangle \otimes |0\rangle), \tag{8.247a}$$

which simplifies to

$$|\Psi_{\text{out}}\rangle = \frac{1}{\sqrt{2}}(|1\rangle \otimes |0\rangle + i|0\rangle \otimes |1\rangle) \tag{8.247b}$$

in the case of a symmetric 50:50 beam splitter with the beam splitter matrix given by Eq. (8.242), where we specialize to the case $\phi = 0$ for simplicity.

This is an *entangled* state of the two output modes, which has no classical analogue.

8.11.2 The Mach-Zehnder Interferometer

The Mach-Zehnder interferometer was discussed in Section 4.7.3.2 from the classical point of view. The Mach-Zehnder interferometer is also a useful and versatile instrument for quantum optics experiments, where it transforms the input field (made up of radiation in the form of beams along two input arms) to the output field (again, along two output arms) by means of two beam splitters and, additionally, by means of the path difference introduced between the two alternative routes between the beam splitters (see Fig. 4.17).

In most quantum optics setups, the Mach-Zehnder interferometer is used with its beam splitters and mirrors in the 45-degree orientation, and with monochromatic (or, more precisely, quasi-monochromatic) plane waves incident along the input arms. In the classical interference experiments, there is only one incident wave, and the other input field drops out of consideration. In the quantum description, however, the vacuum field is to be taken into account so as to make a consistent theory. In a number of quantum optics experiments (such as the one on the *two-photon interference* phenomenon), both input arms are made to carry incident fields.

The crucial role in the operation of the Mach-Zehnder interferometer is played by the two beam splitters, each of which effects a field transformation as indicated in Section 8.11.1.

With reference to Fig. 4.17 and to the description of the phase changes on reflections at the two beam splitters in Section 4.7.3.2 (in Fig. 4.17 the beam splitters are shown to be partially reflecting plates, for which the above analysis applies, with appropriate phase changes in internal and external reflections taken into consideration), the annihilation operators at the output of the beam splitter B may be seen to be related to those at the input as

$$\begin{pmatrix} \hat{a}_3 \\ \hat{a}_4 \end{pmatrix} = \begin{pmatrix} -r & t \\ t & r \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix}, \tag{8.248a}$$

where r and t can both be taken to be real and positive. In expression (8.248a), $\hat{a}_1$ refers to the input beam to beam splitter B in the vertically upward direction in Fig. 4.17, $\hat{a}_2$ to the input beam (not shown in Fig. 4.17) in the horizontally leftward direction, $\hat{a}_3$ to the output beam of beam splitter B in the horizontally leftward direction, and $\hat{a}_4$ to the output beam of beam splitter B in the vertically upward direction.

The fields traveling along paths marked '1' and '2' in Fig. 4.17 between beam splitters B and B$'$ acquire a phase difference by virtue of the difference in path lengths. If $\hat{a}_5$ refers to the input beam to beam splitter B$'$ in the vertically upward direction in Fig. 4.17 and $\hat{a}_6$ refers to the input beam to beam splitter B$'$ in the horizontally leftward direction, then one can write (ignoring an overall phase factor)

$$\begin{pmatrix} \hat{a}_5 \\ \hat{a}_6 \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\delta} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{a}_3 \\ \hat{a}_4 \end{pmatrix}, \tag{8.248b}$$

where $\delta = \frac{2\pi}{\lambda}(b - b')$ (in the notation of Section 4.7.3.2) stands for the phase difference between paths '1' and '2.'

Finally, denoting by $\hat{a}_7$ and $\hat{a}_8$ the annihilation operators for the modes corresponding to the output beams of beam splitter B$'$ along the horizontally leftward and the vertically upward directions, we write

$$\begin{pmatrix} \hat{a}_7 \\ \hat{a}_8 \end{pmatrix} = \begin{pmatrix} -r & t \\ t & r \end{pmatrix} \begin{pmatrix} \hat{a}_5 \\ \hat{a}_6 \end{pmatrix}, \tag{8.248c}$$

where we assume that the two beam splitters are identical. Fig. 8.13 depicts symbolically beam splitters B and B$'$, and the indices $1, \ldots, 8$ label the input and output beams of the two beam splitters. The reflections corresponding to $2 \to 4$ and $6 \to 8$ have been assumed to be internal ones, and those corresponding to $1 \to 3$ and $5 \to 7$ have been assumed to be external ones (see Section 1.14.3, where we looked at phase changes in reflections).

Collecting together the above three transformations, we see the resultant transformation from $\hat{a}_1,\ \hat{a}_2$ to $\hat{a}_7,\ \hat{a}_8$ is

$$\begin{pmatrix} \hat{a}_7 \\ \hat{a}_8 \end{pmatrix} = B_{\mathrm{MZ}} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix}, \tag{8.249a}$$

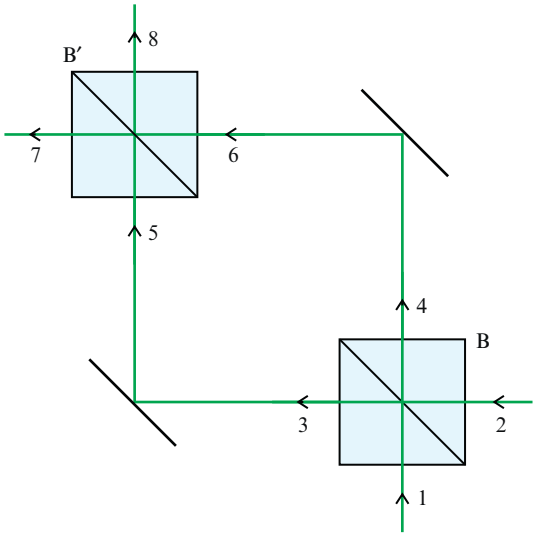

Fig. 8.13
Depicting symbolically the input and output arms of the two beam splitters B and B′ in a Mach-Zehnder interferometer (the complete interferometer setup is depicted schematically in Fig. 4.17 with partially reflecting mirrors used as beam splitters). The arms marked '1' and '2' carry the input beams to B, the output beams of which are directed along the arms marked '3' and '4.' Similarly, the arms marked '5' and '6' represent the input arms of beam splitter B′, of which the arms marked '7' and '8' are the output arms. The output fields (arms '7' and '8') of the whole setup are related to the input fields (arms '1' and '2') by the matrix $B_{\rm MZ}$ in Eq. (8.249b).

where the Mach-Zehnder field transformation matrix $B_{\rm MZ}$ is

$$B_{\rm MZ} = \begin{pmatrix} r_{\rm MZ} & t'_{\rm MZ} \\ t_{\rm MZ} & r'_{\rm MZ} \end{pmatrix} \quad \text{(say)}$$
$$= \begin{pmatrix} r^2 \mathrm{e}^{-\mathrm{i}\delta} + t^2 & rt(1 - \mathrm{e}^{-\mathrm{i}\delta}) \\ rt(1 - \mathrm{e}^{-\mathrm{i}\delta}) & t^2 \mathrm{e}^{-\mathrm{i}\delta} + r^2 \end{pmatrix}. \tag{8.249b}$$

One can now check that the transformation coefficients $r_{\rm MZ}$, $t_{\rm MZ}$, $r'_{\rm MZ}$, and $t'_{\rm MZ}$ satisfy the energy conservation conditions (8.237) (recall that the coefficients r and t here are both real and positive)—that is, $B_{\rm MZ}$ is unitary, which ensures that the set of annihilation operators along with their creation operators $(a_i,\ a_i^\dagger\ (i = 1, \ldots, 8))$ satisfy the commutation relations for independent oscillator modes.

The matrix $B_{\rm MZ}$ gives the field transformations in the classical description as well. For instance, if the field amplitude A_2 is zero and the phase difference δ is also zero, then, for identical beam splitters, one finds $A_8 = 0$, in conformity with what we found in Section 4.7.3.2.

For completeness, let us work out the output states of the Mach-Zehnder interferometer for a one-photon input state to the arm marked '1' in Fig. 8.13. Thus with

$$|\Psi_{\text{in}}\rangle = |1\rangle \otimes |0\rangle = (a_1^\dagger \otimes I)(|0\rangle \otimes |0\rangle) \tag{8.250}$$

in an obvious notation (first factor for arm '1' and second factor for arm '2'), we find, on using the inverse of Eq. (8.249a) and taking the Hermitian adjoint,

$$|\Psi_{\text{out}}\rangle = (r_{\text{MZ}} a_7^\dagger \otimes I + t_{\text{MZ}} I \otimes a_8^\dagger)(|0\rangle \otimes |0\rangle) \tag{8.251a}$$

(first factor for arm '7' and second factor for arm '8'). For a pair of identical 50:50 beam splitters ($r = t = \frac{1}{\sqrt{2}}$), this reduces to

$$|\Psi_{\text{out}}\rangle = \frac{1}{2}\left((1 + \mathrm{e}^{-\mathrm{i}\delta})|1\rangle|0\rangle + (1 - \mathrm{e}^{-\mathrm{i}\delta})|0\rangle|1\rangle\right). \tag{8.251b}$$

As in the single beam splitter, this output state is again an entangled one involving the two output modes, where the vacuum state of either mode is involved. On calculating the partitioning of intensity between the two output arms, however, one finds the same result as in the classical interference theory (eg, a constructive interference in one of the two arms occurs along with a destructive interference in the other). Indeed, the phenomenon of interference relates to the first-order coherence characteristics of a field, where the classical and quantum results coincide, as we will see in Section 8.16.

8.12 Atom-Field Interaction

8.12.1 Matter and Radiation: A and B Coefficients

A number of fundamental features of the interaction of electromagnetic radiation with matter were made clear by Einstein from thermodynamic considerations, where the radiation was assumed to be in equilibrium with matter.

On examining the condition of thermodynamic equilibrium of the system composed of matter and radiation, and by referring to *Planck's formula* for the energy density of blackbody radiation, Einstein came to a number of important conclusions. It transpired that there are *three* basic processes involved in the absorption and emission of radiation by matter: stimulated absorption, spontaneous emission, and *stimulated emission*. The first of these three is common knowledge and occurs, for instance, when the atoms of a gas are excited to higher-energy states as it is heated to high temperatures. The second process is also commonly observed when the excited atoms return, in the absence of external sources of radiation, to their ground states by the emission of radiation.

On the face of it, an external radiation source does not appear to have any essential role in influencing the emission process. However, Einstein's analysis clearly indicated such a role. More precisely, an external electromagnetic field gives rise to *stimulated* emission, a process distinct from spontaneous emission.

Einstein introduced the now famous A and B coefficients to describe the rates of the three processes. In the context of the principal results arrived at by him, it will be convenient to refer to the basic processes at a microscopic level, looking at the interaction of the individual atoms with the radiation field. Considering two atomic energy levels and denoting the two corresponding states of the atom by the subscripts '1' and '2,' if N_2 denotes the number of atoms in the higher energy state ('2') and if this number changes by δN_2 because of de-excitation to state '1' by spontaneous emission in time δt, then the A coefficient (with reference to the two levels) is defined as

$$A = -\frac{1}{N_2}\frac{\delta N_2}{\delta t}. \tag{8.252}$$

A basic feature of the processes of stimulated absorption and stimulated emission is that these are in the nature of *resonant* ones, with appreciable probability of occurrence only when the atoms are made to interact with radiation of frequency $\omega = \omega_0$, where

$$\omega_0 = \frac{E_2 - E_1}{\hbar}, \tag{8.253}$$

with E_1 and E_2 denoting the energies of the two atomic states. Thus while blackbody radiation contains components of all frequencies, only the component with frequency ω_0 is involved in the transition processes.

Though the relation between the A and the B coefficients was derived by Einstein by considerations relating to radiation in equilibrium with a material body, the coefficients themselves are intrinsic properties of the constituents of the matter under consideration.

If $u(\omega_0)$ denotes the energy density (ie, energy per unit volume per unit frequency interval) of the radiation field at transition frequency ω_0, then the B coefficients (there are two of those, with reference to the two energy levels under consideration) are defined as

$$B_{12} = \frac{1}{u(\omega_0)N_1}\frac{\delta N_{12}}{\delta t}, \quad B_{21} = \frac{1}{u(\omega_0)N_2}\frac{\delta N_{21}}{\delta t}. \tag{8.254}$$

Here N_1 (analogous to N_2 introduced above) denotes the number of atoms in the ground state at any given time, δN_{12} denotes the number of transitions from the ground state to the excited state by absorption in time δt, and δN_{21} similarly denotes the number of transitions by

stimulated emission from the excited state to the ground state in time δt. The presence of $u(\omega_0)$ in the above formulae results from the assumption that the rates $\frac{\delta N_{12}}{\delta t}$ and $\frac{\delta N_{21}}{\delta t}$ are proportional to the energy density of radiation, analogous to the rates of a first-order chemical reaction.

With these definitions of the A and B coefficients, Einstein derived the following formulae relating the three coefficients to one another:

$$\frac{A}{B_{12}} = \frac{\hbar\omega^3}{\pi^2 c^3}, \quad B_{12} = B_{21}. \tag{8.255}$$

1. In writing the second of the relations in Eq. (8.255), one assumes the two atomic levels are *nondegenerate*. In the case of degenerate energy levels, the degeneracy factors are to be taken into account. In general, the ground state of an atom is nondegenerate.
2. In formulating the condition of equilibrium of matter in bulk in the presence of radiation, one considers a pair of atomic energy levels and writes out the condition for the populations of these two levels to be time invariant even as transitions occur between the two levels. If the condition of invariant population holds for each and every pair of energy levels of the atoms making up the bulk matter, then the equilibrium of the bulk matter in interaction with the radiation is guaranteed. This constitutes an instance of the principle of *detailed balance*. One thereby obtains a set of A and B coefficients for each pair of energy levels, satisfying (8.255).
3. Strictly speaking, the values of the B coefficients, as defined above, are conditional on the material body being in the form of a fluid, characterized by the property of *isotropy*, as we will see in Section 8.12.2.6. Subject to this condition, the B coefficients are properties intrinsic to the atomic constituents.
4. One other requirement implicit in the definition of the A and B coefficients and in Einstein's theory is that the interaction between the atoms and the radiation (in the case of the A coefficient, it is the *vacuum field* rather than the field of external radiation that is of relevance) is to be a sufficiently *weak* one. Indeed, the notion of a transition rate is not well defined in the case of a strong interaction between the atom and the radiation field. For a sufficiently weak interaction between the field and the atomic constituents of a material body, one speaks of a *linear* regime, to which the considerations relating to the A and B coefficients apply. In the case of the atomic constituents being subjected to a relatively strong field, one speaks of a *nonlinear* regime. I will give a brief outline of *nonlinear optics* in Chapter 9, where the response of the material body to an external field will be seen to involve novel features not to be found in the linear response regime.

In summary, Einstein established that stimulated emission is a fundamental process involved in the interaction of radiation with matter, introduced the A and B coefficients, and derived the relation between these coefficients. The more detailed theory of the atom-field interaction outlined in the following sections lends support to all these results obtained by Einstein.

8.12.2 The Atom Interacting With a Classical Electromagnetic Field

8.12.2.1 Atom-field interaction in semiclassical theory: Introduction

Having had an introduction to the quantum states of the free electromagnetic field in Section 8.6, we are now in a position to look at the interaction of the electromagnetic field with an atom. This gives us a number of fundamental results on the basis of which one can proceed to the consideration of interaction of radiation with matter at large. Even the interaction of radiation with a single atom has important consequences in numerous quantum optics setups, which is why this will be our main concern here.

In describing the processes resulting from the atom-field interaction, we will concentrate on the *transitions* between stationary states of the atom, occurring with the *absorption* or *emission* of radiation (recall from Section 8.12.1 that these are the basic processes with reference to which the A and B coefficients were defined). More precisely, with reference to the quantum description of the electromagnetic field, we will be concerned with processes involving the absorption and emission of a *single* photon.

As mentioned on several occasions earlier in this book, the interaction of matter with radiation can be considered at various levels of description. For instance, one can obtain a number of results on dispersion in material media by looking at the medium as a collection of classical dipole oscillators undergoing forced oscillations induced by the electromagnetic field, where the latter is also described classically. This classical theory also gives useful results in Rayleigh scattering by the atoms in a dilute gas or by density fluctuations in a fluid.

More detailed and complete information on these phenomena is obtained in a semiclassical (or semi*quantum*) description, where the electromagnetic field is described classically but the atoms and molecules are described in quantum terms. This semiclassical description leads to a reasonably complete theory of diverse phenomena such as dispersion, Rayleigh scattering, Raman scattering, and photoelectric emission. This theory has the added simplifying feature that the quantum description of the atom is confined to the limits of the *nonrelativistic* theory.

The basic outline of this semiclassical theory of atom-field interaction is given below, where concrete results can be worked out in the limit of *weak* interactions, giving a precise meaning to the notion of a *transition rate* (see Section 8.12.2.3). A solvable semiclassical model involves a two-level atom interacting with a near-resonant monochromatic field, where the results hold regardless of the strength of the interaction, and where one encounters the interesting phenomenon of *oscillations* between the two quantum states of the atom (see Section 8.12.2.5).

> These oscillations constitute the quantum counterpart of the dipolar oscillations one considers in the classical theory of dispersion and Rayleigh scattering.

Finally, one can refer to a description where *both* the atom and the field are described quantum mechanically. Once again, a model involving a two-level atom (see Section 8.12.3) turns out to be a simple and tractable one, leading to a number of meaningful results.

8.12.2.2 Classical field: The interaction Hamiltonian

Let the Hamiltonian operator describing the atom in isolation (ie, in the absence of the electromagnetic field) be $\hat{H}_{\text{atom}}$. The stationary states of the atom are the eigenstates (say, $|i\rangle,\ i = 1, 2, \ldots$) of this Hamiltonian, with the corresponding eigenvalues (E_i) being the energies of these atomic states. The total Hamiltonian with the atom interacting with the electromagnetic field, the latter described classically, has the form

$$\hat{H} = \hat{H}_{\text{atom}} - \hat{\mathbf{d}} \cdot \mathbf{E}, \tag{8.256a}$$

where a few words of explanation are in order.

> I do not attempt to derive this formula since the derivation involves a number of detailed considerations in electromagnetic theory which will not be needed in the subsequent developments.

In expression (8.256a), $\mathbf{E}$ stands for the electric field strength at the location of the atom, whose size is assumed to be small compared with the typical wavelengths of the plane wave modes of the radiation field relevant in the atomic processes under consideration (the *dipole approximation*). The electromagnetic field is described here in the *Coulomb gauge* (see Eq. 1.28) since, in the Coulomb gauge, the interaction can be effectively described in terms of the vector potential alone, assuming that the electromagnetic field, considered in isolation, is a free one. The operator $\hat{\mathbf{d}}$ stands for the electric dipole moment of the atom in the quantum theoretic formalism, defined as

$$\hat{\mathbf{d}} = \sum q_i \hat{\mathbf{r}}_i, \tag{8.256b}$$

where the summation is over all the charges ($q_1, q_2, \ldots$) in the atom, with $\hat{\mathbf{r}}_i$ denoting the position vector operator of the ith charge. The second term on the right-hand side of Eq. (8.256a) expresses the basic fact of the atom-field interaction relevant to the present context: the interaction arises from the coupling of the dipole moment of the atom to the electric field intensity of the radiation field.

> This is a consequence of a set of assumptions and approximations, not the least of which is the dipole approximation. I omit the details of these.

One can further simplify the theory by assuming that the transitions are between atomic states that can be interpreted as *single-electron* ones, where $\hat{H}_{\text{atom}}$ reduces to the Hamiltonian of a single electron in an effective field due to the nucleus and the remaining electrons, and $\hat{\mathbf{d}}$ reduces to

$$\hat{\mathbf{d}} = -e\hat{\mathbf{r}}, \tag{8.256c}$$

with $-e$ standing for the electronic charge.

8.12.2.3 Weak interaction: Results of perturbation theory

In the case of an atom interacting with a weak electromagnetic field, the consequences of the interaction can be worked out in the framework of quantum mechanical *perturbation theory*, where the atom is assumed to start from an initial unperturbed state $|i\rangle$ and the probability of transition to some other unperturbed state, say, $|f\rangle$, caused by the interaction is worked out in the form of a series. Assuming that the field is a plane monochromatic one, given by the expression

$$\mathbf{E} = \mathbf{E}_0 \cos(\omega t), \tag{8.257}$$

at the location of the atom, with a sufficiently small amplitude $|\mathbf{E}_0|$, the first term of the series gives the leading approximation to the transition probability. The succeeding terms will be ignored as being in the nature of small corrections to the leading-order result. In this leading order, the probability of the atom continuing to stay in the initial state $|i\rangle$ is nearly unity, with the probability of a transition (in a given time t) being relatively small. In accordance with the results of time-dependent perturbation theory, this transition probability is the squared modulus of the so-called transition *amplitude* given (up to an overall phase factor) by

$$A_{\mathrm{i}\to\mathrm{f}}(t) = \frac{1}{2\hbar}\left[\frac{\mathrm{e}^{\mathrm{i}(\omega+\omega_0)t}-1}{\omega+\omega_0} - \frac{\mathrm{e}^{-\mathrm{i}(\omega-\omega_0)t}-1}{\omega-\omega_0}\right]\langle f|\hat{\mathbf{d}}\cdot\mathbf{E}_0|i\rangle, \tag{8.258a}$$

where the last factor on the right-hand side is the matrix element between the initial and the final states of the operator $\hat{\mathbf{d}}\cdot\mathbf{E}_0$ (recall that $\mathbf{E}_0$, unlike $\hat{\mathbf{d}}$, is just a vector whose components have specified numerical values—*c-numbers* in the jargon of quantum theory). In formula (8.258a), ω_0 is given by

$$\omega_0 = \frac{E_\mathrm{f} - E_\mathrm{i}}{\hbar}, \tag{8.258b}$$

being the difference of the energies of the final and initial atomic states in units of $\hbar$ (this can be interpreted as the angular frequency of a photon of energy $E_\mathrm{f} - E_\mathrm{i}$; however, the transition is not explicitly associated with the emission or the absorption of a photon in the present semiclassical context since we have assumed that the electromagnetic field is a classical one).

One observes that the transition amplitude is of an appreciable magnitude only when either of the denominators is small (ie, the process under consideration is in the nature of a *resonant* one) since, in the case of a weak interaction, the matrix element $\langle f|\hat{\mathbf{d}}\cdot\mathbf{E}_0|i\rangle$ is small. With ω assumed to be positive, this can occur in one of two ways: (1) resonant *absorption*, $E_\mathrm{f} > E_\mathrm{i}$, $\omega \approx \omega_0$ (here the atom absorbs energy from the field in making a transition from an initial state of lower energy to a final state of relatively higher energy); (b) resonant *emission*, $E_\mathrm{f} < E_\mathrm{i}$, $\omega \approx -\omega_0$ (here the atom sheds energy in the presence of the electromagnetic field to make a transition to a final state of lower energy compared with the initial state). For either of the processes, only one of the two terms within the brackets in Eq. (8.258a) needs be retained (the so-called *rotating wave approximation*), whereby one obtains, for any given small *detuning*

$$\Omega \equiv \omega - |\omega_0|\,, \tag{8.259}$$

the transition *probability* (the modulus squared of the transition amplitude) in time t as

$$P_{\mathrm{i}\to\mathrm{f}}(t) = |A_{\mathrm{i}\to\mathrm{f}}(t)|^{\,2} = \frac{\left|\langle f|\hat{\mathbf{d}}\cdot\mathbf{E}_0|i\rangle\right|^{\,2}}{\hbar^2}\frac{\sin^2\frac{\Omega t}{2}}{\Omega^2}. \tag{8.260}$$

A quantity of greater interest is the *transition rate*, which can be defined from this expression by noting that, for large t, the factor $\frac{\sin^2\frac{\Omega t}{2}}{\Omega^2}$ is a sharply peaked function of the detuning Ω, and can be expressed as $\beta\delta(\Omega)$, where β is a coefficient that works out to $\frac{\pi}{2}t$.

One can see this by evaluating (with arbitrarily specified $\delta > 0$)

$$\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \frac{\sin^2\frac{\Omega t}{2}}{\Omega^2}d\Omega,$$

which for large t can be approximated as

$$\frac{t}{2}\int_{-\infty}^{\infty}\frac{\sin^2 u}{u^2}du = \frac{\pi t}{2}.$$

Before we proceed further, a word is in order regarding the proportionality of the transition probability with the time interval t. This, of course, is a result that cannot be taken without qualification since, with increasing time, $P_{\mathrm{i}\to\mathrm{f}}(t)$ diverges, which goes against the fundamental assumption of a small transition probability on which the perturbative theory rests. In reality, with $\Omega \to 0$, the above result holds only for t sufficiently large, but still small compared with $\frac{1}{\Omega}$. For larger values of t the perturbation approximation does not work, and the transition probability exhibits an *oscillatory behavior* instead of increasing unboundedly, as we will see in Section 8.12.2.5, where we take up a *nonperturbative* analysis of the atom-field interaction for a two-level atom under the rotating wave approximation.

Thus for sufficiently small detuning, the transition probability in the perturbative regime effectively varies linearly with t for large t, and one can define a transition *rate*

$$W_{\mathrm{i}\to\mathrm{f}} = \frac{1}{t}P_{\mathrm{i}\to\mathrm{f}}, \tag{8.261}$$

which for sharply defined atomic energy levels and a strictly monochromatic radiation field is given by

$$W_{\mathrm{i}\to\mathrm{f}} = \frac{\pi}{2\hbar^2}\left|\langle f|\hat{\mathbf{d}}\cdot\mathbf{E}_0|i\rangle\right|^{\,2}\delta(\Omega). \tag{8.262}$$

This expression, however, is to be modified a bit before it can be applied to a concrete problem, since the radiation inducing the transition cannot, in reality, be strictly monochromatic.

In addition, the atomic energy levels do not correspond to sharply defined energies but have a certain width because of the spontaneous emission process and other line broadening mechanisms. This will be included later in our considerations.

Formula (8.262) appears as the contribution of just a single *mode* of the electromagnetic field to the transition probability, where a mode corresponds to a monochromatic plane wave in a state of linear polarization (other states of polarization, such as circular polarization, can also be considered). If there are several such modes contributing to the transition probability, and if all these modes are uncorrelated to each other (ie, the radiation is an incoherent mixture of the modes), one has to sum up the contributions of all the modes, where the modes are, in general, spread over a certain range of frequencies. The summation then amounts to multiplication of expression (8.262) by the *density of modes* (ie, the number of modes per unit frequency interval) corresponding to a variable frequency ω followed by integration over ω (for a coherent superposition, on the other hand, one has to integrate over the transition amplitude first and then work out the resultant transition probability).

In other words, a more appropriate formula for the transition rate is

$$W_{\mathrm{i}\to\mathrm{f}} = \frac{\pi}{2\hbar^2}\int \left|\langle f|\hat{\mathbf{d}}\cdot\mathbf{E}_0|i\rangle\right|^2 \rho(\omega)\delta(\omega-\omega_0), \tag{8.263}$$

where $\rho(\omega)$ stands for the density of modes at frequency ω, and the integration is over a range of frequencies, centered around ω_0, making up the spectrum of the radiation inducing the transition, and where the range of frequencies in question is necessarily small so that the rotating wave approximation can be employed for each of the modes.

In writing formula (8.263), we have assumed for concreteness an absorption process, where $E_\mathrm{f} > E_\mathrm{i}$. It can be employed for an emission process ($\omega_0 < 0$) by replacement of ω_0 with $-\omega_0$ in the argument of the delta function.

An alternative form of formula (8.263) uses the *energy density* of the radiation instead of the density of modes. The squared matrix element for any one mode in this formula can be written as

$$\left|\langle f|\hat{\mathbf{d}}\cdot\mathbf{E}_0|i\rangle\right|^2 = \left|\langle f|\hat{\mathbf{d}}\cdot\hat{e}_0|i\rangle\right|^2 E_0^2, \tag{8.264a}$$

where $\hat{e}_0$ denotes the unit vector along the direction of polarization of the mode under consideration, and E_0 is the amplitude of the electric field vector (in general, a function of the frequency ω). The energy density for such a mode is given by $\frac{1}{2}\epsilon_0 E_0^2$, which means that the product $\frac{1}{2}\epsilon_0 E_0^2\rho(\omega)$ stands for the energy density per unit frequency interval of the radiation causing the atomic transition,

$$u(\omega) = \frac{1}{2}\epsilon_0 E_0(\omega)^2\rho(\omega), \tag{8.264b}$$

and that the formula for the transition rate can be written in the form

$$W_{\mathrm{i}\to\mathrm{f}} = \frac{\pi}{\epsilon_0\hbar^2}\int \left|\langle f|\hat{\mathbf{d}}\cdot\hat{e}_0|i\rangle\right|^2 u(\omega)\delta(\omega-\omega_0). \tag{8.265}$$

Finally, we assume the density of states $\rho(\omega)$ and the energy density $u(\omega)$ to be smoothly varying functions in the vicinity of $\omega = \omega_0$, and integrate out the delta function in Eqs. (8.263) and (8.265), so as to obtain

$$W_{\mathrm{i}\to\mathrm{f}} = \frac{\pi}{2\hbar^2}\left|\langle f|\hat{\mathbf{d}}\cdot\mathbf{E}_0|i\rangle\right|^2\rho(\omega_0) \tag{8.266a}$$

or, alternatively,

$$W_{\mathrm{i}\to\mathrm{f}} = \frac{\pi}{\epsilon_0\hbar^2}\left|\langle f|\hat{\mathbf{d}}\cdot\hat{e}_0|i\rangle\right|^2 u(\omega_0). \tag{8.266b}$$

It is this last expression for the transition rate that features in the derivation of Einstein's B coefficients. I will outline this in Section 8.12.2.6.

Referring to formula (8.265), one notes that, in the absence of the field ($u = 0$), the transition rate drops to zero. In other words, neither absorption nor emission of radiation can occur in the absence of the electromagnetic field, which is only to be expected within the confines of the semiclassical theory. In the fully quantum theoretic treatment, however, *spontaneous emission* occurs even when the photon number in the initial state of the field is zero.

In the present approximate theory of the atom-field interaction, the atom is assumed to be coupled to the electromagnetic field only through its electric dipole moment $\hat{\mathbf{d}}$. For certain pairs of initial and final states $|i\rangle$ and $|f\rangle$, the matrix element of the dipole moment (or its component along $\mathbf{E}_0$) may vanish. Transitions may, however, occur even between such states by means of the *magnetic* coupling with the field, though with a much lower transition rate.

Formulae (8.266a), (8.266b) constitute the main result of the perturbative theory of the atom-field interaction. In what follows we first look at a nonperturbative treatment of the atom-field interaction where the field is still described classically, while the atom is assumed to have just two stationary states (the 'two-level atom'). The theory will then be improved upon by making use of a quantum description of the field. As we will see, at each of these successive stages of theory building, new features of the absorption and emission processes emerge.

Incidentally, it is of quite considerable practical relevance to develop a formula for the transition rate from any given initial state $|i\rangle$ to a *range of final states* $|f\rangle$, for which we write, from (8.262),

$$W_{\mathrm{i}\to\{\mathrm{f}\}} \approx \frac{\pi}{2\hbar^2}\sum \left|\langle f|\hat{\mathbf{d}}\cdot\mathbf{E}_0|i\rangle\right|^2\delta(\omega-\omega_0), \tag{8.267}$$

where we continue to consider, for the sake of concreteness, the process of stimulated emission. In this expression the summation is over the range of final states $\{|f\rangle\}$ and also over the modes that are relevant for the transitions corresponding to a range of Bohr frequencies

ω_0. The summation, moreover, is to be replaced with an integration involving the density of final atomic states and the density of modes when these final states and the relevant modes are distributed continuously.

8.12.2.4 Digression: Fermi's golden rule

A more general form of the expression for the transition rate, which can be used in perturbation theoretic expressions for arbitrarily specified time-dependent interactions, can be arrived at by noting that the effective interaction Hamiltonian in the present dipolar interaction problem is (refer to equations 8.256a, 8.257)

$$\hat{H}_{\text{int}} = -\frac{1}{2}\hat{\mathbf{d}} \cdot \mathbf{E}_0 \mathrm{e}^{-\mathrm{i}\omega t} \tag{8.268}$$

(recall the rotating wave approximation that we have invoked for $\omega \approx \omega_0$). Referring to formula (8.262), one finds that this leads to

$$W_{\mathrm{i}\to\{\mathrm{f}\}} \approx \frac{2\pi}{\hbar^2} \sum \left| \langle f|\hat{H}_{\text{int}}|i\rangle \right|^2 \delta(\omega - \omega_0), \tag{8.269a}$$

where we have generalized from (8.262) in including a summation over final states $|f\rangle$ of the system undergoing the transition. In this expression, ω characterizes the harmonic variation of the interaction Hamiltonian and ω_0 is the transition frequency corresponding to a typical final state $|f\rangle$, for a given initial state $|i\rangle$.

While this is the form in which *Fermi's golden rule* appears in the context of transitions in a system in a weak harmonic field, the rule itself is broader in scope than the above generalization suggests. Instead of focusing on the transitions in a quantum system in a harmonically varying classical potential, one can more generally consider any weak interaction problem involving two given systems. The perturbation theoretic expression for the transition rate is then given by formula (8.269a) with the delta function now being replaced with $\delta(\omega_{\mathrm{f}} - \omega_{\mathrm{i}})$, where $|i\rangle$ and $|f\rangle$ represent joint initial and final states of the interacting systems having energies $\hbar\omega_{\mathrm{i}}$ and $\hbar\omega_{\mathrm{f}}$. Thus this more general form of the golden rule formula is

$$W_{\mathrm{i}\to\{\mathrm{f}\}} \approx \frac{2\pi}{\hbar^2} \sum \left| \langle f|\hat{H}_{\text{int}}|i\rangle \right|^2 \delta(\omega_{\mathrm{f}} - \omega_{\mathrm{i}}). \tag{8.269b}$$

The golden rule formula (8.269b) is commonly employed in the *interaction picture* description of quantum dynamics.

8.12.2.5 Rabi oscillations: Two-level atom in a classical field

The theory outlined in Section 8.12.2.3 essentially relates to transitions between two given states ($|i\rangle$, $|f\rangle$) of an atom, though one can use it in working out the transition rate between any given combinations of initial and final states. One can, from the very beginning, consider the particular case of a two-level atom and then, taking into account its interaction with a monochromatic field, attempt an *exact* solution of the Schrödinger equation, giving the wave

function of the atom at any given instant of time. Such an exact solution can indeed be obtained under the rotating wave approximation mentioned in Section 8.12.2.3.

The two-level atom is characterized by a pair of orthonormal states, say, $|1\rangle$ and $|2\rangle$, with energies E_1 and E_2 ($E_1 < E_2$), where $|1\rangle$ is referred to as the 'ground state' and $|2\rangle$ as the 'excited state.' These are eigenstates of the free Hamiltonian $\hat{H}_{\text{atom}}$ describing the atom in the absence of the electromagnetic field.

> More precisely, $|1\rangle$ and $|2\rangle$ denote the ground state and the excited state at a fixed time, say, $t = 0$. In the Schrödinger picture (one of a number of possible schemes for describing the time evolution of states and observables in quantum theory) the states at any other time t are given by $\mathrm{e}^{-\mathrm{i}\frac{E_1 t}{\hbar}}|1\rangle$ and $\mathrm{e}^{-\mathrm{i}\frac{E_2 t}{\hbar}}|2\rangle$—that is, the states remain the same (stationary states), while acquiring time-dependent phase factors.

The atom is now assumed to be acted on by an electromagnetic field in the form of a monochromatic plane wave. As a result of the interaction, the expression for the state at time t assumes the form

$$|\psi(t)\rangle = c_1(t)\mathrm{e}^{-\mathrm{i}\frac{E_1 t}{\hbar}}|1\rangle + c_2(t)\mathrm{e}^{-\mathrm{i}\frac{E_2 t}{\hbar}}|2\rangle, \tag{8.270}$$

where $c_1(t)$ and $c_2(t)$ are time-dependent coefficients to be determined, depending on the Hamiltonian $\hat{H}$, which involves, apart from the free term $\hat{H}_{\text{atom}}$, the interaction term as well (see Eqs. 8.256a, 8.256c, and 8.257):

$$\begin{aligned} \hat{H} &= \hat{H}_{\text{atom}} - \hat{\mathbf{d}} \cdot \mathbf{E}_0 \cos(\omega t), \\ &= \hat{H}_{\text{atom}} - \hat{\mathbf{d}} \cdot \hat{e}_0 E_0 \cos(\omega t) \quad \text{(say)}. \end{aligned} \tag{8.271}$$

For simplicity, we assume here that the electric field is linearly polarized with the electric vector oscillating along the unit vector $\hat{e}_0$. Thus $\mathbf{E} = \hat{e}_0 E_0 \cos(\omega t)$, where E_0 is the real amplitude of the electric field vector.

> The caret over a symbol has been used here to denote a unit vector and also to denote an operator. This need not cause confusion if one remains aware of the context.

The principle underlying the determination of $c_1(t)$ and $c_2(t)$ is to require that $|\psi(t)\rangle$ satisfies the *Schrödinger equation*:

$$\mathrm{i}\hbar\frac{d|\psi\rangle}{dt} = \hat{H}|\psi\rangle. \tag{8.272}$$

> In the absence of interactions, expression (8.270) satisfies the Schrödinger equation for arbitrarily chosen constant (ie, time-independent) values of the coefficients c_i ($i = 1, 2$).

Inserting expression (8.270) in Eq. (8.272), one arrives at the following pair of coupled first-order differential equations for $c_1(t)$ and $c_2(t)$ which one has to solve subject to given initial conditions:

$$\dot{c}_1 = -\frac{\mathrm{i}}{\hbar} E_0 d_{12} \cos(\omega t) \mathrm{e}^{-\mathrm{i}\omega_0 t} c_2, \tag{8.273a}$$

$$\dot{c}_2 = -\frac{i}{\hbar} E_0 d_{21} \cos(\omega t) \mathrm{e}^{\mathrm{i}\omega_0 t} c_1. \tag{8.273b}$$

Here ω_0 is given by

$$\omega_0 = \frac{E_2 - E_1}{\hbar}, \tag{8.273c}$$

and d_{12}, d_{21} stand for the matrix elements (see Eq. 8.256c)

$$d_{12} = \langle 1| - \hat{\mathbf{d}} \cdot \hat{e}_0 |2\rangle, \quad d_{21} = \langle 2| - \hat{\mathbf{d}} \cdot \hat{e}_0 |1\rangle. \tag{8.273d}$$

Since d_{12} and d_{21} are matrix elements of a Hermitian operator, they satisfy

$$d_{12} = d_{21}^* = p = |p| e^{i\phi} \quad \text{(say)}, \tag{8.274}$$

where ϕ is the phase of p.

1. The matrix element d_{12} $(= d_{21}^*)$ depends on the nature of the transition (ie, by the *selection rule* obeyed by it). In the case of a $\Delta m = 0$ transition (ie, one for which the magnetic quantum number characterizing the atomic state remains unaltered), one can choose it to be real, with the unit vector $\hat{e}_0$ corresponding to a state of linear polarization. In the case of a $\Delta m = \pm 1$ transition, on the other hand, the matrix elements assume a simple form when $\hat{e}_0$ is taken to be a complex unit vector, corresponding to a state of *circular polarization*. The conclusions, though, remain the same.
2. The evolving atomic state is here assumed to be a pure one for simplicity. More generally, the state of the atom interacting with the field is to be assumed to be a *mixed* one, represented in terms of a 2×2 *density matrix*. The latter is described completely by means of three real parameters. The equations describing the time evolution of these three parameters are referred to as *Bloch equations*. While our considerations here are confined to the case of an evolving pure state, similar conclusions are arrived at from the more general Bloch equations.

In arriving at Eqs. (8.273a) and (8.273b), we used the fact that the matrix elements $\langle 1| - \hat{\mathbf{d}} \cdot \hat{e}_0 |1\rangle$ and $\langle 2| - \hat{\mathbf{d}} \cdot \hat{e}_0 |2\rangle$ are zero, which follows from the parity properties of the integrands in the integral expressions for these matrix elements (check this out).

We now make the *rotating wave approximation* in Eqs. (8.273a) and (8.273b), where we note that the time variation of $\cos(\omega t)\mathrm{e}^{\pm \mathrm{i}\omega_0 t}$ involves two frequencies—namely, $\omega + \omega_0$ and $\omega - \omega_0$. For a small detuning $\Omega \equiv \omega - \omega_0$, these correspond to *fast* and *slow* variations (in the case of absorption, which we consider here for the sake of concreteness), respectively, of which the former average out to zero over sufficiently large intervals of time, and can be left out of consideration. Here the terms 'fast' and 'slow' are used with reference to the field frequency ω and to the *Rabi frequency* (see below), which represents the frequency of variation of the transition probabilities $|c_1|^2$ and $|c_2|^2$ under zero detuning.

On making the above approximation, and on defining the Rabi frequency

$$\Omega_{\mathrm{R}} = \frac{|p|E_0}{\hbar}, \tag{8.275}$$

one writes Eqs. (8.273a) and (8.273b) in the form

$$\dot{c}_1 = -\frac{\mathrm{i}\Omega_{\mathrm{R}}}{2} e^{i(\Omega t+\phi)} c_2, \quad \dot{c}_2 = -\frac{\mathrm{i}\Omega_{\mathrm{R}}}{2} e^{-i(\Omega t+\phi)} c_1 \tag{8.276a}$$

or, equivalently,

$$\ddot{c}_1 - \mathrm{i}\Omega\dot{c}_1 + \frac{\Omega_{\mathrm{R}}^2}{4} c_1 = 0, \quad \ddot{c}_2 + \mathrm{i}\Omega\dot{c}_2 + \frac{\Omega_{\mathrm{R}}^2}{4} c_2 = 0. \tag{8.276b}$$

We consider, for concreteness, the initial conditions

$$|c_1|(t=0) = 1, \quad c_2(t=0) = 0, \tag{8.277}$$

which corresponds to the atom initially in the ground state, wherefrom the probability for transition to the excited state can now be worked out. As we see below, the variations of $c_1(t)$ and $c_2(t)$ are oscillatory in nature, which is a characteristic feature of state variation of the atom for more general initial conditions as well. Subject to Eq. (8.277), the solution to Eqs. (8.276b) reads

$$\begin{aligned} c_1(t) &= e^{\frac{\mathrm{i}}{2}(\Omega t+\phi)} \left(\cos\Omega' t - \frac{\mathrm{i}\Omega}{2\Omega'} \sin\Omega' t \right), \\ c_2(t) &= -\mathrm{i}\frac{\Omega_{\mathrm{R}}}{2\Omega'} e^{-\frac{\mathrm{i}}{2}(\Omega t+\phi)} \sin\Omega' t, \end{aligned} \tag{8.278a}$$

where

$$\Omega' \equiv \frac{1}{2}\sqrt{\Omega^2 + \Omega_{\mathrm{R}}^2}. \tag{8.278b}$$

Having obtained the time variation of the coefficients $c_1(t)$ and $c_2(t)$, one can, by referring to the initial conditions (8.277), identify the transition probability $P_{1\to 2}(t)$ for transition from the ground state to the excited state as

$$P_{1\to 2}(t) = |c_2(t)|^2 = \frac{\Omega_{\mathrm{R}}^2}{8\Omega'^2}(1 - \cos 2\Omega' t), \tag{8.279}$$

which varies sinusoidally with frequency $2\Omega' = \sqrt{\Omega^2 + \Omega_{\mathrm{R}}^2}$. This reduces to Ω_{R}, the Rabi frequency, at resonance ($\Omega = 0$). A similar oscillatory time variation characterizes $|c_1(t)|^2$, the probability of the atom remaining in the ground state at time t.

I give here a list of the relevant (angular) frequencies appearing in the problem under consideration, for your ready reference: (a) ω is the frequency of the electromagnetic field causing the transitions, (b) ω_0 is the Bohr frequency pertaining to the two atomic levels,

(c) $\Omega = \omega - \omega_0$ is the difference between the two, referred to as the *detuning*, (d) Ω_R is the *Rabi frequency* (Eq. 8.275), being the frequency of oscillation of $|c_1|^2$ and $|c_2|^2$ under exact resonance (zero detuning; at times the Rabi frequency is defined as $\frac{|p|E_0}{2\hbar}$, ie, half the expression on the right-hand side of Eq. (8.275)), and (e) and Ω' is half the oscillation frequency of $|c_1|^2$ and $|c_2|^2$ in the presence of a small nonzero detuning.

Recall that we have been considering here the processes of field-induced absorption and emission where the field is described classically and the coupling (under the dipole approximation) is *not necessarily a weak one*. Moreover, the processes are assumed to be *resonant* in that the detuning Ω is small, and we have assumed the rotating wave approximation holds ($\Omega \ll \omega$). Implied in our derivation is the further assumption that the Rabi frequency is small compared with the field frequency ($\Omega_R \ll \omega$).

Fig. 8.14 depicts schematically the nature of the variation of the transition probability $P_{1\to 2}(t)$ as a function of time for two different values of the detuning Ω. In the case of exact resonance ($\Omega = 0$), the transition probability oscillates between the values 0 and 1 with frequency Ω_R, the Rabi frequency for the two levels under consideration. This means that if instead of a

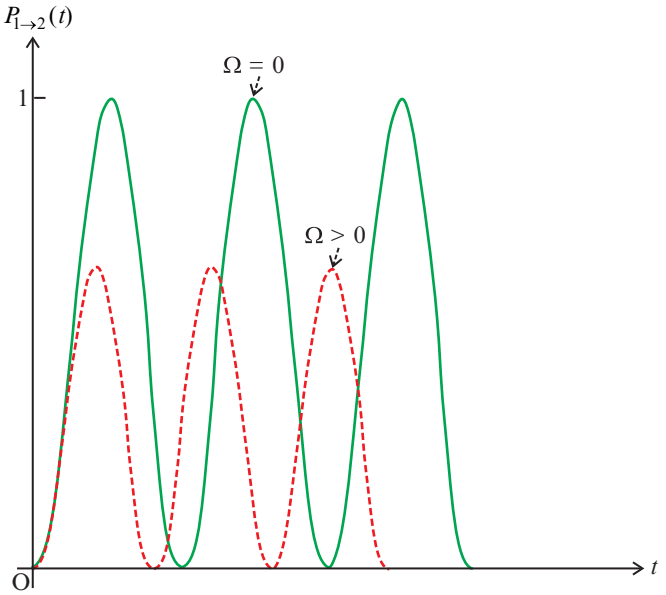

Fig. 8.14

Depicting schematically the variation of $P_{1\to 2}$ with time with an arbitrary strength of coupling (characterized by E_0). A two-level atom is considered, interacting with a monochromatic field in the dipole approximation. In a near-resonance situation where the detuning Ω is small, the rotating wave approximation can be employed. For exact resonance ($\Omega = 0$) the transition probability oscillates between 0 and 1 with frequency Ω_R. In the presence of detuning the amplitude of oscillation is less than unity.

single atom we consider a population of N atoms, all initially in the ground state, then the populations in the two levels will oscillate periodically, with all the atoms flipping over to the excited state in a time $\tau_R = \frac{\pi}{\Omega_R}$ (with this complete population inversion being repeated subsequently with a time period $T_R = \frac{2\pi}{\Omega_R}$). In the presence of a small detuning ($\Omega > 0$), on the other hand, the transition probability oscillates with a frequency $2\Omega'$ and with an amplitude less than unity, and complete population inversion is never achieved.

The perturbative result of the weak coupling limit (Eq. 8.260) is recovered, as expected, from the results derived in the present section (Eqs. 8.278a and 8.279) by going over to the limit of small E_0, when the oscillation frequency ($2\Omega'$) of $|c_1|^2$ and $|c_2|^2$ reduces to Ω (check this out).

> As we saw in Section 8.12.2.3, the singular nature of the transition probability in the weak interaction limit effectively leads to an expression linear in t. In the exact theory, however, the singularity is tempered because of the nonzero value of Ω_R in the expression for Ω', and the transition probability oscillates with time.

8.12.2.6 Induced transition: Einstein's B coefficients

In the weak coupling limit one can work out from the theory outlined above the values of the B coefficients and hence the value of the A coefficient as well (recall that the A coefficient and the two B coefficients are related as in Eq. 8.255).

For this, I refer you to formula (8.265) and to the fact that the B coefficient (we specifically consider here the coefficient B_{12}, ie, the one involved in stimulated absorption; the two B coefficients are equal for nondegenerate initial and final states) is defined in Einstein's theory for a gas (or a fluid) made up of the atoms in which the transitions are induced by the external field. Since the dipole moments of the atoms in a gas are oriented at random in all possible directions, one has to perform an averaging with respect to the dipolar orientations. Referring to a linearly polarized parallel beam of narrowband incoherent radiation, for which $\hat{e}_0$ is a fixed direction in space, one can write

$$|\langle f| - \hat{\mathbf{d}} \cdot \hat{e}_0 |i\rangle|^2_{\text{av}} = |\mathbf{p}|^2 (\cos^2\theta)_{\text{av}}, \tag{8.280a}$$

where

$$\mathbf{p} = \langle f|\hat{\mathbf{d}}|i\rangle \tag{8.280b}$$

is the vector dipole moment matrix element between the initial and final atomic states and θ stands for the angle between the vector $\mathbf{p}$ and $\hat{e}_0$.

Since the average value of $\cos^2\theta$ is $\frac{1}{3}$ (reason out why), one finally obtains

$$W_{\text{i}\to\text{f}} = \frac{\pi p^2}{3\epsilon_0 \hbar^2} u(\omega_0), \tag{8.280c}$$

where p stands for the magnitude of the dipole moment matrix element and the energy density $u(\omega)$ of the radiation field is evaluated at $\omega = \omega_0$, the transition frequency.

Since the transition rate is, by definition, the transition probability per unit time, one can now compare Eq. (8.280c) with Eq. (8.254) and write

$$B = \frac{\pi p^2}{3\epsilon_0 \hbar^2}, \tag{8.281}$$

where this value refers to B_{12} in the notation of Section 8.12.1. Assuming the initial and final atomic levels to be nondegenerate, and using the Einstein relations (8.255), one finally obtains

$$B_{12} = B_{21} = \frac{\pi p^2}{3\epsilon_0 \hbar^2}, \quad A = \frac{\omega^3 p^2}{3\pi\epsilon_0 \hbar c^3}. \tag{8.282}$$

Thus while Einstein's theory gave only the ratios between the coefficients, the more detailed semiclassical theory (where the atom is described quantum mechanically, while the electromagnetic field is described classically) gives the values of the coefficients themselves. However, in this theory, only the B coefficients are obtained independently (the coefficient B_{21} is worked out in a manner analogous to the derivation of B_{12}), while the A coefficient is inferred from the first relation in Eq. (8.255).

The semiclassical theory cannot give the value of the A coefficient independently since it cannot account for spontaneous emission. The latter can be explained only when the electromagnetic field is also described quantum mechanically. The subject of spontaneous emission will be taken up briefly in Section 8.12.4. Incidentally, in the fully quantum theory of the atom-field interaction where the electromagnetic field is also treated quantum mechanically, the B coefficients can be evaluated in a consistent manner as well, where one finds that these are given by the same expressions as found above. I will outline this more complete quantum theory of the A and B coefficients in Section 8.12.4.

8.12.3 The Atom and the Quantized Field: The Jaynes-Cummings Model

8.12.3.1 The Hamiltonian

We now undertake the task of describing the interaction of an atom, considered as a quantum mechanical object, with an electromagnetic field, where the latter is also regarded as a quantum system. In this, however, we make a number of approximations relevant in the context of a particular type of process—namely, one where the field is confined in a 1D cavity and where, among the possible eigenmodes in the cavity, only one particular eigenmode of frequency, say, ω is effective in causing transitions between a certain pair of atomic states with

energies, say, E_a and E_b. Of the two atomic states, one (with energy E_a) is commonly the ground state ($|a\rangle$) of the atom, while the other ($|b\rangle$, energy $E_b (> E_a)$) is an excited state, commonly the first excited state.

In common with the theory outlined in Sections 8.12.2.3 and 8.12.2.5, our approach will be to adopt the dipolar approximation, which assumes, among other things, that the wavelength of the relevant mode is large compared with the size of the atom. The Hamiltonian of the system made up of the atom and the radiation is of the form

$$\hat{H} = \hat{H}_{\text{atom}} + \hat{H}_{\text{radiation}} + \hat{H}_{\text{interaction}}, \tag{8.283}$$

where the first two terms represent the Hamiltonians of the atom and the radiation considered independently of each other, and the third term results from the interaction of the atom and the radiation. More precisely (see Eq. 8.256a),

$$\hat{H} = \frac{1}{2}\hbar\omega_0(|b\rangle\langle b| - |a\rangle\langle a|) + \hbar\omega\hat{a}^\dagger\hat{a} - \hat{\mathbf{d}}\cdot\hat{\mathbf{E}}. \tag{8.284}$$

This requires explanation. The first pair of terms stands for the Hamiltonian of the two-level atom considered in isolation from the radiation, where the zero of the energy scale is chosen to lie midway between the energies E_a and E_b and where

$$\omega_0 = \frac{E_b - E_a}{\hbar}, \tag{8.285a}$$

so that, on the resulting energy scale, the two energy levels are

$$E_a = -\frac{1}{2}\hbar\omega_0, \quad E_b = \frac{1}{2}\hbar\omega_0. \tag{8.285b}$$

The form of the atomic Hamiltonian chosen above ensures that $|a\rangle$ and $|b\rangle$ are indeed the eigenstates of $\hat{H}_{\text{atom}}$:

$$\hat{H}_{\text{atom}}|a\rangle = -\frac{1}{2}\hbar\omega_0|a\rangle, \quad H_{\text{atom}}|b\rangle = \frac{1}{2}\hbar\omega_0|b\rangle. \tag{8.285c}$$

> Though the first pair of terms in Eq. (8.284) formally represents the Hamiltonian of the free atom, its actual significance in the atom-field system is different since the time evolution of the operators is altered because of the interaction.

The second term ($\hbar\omega\hat{a}^\dagger\hat{a}$) in Eq. (8.284) is the Hamiltonian of the harmonic oscillator representing the single effective eigenmode of frequency ω in the cavity field, with $\hat{a}^\dagger$ and $\hat{a}$ as the associated creation and annihilation operators, where a constant term $\frac{1}{2}\hbar\omega$ has been dropped because it is of no relevance in the present context.

Finally, the interaction Hamiltonian is of the same form as that in Eq. (8.256a), now written in the quantum context, where $\hat{\mathbf{d}}$ stands for the atomic dipole operator as before, while the electric field vector also appears as an operator, being related to the creation and annihilation operators as

$$\hat{\mathbf{E}} = \mathrm{i}\sqrt{\frac{\hbar\omega}{\epsilon_0 V}}\sin(kz_0)\hat{e}_0\left(\hat{a} - \hat{a}^\dagger\right). \tag{8.286}$$

This expression derives from formula (8.166b), but is written in the Schrödinger picture instead of in the Heisenberg picture. We assume that the cavity field is linearly polarized along the unit vector $\hat{e}_0$ in the x-y plane and the atom is located at the point z_0 along the cavity axis. As explained in Section 8.12.2.5, the atomic operator $-\hat{\mathbf{d}}\cdot\hat{e}_0$ has only the off-diagonal matrix elements $\langle a|-\hat{\mathbf{d}}\cdot\hat{e}_0|b\rangle = \langle b|-\hat{\mathbf{d}}\cdot\hat{e}_0|a\rangle = p$, where p is be assumed to be real, which implies that the operator can be written as

$$-\hat{\mathbf{d}}\cdot\hat{e}_0 = p(|a\rangle\langle b| + |b\rangle\langle a|) \tag{8.287}$$

(check this out; a complex value of p leads to essentially the same results by a redefinition of the phase of $|a\rangle$ or $|b\rangle$, in which case one has to replace p with $|p|$; in other words, it is $|p|$ that is of physical relevance). At this point we introduce the following atomic operators:

$$\begin{aligned}&\sigma_+ = |a\rangle\langle b|, \quad \sigma_- = |b\rangle\langle a|,\\ &\sigma_1 = (\sigma_+ + \sigma_-), \quad \sigma_2 = -\mathrm{i}(\sigma_+ - \sigma_-), \quad \sigma_3 = |a\rangle\langle a| - |b\rangle\langle b|.\end{aligned} \tag{8.288a}$$

These satisfy the algebra of the Pauli spin matrices (however, the indexing of the matrices differs from that in Eq. 7.106b):

$$[\sigma_i, \sigma_j] = 2\mathrm{i}\sum_{k=1}^{3}\epsilon_{ijk}\sigma_k \quad (i,j=1,2,3), \tag{8.288b}$$

where ϵ_{ijk} $(i,j,k=1,2,3)$ stands for the Levi-Civitta symbol with three indices.

One can then express the operator $-\hat{\mathbf{d}}\cdot\hat{e}_0$ in terms of these as

$$-\hat{\mathbf{d}}\cdot\hat{e}_0 = p(\sigma_+ + \sigma_-) = p\sigma_1, \tag{8.289}$$

and the third term in Eq. (8.284) assumes the form

$$-\hat{\mathbf{d}}\cdot\hat{\mathbf{E}} = \mathrm{i}p\gamma(\sigma_+ + \sigma_-)(\hat{a} - \hat{a}^\dagger), \tag{8.290a}$$

where

$$\gamma = \sqrt{\frac{\hbar\omega}{\epsilon_0 V}}\sin(kz_0). \tag{8.290b}$$

Thus we arrive at

$$H = -\frac{1}{2}\hbar\omega_0\sigma_3 + \hbar\omega\hat{a}^\dagger\hat{a} + \mathrm{i}p\gamma(\sigma_+ + \sigma_-)(a - a^\dagger). \tag{8.291}$$

The time evolution of the operators σ_+ and σ_- in the absence of the interaction can be described as

$$\sigma_\pm(t) = \sigma_\pm(0)\mathrm{e}^{\mp\mathrm{i}\omega_0 t} \tag{8.292a}$$

(check this out), while the photon creation and annihilation operators evolve as

$$\hat{a}(t) = \hat{a}(0)\mathrm{e}^{-\mathrm{i}\omega t}, \quad \hat{a}^\dagger(t) = \hat{a}^\dagger(0)\mathrm{e}^{\mathrm{i}\omega t}. \tag{8.292b}$$

Hence the time dependence of the four terms in the product $(\sigma_+ + \sigma_-)(a - a^\dagger)$ can be described, in approximate terms, as follows:

$$\begin{aligned} &\sigma_-\hat{a} \to \exp(\mathrm{i}(\omega_0 - \omega)t), \quad \sigma_+\hat{a}^\dagger \to \exp(-\mathrm{i}(\omega_0 - \omega)t), \\ &\sigma_+\hat{a} \to \exp(-\mathrm{i}(\omega_0 + \omega)t), \quad \sigma_-\hat{a}^\dagger \to \exp(\mathrm{i}(\omega_0 + \omega)t). \end{aligned} \tag{8.292c}$$

In other words, for a small detuning $\Omega = \omega - \omega_0$, the operators and their matrix elements in the second line in Eq. (8.292c) evolve rapidly compared with those occurring in the first line, and hence the effects of the operators $\sigma_+ a$ and $\sigma_- a^\dagger$ on the time evolution of states of the system can be assumed to be small in the sense of an *average* compared with those of the operators $\sigma_- a$ and $\sigma_+ a^\dagger$.

Thus in the spirit of the rotating wave approximation adopted in Sections 8.12.2.3 and 8.12.2.5, we disregard the terms involving $\sigma_-\hat{a}$ and $\sigma_+\hat{a}^\dagger$ in H, and arrive at the *Jaynes-Cummings Hamiltonian*:

$$\hat{H} = -\frac{1}{2}\hbar\omega_0\sigma_3 + \hbar\omega\hat{a}^\dagger\hat{a} + \mathrm{i}\hbar g(\sigma_-\hat{a} - \sigma_+\hat{a}^\dagger), \tag{8.293a}$$

where

$$g = \frac{1}{\hbar}p\gamma = \sqrt{\frac{\omega}{\hbar\epsilon_0 V}}\sin(kz_0)\langle a| - (\hat{\mathbf{d}}\cdot\hat{e}_0)|b\rangle. \tag{8.293b}$$

This can be written as a sum of two parts:

$$H = H' + H'', \tag{8.294a}$$

where

$$H' = \hbar\omega(\hat{a}^\dagger\hat{a} + |b\rangle\langle b|) + \hbar\left(\frac{\omega_0}{2} - \omega\right)(|a\rangle\langle a| + |b\rangle\langle b|) \tag{8.294b}$$

is a sum of two conserved operators, while

$$H'' = \hbar\Omega|a\rangle\langle a| + \mathrm{i}\hbar g(\sigma_-\hat{a} - \sigma_+\hat{a}^\dagger). \tag{8.294c}$$

In this decomposition, the two parts H' and H'' *commute* with each other.

The Jaynes-Cummings Hamiltonian can have different forms depending on the definitions of the operators involved. For instance, a commonly used form is

$$H = \frac{1}{2}\hbar\omega_0\sigma_3 + \hbar\omega\hat{a}^\dagger\hat{a} + \hbar g(\sigma_+\hat{a} + \sigma_-\hat{a}^\dagger), \tag{8.295}$$

where the definitions of $\sigma_\pm$ and σ_3 differ and where, moreover, the formula (8.158a) from which expression (8.164b) is derived by quantization differs by a different choice for the phase of A_α. The physics of the problem, however, remains the same.

8.12.3.2 The oscillations

The Jaynes-Cummings Hamiltonian gives rise to a host of interesting results relating to the interaction of an atom with a radiation field, many of which have been experimentally verified, leading to a number of applications of great relevance.

The most direct conclusion from the model relates to *oscillations* between the states $|a\rangle$ and $|b\rangle$, analogous to those found in the semiclassical Rabi model. For instance, assuming that the detuning Ω is zero (ie, the field is in resonance with respect to the two atomic states) and that the atom-field system is initially in the pure product state

$$|\psi(0)\rangle = |b\rangle \otimes |n\rangle, \tag{8.296a}$$

with the atom in the excited state $|b\rangle$ and the field in the number state $|n\rangle$ ($n = 0, 1, 2, \ldots$), the state at time t can be seen to be a superposition of the form

$$|\psi(t)\rangle = c_1(t)|b\rangle|n\rangle + c_2(t)|a\rangle\ |n+1\rangle \tag{8.296b}$$

(I drop the symbol $\otimes$ depicting a direct product for brevity but will use it whenever it is needed for the sake of clarity).

Here the coefficients $c_1(t)$ and $c_2(t)$ can be worked out from the Schrödinger equation and are seen to be

$$c_1(t) = \mathrm{e}^{-\mathrm{i}\chi t}\cos\left(\frac{\Omega_{n+1}}{2}t\right), \quad c_2(t) = -\mathrm{e}^{-\mathrm{i}\chi t}\sin\left(\frac{\Omega_{n+1}}{2}t\right), \tag{8.297a}$$

where

$$\chi = \hbar\left(\frac{\omega_0}{2} + n\omega\right), \quad \Omega_n = 2g\sqrt{n} \quad (n = 1, 2, \ldots). \tag{8.297b}$$

The *transition probability* for transition from the state $|b\rangle|n\rangle$ to the state $|a\rangle|n+1\rangle$ (ie, the probability for the atom to be deexcited with the emission of a photon) in time t is thus

$$|c_2(t)|^2 = \sin^2\left(\frac{\Omega_{n+1}}{2}t\right), \tag{8.297c}$$

while the probability that the atom stays in the initial state at time t is

$$|c_1(t)|^2 = \cos^2\left(\frac{\Omega_{n+1}}{2}t\right). \tag{8.297d}$$

Thus $|c_1|^2$ and $|c_2|^2$ oscillate periodically with frequency $\Omega_\mathrm{R} = 2|g|\sqrt{n+1}$, which in this instance can be interpreted as the analogue of the Rabi frequency characterizing the atomic oscillations in the classical field (compare the factor $\sqrt{n+1}$ in Ω_{n+1} here with the factor E_0, the classical field amplitude, in formula (8.275); recall, in this context, the second relation in Eq. (8.181), where the definition of γ differs because the field is assumed to be in a propagating mode).

1. In formula (8.297a), χ is an inessential phase factor that makes its appearance because I have written out the expressions for $c_1(t)$ and $c_2(t)$ as obtained in the Schrödinger picture. In the interaction picture, on the other hand, this phase factor does not appear (check this out; the basic principles underlying the interaction picture are outlined in Section 8.3.11). The physics of the problem, embodied in the probabilities $|c_1(t)|^2$ and $|c_2(t)|^2$, remains the same, as it should.
2. The second formula in Eq. (8.297b) defines a sequence of characteristic frequencies Ω_n $(n = 1, 2 \ldots)$, of which only one is involved in the oscillations of $|c_1|^2$ and $|c_2|^2$ here—namely, the one relevant to the situation corresponding to the initial state (8.296a)

Interestingly, the state (8.296b) is, in general, an *entangled* one even though the initial state (8.296a) is not (check this out; show that the density operator corresponding to Eq. (8.296b) is not of the form (8.80b)). This expresses, in the present context, a simple but profound fact of quantum theory: *the interaction of two quantum systems produces, in general, an entangled state of the composite system made up of the two quantum systems*.

> It is then the job of applied quantum theory to *make use* of this entanglement. This is an area of vast potentialities.

The Jaynes-Cummings model describes single-photon emission and absorption processes where the entire dynamics ultimately relates to elementary processes of the form $|b\rangle \otimes |n\rangle \leftrightarrow |a\rangle \otimes |n+1\rangle$. This is why the model is a tractable one. At the same time it enjoys good contact with reality since multiphoton processes (ie, ones where more than one photon is emitted or absorbed in a single event) arise under relatively rare circumstances. These two features make the model so successful.

8.12.3.3 Collapse and revival

Once one recognizes the role of the elementary process $|b\rangle \otimes |n\rangle \leftrightarrow |a\rangle \otimes |n+1\rangle$ in Jaynes-Cummings dynamics, one can progress to the consideration of time evolution of states of a relatively more general description. For instance, one can start from an initial state of the direct product form

$$|\psi(0)\rangle = |\psi_{\text{atom}}\rangle \otimes |\psi_{\text{field}}\rangle, \tag{8.298a}$$

where both $|\psi_{\text{atom}}\rangle$ and $|\psi_{\text{field}}\rangle$ are superpositions of the form

$$|\psi_{\text{atom}}\rangle = a_1|a\rangle + a_2|b\rangle, \quad |\psi_{\text{field}}\rangle = \sum_{n=0}^{\infty} b_n|n\rangle. \tag{8.298b}$$

In this expression a_1, a_2, and b_n $(n = 0, 1, 2, \ldots)$ are superposition coefficients satisfying the conditions of normalization of the atomic and field states. We again assume for simplicity that the detuning Ω is zero.

The Jaynes-Cummings dynamics causes the state $|\psi(0)\rangle$ to evolve through a cascade of elementary processes of the type indicated above, and the resulting state at any time t is

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} \left(\left(a_1 b_n \cos \frac{\Omega_n}{2} t - a_2 b_{n-1} \sin \frac{\Omega_n}{2} t \right) |a\rangle \otimes |n\rangle + \left(a_1 b_{n+1} \sin \frac{\Omega_{n+1}}{2} t + a_2 b_n \cos \frac{\Omega_{n+1}}{2} t \right) |b\rangle \otimes |n\rangle \right) \tag{8.299}$$

(check this out; it is convenient to work in the interaction picture, since the expressions in the Schrödinger picture involve inessential phase factors as in Eq. (8.297a); note that the term with coefficient $a_2 b_{-1}$ is only a formal one since $\Omega_0 = 0$).

What is important to note here is that the time evolution of the state $|\psi(t)\rangle$ is now *multiperiodic* in nature, involving frequencies $\frac{\Omega_n}{2}$ $(n = 0, 1, 2, \ldots)$ (there is a constant term corresponding to $\Omega_0 = 0$). Once again $|\psi(t)\rangle$, though a pure state of the atom-field system, is an *entangled* one, as can be seen by construction of the density operator $\hat{\rho}(t) = |\psi(t)\rangle\langle\psi(t)|$. The density operator gives *quantitatively verifiable* results relating to a number of phenomena in atom-field-interaction—results that can made use of in various significant ways.

A phenomenon of remarkable significance resulting from the multiperiodic nature of the time evolution is the one of *collapse and revival*. This can be illustrated by working out the expectation value of the atomic operator $-\sigma_3 = |b\rangle\langle b| - |a\rangle\langle a|$ in the state $\hat{\rho}(t)$, which gives the difference (f) between the probabilities of the atom being in states $|b\rangle$ and $|a\rangle$, respectively, at time t, and is referred to as the *atomic inversion*. Assuming for simplicity that $a_1 = 0$ $(a_2 = 1)$—that is, the atom is initially in the excited state—one obtains

$$f(t) = \sum_{n=0}^{\infty} |b_n|^2 \cos\left(2g\sqrt{n+1}t\right) \tag{8.300a}$$

(check this out). As a particular instance, consider the case when the state of the field at $t = 0$ is a *coherent* one with parameter λ—that is (see Eq. 8.298b),

$$b_n = \frac{\mathrm{e}^{-\frac{|\lambda|^2}{2}} \lambda^n}{\sqrt{n!}}, \tag{8.300b}$$

in which case the atomic inversion is

$$f(t) = \sum_{n=0}^{\infty} \frac{\mathrm{e}^{-|\lambda|^2} |\lambda|^{2n}}{n!} \cos(2g\sqrt{n+1}t). \tag{8.300c}$$

This is a multiperiodic function involving frequencies Ω_n $(n = 1, 2, \ldots)$, whose variation is depicted schematically in Fig. 8.15, where one observes the feature of *collapse and revival* of the oscillations. After an initial interval, the oscillations die down, and one finds a quiescent state, where the oscillation amplitude is minimal.

The result (8.300a) can also be derived by first considering the *reduced* state of the atom (see Section 8.3.12) by means of partial tracing over the field states and then working out the expectation value of the atomic operator $-\sigma_3$.

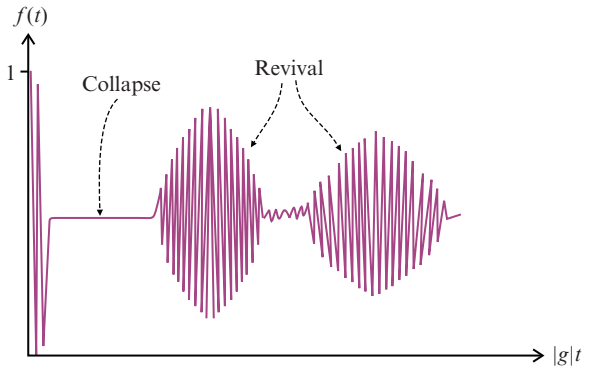

Fig. 8.15
Depicting schematically the phenomenon of collapse and revival of the state of the interacting atom-field system in the Jaynes-Cummings model. The variation of the atomic inversion ($f(t)$; Eq. 8.300c) is shown as a function of the scaled time $|g|t$. The variation is of the oscillatory type but is marked by a succession of collapses and revivals. During a collapse the system remains in a quiescent state for some time interval, after which it starts oscillating again. The initial state of the system is taken to be one of the pure product type, with the atom in the excited state and the field in a coherent state. The diagram is not to scale, and the relevant parameters are chosen arbitrarily. After the first collapse the time intervals for the successive collapsed states to last become smaller because of various disturbances acting on the system.

This state of collapse is brought about by a cancellation of oscillations caused by the various different frequencies involved in the multiperiodic variation, which are, in general, *incommensurate* with one another. As the oscillations die down, the atom-field system remains in an approximately time-invariant entangled state. For a field with a relatively large value of the mean photon number, this state is interesting in that here the field is maintained for a considerable time interval in an entangled quantum state with the atom, even though the initial coherent state of the field is a nearly *classical* one. An effective parameter that can be used to manipulate such states is the detuning Ω (recall that we have for so long assumed Ω to be zero for simplicity).

Looking at the Jaynes-Cummings dynamics more closely, one observes that for an initial state of the form (8.298a), the dominant, or effective, Rabi frequency is $\Omega_{\text{eff}} = 2g\sqrt{n+1}$, with $n = \bar{n}$, the mean photon number. If the field is in a coherent state characterized by parameter λ, then $\bar{n} = |\lambda|^2$, and thus one obtains

$$\Omega_{\text{eff}} \sim 2g|\lambda|, \tag{8.301}$$

an estimate whose accuracy improves with increasing value of $|\lambda|$. One can also estimate the first collapse time t_c and the first revival time t_r, obtaining

$$t_c \sim \frac{1}{g}, \quad t_r \sim \frac{2\pi}{g}|\lambda|. \tag{8.302}$$

There occurs a revival at every integer multiple of t_r, which is large compared with the effective Rabi period $T_{\text{eff}} = \frac{2\pi}{\Omega_{\text{eff}}}$. At the same time, t_r is large compared with t_c as well, where the latter is independent of the mean photon number. Moreover, the time interval during which the state of the first collapse continues is large compared with T_{eff}. In comparison, subsequent collapses are quite short-lived. Finally, in a real-life situation, the oscillations tend to die down because of various damping and *decoherence* effects.

The collapse and revival of oscillations in the Jaynes-Cummings model is a specifically quantum effect since it does not occur in the semiclassical model (in which the field is treated classically; see Section 8.12.2.5), where the Rabi oscillations continue indefinitely.

8.12.3.4 The dressed states

In Sections 8.12.3.2 and 8.12.3.3 we took the detuning Ω $(= \omega - \omega_0)$ to be zero for simplicity. More generally, one can work out the solution for a small but nonzero value of Ω (recall that we invoked the rotating wave approximation in writing the Jaynes-Cummings Hamiltonian (8.293a); in principle, however, the latter can be considered for arbitrary values of the detuning as well). The basic approach remains the same as in Section 8.12.3.2, where we made use of the fact that the states $|a\rangle \otimes |n+1\rangle$ and $|b\rangle \otimes |n\rangle$ form an *invariant* subspace of the Hamiltonian. In the case of a nonzero value of the detuning, one can simplify things by working out the so-called *dressed states*, the latter being the stationary states in the presence of the atom-field interaction.

For a given pair of *bare* states (ie, stationary states in the absence of interactions) $|\tilde{a}\rangle = |a\rangle \otimes |n+1\rangle$ and $|\tilde{b}\rangle = |b\rangle \otimes |n\rangle$, a dressed state is of the form

$$|E\rangle = \alpha|\tilde{a}\rangle + \beta|\tilde{b}\rangle, \tag{8.303a}$$

and satisfies

$$H|E\rangle = E|E\rangle, \tag{8.303b}$$

where E is the energy eigenvalue, to be determined along with the coefficients α and β (or, more precisely, the ratio $\frac{\beta}{\alpha}$; the two are to satisfy the normalization condition $|\alpha|^2 + |\beta|^2 = 1$) so as to determine the dressed states and their energies. For this, one uses the Hamiltonian (8.293a) and the orthogonality of $|\tilde{a}\rangle$ and $|\tilde{b}\rangle$ to obtain

$$\frac{\alpha}{\beta} = \frac{\mathrm{i}}{2g\sqrt{n+1}}\left(-\Omega \mp \sqrt{\Omega^2 + 4g^2(n+1)}\right), \tag{8.304a}$$

which give us two stationary states $|E_1\rangle$ and $|E_2\rangle$, with energies

$$E_{1,2} = \frac{1}{2}\hbar\omega_0 + n\hbar\omega + \frac{\hbar}{2}\left(\Omega \pm \sqrt{\Omega^2 + 4g^2(n+1)}\right)$$
$$= \left(n + \frac{1}{2}\right)\hbar\omega \pm \frac{\hbar}{2}\sqrt{\Omega^2 + 4g^2(n+1)}. \tag{8.304b}$$

In the limit $g \to 0$, one gets the two bare states $|\tilde{a}\rangle$ and $|\tilde{b}\rangle$ with energies $E_{\tilde{a}} = -\frac{1}{2}\hbar\omega_0 + (n+1)\hbar\omega$ and $E_{\tilde{b}} = \frac{1}{2}\hbar\omega_0 + n\hbar\omega$, respectively, as expected (check this out). The variation of E_1 and E_2 with the detuning Ω is shown in Fig. 8.16 for a small nonzero coupling g, while the case of zero coupling is also shown, for which E_1 and E_2 coincide with the bare energies $E_{\tilde{a}}$ and $E_{\tilde{b}}$. The two bare states are degenerate at $\Omega = 0$, while a small detuning lifts the degeneracy. For relatively large values of the detuning Ω (more precisely, for large values of $\frac{\Omega}{g\sqrt{n+1}}$; for sufficiently small coupling strength g, this is compatible with values of Ω for which the rotating wave approximation is applicable), the dressed energies $E_{1,2}$ tend to the bare values $E_{\tilde{a},\tilde{b}}$.

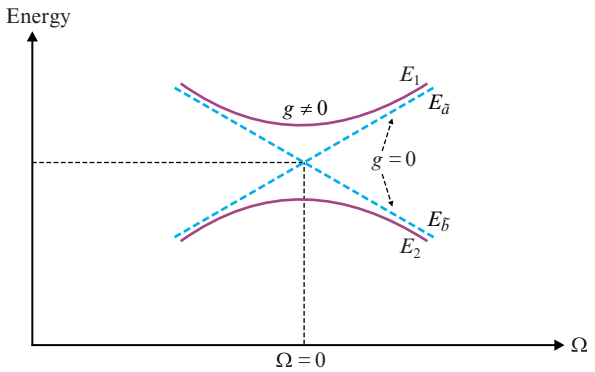

Fig. 8.16
Depicting schematically the energies E_1 and E_2 of the dressed states $|E_1\rangle$ and $|E_2\rangle$ for a given pair of bare states $|\tilde{a}\rangle = |a\rangle \otimes |n+1\rangle$ and $|\tilde{b}\rangle = |b\rangle \otimes |n\rangle$. The solid curves show the variation of E_1 and E_2 with the detuning (Ω) for a small nonzero coupling strength g. The case of zero coupling is shown by the dotted lines, which corresponds to the bare energies $E_{\tilde{a}}$ and $E_{\tilde{b}}$. The dressed and bare energies tend to coincide for large values of $\frac{\Omega}{g\sqrt{n+1}}$.

Having obtained the states $|E_1\rangle$ and $|E_2\rangle$ in terms of the bare states $|\tilde{a}\rangle = |a\rangle \otimes |n+1\rangle$ and $|\tilde{b}\rangle = |b\rangle \otimes |n\rangle$, one can work out the time evolution in the subspace of the states $|\tilde{a}\rangle$ and $|\tilde{b}\rangle$ by expressing the initial state in terms of $|E_1\rangle$ and $|E_2\rangle$ and by noting that the latter evolve in a trivial manner by acquiring phases $\mathrm{e}^{-\frac{\mathrm{i}E_1 t}{\hbar}}$ and $\mathrm{e}^{-\frac{\mathrm{i}E_2 t}{\hbar}}$. Thus if the initial state is of the form

$$|\psi(0)\rangle = a_1|E_1\rangle + a_2|E_2\rangle, \tag{8.305a}$$

then the state at time t will be

$$|\psi(t)\rangle = a_1 \mathrm{e}^{-\frac{\mathrm{i}E_1 t}{\hbar}} |E_1\rangle + a_2 \mathrm{e}^{-\frac{-\mathrm{i}E_2 t}{\hbar}} |E_2\rangle. \tag{8.305b}$$

This can then be expressed in terms of the bare states $|\tilde{a}\rangle$ and $|\tilde{b}\rangle$. One again obtains an oscillatory evolution, but the oscillations involve two frequencies instead of one (as was the case in Section 8.12.3.2 for zero detuning) obtained by a splitting of the Rabi frequency Ω_n.

More generally, considering an initial state as in Section 8.12.3.3, one obtains the collapse-and-revival dynamics, but now for a nonzero detuning $\Omega (= \omega - \omega_0)$. I will not give the details of these results here since these are not of basic relevance for our purpose.

One can also apply similar considerations to an initial *mixed state* of the form

$$\hat{\rho}(0) = \hat{\rho}_{\mathrm{atom}}(0) \otimes \hat{\rho}_{\mathrm{field}}(0) \tag{8.306a}$$

so as to work out the state $\hat{\rho}(t)$ at time t, which is once again an entangled one. In expression (8.306a), $\hat{\rho}_{\mathrm{atom}}(0)$ stands for the initial state of the atom, while $\hat{\rho}_{\mathrm{field}}(0)$ represents the initial field state. By tracing over the field states, one can obtain the reduced state of the atom at time t:

$$\hat{\tilde{\rho}}_{\mathrm{atom}}(t) = \sum_{n=0}^{\infty} \langle n|\hat{\rho}(t)|n\rangle. \tag{8.306b}$$

Finally, one can work out the values of observable quantities such as the atomic inversion $f(t)$ (see Section 8.12.3.3). For instance, if the atom is initially in the excited state $|b\rangle$ and the field is initially in the *thermal state* at temperature T (see formulae (8.172a) and (8.172b), then the atomic inversion is

$$f(t) = \sum_{n=0}^{\infty} \frac{1}{Z} \mathrm{e}^{-\frac{n\hbar\omega}{k_{\mathrm{B}}T}} \cos\left(2g\sqrt{n+1}t\right). \tag{8.307}$$

A plot of the atomic inversion as a function of time looks as in Fig. 8.17, where there is no fixed pattern of collapse and revival. The initial collapse is short-lived compared with that in the case of the field being initially in a coherent state (with the same value of the mean photon number), and the dynamics involves a relatively higher degree of fluctuations.

The Jaynes-Cummings model is of great relevance in quantum optics since it is a model of quite considerable simplicity and, at the same time, is capable of producing results of practical value, especially in the context of *cavity quantum electrodynamics* (CQED). The latter is a large field of study with an interplay of fundamental theoretical concepts and remarkable application potentialities, a number of which have already been realized in practice. I will give a brief outline of CQED in Section 8.20.

Fig. 8.17
Depicting schematically the variation of atomic inversion $f(t)$ as a function of time for the case when the field is initially in a thermal state. There is no fixed pattern of collapse and revival. The initial collapse is short-lived compared with the case of the field in a coherent state (with the same mean photon number), and the time series involves a relatively higher degree of fluctuations.

8.12.4 Quantum Theory of the A and B Coefficients

Interestingly, the Jaynes-Cummings dynamics includes the possibility of *spontaneous emission*, since for an initial pure product state $|b\rangle \otimes |n = 0\rangle$ with the atom in the excited state $|b\rangle$ and the field in the vacuum state $|n = 0\rangle$, there occurs a transition to the state $|a\rangle \otimes |n = 1\rangle$ with the emission of a photon. However, the Jaynes-Cummings model describes a reversible interaction between the atom and the field with energy flowing back and forth between the atom and the field because the former is assumed to interact with just a single field mode in a cavity and no energy is assumed to leak away to the environment exterior to the latter.

Thus the Jaynes-Cummings model does not describe the process of spontaneous *decay* from the excited state to the ground state of the atom, which is an irreversible process, and arises only when one takes into account *all* the possible modes of the field labeled with mode indices $\mathbf{k}s$ (see Sections 8.5.1 and 8.5.2) relevant for the atomic transition under consideration and includes the possibility of energy being carried away by propagating modes. For a field in open space (ie, in an infinitely large volume V), there arises a *continuum* of modes, and the spontaneous emission can occur with the photon being emitted into any one of the vacuum states of these field modes. For a weak interaction between the atom and the field, one can work out the transition rate relating to spontaneous decay and the value of *Einstein's A coefficient*. Indeed, the quantum theory of atom-field interaction leads us to expressions for the

A and B coefficients, wherein one finds that these expressions conform to Einstein's relations stated in Section 8.12.1. I will now outline this theory.

We begin by considering the decay of an excited atomic state $|b\rangle$ to the ground state $|a\rangle$ by means of dipolar coupling to a field mode of frequency ω. The coupling is described by the interaction Hamiltonian having the Jaynes-Cummings form in Section 8.12.3.1 (see Eq. 8.293a)

$$\hat{H}_{\text{int}} = \mathrm{i}\hbar g(\sigma_- \hat{a} - \sigma_+ \hat{a}^\dagger), \tag{8.308a}$$

where the notation is the same as in the Jaynes-Cummings model, with the only difference that g is now given by

$$g = \sqrt{\frac{\omega}{2\hbar\epsilon_0 V}}\langle a| - \hat{\mathbf{d}} \cdot \hat{e}_0 |b\rangle. \tag{8.308b}$$

In writing formulae (8.308a) and (8.308b), we have assumed that the relevant field mode is in the form of a propagating plane wave as in Eq. (8.166a) but have used the Schrödinger picture expressions for the operators and have assumed the atom to be located at $z = 0$ without any loss in generality.

The transition rate in the weak coupling limit (recall that it is only in the weak coupling limit that the notion of a transition rate is meaningful) can be obtained by the use of the Fermi golden rule (Eq. 8.269b), for which one needs the squared transition matrix element $|\langle f|\hat{H}_{\text{int}}|i\rangle|^2$. For an initial state $|i\rangle = |b\rangle|n\rangle$ of the combined atom-field system, the final state corresponding to a transition to the atomic state $|a\rangle$ caused by a photon emission will be $|a\rangle|n+1\rangle$.

The required squared matrix element is then (in an obvious notation)

$$|\langle a, n+1|\mathrm{i}\hbar g(\sigma_- \hat{a} - \sigma_+ \hat{a}^\dagger)|b, n\rangle|^2 = \hbar^2 g^2 (n+1). \tag{8.309}$$

In the transition problem under consideration, we are interested in a given initial atomic state $|b\rangle$, while the initial field state is not specified. This means that one has to perform an *averaging* over all possible initial field states (ie, over n for any given mode of frequency ω). Further, the δ-function in the golden rule formula (8.269b) reduces to $\delta(\omega - \omega_0)$, where ω_0 is given by Eq. (8.285a) (check this out) and the sum over final states reduces to a multiplication by the density of modes $\rho(\omega)$ along with an integration over ω. This integral now contains a factor $(\langle n\rangle + 1)\rho(\omega)$, in which $\langle n\rangle\rho(\omega)$ can be interpreted as $(\hbar\omega)^{-1}U(\omega)\rho(\omega)$, where $U(\omega)$ is the mean field energy per mode of frequency ω.

Thus the above factor $((\langle n\rangle + 1)\rho(\omega))$ is the sum of two parts—namely, $(\hbar\omega)^{-1}U(\omega)\rho(\omega)$ and $\rho(\omega)$—and, accordingly, the transition rate

$$W = \frac{\pi|\langle b|\hat{\mathbf{d}} \cdot \hat{e}_0|a\rangle|^2}{\epsilon_0 V\hbar} \int d\omega\omega\delta(\omega - \omega_0)[(\hbar\omega)^{-1}U(\omega)\rho(\omega) + \rho(\omega)] \tag{8.310}$$

decomposes into two parts:

$$W = W^{\text{field}} + W^{\text{vacuum}} \quad \text{(say)}. \tag{8.311a}$$

Here W^{field} contains the factor $U(\omega)$ in the integrand, and can thus be interpreted as the field-induced transition rate, while W^{vacuum} contains no field-dependent factor in the integrand, and can thus be interpreted as the *vacuum* contribution to the transition rate. In other words, these correspond precisely to the transition rates corresponding to *stimulated* and *spontaneous* emission, respectively.

Before arriving at the final results linking W^{vacuum} and W^{field} to the A and B coefficients, however, one needs a further averaging in $|\langle b|\hat{\mathbf{d}} \cdot \hat{e}_0|a\rangle|^2$, where the angle between the atomic dipole moment matrix element $\langle b|\hat{\mathbf{d}}|a\rangle$ and the unit vector $\hat{e}_0$ is to be treated as a random variable. In the case of the field-induced transition, this averaging is needed since the dipole moment matrix element in a gaseous medium is oriented at random with respect to the field polarization direction, as explained in Section 8.12.2.6, while in the case of spontaneous emission, this averaging is needed because of the isotropy of the vacuum state, because of which one has to average for all possible orientations of $\hat{e}_0$ relative to the dipole moment matrix element. In either case, if we denote the magnitude of the dipole moment matrix element by p, the result of the averaging gives the factor $\frac{p^2}{3}$ in the required expressions.

One thereby obtains, with the directional averaging performed as indicated,

$$W^{\text{field}} = \frac{\pi p^2}{3\hbar^2\epsilon_0} u(\omega_0), \tag{8.311b}$$

$$W^{\text{vacuum}} = \frac{\omega_0^3 p^2}{3\pi\hbar\epsilon_0 c^3}, \tag{8.311c}$$

where in Eq. (8.311b) the energy density (field energy per unit volume per unit frequency interval) $u(\omega_0)$ at the transition frequency ω_0 has been introduced in the place of $\frac{1}{V}\int U(\omega)\rho(\omega)\delta(\omega - \omega_0)$, while in Eq. (8.311c) the expression for the density of modes ($\rho(\omega_0) = \frac{\omega_0^2 V}{\pi^2 c^3}$; I skip the derivation of this standard result in electromagnetic theory) has been used.

From a comparison with the definitions of the A and B coefficients, we finally obtain, in the notation of Section 8.12.1,

$$A = \frac{\omega_0^3 p^2}{3\pi\epsilon_0\hbar c^3}, \quad B_{21} = \frac{\pi p^2}{3\epsilon_0\hbar^2}, \tag{8.312}$$

in agreement with the results in Eq. (8.282).

While the B coefficient obtained here applies to stimulated emission, the coefficient B_{12} pertaining to stimulated absorption can also be obtained by a similar derivation by taking $|a\rangle$ and $|b\rangle$ as the initial and final atomic states, respectively. If the states $|a\rangle$ and $|b\rangle$ are

nondegenerate, one simply obtains $B_{12} = B_{21}$. If, on the other hand, the degeneracies of these two states are g_1 and g_2, respectively, then the sum over final states in the Fermi golden rule yields, in the two cases,

$$B_{12} = \frac{\pi p^2}{3\hbar^2\epsilon_0} g_2, \quad B_{21} = \frac{\pi p^2}{3\hbar^2\epsilon_0} g_1, \tag{8.313a}$$

and the relation between the two B coefficients reads

$$g_1 B_{12} = g_2 B_{21}. \tag{8.313b}$$

In summary, while the semiclassical theory of Section 8.12.2.6 yields only the values of the B coefficients, the quantum theory of the atom-field interaction gives us the A coefficient as well.

> In our derivation above we considered a two-level atom and referred to the state $|a\rangle$ as the ground state. More generally, the theory applies to the resonant transition between any pair of states in a real atom. If $|a\rangle$ is the ground state, its degeneracy is to be taken as $g_1 = 1$.

Incidentally, the vacuum transition rate W^{vacuum} is commonly referred to as the *spontaneous decay rate* of the atom, and is denoted by Γ. While this is the decay rate obtained for the atom placed in open space, the decay rate is modified when the atom is placed within a *resonant cavity*, as we will see in Section 8.20.3. To distinguish the decay rate in open space from the modified decay rate within a cavity, we will refer to the former as the *natural decay rate* and to the latter as the *cavity-induced decay rate* (Γ_{cav}).

8.13 The Laser: Principles of Operation

8.13.1 The Basic Idea of the Laser

The laser (device for *l*ight *a*mplification by *s*timulated *e*mission of *r*adiation) is the one thing that has changed the face of the science of optics. Among its multifarious achievements, it has made possible the development of *nonlinear optics*. A child of quantum optics, it has made possible the blossoming of quantum optics itself. Fittingly, it is one of the most widely used devices in present-day technology.

The basic idea underlying the working of the laser is that of *population inversion*. The energy levels of the atoms making up the medium in which the light amplification is to occur include a pair of levels (the *lasing* levels) with respect to which a *nonequilibrium* situation is made to prevail such that the upper level has a higher population compared with the lower one. The medium then acts as an *active* one in which the emission of radiation in the transition from the upper to the lower lasing level acquires the nature of a self-reinforcing process. As the radiation propagates through the medium, its intensity keeps on growing instead of being attenuated, the latter being the commonly observed feature in the absence of population inversion.

The amplified radiation is, by virtue of its origin, monochromatic (laser radiation made up of more than one monochromatic component can also be produced) and can be made to have a high degree of directivity. Most importantly, it is in the nature of highly *coherent* radiation.

However, the operation of the laser requires another essential arrangement—namely, a *feedback* mechanism—so that the laser radiation may build up to a desired level even without any initial input, as in an electronic oscillator. The oscillations (the term 'oscillation' refers to the oscillating electromagnetic field of a sharply defined frequency and also the oscillatory growth of the energy density in the medium up to a steady level) are initiated spontaneously and then build up by means of a positive feedback mechanism, while the oscillations are rendered stable by a self-adjusting process where a negative feedback comes into play, involving losses that eat up the gain resulting from the population inversion. The feedback is provided by a *resonant optical cavity*, in which a number of loss mechanisms are present along with the mechanism for a positive feedback.

While the lasing action involves a pair of energy levels of the atomic constituents of the medium under consideration, the production and maintenance of the population inversion requires that more than two levels be involved. The time-dependent populations of these levels appear as essential variables in the description and analysis of the time-dependent state of the radiation field within the cavity. The latter is expressed in terms of the energy density of the laser radiation. The time evolution of all these variables (the populations of the energy levels, and the energy density of the laser radiation) are described by means of a set of coupled differential equations arrived at from detailed considerations of the relevant processes occurring in the cavity.

The time-independent solutions for these variables, obtained from the differential equations, then constitute the description of the *steady-state* laser oscillations. From the practical point of view, however, the steady-state solution is nothing more than an idealization since, in reality, there appear *fluctuations* around the steady state. The description of the fluctuations, which constitute an essential aspect of the laser operation, require more detailed considerations and may be undertaken at various levels of completeness. The most general approach in the analysis of the fluctuations and, indeed, of the entire dynamics of the laser operation, involves the use of the *density operator* of the radiation field and the system of radiating atoms, while relatively simpler descriptions are also possible at some cost of accuracy and generality.

The dynamics of the atomic states, and of the radiation field, thus constitutes a complex problem, with reference to which the differential equations for the atomic populations and the energy density of the radiation field provide a first level of simplification, representing the time evolution of the *mean* values of the relevant variables.

In the present brief exposition of the basic principles of the laser, I opt for simplicity at the cost of generality, and will outline only a few of the more important aspects of laser operation

in terms of simplified rate equations, while skipping the derivations of the results I quote. A brief exposition of fluctuations in laser radiation will also be included.

The setup for a laser involves three principal components: (1) the *pump*, which is an external source of radiation required for the production of population inversion between a pair of lasing levels; (2) the *active medium* made up of the atoms or molecules that provide the lasing levels responsible for the stimulated emission of radiation; and (3) the resonant cavity, which provides the feedback mechanism and sends out the laser beam through an *output coupler*. These are shown schematically in Fig. 8.18.

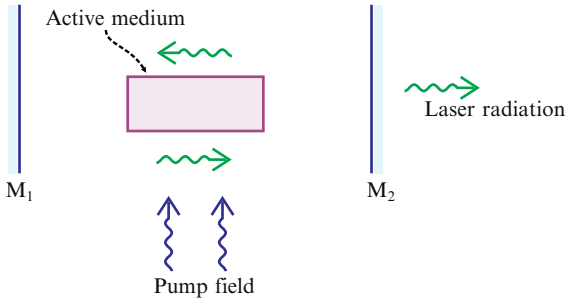

Fig. 8.18
Depicting schematically the principal components in a laser setup. The sample of the active medium is placed in an optical cavity bounded by mirrors M_1 and M_2. The pump field leads to the production of population inversion in the atoms of the medium, which in turn causes an amplification of the field produced by the lasing transition (see Fig. 8.19). The photons generated in the lasing transition make repeated circuits between the two mirrors, producing the gain at the laser frequency, which is also the resonant frequency of the cavity. The laser radiation comes out of the cavity by partial transmission through the mirror M_2, which in this instance represents the output coupler.

In the case of microwave radiation (for which the device is referred to as a *maser*), the cavity is made up of conducting walls. In a laser, on the other hand, the cavity is of the Fabry-Pérot type, with a pair of highly reflecting parallel mirrors forming its boundary, the active medium being placed between the mirrors. One of the two mirrors—the output coupler—has a relatively low reflectivity so as to allow the laser beam to come out of the cavity once the beam has acquired sufficient intensity. The output coupling mechanism provides a field-dependent transmission out from the cavity.

The design of the pump mechanism, the resonator cavity, and the output coupler can differ from one laser device to another, resulting in two distinct *patterns* of laser activity—namely, *continuous wave* laser operation and *pulsed* laser operation. While I will focus here on the principles of continuous wave laser action, pulsed laser operation is also of great importance in laser technology, especially in respect of optical communications.

8.13.2 The Three-Level and Four-Level Schemes

Lasing activity cannot be made to occur with just two energy levels because a population inversion cannot be achieved in a two-level scheme of operation.

> In a state of thermal equilibrium, the population of the upper of the two levels is less than that of the lower one by virtue of the Boltzmann distribution. Away from equilibrium, the population of the upper level can at most be equal to that of the lower level when the transitions from and to the upper (or the lower) level occur at an equal rate, and no further net transfer of the population can occur (*saturation*).

On the other hand, laser operation is possible with a *three-level* or a *four-level* scheme with a pump field of frequency, say, ω_p, different from the laser frequency ω, the latter being less than ω_p.

Fig. 8.19 shows the energy levels of a three-level laser, where '0,' '1,' and '2' denote, respectively, the ground-state energy level of the atom, the lasing level which is made to acquire a population higher than the ground level, and the pump level to which the pump field raises the atom from the ground level so that the population inversion between levels '0' and '1' can be achieved (thus the ground state constitutes the lower of the two lasing levels in this case). The atom in question possesses energy levels other than the three indicated, but those are not of relevance in respect of the laser operation.

In this three-level scheme of laser operation the pump field is used to raise the atoms from the ground level '0' to the pump level '2,' from which there occurs a rapid transition (which may be a radiative or a nonradiative one) to the lasing level '1.' The latter is so chosen that the rate of spontaneous decay from it to the ground level is small.

> That is to say, the decay by a dipole transition is prohibited by a selection rule. In reality, the possibility of decay is not completely ruled out, and the decay does occur at a slow rate by means of an electric quadrupole interaction or a magnetic dipole one; '1' then corresponds to a *metastable* state.

Thus, there occurs an accumulation of atoms in level '1' until a state of population inversion with respect to the ground state is arrived at. However, the pump field has to conform to rather stringent requirements for this, since the ground level has a high population to start with, by virtue of the Boltzmann distribution, which is why the four-level scheme is a more convenient and preferred one for lasing action to occur.

Fig. 8.20 depicts the level scheme of a four-level laser where the lasing levels '1' and '2' are situated on the energy scale between the ground level '0' and the pump level '3.' Compared with the three-level scheme, population inversion is achieved more easily in this case. The pump field raises atoms from the ground level to the pump level, but the ground level, which has a relatively high population, is not involved in the lasing action. As a fast transition occurs

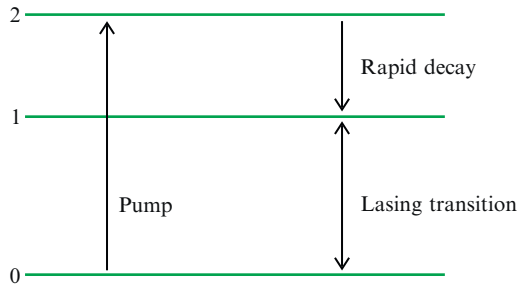

Fig. 8.19
The three-level scheme of laser operation. A pump field raises the atoms of the active medium from the ground level '0' to the pump level '2,' from which there occurs a rapid transition to level '1.' The lasing transition occurs from '1' to the ground level '0.'

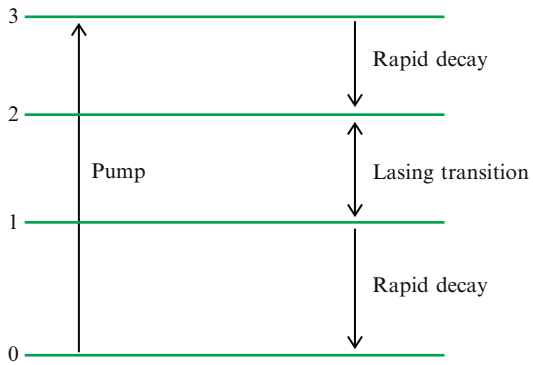

Fig. 8.20
The four-level scheme of laser operation. The pump field raises atoms of the medium from the ground level '0' to the pump level '3,' from which a rapid transition occurs to level '2.' The lasing transition occurs between '2' and '1.' Because the initial population of level '1' is very small, a population inversion is easily produced between the lasing levels.

from level '3' to level '2,' there occurs an accumulation in the latter, and since level '1' is thinly populated to start with (Boltzmann distribution again), population inversion between '1' and '2' is achieved easily.

In reality, one or more of the levels may have a band structure, where a band is made up of a large number of levels bunched together. Moreover, there exist variants of the lasing schemes that may differ in one or more respects from the ones outlined above. However, the basic principle of achieving population inversion remains the same.

Incidentally, the term 'four level' is commonly used to refer to a broad class of energy level schemes—namely, the ones in which the lower of the two lasing levels is almost empty, so that a population inversion can be achieved easily. If, for instance, the pump level '3' and the upper lasing level '2' in Fig. 8.20 are the same, then the laser operation will effectively resemble a four-level one, even though only three levels are involved in the scheme.

Before we progress to the rate equations describing laser operation in Section 8.13.3 below, it is of some relevance to note that, though the laser oscillations grow to a desired level by means of stimulated emission from the upper of the two lasing levels to the lower one, the process of buildup is *initiated by means of spontaneous emission* from the former to the latter. The photons produced in spontaneous emission are completely uncorrelated in respect of their directions of motion. Most of these photons are ineffective in the process that builds up the laser oscillations. Only a few of these—namely, the ones with their wave vectors along the axis of the optical cavity—get trapped in it and initiate the lasing action. Once the lasing action is initiated, the process of stimulated emission and the mechanism of positive feedback in the cavity take over, leading to steady-state laser oscillations.

8.13.3 Continuous Wave Operation: Rate Equations

As an illustration of the rate equation analysis of laser action, we consider below the *continuous wave* operation of a single-mode laser based on the level scheme shown in Fig. 8.21, where the level marked '2' is the pump level and is also the upper of the two lasing levels, the lower lasing level being the one marked '1.' As mentioned previously, the scheme is effectively similar to a four-level one, though only three levels are involved.

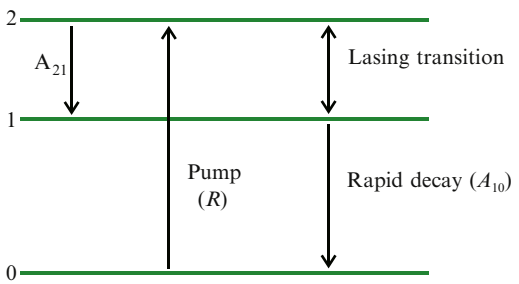

Fig. 8.21
Level scheme involving three energy levels, where the laser operation is effectively that of a four-level laser. The pump transition occurs from the ground level '0' to level '2,' while the lasing transition occurs between '2' and '1.' Because the initial population in level '1' is small, a population inversion is produced easily. The energy level scheme shown can be regarded as a special case of the four-level scheme in Fig. 8.20, with levels '3' and '2' being the same.

Let the instantaneous populations of levels '0,' '1,' and '2' be N_0, N_1, and N_2, respectively. The rates of change of the populations can be expressed in terms of the relevant transition rates and the energy density of the radiation in the cavity causing the lasing transition. One transition rate of relevance is the probability of the transition per unit time from '0' to '2' caused by the pump field, which we denote by R. The probability of stimulated emission from '2' to '1' is proportional to the energy density per unit frequency interval of the radiation in the cavity, the constant of proportionality being $B_{21}(= B_{12})$ (we assume the three levels to be nondegenerate for simplicity). In the quantum description the energy density is a fluctuating quantity depending on the photon number distribution ($P(n)$) in the cavity. However, for the present we write the probability of stimulated emission per unit time as $\bar{\Gamma}n$, where n stands for an assumed value of the photon number that can be taken as the mean photon number in the cavity or can, subsequently, be considered to be a fluctuating quantity, and where $\bar{\Gamma}$ is a constant proportional to the B coefficient given in Eq. (8.282) (we consider here a gas laser for simplicity; the expression for $\bar{\Gamma}$ depends on the line width of the radiation produced in the transition from '2' to '1').

In addition, the rate equations involve the spontaneous decay rates A_{20}, A_{10}, and A_{21}, where the subscripts indicate the levels that these transition rates are associated with. The rate equations for the three atomic populations then read

$$\begin{aligned}
\frac{dN_0}{dt} &= -RN_0 + A_{10}N_1 + A_{20}N_2, \\
\frac{dN_1}{dt} &= -A_{10}N_1 + A_{21}N_2 + n\bar{\Gamma}(N_2 - N_1), \\
\frac{dN_2}{dt} &= RN_0 - A_{20}N_2 - A_{21}N_2 - n\bar{\Gamma}(N_2 - N_1)
\end{aligned} \tag{8.314}$$

(check these formulae out). These rate equations can be simplified to some extent since, in typical laser operation, the spontaneous rate A_{20} is commonly of negligible relevance, while A_{10} is large compared with the pumping rate R and the rates A_{21} and $\bar{\Gamma}n$. As a consequence, if N_1 is initially small (Boltzmann distribution; we assume that the levels are separated by energy gaps large compared with $k_{\mathrm{B}}T$), it continues to be small, which is why population inversion is achieved easily in this scheme. Further, the ground-state population $N_0(t)$ remains close to its initial value (ie, under the assumption of large energy gaps, close to the total number of atoms N). For any specified value of the photon number n, the system evolves to a steady state where all the time derivatives $\frac{dN_i}{dt}$ $(i = 0, 1, 2)$ are zero. Denoting the steady-state populations by $\bar{N}_i$ $(i = 0, 1, 2)$, one obtains the following approximate expression for the steady-state population inversion:

$$\Delta N \equiv \bar{N}_2 - \bar{N}_1 \approx \bar{N}_2 \approx \frac{RN}{A_{21} + n\bar{\Gamma}}. \tag{8.315}$$

One observes here the inverse relationship between the population inversion ΔN and the photon number n, which explains the stability of the steady state: for a small initial value of n, the population inversion tends to a large steady value. This, in turn, tends to increase n (we will presently look at the evolution of the photon number distribution in the cavity) and stabilize the population inversion.

The rate analysis of the atomic populations remains incomplete unless the evolution of the photon number distribution is taken into consideration. The photon number distribution, in turn, depends on the atomic populations, and the analysis of the atom-radiation composite system becomes a complex problem.

In the approach adopted here, one can set up an approximation scheme by replacing n in formula (8.315) with the mean photon number $\langle n \rangle$ and, at the same time, expressing $\langle n \rangle$ in terms of the steady-state atomic populations. This gives a system of coupled equations for the steady state of the atom-radiation composite system, from which one can solve for $\langle n \rangle$.

> Once the steady state has been solved for, one can address the fluctuation problem, where the inverse relation between the population inversion and the mean photon number mentioned above assumes relevance, resulting in a suppression of the fluctuations in the photon number distribution compared with the situation for classical chaotic light.

We now look at the evolution of the photon number distribution $P(n)$ $(n = 0, 1, 2, \ldots)$—that is, work out an expression for the rate of change of the probability of the optical cavity containing n photons in the lasing mode. The expression for the rate of change of $P(n)$ is made up of four terms. For instance, assuming that the cavity contains n photons (for which the probability is $P(n)$), the photon number can change by emission from level '2' that occurs at the rate $(n+1)\bar{\Gamma}N_2$, as a result of which one obtains a negative contribution to $\frac{dP(n)}{dt}$ of magnitude $(n+1)\bar{\Gamma}N_2P(n)$.

> The factor $(n+1)\bar{\Gamma}N_2$ appearing here contrasts with the factor $n\bar{\Gamma}N_2$ in the second and third relations in Eq. (8.314). This factor appears because of two distinct processes—a field-induced emission, which is proportional to n, and a spontaneous emission into the lasing mode vacuum, which is independent of n.

With the cavity containing n photons in the lasing mode, there is another mechanism by which the number of photons can change—namely, by *transmission* through the cavity mirrors, where the transmission probability $\bar{\Gamma}_{\text{cav}}$ is a cavity-dependent constant. The contribution of this process of transmission to $\frac{dP(n)}{dt}$ is again negative, and is of magnitude $\bar{\Gamma}_{\text{cav}}nP(n)$.

The other two contributions to the total rate of change of $P(n)$ arise in analogous manners, and one obtains

$$\frac{dP(n)}{dt} = \bar{\Gamma}N_2\,(nP(n-1) - (n+1)P(n)) - \bar{\Gamma}_{\text{cav}}\,(nP(n) - (n+1)P(n+1)) \qquad (n \neq 0), \tag{8.316a}$$

while the rate of change of $P(0)$ is given by

$$\frac{dP(0)}{dt} = -\bar{\Gamma} N_2 P(0) + \bar{\Gamma}_{\text{cav}} P(1)) \tag{8.316b}$$

(check these formulae out).

Eqs. (8.314), (8.316a), and (8.316b) form a system of coupled differential equations for the atomic populations and the photon number distribution, from which it is not difficult to extract useful information under the assumptions made above.

Strictly speaking, Eq. (8.314) makes sense only when it is weighted with the probability $P(n)$ for n photons within the cavity. However, when N_2 varies slowly with n, one can replace n with $\langle n \rangle$ (this approximation is valid for relatively large $\langle n \rangle$) and write, for the steady state value of N_2 (see Eq. 8.315),

$$\bar{N}_2 = \frac{RN}{A_{21} + \langle n \rangle \bar{\Gamma}}. \tag{8.317}$$

On the other hand, the rate equation for the photon number distribution gives, for any specified value of N_2, which we now take to be the steady-state value (8.317),

$$\frac{d\langle n \rangle}{dt} = \bar{\Gamma} \bar{N}_2 (1 + \langle n \rangle) - \bar{\Gamma}_{\text{cav}} \langle n \rangle \tag{8.318a}$$

(check this out), giving the following relation between $\bar{N}_2$ and the steady-state mean photon number ($\langle \bar{n} \rangle$):

$$\bar{\Gamma} \bar{N}_2 (1 + \langle \bar{n} \rangle) = \bar{\Gamma}_{\text{cav}} \langle \bar{n} \rangle. \tag{8.318b}$$

One can now solve for $\bar{N}_2$ and $\langle \bar{n} \rangle$ from Eqs. (8.317) and (8.318b) taken together (where, in the former, $\langle n \rangle$ is replaced with $\langle \bar{n} \rangle$), thereby obtaining a reasonably accurate description of the steady-state laser operation. We introduce, for convenience, the parameters

$$C = \frac{NR\bar{\Gamma}}{A_{21}\bar{\Gamma}_{\text{cav}}}, \quad n_{\text{s}} = \frac{A_{21}}{\bar{\Gamma}}, \tag{8.319}$$

referred to as the *co-operation parameter* and the *saturation photon number*, respectively.

> The expression for the co-operation parameter relates to the line shape function for the lasing transition, which we assume to be of the Lorentzian form. Moreover, the transition is assumed to be in resonance with the cavity housing the active material.

In terms of these parameters, $\langle \bar{n} \rangle$ is seen to satisfy a quadratic equation, of which one solution is negative, while the positive solution reads

$$\langle \bar{n} \rangle = \frac{1}{2} \left[(C-1) n_{\text{s}} + \left((C-1)^2 n_{\text{s}}^2 + 4 C n_{\text{s}} \right)^{1/2} \right], \tag{8.320a}$$

corresponding to which $\bar{N}_2$ is given by

$$\bar{N}_2 = \frac{\bar{\Gamma}_{\text{cav}}}{\bar{\Gamma}} \frac{Cn_s}{n_s + \langle \bar{n} \rangle}. \tag{8.320b}$$

An analysis of the power output from the laser, by taking into account the transmission loss at the end mirrors, tells us that the value $C = 1$ corresponds to the *threshold* of laser operation, since for $C < 1$ the power output is negligibly small, while a substantial power output results for $C > 1$ (see formula (8.323)). The minimum pump rate $R_{\text{threshold}}$ necessary to achieve this threshold is given by

$$R_{\text{threshold}} = \frac{A_{21}\bar{\Gamma}_{\text{cav}}}{N\bar{\Gamma}}. \tag{8.321}$$

As the co-operation parameter is made to increase by an increase in the pump rate R, there occurs a steep rise in the steady-state mean photon number $\langle \bar{n} \rangle$ when C crosses the threshold value unity (this steep rise appears to be tempered in our approximate analysis). For $C > 1$, $\langle \bar{n} \rangle$ attains a value on the order of n_s as seen from the formula

$$\langle \bar{n} \rangle \approx (C - 1)n_s, \tag{8.322}$$

which is obtained from Eq. (8.320a) by disregarding terms $\sim \frac{1}{(C-1)n_s}$ (this explains the significance of the saturation number), while for $C < 1$, the value of $\langle \bar{n} \rangle$ remains far below n_s (typically, $n_s \sim 10^7$ for a gas laser). In other words, there occurs a change analogous to a *phase transition* as the pump rate crosses its threshold value. Fig. 8.22 shows the variation of $\langle \bar{n} \rangle$ with C, where the steep rise at $C = 1$ is evident. $\bar{N}_2$ increases linearly with the pump rate below the threshold, while it levels off to a constant value as the threshold is crossed.

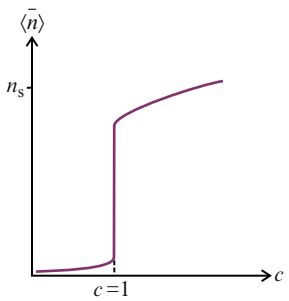

Fig. 8.22

Depicting the variation of the steady-state mean photon number $\langle \bar{n} \rangle$ with the cooperation parameter C. A steep rise in $\langle \bar{n} \rangle$ occurs as C is made to cross the threshold value unity, corresponding to the threshold pump rate given by Eq. (8.321). For $C > 1$, $\langle \bar{n} \rangle$ attains a value of the order of n_s. The vertical scale is logarithmic so as to make apparent the variation of $\langle \bar{n} \rangle$ below the threshold.

The rate equation analysis presented here is only an approximate one since it ignores the correlation between the fluctuations in the atomic populations and the photon number fluctuations, while the two fluctuations, considered independently of each other, are averaged away, resulting in the rate equations of the respective mean values. A more complete analysis for the steady-state mean photon numbers below and above the threshold takes into account the joint probability distribution for the photon number and the atomic populations, but the simpler theory presented here captures the essential aspects of steady-state laser operation.

The fluctuation properties of laser light will be briefly taken up in Section 8.13.5.

8.13.4 Output Photon Flux

The Fabry-Pérot cavity is instrumental in increasing the effective gain and setting up the laser oscillations even in the absence of an input field by causing the photons emitted into the lasing mode to make repeated traversals of the active material. The classical theory of the Fabry-Pérot cavity gives a nonzero output intensity for zero input intensity provided the effective gain in the cavity exceeds a certain threshold value. This result carries over to the quantum theory, where one obtains a nonzero output photon flux by virtue of the lasing action provided the pump rate is above the threshold value (8.321). We assume for simplicity that one of the two end mirrors is a perfectly reflecting one. In that case, for a steady-state mean photon number $\langle \bar{n} \rangle$ in the cavity, the photon flux coming out through the other end mirror is, by the definition of the cavity loss constant $\bar{\Gamma}_{\text{cav}}$,

$$F = \bar{\Gamma}_{\text{cav}} \langle \bar{n} \rangle = \bar{\Gamma}_{\text{cav}} (C - 1) n_{\text{s}} \quad \text{(above threshold)}, \tag{8.323}$$

where the last equality is obtained by Eq. (8.322).

> The cavity loss constant $\bar{\Gamma}_{\text{cav}}$ corresponds to the classical quantity $\frac{c|T|^2}{L}$, where L stands for the distance between the two end mirrors and T stands for the amplitude transmission coefficient of the output coupler (ie, the mirror through which the output beam emerges).

8.13.5 Fluctuation Properties of Laser Light

The fluctuations in the photon number in the cavity are described by the distribution function $P(n)$, which evolves in time in accordance with Eqs. (8.316a) and (8.316b). The atomic populations are also characterized by fluctuations, where these are related to the photon number fluctuations. In the rate analysis outlined in Section 8.13.3, these fluctuations are averaged away by replacement of n in Eq. (8.314) with $\langle n \rangle$, where the rate equation for $\langle n \rangle$, as given by Eq. (8.318a), is obtained by averaging from the rate equation for $P(n)$.

One obtains a reasonably accurate description of the fluctuations in the steady-state laser radiation by noting, from Eq. (8.316a), that in the steady state, $P(n)$ satisfies the population-dependent recursion relation,

$$\bar{N}_2\bar{\Gamma}P(n-1) = \bar{\Gamma}_{\text{cav}}P(n), \tag{8.324}$$

where the steady-state value of N_2 for any given photon number is given by the last equality in Eq. (8.315). Substituting the value of $\bar{N}_2$ corresponding to the photon number $n-1$ in formula (8.324), one obtains the recursion relation

$$P(n) = \frac{\bar{\Gamma}}{\bar{\Gamma}_{\text{cav}}}\frac{RN}{A_{21}+(n-1)\bar{\Gamma}}P(n-1) = \frac{Cn_{\text{s}}}{n_{\text{s}}+n-1}P(n-1), \tag{8.325}$$

which no longer involves the population $\bar{N}_2$. On iterating this relation, one obtains the following expression for $P(n)$ in terms of $P(0)$:

$$P(n) = \frac{(Cn_{\text{s}})^n(n_{\text{s}}-1)!}{(n_{\text{s}}+n-1)!}P(0), \tag{8.326}$$

where $P(0)$ is to be determined from the normalization condition on the probability distribution. For simplicity we consider the distribution for small and large values of C (ie, for values below and above the threshold, respectively).

For C small compared with unity ($\langle\bar{n}\rangle \ll n_{\text{s}}$), one obtains

$$P(n) \approx (1-C)C^n \quad \text{(below threshold)}, \tag{8.327a}$$

which corresponds to a geometric distribution, characteristic of classical chaotic light as in Eq. (8.176b). The classical nature of laser radiation below the threshold becomes apparent from the *second-order degree of coherence*, which is seen to have the value

$$g^{(2)}(0) = 2, \tag{8.327b}$$

(see Eq. 8.377; you will find the basic ideas relating to the second-order degree of coherence outlined in Section 8.17.3, where the second-order degree of coherence distinguishes between the classical and quantum states of radiation).

On the other hand, the steady-state photon number distribution assumes a different form when the pump rate is above the threshold ($\langle\bar{n}\rangle \sim n_{\text{s}}$):

$$P(n) \approx \frac{1}{\left(2\pi(\langle\bar{n}\rangle+n_{\text{s}})\right)^{1/2}}\exp\left[-\frac{(n-\langle\bar{n}\rangle)^2}{2(\langle\bar{n}\rangle+n_{\text{s}})}\right] \quad \text{(above threshold)}. \tag{8.328a}$$

This is a Gaussian distribution analogous to the photon number distribution of a coherent state having a large mean photon number (see formula (8.175); for large values of $\langle n\rangle$ the Poisson distribution is approximated by a Gaussian one), with the difference that the variance exceeds the mean by n_{s}:

$$(\Delta n)^2 = \langle\bar{n}\rangle + n_{\text{s}}. \tag{8.328b}$$

In other words, the photon number distribution of laser light above the threshold, though analogous to that of radiation in a coherent state, is a super-Poissonian one. For values of C substantially larger than unity (ie, for laser radiation much above the threshold), the photon number distribution tends to a Poissonian form, and the second-order degree of coherence approaches the value unity, characteristic of coherent radiation (refer to Section 8.17.3.1):

$$g^{(2)}(0) = 1 + \frac{n_s}{(\langle \bar{n} \rangle)^2} \approx 1. \tag{8.328c}$$

One can also work out the fluctuations in the electric field strength in the laser radiation above the threshold, where one finds that the squared field amplitude is characterized by a Gaussian distribution with a narrow width, while the phase is uncertain in the entire range from 0 to 2π. As seen in Eq. (8.328b), the amplitude fluctuation is slightly larger than that in coherent radiation, because of the small degree of spontaneous emission mixed in laser light.

The above analysis of fluctuations in laser light is not a complete one since it is based on only the number distribution of the lasing mode photons. More generally, one has to consider the *density operator* for the laser field, where, referred to the photon number basis, *off-diagonal* elements (the so-called 'coherences') are present in addition to the diagonal elements $P(n)$. However, the results derived above are seen to remain valid in the more general analysis, with only minor modifications.

Laser physics is a vast subject having innumerable aspects integrated into a complex whole. I have, in the preceding paragraphs, touched on only a few basic principles of laser operation relating to *continuous wave* laser action, where the rate equation analysis is of considerable relevance. Another important type of laser action involves transient and *pulsed* laser output. The pattern of laser activity in the time domain and in the frequency domain can show a wide variation due to such means as pulsed pumping, Q switching, and various external perturbations. These complex patterns of activity require a number of further considerations for an appropriate description, which I do not enter into in this introductory exposition.

8.14 Quantum Theory of Photocounting

8.14.1 Photodetection Probability

Optical setups use detectors in effecting measurements in optical fields, where these detectors mostly work on the principle of *photon absorption* by material particles or aggregates of particles. The material particles in the detector interact with the electromagnetic field, as a result of which there occur transitions in their quantum states along with annihilation of photons. The absorption of a photon causes the emission of an electron with a certain probability, and one makes use of these electrons in inferring the characteristics of the optical field. Appropriate circuitry is used to either count the number of electrons emitted per unit

time or measure the photocurrent generated in the circuit by means of these electrons. This method is referred to as 'direct detection' since the detector responds to the intensity (or, equivalently, the photon flux) of the field directly incident on it. An alternative method—namely, *homodyne detection*—will be considered in Section 8.19

Under certain commonly encountered conditions, the interaction between an atom or a molecule in the detector and the electromagnetic field can be assumed to be of the dipole type (see Section 8.12), and the probability of photon absorption along with a transition in the atomic state in a small time interval is proportional to $|\langle\Psi_f| - \hat{\mathbf{d}} \cdot \hat{\mathbf{E}}|\Psi_i\rangle|^2$. In this expression, $|\Psi_i\rangle$ and $|\Psi_f\rangle$ stand for the initial and final states (assumed to be pure ones for the time being) of the atom-field composite system, $\hat{\mathbf{d}}$ stands for the atomic dipole operator, and $\hat{\mathbf{E}}$ stands for the electric vector operator of the electromagnetic field.

With reference to an actual detection or measurement process, the initial state $|\Psi_i\rangle$ may be assumed to be given, while transitions to all the accessible final states $|\Psi_f\rangle$ are to be considered in working out the total probability of photon absorption. In reality, only a given set of final states $|\Psi_f\rangle$ are allowed (ie, are accessible from the initial state by the *selection rules* governing the process), but one can sum over *all* the possible final states since the selection rules ensure that the contribution of the inaccessible states to the sum is zero anyway.

The expression for the electric field operator $\hat{\mathbf{E}}$ at the point $\mathbf{r}$ and at time t for a single mode can be obtained from Eq. (8.166a) in the Heisenberg picture. In general, even for a multimode field, the electric field operator is seen to be made up (up to a factor $\frac{1}{2}$) of a *positive frequency* part ($\hat{\mathbf{E}}^{(+)}$) involving only the annihilation operators for the various modes, and a *negative frequency* one $\mathbf{E}^{(-)}$ involving only the creation operators. Since $|\Psi_f\rangle$ in the above expression results from $|\Psi_i\rangle$ by the absorption of a single photon, it suffices to retain only the positive frequency part of $\hat{\mathbf{E}}$ in the above transition matrix element, since the negative frequency part, involving only creation operators, gives a zero contribution.

The probability of photodetection per unit time at a point $\mathbf{r}$ and at time t then assumes the form

$$P(\mathbf{r}, t) = A\langle\Psi_i||\hat{\mathbf{d}} \cdot \hat{\mathbf{E}}^{(+)}(\mathbf{r}, t)|^2|\Psi_i\rangle, \tag{8.329a}$$

where A is a constant depending on the detector (check this result out; use the *completeness* of the set of all possible final states $|\Psi_f\rangle$).

Under commonly encountered experimental situations this expression can be further transformed to one of the form

$$P(\mathbf{r}, t) = A|\mathbf{p} \cdot \hat{e}|^2\langle\psi_i||\hat{E}^{(+)}(\mathbf{r}, t)|^2|\psi_i\rangle, \tag{8.329b}$$

where $\mathbf{p}$ stands for the expectation value of the atomic dipole moment in the initial atomic state and $|\psi_i\rangle$ denotes the initial field state. Here we have assumed that the electric field is polarized along a given unit vector $\hat{e}$ and have introduced a scalar electric field operator $\hat{E}(\mathbf{r}, t)$

(for a field in an arbitrary state of polarization, one has to perform a sum over a pair of orthogonal polarization vectors).

Since the positive and negative frequency parts of the field operator are adjoints of each other, one finally obtains

$$P(\mathbf{r},t) = \eta' \langle \psi_{\mathrm{i}} | \hat{E}^{(-)}(\mathbf{r},t) \hat{E}^{(+)}(\mathbf{r},t) | \psi_{\mathrm{i}} \rangle, \tag{8.330}$$

where η' is a new constant (analogous to the constant η introduced in Section 7.19.1) depending on the detector and involving the atomic dipole moment in the initial state.

More generally, if the initial field state is a mixed one described by the density operator $\hat{\rho}$, then the rate of emission of photoelectrons at $(\mathbf{r},t)$ is of the form

$$P(\mathbf{r},t) = \eta' \langle \hat{E}^{(-)}(\mathbf{r},t) \hat{E}^{(+)}(\mathbf{r},t) \rangle = \eta' \mathrm{Tr}(\hat{E}^{(-)}(\mathbf{r},t) \hat{E}^{(+)}(\mathbf{r},t) \hat{\rho}). \tag{8.331}$$

In looking closely at the above expression, one observes that it involves the quantum mechanical average of the product $\hat{E}^{(-)}(\mathbf{r},t)\hat{E}^{(+)}(\mathbf{r},t)$, which is analogous to a classical ensemble average of the form

$$\langle E^*(\mathbf{r},t) E(\mathbf{r},t) \rangle,$$

where E stands for the scalar analytic signal corresponding to the real random variable describing the fluctuating electric field at $(\mathbf{r},t)$, and where one is to recall that the analytic signal relates only to the positive frequency components of the real electric field (while, correspondingly, E^* relates to the negative frequency part).

Note that, since the positive and negative frequency parts of the field operator involve only the annihilation operators and only the creation operators, respectively, the product appearing in Eq. (8.331) is a *normal ordered* one. The normal ordering here is a consequence of the fact that the photodetection process is based on photon absorption.

In analogy with the classical case, the above expectation value may be seen to constitute a particular instance of what may be termed a quantum *correlation* function. Quantum correlation functions will be discussed in Section 8.15.

The semiclassical theory of photodetection, briefly mentioned in Section 7.19, tells us that the photocount rate is proportional to the instantaneous intensity I of the field at the location of the photodetector, where the field is described in classical terms. The theory outlined in the present section is a quantum version of the semiclassical theory, since the operator $\hat{E}^{(-)}(\mathbf{r},t)\hat{E}^{(+)}(\mathbf{r},t)$ occurring in formula (8.331) can be interpreted as the quantum mechanical intensity operator (up to a scale factor) at the point $\mathbf{r}$ at time t.

1. The quantum mechanical Poynting vector operator is defined in analogy with the classical expression, with appropriate normal ordering of the field operators. For a parallel beam of linearly polarized radiation, the Poynting vector operator points along

the direction of the beam, and its magnitude is proportional to $\hat{E}^{(-)}\hat{E}^{(+)}$ as mentioned above (with a constant of proportionality $\frac{1}{4}\epsilon_0 c$).

2. The constant η in formula (7.159) differs from the constant η' introduced above since these two appear in two different contexts. Identifying the quantum expectation value of the intensity operator with the classical intensity, one observes that these can be taken to be proportional to each other.

We now specialize to the case, corresponding to numerous situations of practical interest, of a parallel beam of linearly polarized narrowband light incident on the detector, where one can use expression (8.201) for the intensity operator. Taking note of the definition of the photon flux operator in Eq. (8.202), one can relate the instantaneous photocount rate to the photon flux. In an actual experiment, one determines the number of photocounts in some finite time interval t' (analogous to the integrated intensity in the classical theory), which relates to the number of photons arriving at the detector in that time, for which the operator is

$$\hat{N}(t,t') = \int_t^{t+t'} dt'' \hat{a}^\dagger(t'')\hat{a}(t''). \tag{8.332}$$

The expectation value of the above expression is independent of the location of the photodetector for a stationary light beam. However, for a nonstationary field, the position of the detector in the beam enters as in Eq. (8.201).

The photodetector, however, does not directly measure the expectation value of this quantity since, in practice, one cannot achieve a complete conversion of photons to photocounts for a variety of reasons, one of which is the *recovery time* of the detector (the time during which the detector remains nonresponsive immediately following a photoemission event). The photon flux arriving at the detector and the *effective* flux causing photoemission events are related as in the incident beam and the transmitted (or reflected) beam in a beam splitter. Recall that in a beam splitter the output annihilation operators are related to the input operators by the beam splitter matrix (see Section 8.11.1). In contrast to the classical beam splitter action (where the field amplitudes are related by the same beam splitter matrix), this introduces additional *quantum fluctuations* in the output beam. In this quantum context, only one of the two input arms of the equivalent beam splitter is assumed to carry the input photon flux, while the other input arm carries the vacuum field.

Taking note of this relation between the incident photon flux and the *effective* photon flux, one defines a *photocount* operator for counting time interval t' as

$$\hat{N}_{\mathrm{D}}(t,t') = \int_t^{t+t'} dt'' \hat{d}^\dagger(t'')\hat{d}(t''), \tag{8.333}$$

where the subscript 'D' is used to refer to quantities actually measured by the detector, and $\hat{d}$ stands the annihilation operator for the transmitted beam of the equivalent beam splitter.

The *mean* photocount in time t' is then given by

$$\langle \hat{N}_{\mathrm{D}}(t,t')\rangle = \eta_{\mathrm{Q}}\langle \hat{N}(t,t')\rangle, \tag{8.334a}$$

where η_{Q} is the *quantum efficiency* of the detector, related to the transmission coefficient T (refer to the beam splitter matrix in Eq. (8.243); the symbol t in that equation is replaced with T here) of the equivalent beam splitter as

$$\eta_{\mathrm{Q}} = |T|^2. \tag{8.334b}$$

8.14.2 Photocount Distribution

One can similarly work out the mean squared photocount rate and also the higher moments of the photocount distribution, taking care to use only normal ordered products in the calculations so as to eliminate the effects of the vacuum field, which is a requirement since the photocounting process works by photon absorption. For instance, the expectation value of the squared photocount operator is

$$\langle \hat{N}_{\mathrm{D}}(t,t')^2\rangle = \eta_{\mathrm{Q}}^2\langle \hat{N}(t,t')^2\rangle + \eta_{\mathrm{Q}}(1-\eta_{\mathrm{Q}})\langle \hat{N}(t,t')\rangle, \tag{8.335}$$

from which one obtains

$$\begin{aligned}\langle \Delta\hat{N}_{\mathrm{D}}(t,t')^2\rangle &= \eta_{\mathrm{Q}}^2\langle \Delta\hat{N}(t,t')^2\rangle + \eta_{\mathrm{Q}}(1-\eta_{\mathrm{Q}})\langle \hat{N}(t,t')\rangle \\ &= \langle \hat{N}_{\mathrm{D}}(t,t')\rangle + \eta_{\mathrm{Q}}^2(\langle \Delta\hat{N}(t,t')^2\rangle - \langle \hat{N}(t,t')\rangle).\end{aligned} \tag{8.336}$$

The quantum efficiency can be expressed as a multiplicative scale factor times a dimensionless constant intrinsic to the detector that can have a value from 0 to 1. For $\eta_{\mathrm{Q}} \approx 0$, one will always have $\langle \Delta\hat{N}_{\mathrm{D}}(t,t')^2\rangle \approx \langle \hat{N}_{\mathrm{D}}(t,t')\rangle$—that is, the photocount statistics will show a Poisson distribution regardless of the state of the field and will fail to distinguish classical from nonclassical states. The latter two are distinguished by the statistics of $\hat{N}(t,t')$, which is Poissonian or super-Poissonian for a classical state, and sub-Poissonian for a number of nonclassical states. Expression (8.336) shows that the statistics of $\hat{N}_{\mathrm{D}}(t,t')$ can reflect that of $\hat{N}(t,t')$, indicating faithfully whether it is sub-Poissonian or not, only for relatively high values of η_{Q} (ie, for efficient detectors). Indeed, the conclusive proof of the quantum nature of electromagnetic radiation came relatively late in the day, only when detectors with high quantum efficiency were designed, and effects relating to sub-Poissonian photon statistics and antibunching were observed experimentally.

I will now give you the *photocount distribution formula*—that is, the probability of obtaining n photocounts in time t' ($n = 0, 1, 2, \ldots$)—which is the quantum analogue of the semiclassical Mandel formula (7.165):

$$P_n(t,t') = \left\langle \mathcal{N}\left[\frac{(\eta_{\mathrm{Q}}\hat{N}(t,t'))^n}{n!}\mathrm{e}^{-\eta_{\mathrm{Q}}\hat{N}(t,t')}\right]\right\rangle, \tag{8.337}$$

where the symbol $\mathcal{N}[\cdots]$ signifies a normal ordering of the operator expression contained within the brackets.

For instance, $\mathcal{N}[(\hat{a}^\dagger\hat{a})^2] = \hat{a}^{\dagger 2}\hat{a}^2 + \hat{a}^\dagger\hat{a}$.

While the master formula (8.337) has been stated for a parallel beam of polarized light (formulae (8.334a) and (8.335) can be deduced as corollaries), one can derive the photocount distribution formula for more general field states by use of the (scaled) quantum intensity operator $\hat{I}(\mathbf{r},t) = \hat{E}^{(-)}(\mathbf{r},t)\hat{E}^{(+)}(\mathbf{r},t)$ instead of the photon flux operator for the parallel beam, thereby arriving at:

$$P_n(t,t') = \left\langle \mathcal{N}\left[\frac{(\eta'_{\mathrm{Q}}\hat{J}(t,t'))^n}{n!}\mathrm{e}^{-\eta'_{\mathrm{Q}}\hat{J}(t,t')}\right]\right\rangle, \tag{8.338a}$$

where

$$\hat{J}(t,t') = \int_t^{t+t'} \hat{I}(t'')dt'' = \int_t^{t+t'} dt''\hat{E}^{(-)}(t'')\hat{E}^{(+)}(t''), \tag{8.338b}$$

and where the spatial coordinate $\mathbf{r}$ is left understood. The constant η'_{Q} differs from η_{Q} by a scale factor. More precisely, each of the two constants η_{Q} and η'_{Q} is of the form $\mu\eta_0$, where η_0 is a dimensionless constant characterizing the detector efficiency, while μ is an appropriate scale factor. As explained above, one needs a detector with a relatively large value of η_0 to identify nonclassical field states by means of photocounting, while a low value of η_0 always produces a photocount close to a Poissonian one.

8.15 Quantum Correlation Functions

The photocount variance formula (8.336) relates to a quantity that can be determined experimentally and tells us that the theoretical expression for this quantity (the right-hand side of the equation) involves the expectation value of a product of four field functions, in addition to that of a product of two field functions, where the factors in the product may correspond to different space-time points (this is seen more directly from the expression of the photocount variance in terms of the field strength operator, ie, one where $\hat{N}_{\mathrm{D}}(t,t')$ is replaced with $\hat{J}(t,t')$ and the constant η_{Q} is appropriately rescaled).

More generally, one is led to the consideration of quantum *correlation functions* involving normal ordered products of field operators evaluated at distinct space-time points, where there may be an arbitrary number of factors in the products (with the number of positive frequency factors, however, being the same as the number of negative frequency factors). For instance, the first-order correlation function in formula (8.331) defines (up to an appropriate scale factor) the average intensity at $(\mathbf{r},t)$,

$$I(\mathbf{r},t) = G^{(1)}(\mathbf{r},t;\mathbf{r},t) = \langle\hat{E}^{(-)}(\mathbf{r},t)\hat{E}^{(+)}(\mathbf{r},t)\rangle, \tag{8.339a}$$

while the first-order field correlation between space-time points $(\mathbf{r}_1, t_1)$ and $(\mathbf{r}_2, t_2)$ is of the form

$$G^{(1)}(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2) = \langle \hat{E}^{(-)}(\mathbf{r}_1, t_1)\hat{E}^{(+)}(\mathbf{r}_2, t_2)\rangle. \tag{8.339b}$$

A correlation function of the second order, on the other hand, looks like

$$G^{(2)}(\mathbf{r}_1, t_1, \mathbf{r}_2, t_2; \mathbf{r}_3, t_3, \mathbf{r}_4, t_4) = \langle \hat{E}^{(-)}(\mathbf{r}_1, t_1)\hat{E}^{(-)}(\mathbf{r}_2, t_2)\hat{E}^{(+)}(\mathbf{r}_3, t_3)\hat{E}^{(+)}(\mathbf{r}_4, t_4)\rangle. \tag{8.339c}$$

These field correlation functions define the *coherence characteristics* of an optical field, where, as in the case of a classical field, correlation functions of relatively low order (mostly, those of order 1 and 2) relate to field characteristics that are commonly determined experimentally. As in Eq. (8.331), all the correlation functions involve products of field operators in the normal order.

The first-order correlation in Eq. (8.339a) is determined by a device placed at the point $\mathbf{r}$ that measures the ensemble-averaged instantaneous intensity (compare Eqs. 8.331 and 8.339a with the semiclassical formula (7.159)) by means of the mean photocount rate.

On the other hand, the second-order correlation function $G^{(2)}(\mathbf{r}_1 t_1, \mathbf{r}_2 t_2; \mathbf{r}_1 t_1, \mathbf{r}_2, t_2)$ gives the correlation between the photocount rate at $(\mathbf{r}_1, t_1)$ and that at $(\mathbf{r}_2, t_2)$. All these quantum correlation functions are analogous to classical correlation functions of various orders. Indeed, the optical equivalence theorem allows us to interpret the quantum correlations formally in terms of corresponding correlations of an equivalent classical field in a mixed classical state defined by a distribution function in a surrogate phase space.

This is seen by expressing the electric field operators occurring in the expressions for the correlation functions in terms of the creation and the annihilation operators and then invoking the P-representation of the field state in which the expectation value is sought to be evaluated (see Section 8.10.2).

However, as I have already mentioned, this does not *reduce* the quantum correlations to classical ones. As regards the features based on the first-order correlation functions, though, there indeed is a convergence between the quantum and the classical coherence characteristics. On the other hand, the coherence characteristics based on the second-order correlation functions clearly distinguish between the classical and the quantum descriptions. I will now briefly outline what this means.

8.16 First-Order Coherence

The classical description of the coherence characteristics based on the first-order correlation function was taken up in the explanation of interference phenomena (see Chapter 4) and again, at a more fundamental theoretical level, in Section 7.14.

8.16.1 First-Order Correlation: Classical Coherence Characteristics

The classical coherence characteristics in question are all explained in terms of the mutual coherence function $\Gamma_{12}(\tau)$ (see formula (7.125)), which is directly related to the correlation function

$$\Gamma(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2) = \langle E^*(\mathbf{r}_1, t_1) E(\mathbf{r}_2, t_2) \rangle \tag{8.340a}$$

as

$$\Gamma(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2) = \Gamma_{12}(t_2 - t_1). \tag{8.340b}$$

The basic coherence characteristic emerging from the first-order classical correlation function $\Gamma(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2)$ relates to the intensity distribution in an interference pattern such as in Young's double-hole setup shown in Fig. 7.8. I will now briefly recall, for your easy reference, how this comes about.

Under a set of simplifying assumptions explained in Sections 4.2.5 and 4.5, one can use scalar field variables in explaining the intensity distribution, where the intensity at any observation point P is worked out in terms of a superposed field relating to the fields (more precisely, the relevant analytic signals) at the two pinholes as (in a by now familiar notation)

$$E(\mathbf{r}, t) = E\left(\mathbf{r}_1, t - \frac{u_1}{c}\right) + E\left(\mathbf{r}_2, t - \frac{u_2}{c}\right). \tag{8.341}$$

This gives the instantaneous intensity (up to an inessential scale factor)

$$I(\mathbf{r}, t) = \langle E^*(\mathbf{r}, t) E(\mathbf{r}, t) \rangle = \Gamma_{11} + \Gamma_{22} + 2\mathrm{Re}\left(\Gamma_{12}(\tau)\right) \quad \left(\tau = \frac{u_1 - u_2}{c}\right), \tag{8.342}$$

the derivation of which is outlined in Section 7.14.5.1 in a slightly different notation and in a slightly more general form involving the constants $|N_1|^2$, $|N_2|^2$, and $|N_1 N_2|$, all of which have been set at unity, consistent with our level of approximation. For a stationary field the instantaneous intensity is independent of t, as seen from the right-hand side of expression (8.342).

The intensity variation in the double-hole interference pattern is explained entirely by the third term in formula (8.342), where the mutual coherence function Γ_{12} makes an appearance. More precisely, the intensity variation occurs principally through the *phase* of the mutual coherence function or, for a monochromatic field, through a term of the form $2|\Gamma_{12}| \cos(\delta_0 + \delta)$, where $\delta = \frac{\omega}{c}(u_1 - u_2)$ and where δ_0 is a phase independent of the position of the observation point P, which can be taken to be zero for simplicity. Thus, in summary, the variable part of the intensity distribution is entirely determined by the first-order correlation function.

One needs to mention in this context the significance of the complex degree of coherence $\gamma^{(1)}$, which is obtained from Γ_{12} by means of a normalization (Eq. 7.127a) and which determines the visibility of the interference fringes as in Eq. (7.132). The magnitude of the degree of

coherence lies in the range from zero to unity (see Eq. 7.119; Eq. 7.127b is a restatement of the same bounds):

$$0 \leq |\gamma^{(1)}| \leq 1, \tag{8.343}$$

where the upper limit of the range corresponds to the case of complete coherence, for which a sufficient condition is the factorizability of $|\Gamma_{12}|$ as $\sqrt{\Gamma_{11}\Gamma_{22}}$. We use here a slightly different notation, suppressing the reference to the spatial points and adding the superscript '(1)' to denote the order of the degree of coherence (where we follow a convention analogous to the one commonly adopted in quantum optics). The lower limit ($\gamma^{(1)} = 0$), on the other hand, corresponds to complete incoherence, while intermediate values indicate various intermediate degrees of first-order coherence.

The case of a linearly polarized plane wave incident on the pinholes deserves mention since this corresponds to complete coherence. For a plane wave of the form

$$E_{\text{incident}} = E_0 \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \tag{8.344a}$$

one obtains, from Eq. (8.342), the intensity at the point of observation in the form

$$I = 2|E_0|^2(1 + \cos\delta), \tag{8.344b}$$

where a number of simplifying assumptions are made (such as the incident wave vector is normal to the screen containing the pinholes, the lines joining the pinholes to the observation point are almost parallel to the incident wave vector, and the two pinholes transmit identically), and where δ has been defined above.

8.16.2 First-Order Quantum Coherence

8.16.2.1 Interference: The intensity formula

In the quantum description the first-order coherence effects appear to be substantially similar to the corresponding classical effects (see Section 4.11 for a brief introduction). In particular, the intensity distribution in an interference setup with monochromatic light shows a $\cos\delta$ variation analogous to what we find in the classical description and also in experimental observations, as we will now see.

Confining ourselves to a description in terms of a scalar field in a double-hole setup (as in the case of a field with the electric vector linearly polarized along a fixed direction; in the context of the double-hole setup, this requires the two lines joining the pinholes to the point of observation to be in near-forward directions), the positive frequency field operator at the point of observation can be expressed in the form (see Eq. 8.341 for the purpose of comparison)

$$\hat{E}^{(+)}(\mathbf{r}, t) = \hat{E}^{(+)}\left(\mathbf{r}_1, t - \frac{u_1}{c}\right) + \hat{E}^{(+)}\left(\mathbf{r}_2, t - \frac{u_2}{c}\right). \tag{8.345}$$

The instantaneous intensity (see Eq. 8.339a), which is independent of time for a stationary field, is then

$$I(\mathbf{r},t) = G^{(1)}(x_1,x_1) + G^{(1)}(x_2,x_2) + 2\mathrm{Re}(G^{(1)}(x_1,x_2)), \tag{8.346}$$

where x_i $(i = 1, 2)$ stands for the pair $(\mathbf{r}_i, t - \frac{u_i}{c})$ and where, for a stationary field, each of the terms in expression (8.346) is actually independent of the origin of time.

In this formula for the intensity, which is analogous to the classical expression (8.342), the first two terms represent the intensities due to the two pinholes each considered independently of the other, while the last term stands for the interference effect due to the two pinholes transmitting simultaneously. As in the classical case, the intensity variation occurs principally through the cosine of the phase angle of $G^{(1)}(x_1, x_2)$, which is the quantum analogue of the classical mutual coherence function. It now remains to work out the relevant first-order correlation functions for the state of the field incident on the two pinholes.

For this, recall that formula (8.345) relates the field operators at an observation point in the region to the right of the screen in Fig. 7.8 to those in the immediate neighborhood of the two holes. This can be regarded as a field transformation from a pair of 'input ports' (the two pinholes) to an 'output port' (the observation point). For an appropriate transformation to be defined one needs, however, two output ports as in a beam splitter or a Mach-Zehnder interferometer, where in the present instance the second output port may be taken to be a redundant one. The output field operators are related to the input ones in a manner analogous to Eq. (8.236).

For a pair of pinholes transmitting identically to the observation point at the same phase, the relevant output creation operator is related to the input ones as

$$\hat{b}^\dagger = \frac{1}{\sqrt{2}}(\hat{a}_1^\dagger + \hat{a}_2^\dagger), \tag{8.347}$$

the corresponding annihilation operators being related by the adjoint formula (the formulae for the operators for the redundant output port need not be considered). A pure state of the field at the observation point can be obtained with the help of an appropriate function of the output creation and annihilation operators on the vacuum state, while a mixture of such pure states is also possible.

In the following, we first consider a *one-photon* field incident on the pinholes as an instance of interference in a nonclassical field state. A one-photon field state at the observation point is given by

$$|\psi\rangle = \hat{b}^\dagger|0\rangle = \frac{1}{\sqrt{2}}(\hat{a}_1^\dagger + \hat{a}_2^\dagger)|0\rangle. \tag{8.348}$$

This formula needs a bit of explanation since the input field operators correspond to two distinct modes (corresponding to the pinholes) that constitute a composite system, of which $|0\rangle$ is the joint vacuum state. The expression $\hat{a}_1^\dagger$ then stands as an abbreviation for the direct product $\hat{a}_1^\dagger \otimes \hat{I}$, where $\hat{I}$ denotes the unit operator for the second mode (pinhole P_2 in Fig. 7.8; the present context differs from the one shown in Fig. 7.8 in that a plane monochromatic field is assumed to be incident on the screen). Similarly $\hat{a}_2^\dagger$ should more properly be expressed as $\hat{I} \otimes \hat{a}_2^\dagger$, where $\hat{I}$ now stands for the unit operator for the first mode (pinhole P_1). We will, however, continue to use the shorter forms of the expressions for brevity.

Eq. (8.348) evaluates to the following superposition of direct products of states for the two modes:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|1\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle), \tag{8.349}$$

which is actually a two-mode *entangled* state of the field. However, in spite of the nonclassical nature of this state, the first-order coherence characteristics of the field as revealed through the intensity expression (8.346) resemble the classical ones, as we will see below.

The first-order correlation functions in the intensity formula (8.346) can all be evaluated now by working out the relevant expectation values in the state (8.349) and by using the following expression for the positive frequency field operators occurring on the right-hand side of Eq. (8.345):

$$\hat{E}^{(+)}\left(\mathbf{r}_i, t - \frac{u_i}{c}\right) = \gamma \hat{a}_i \exp\left(-\mathrm{i}\omega\left(t - \frac{u_i}{c}\right)\right) \quad (i = 1, 2). \tag{8.350}$$

In writing this last expression, we have assumed the propagation vectors of the plane wave modes under consideration to be along the z-axis of a Cartesian system and the screen containing the pinholes to lie in the plane $z = 0$, and have used the constant γ defined in Eq. (8.179c). There are, in addition, the usual simplifying assumptions (see Section 8.16.1) such as the one where the lines P_1P and P_2P in Fig. 7.8 are assumed to be nearly parallel to the z-axis, and also the one where the pinholes are assumed to transmit identically.

The rest of the calculation is now straightforward, involving a few steps of algebra, and gives the following results:

$$G^{(1)}(x_1, x_1) = G^{(1)}(x_2, x_2) = \frac{\gamma^2}{2}, \quad G^{(1)}(x_1, x_2) = G^{(1)*}(x_2, x_1) = \frac{1}{2}\gamma^2 \mathrm{e}^{\mathrm{i}\delta} \quad \left(\delta = \frac{\omega}{c}(u_1 - u_2)\right) \tag{8.351}$$

(check this out). The intensity expression then works out, up to a scale factor, to be

$$I = \gamma^2(1 + \cos\delta), \tag{8.352}$$

in complete analogy with Eq. (8.344b), since the assumption of monochromatic plane wave modes ensures complete coherence, because of the factorizability of $|G^{(1)}(x_1, x_2)|$, which is seen to be trivially satisfied in the present case.

This corroborates the statements made in Section 4.11 to the effect that the phenomenon of interference admits of an explanation from both the classical point of view and the quantum point of view since it relates to the first-order coherence characteristics of the electromagnetic field. Even the nonclassical entangled one-photon state is seen to give rise to the same intensity formula as obtained in the classical theory. As we will see later, an identical intensity distribution formula is obtained for the classical coherent state of the field. The distinction between the nonclassical and the classical states becomes apparent when one considers the *second*-order coherence characteristics, such as the ones relating to the Hanbury Brown-Twiss effect.

A derivation similar to the one leading to Eq. (8.352) also works for the case of an incident field in an n-photon state ($n = 2, 3, \ldots$), where the number of photons (n) may be variously distributed between the two modes corresponding to the pinholes, making up an entangled state of a more complex structure than the one in Eq. (8.349). The intensity expression in this case is

$$I = 2\gamma^2 n^2 (1 + \cos\delta). \tag{8.353}$$

On the other hand, for a state such as $|\psi\rangle = |1\rangle \otimes |1\rangle$, which contains one photon in each of the two relevant modes, there is no phase correlation between the fields due to the two pinholes, and the interference fringes disappear (check this out).

We next turn our attention to the case of a *coherent field*. For a coherence parameter λ, the state of the field at the point of observation is obtained by considering the action of the displacement operator $\hat{D}(\lambda) = \exp(\lambda \hat{b}^\dagger - \lambda^* \hat{b})$ on the vacuum state—that is, from Eq. (8.347),

$$|\psi\rangle = \hat{D}(\lambda)|0\rangle = \left|\frac{\lambda}{\sqrt{2}}\right\rangle \otimes \left|\frac{\lambda}{\sqrt{2}}\right\rangle, \tag{8.354}$$

where we have used the fact that the operators $\hat{a}_1$ and $\hat{a}_1^\dagger$ commute with $\hat{a}_2$ and $\hat{a}_2^\dagger$. In contrast to the single-photon case, this a direct product of two coherent states, each with a parameter $\frac{\lambda}{\sqrt{2}}$, and one can again work out the intensity expression with little difficulty, obtaining the result

$$I = \gamma^2 |\lambda|^2 (1 + \cos\delta) \tag{8.355}$$

(check out the results (8.352) and (8.355)).

A similar result is obtained for a state of the form $|\lambda\rangle \otimes |\lambda'\rangle$, though in this case there is only partial coherence for $\lambda \neq \lambda'$, with an associated impairment of the visibility.

While we have considered first-order quantum coherence in the preceding paragraphs in the context of the double-hole interference setup, similar results are obtained for other setups as well, such as the Mach-Zehnder interferometer, of which the quantum theory was discussed in Section 8.11.2. Thus for a one-photon input state, the mean intensities in the output ports '7' and '8' can be worked out from Eq. (8.251b) and are given by (up to a multiplicative constant each)

$$I_7 = \frac{1}{2}(1 + \cos\delta), \quad I_8 = \frac{1}{2}(1 - \cos\delta), \tag{8.356}$$

in conformity with the classical result (check this out).

8.16.2.2 Quantum first-order degree of coherence

We will now look into expressions, in respect of a number of chosen field states, for the quantum first order degree of coherence $g^{(1)}$ derived from the first-order correlation function $G^{(1)}$ as

$$g^{(1)}(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2) = \frac{G^{(1)}(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2)}{[G^{(1)}(\mathbf{r}_1, t_1; \mathbf{r}_1, t_1) G^{(1)}(\mathbf{r}_2, t_2; \mathbf{r}_2, t_2)]^{1/2}}; \tag{8.357}$$

that is, in a manner analogous to the way the classical degree of coherence is related to the correlation function (see Eq. 7.127a, where the space-time points are indicated in a different manner and where a stationary field is assumed). As in the classical context, the quantum degree of coherence determines the relative variation of intensity in an interference pattern (check this out; see Eq. 8.346). For each of the various field states considered below, we will compare the range of variation of the magnitude of the quantum degree of coherence ($|g^{(1)}|$) with the corresponding range (of $|\gamma^{(1)}|$) for a classical field, where the latter extends from 0 to 1, as indicated in Eq. (8.343).

In the following we specialize to linearly polarized light beams with their propagation vectors parallel to the z-axis of a coordinate system, in which case the expression for the quantum first order degree of coherence simplifies to

$$\begin{aligned} g^{(1)}(z_1, t_1; z_2, t_2) &= \frac{G^{(1)}(z_1, t_1; z_2, t_2)}{[G^{(1)}(z_1, t_1; z_1, t_1) G^{(1)}(z_2, t_2; z_2, t_2)]^{1/2}} \\ &= \frac{\langle \hat{E}^{(-)}(z_1, t_1) \hat{E}^{(+)}(z_2, t_2) \rangle}{[\langle \hat{E}^{(-)}(z_1, t_1) \hat{E}^{(+)}(z_1, t_1) \rangle \langle \hat{E}^{(-)}(z_2, t_2) \hat{E}^{(+)}(z_2, t_2) \rangle]^{1/2}}. \end{aligned} \tag{8.358}$$

The assumption of linear polarization allows us to use a scalar electric field strength operator, which we have been doing for the most of this chapter, for the sake of simplicity.

In the particular case of a stationary beam propagating in free space, the dependence on the spatial coordinate z and time t in all the relevant expressions occurs through the combination $t - \frac{z}{c}$, and $g^{(1)}$ becomes a function of $\tau = t_2 - t_1 - \frac{z_2 - z_1}{c}$ alone, as

$$g^{(1)}(\tau) = \frac{\langle E^{(-)}(t)E^{(+)}(t+\tau)\rangle}{\langle E^{(-)}(t)E^{(+)}(t)\rangle}, \tag{8.359}$$

where, on the right-hand side, t can be assigned any arbitrarily chosen value.

In close analogy to the classical first order degree of coherence $\gamma^{(1)}$, the value of $|g^{(1)}|$ is seen to lie in the range

$$0 \leq |g^{(1)}| \leq 1 \tag{8.360}$$

(compare with Eq. 8.343), which testifies to the resemblance between first-order quantum and classical coherence characteristics, as we saw in Section 8.16.2.1. This will be borne out by the expressions for $g^{(1)}$ for the various field states considered below. The resemblance, however, is lost as one looks at the *second-order* coherence characteristics, as we will see in Section 8.17.2

Single-mode field states

For a single-mode light beam, not necessarily a stationary one, one can use the expression (8.179a) for the electric field strength operator, in which case the expression for $g^{(1)}$ assumes the simple form

$$g^{(1)}(z_1, t_1; z_2, t_2) = g^{(1)}(\tau) = \mathrm{e}^{\mathrm{i}(\chi_1 - \chi_2)}, \tag{8.361}$$

where $\chi_i = \omega t_i - k z_i - \frac{\pi}{2}$ $(i = 1, 2)$ and $\omega\tau = \chi_2 - \chi_1$. A single-mode field corresponding to a plane parallel light beam is thus first-order coherent ($|g^{(1)}| = 1$) regardless of which state it is in.

Multimode field states

Unlike the single-mode states, the first-order degree of coherence for a multimode field does depend on the state under consideration. For a multimode coherent state, $|g^{(1)}|$ is unity for all pairs of space-time points (ie, the field is first-order coherent).

For a polarized parallel beam of multimode chaotic light with discretely distributed modes, the first-order degree of coherence is given by

$$g^{(1)}(\tau) = \frac{\sum_k \omega_k \langle n_k\rangle \mathrm{e}^{-\mathrm{i}\omega_k \tau}}{\sum_k \omega_k \langle n_k\rangle}, \tag{8.362}$$

where ω_k stands for the frequency of a mode with propagation constant k (the propagation vectors are all parallel to one another) and $\langle n_k\rangle$ is the mean photon number for the mode. Thus for a state comprising more than one mode, $|g^{(1)}(\tau)| < 1$, consistent with the range of variation of $|\gamma^{(1)}(\tau)|$ for classical light.

Continuous-mode states

In the limit of the modes being continuously distributed, one can adopt a continuous-mode description as in Section 8.9, in which case the degree of coherence depends on the spectral function. The general expression for the first-order degree of coherence for a continuous-mode state is

$$g^{(1)}(\tau) = \frac{\langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle}{\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle}, \tag{8.363}$$

where $\hat{a}(t)$ and $\hat{a}^\dagger(t)$ are the time-dependent continuous-mode annihilation and creation operators defined in Eq. (8.197).

In the case of a Lorentzian spectral function, one obtains for narrowband chaotic light with mean frequency ω_0,

$$g^{(1)}(\tau) = \exp\left(-\mathrm{i}\omega_0\tau - \frac{|\tau|}{\tau_0}\right), \tag{8.364}$$

where τ_0 is a constant characterizing the spectral width. This expression is completely analogous to the classical formula (7.80), corroborating that the classical and the quantum descriptions of the electromagnetic field closely resemble each other when regarded from the point of view of first-order coherence.

We consider next, as an application of formula (8.363), the first-order degree of coherence for the continuous-mode single-photon state defined in Eq. (8.209). This being a wave packet state, a parallel beam does not represent a stationary field, and the first-order degree of coherence for a wave packet defined by the spectral function $\xi(\omega)$, with $\tilde{\xi}(t)$ as its Fourier transform, is

$$g^{(1)}(z_1, t_1; z, t_2) = \frac{\tilde{\xi}^*(t_1 - \frac{z_1}{c})\tilde{\xi}(t_2 - \frac{z_2}{c})}{|\tilde{\xi}^*(t_1 - \frac{z_1}{c})\tilde{\xi}(t_2 - \frac{z_2}{c})|}. \tag{8.365}$$

Evidently, the field is first-order coherent like its single-mode counterpart.

In summary, nonclassical states are not distinguished from classical ones by the values of their quantum mechanical first order degree of coherence.

8.17 Second-Order Coherence

8.17.1 Introduction: Classical Coherence and Its Limitations

Second-order coherence effects are the ones relating to *intensity correlations* between pairs of space-time points in an optical field. These correlations are, in general, expressed in terms of mean values of products of four field functions evaluated at appropriate space-time points, of the form

$$\Gamma(\mathbf{r}_1 t_1, \mathbf{r}_2 t_2; \mathbf{r}_3 t_3, \mathbf{r}_4 t_4) = \langle E^*(\mathbf{r}_1, t_1) E^*(\mathbf{r}_2, t_2) E(\mathbf{r}_3, t_3) E(\mathbf{r}_4, t_4) \rangle. \tag{8.366}$$

I outlined the classical analysis of intensity fluctuations at a single point in Sections 7.12 and 7.13, where the probability distribution of the instantaneous intensity was worked out for polarized and partially polarized radiation, from which one can obtain the variance of the instantaneous intensity and also all the higher moments thereof.

More generally, the second-order coherence characteristics are expressed in terms of the intensity correlations between pairs of distinct points without or with a time delay, where these provide the basis of the experimental observations in setups of the Hanbury Brown-Twiss type. For a pair of points $\mathbf{r}_1$ and $\mathbf{r}_2$ and for a time delay t', the expression for the intensity correlation assumes the form

$$\Gamma^{(2)}(\mathbf{r}_1, \mathbf{r}_2; t') = \langle E^*(\mathbf{r}_1, t) E^*(\mathbf{r}_2, t + t') E(\mathbf{r}_1, t) E(\mathbf{r}_2, t + t') \rangle, \tag{8.367a}$$

where the field has been assumed to be stationary. This expression can be normalized to the classical second-order degree of coherence which reads, with a convenient abbreviation in notation,

$$\gamma^{(2)}(\mathbf{r}_1, \mathbf{r}_2; t') = \frac{\Gamma^{(2)}(\mathbf{r}_1, \mathbf{r}_2, t')}{\Gamma^{(1)}(\mathbf{r}_1) \Gamma^{(1)}(\mathbf{r}_2)}. \tag{8.367b}$$

In the commonly encountered situation of a field in the form of a polarized plane parallel beam of radiation in free space along, say, the z-axis, a spatial separation of $z_2 - z_1$ is equivalent to a time delay $\frac{z_1 - z_2}{c}$, and expression (8.367b) simplifies to the form

$$\gamma^{(2)}(\tau) = \frac{\Gamma^{(2)}(\tau)}{|\Gamma^{(1)}|^2} \quad \left(\tau = t' + \frac{z_1 - z_2}{c} \right), \tag{8.367c}$$

where the dependence on the spatial coordinates occurs through the effective delay τ, and where the value of the degree of coherence satisfies the inequalities

$$1 \leq \gamma^{(2)}(0) \leq \infty, \quad 0 \leq \gamma^{(2)}(\tau) \leq \infty \ (\tau \neq 0), \quad \gamma^{(2)}(\tau) \leq \gamma^{(2)}(0). \tag{8.367d}$$

In the case of chaotic light, the second-order degree of coherence is related to the first-order degree of coherence as

$$\gamma^{(2)}(\tau) = 1 + |\gamma^{(1)}(\tau)|^2, \tag{8.368a}$$

and, in the limits of very small and very large delays, attains the limiting values

$$\gamma^{(2)}(0) = 2, \quad \gamma^{(2)}(\infty) = 1. \tag{8.368b}$$

As we will see in Section 8.17.2, the first and third inequalities in Eq. (8.367d) are not satisfied by the quantum analogue of the second-order degree of coherence. In other words, while the first-order coherence characteristics of an optical field in the classical description

resemble those in the quantum description, the two descriptions are distinguished from each other when the second-order coherence characteristics are taken into consideration.

From the point of view of observations, one is led to consider the theory of *photodetection* since the experimental setups generally make use of the process of photon absorption by material particles.

The semiclassical theory of photodetection was briefly outlined in Section 7.19, where the electromagnetic field was described classically. What is observed with a photodetector is the instantaneous rate of photocount generation or, more generally, the number of photocounts generated in some specified time interval T. The probability distribution of this number for a classical field is given in terms of the probability distribution of the integrated intensity by the Mandel formula (7.165). In the special case of the observation time (T) being short compared with the coherence time of the radiation, one obtains the Bose-Einstein distribution (7.170), which is, in general, a *super-Poissonian* one (Eq. 7.171).

Experimentally, one may observe either the number of photocounts in any given interval or the *photocurrent* caused by the emitted photoelectrons. In the latter case, one infers the intensity correlations from the observed correlation of photocurrents. In Section 7.20.4 I briefly sketched the idea as to how the observed photocurrent correlations can be used to arrive at the coherence characteristics of the radiation field.

The experiments of the Hanbury Brown-Twiss type in the early days used traditional sources of radiation where only the classical field states were realized and only a limited range of second-order coherence characteristics came under the purview of experimental observations. Later-day technologies made possible the use of sources producing fields in nonclassical states as well, where a larger spectrum of photocount distributions was revealed.

I will now outline how, in the quantum description, the intensity- or photocount correlations relate to the quantum mechanical second-order correlation function, and how the normalized correlation (ie, the second-order degree of coherence) can have a larger range of values compared with the classical case, thereby engendering *nonclassical* second-order coherence effects.

8.17.2 Second-Order Quantum Coherence

The quantum mechanical operator for instantaneous intensity (up to a multiplicative factor, which we take to be unity for simplicity) for polarized light at a point $\mathbf{r}$ is

$$\hat{I}(\mathbf{r},t) = \hat{E}^{(-)}(\mathbf{r},t)\hat{E}^{(+)}(\mathbf{r},t), \tag{8.369}$$

while the intensity correlation between points $\mathbf{r}_1$ and $\mathbf{r}_2$ with a time delay t' is given by

$$G^{(2)}(\mathbf{r}_1,\mathbf{r}_2;t') = \langle \mathcal{N}[\hat{I}(\mathbf{r}_1,t)\hat{I}(\mathbf{r}_2,t+t')]\rangle; \tag{8.370}$$

that is, by the second-order correlation function defined in Eq. (8.339c) with an appropriate re-labeling of the space and time coordinates, where the normal ordering of the relevant operators is indicated (explicit normal ordering is not required in Eq. (8.369) since it is already in the normal ordered form). The dependence of $G^{(2)}$ on t is not indicated explicitly, assuming that the field is stationary.

The typical experimental arrangement measuring $G^{(2)}(\mathbf{r}_1, \mathbf{r}_2; \tau)$ for an optical field is a setup of the Hanbury Brown-Twiss type where a beam of radiation is split into two by a beam splitter and the two beams are fed to photodetectors (see Fig. 7.13), each with a small value of the resolution time and high quantum efficiency, while the correlation between the photocounts registered by the two is recorded for various different values of the delay introduced between the two counting times. What the setup actually measures is the *interbeam* degree of coherence, which, however, can be expressed in terms of the degree of coherence of the beam incident on the beam splitter, with the help of the beam splitter matrix elements.

The quantum mechanical analysis of joint photocount probability at two distinct points in space and at two distinct time instants, analogous to the analysis outlined in Section 8.14.1, leads to the result that the photocount correlation between the two detectors is proportional to $G^{(2)}$. With proper normalization, this relates to the normalized correlation function (the second-order *degree of coherence*), defined as

$$g^{(2)}(\mathbf{r}_1, \mathbf{r}_2; t') = \frac{G^{(2)}(\mathbf{r}_1, \mathbf{r}_2; t')}{G^{(1)}(\mathbf{r}_1) G^{(1)}(\mathbf{r}_2)}, \tag{8.371}$$

with a convenient abbreviation in notation. In expression (8.371), which defines a real positive quantity, the dependence of $G^{(1)}$ (in either of the factors in the denominator on the right-hand side) on the time variable is not mentioned, since the field is assumed to be stationary. Moreover, for a field in the form of a linearly polarized plane parallel beam along the z-axis, the spatial coordinates in formula (8.371) can also be done away with while, at the same time, we change the delay from t' to $\tau = t' + \frac{z_1 - z_2}{c}$ (check this out). This gives the second-order degree of coherence $g^{(2)}$ as a function of the effective delay τ alone, analogous to the classical degree of coherence $\gamma^{(2)}$.

The important thing to mention about the quantum second-order degree of coherence is that the possible values of $g^{(2)}(\tau)$ and $g^{(2)}(0)$ do not satisfy the first and third constraints in Eq. (8.367d) satisfied by the classical second-order degree of coherence, and the only statement that can be made about $g^{(2)}(\tau)$ in general is that

$$0 \leq g^{(2)}(\tau) < \infty. \tag{8.372}$$

One therefore observes that the nonclassical feature of an optical field, as revealed by the second-order degree of coherence, is conditional on either or both of the following two inequalities:

$$0 \leq g^{(2)}(0) < 1, \quad g^{(2)}(\tau) > g^{(2)}(0) \quad (\tau \neq 0). \tag{8.373}$$

An optical field for which either of these inequalities is satisfied is referred to as *nonclassical light*.

As we see below, it is possible for $g^{(2)}(0)$ to lie in the range from 0 to 1 even in the simplest case of a single-mode field, for which $g^{(2)}(\tau)$ is actually independent of τ. More generally, nonclassicality can appear by way of the second inequality in relation (8.373) being satisfied.

8.17.3 Second-Order Degree of Coherence

I will state below a number of results relating to the quantum second-order degree of coherence while skipping the derivations. These will give you an idea as to how the coherence properties of optical fields, described from the quantum point of view, compare with those in the classical description.

8.17.3.1 Single-mode states

If the state of the field in which the relevant averages are evaluated is a single-mode one, the expression for the second-order degree of coherence becomes independent of τ and assumes the simpler form

$$g^{(2)}(\tau) = g^{(2)}(0) = \frac{\langle a^{\dagger 2} a^2 \rangle}{\langle a^\dagger a \rangle^2} \tag{8.374a}$$

(check this out; see Eq. 8.166a). This can be expressed in terms of the photon number distribution in the relevant single-mode state as

$$g^{(2)} \equiv g^{(2)}(0) = \frac{\langle \hat{N}^2 \rangle}{\langle \hat{N} \rangle^2} - \frac{1}{\langle \hat{N} \rangle} = 1 + \frac{(\Delta \hat{N})^2 - \langle \hat{N} \rangle}{\langle \hat{N} \rangle^2} \tag{8.374b}$$

(check this out). This means that it is possible for the value of the second-order degree of coherence for a single mode to lie in the range

$$1 - \frac{1}{\langle \hat{N} \rangle} \leq g^{(2)}(0) < 1 \ (\langle \hat{N} \rangle \geq 1), \tag{8.375}$$

where, for a single-mode state with $\langle \hat{N} \rangle < 1$, the lower limit in the above inequality is to be set to 0. What is more, formula (8.374b) tells us that the quantum second-order degree of coherence for a single-mode state has a value lower than unity precisely when the photon number distribution is sub-Poissonian.

The above possibility of the second-order coherence function showing nonclassical values is actually realized in the case of a single-mode number state $|n\rangle$ with $n > 0$ since one has

$$g^{(2)}(0) = 1 - \frac{1}{n} \quad (n > 0) \tag{8.376}$$

for such a state, which indicates manifest nonclassicality. The photon number distribution in a number state $|n\rangle$ with $n > 0$ is, of course, sub-Poissonian, since $(\Delta\hat{N})^2 = 0$. On the other hand, the second-order degree of coherence for a single-mode chaotic state lies within the classical range since Eqs. (8.176a) and (8.374b) yield the result

$$g^{(2)}(0) = 2. \tag{8.377}$$

This is consistent with the super-Poissonian photon number distribution ($(\Delta n)^2 > \langle n \rangle$) of such a state.

A single-mode coherent state $|\lambda\rangle$, which has a Poissonian photon number distribution, has $g^{(2)}(0)(= g^{(2)}(\tau)) = 1$ and, since $|g^{(1)}(\tau)|$ is also unity, it is second-order coherent for all τ.

Finally, for a single-mode squeezed state $|\lambda; \xi\rangle$, the photon number distribution is sub-Poissonian for certain ranges of the parameters λ and ξ as indicated in Sections 8.4.3.3 and 8.8.1.3, which means that, for such values of the two parameters, the quantum second-order degree of coherence attains a value less than unity. The squeezed vacuum, however, has a super-Poissonian photon number distribution and satisfies $g^{(2)}(0) > 1$.

In summary, single-mode states with sub-Poissonian photon statistics are characterized by the nonclassical feature that the quantum second-order degree of coherence for these states attains a value less than unity.

The second inequality in Eq. (8.373) indicates another condition by which a field state can qualify as a nonclassical one. However, this condition, referred to as *photon antibunching*, cannot be satisfied in the case of a single-mode state, for which $g^{(2)}(\tau)$ is identically equal to $g^{(2)}(0)$.

8.17.3.2 Continuous-mode states

The second-order degree of coherence for a discrete multimode state that can be expressed as a direct product of states corresponding to independent modes can be worked out in a straightforward manner from the single-mode expressions (refer to the expression for the first-order degree of coherence for the discrete multimode chaotic state in Eq. (8.362)). Of greater generality is a continuous-mode state, for which the second-order degree of coherence is given, in the case of a stationary field, by

$$g^{(2)}(\tau) = \frac{\langle \hat{a}^\dagger(t)\hat{a}^\dagger(t+\tau)\hat{a}(t+\tau)\hat{a}(t)\rangle}{\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle^2}. \tag{8.378}$$

For a continuous-mode n-photon number state defined in Eqs. (8.209) and (8.211), which corresponds to a nonstationary field, the expression for $g^{(2)}$ (see Eq. 8.367b, with simplifications appropriate to a parallel beam), evaluates to

$$g^{(2)}(\tau) = 1 - \frac{1}{n}, \tag{8.379}$$

which is identical in form to the one for the single-mode number state $|n\rangle$, indicating a violation of the classical inequality $\gamma^{(2)}(0) > 1$.

For a wave packet coherent state one has $g^{(2)} = 1$ along with $g^{(1)} = 1$, which tells us that this state is second-order coherent.

For a continuous-mode chaotic state, one can work out the second-order degree of coherence for any given form of the spectral function. For instance, with chaotic light with a Lorentzian spectrum of width Γ, one obtains

$$g^{(2)}(\tau) = 1 + \mathrm{e}^{-2\Gamma|\tau|}. \tag{8.380}$$

It turns out that the quantum second-order degree of coherence is analogous to the corresponding classical quantity in that it satisfies the relation

$$g^{(2)}(\tau) = 1 + |g^{(1)}(\tau)|^2, \tag{8.381}$$

where the first-order degree of coherence is obtained as a continuous-mode limit of Eq. (8.362), as in Eq. (8.364).

In other words, the continuous-mode chaotic state resembles the classical chaotic state (see Eq. 7.174e) in its second-order coherence properties.

One can also work out the expressions for the second-order degree of coherence for the photon pair states defined in Section 8.9.5 and the continuous-mode squeezed states, but these require more detailed considerations. The single-beam and two-beam photon pair states, for instance, appear as nonclassical states by violating the classical constraint, while squeezed states of certain descriptions are also nonclassical.

8.17.4 Photon Antibunching

The concept of *antibunching* was briefly introduced in Section 8.17.3.1. A radiation field is said to be characterized by the feature of antibunching if the inequality

$$g^{(2)}(\tau) > g^{(2)}(0) \tag{8.382}$$

is satisfied by the second-order degree of coherence for some time delay τ. Antibunching is a specifically quantum characteristic since the second-order degree of coherence of a field described in classical terms necessarily satisfies the third inequality in Eq. (8.367d).

Fig. 8.23 illustrates schematically the feature of antibunching in terms of the photocount time series. Each vertical line signifies a photocount registered in the field under consideration, where the time of the count increases uniformly from left to right. Fig. 8.23A depicts the photocount series for chaotic light, for which $g^{(2)}(\tau) < g^{(2)}(0)$ for all $\tau > 0$, in which case the

photocounts for any given $\tau(>0)$ occasionally appear in bunches, so there is a relative clustering in the time series compared with the coherent case. The time series for the latter is shown in the Fig. 8.23B, where the counts arrive in random succession ($g^{(2)}(\tau) = 1$ for all τ).Fig. 8.23C is for antibunched light, where the successive photocounts are spaced in time compared with the time series for coherent light.

The condition (8.382) for antibunching is distinct from the condition $g^{(2)}(0) < 1$, which is again a criterion for nonclassical behavior. For a single-mode state, for instance, the condition $g^{(2)}(0) < 1$ signifies sub-Poissonian statistics, and excludes antibunching.

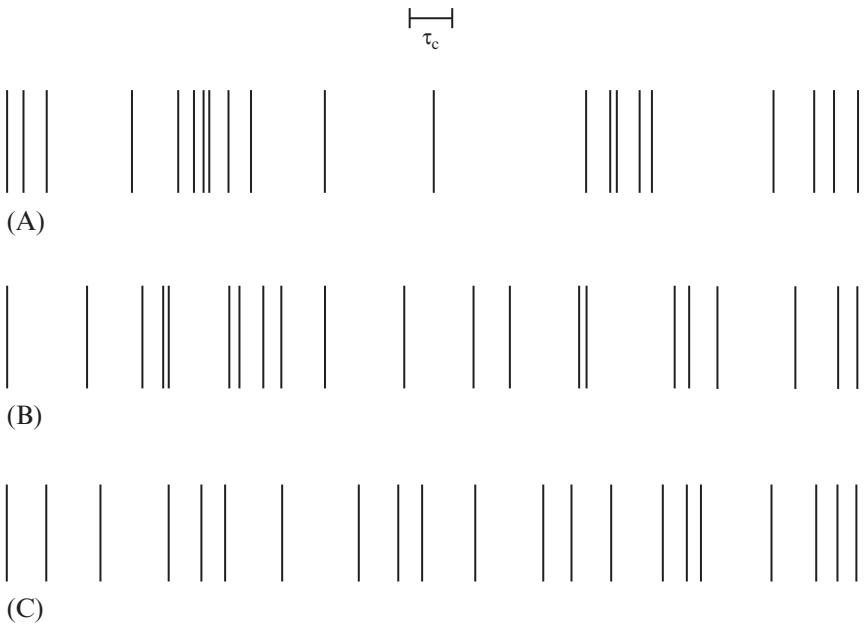

Fig. 8.23
Illustrating the idea of antibunching in terms of photocount time series. The vertical lines signify successive photocounts registered in radiation fields of three different descriptions, shown in (A), (B), and (C). The time of count increases uniformly from left to right. (A) The time series for chaotic light, where one observes the phenomenon of bunching, in which there is a tendency for the successive counts to appear in clusters (on a timescale on the order of the coherence time τ_c, shown at the top) as compared with a coherent field. The time series for the latter is shown in (B), where the successive counts appear randomly. (C) The time series for a field showing antibunching, where the successive photocounts are spaced compared with the coherent field. (Reproduced by permission of Oxford University Press, USA, from *The Quantum Theory of Light*, 3rd edition by R. Loudon (2000): Fig. 6.1, p. 250.)

Photon antibunching was first demonstrated in *resonance fluorescence* by Kimble, Dagenais, and Mandel. Resonance fluorescence is a special type of inelastic scattering where a photon

with a frequency ω matching the energy difference ($\omega \approx \omega_0 = \frac{E_2 - E_1}{\hbar}$) between two stationary states of an atom or a molecule is absorbed by the latter, and is then re-emitted after a time interval, when the atom returns to its initial state (commonly, the ground state), as depicted in Fig. 8.24. The scattered light is approximately of the same frequency as the light absorbed by the atom ($\omega_{sc} \approx \omega$) but has a frequency spread reflecting the line width of the excited atomic state.

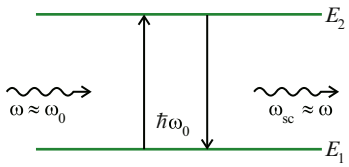

Fig. 8.24

Illustrating the process of resonance fluorescence. A two-level atom with energy levels E_1 and E_2 is raised from the ground state to the excited state by absorbing a photon from a monochromatic radiation field of frequency $\omega \approx \omega_0 = \frac{E_2 - E_1}{\hbar}$. The atom re-emits a photon while returning to the ground state after a time interval $\delta t \approx \tau$, where τ is the mean lifetime of the excited state. The frequency of the emitted photon is $\omega_{sc} \approx \omega$, where the frequency spread reflects the line width of the excited state. The stream of photons resulting from a succession of transitions corresponds to a field characterized by photon *antibunching*.

The coherence characteristics of the field radiated in resonance fluorescence can be worked out by making a number of simple but realistic approximations, such as the one of electric dipole interaction between the atom and the field. The basic approach is similar to that in the Jaynes-Cummings model discussed in Section 8.12.3, in which a number of modifications are to be included such as the ones relating to the irreversible decay modes of the atom and to the fact that the scattered radiation is emitted into space in the form of propagating modes. The state of the atom in the radiation field is described in terms of the *optical Bloch equations* giving the dynamics of the atomic density operator, and one obtains the characteristics of the emitted radiation by treating the atom as an oscillating dipole.

These characteristics of the emitted radiation are principally determined by the following three quantities: (1) the *Rabi frequency* Ω_R, which depends on the intensity of the incident field (the photon number in the quantum description; see Sections 8.12.2 and 8.12.3; I skip the exact definition of Ω_R in the present context) and, additionally, on the dipole matrix element characterizing the atomic transition; (2) the atomic decay rate of the excited state; and (3) the detuning $\Omega = \omega - \omega_0$, where $\Omega = 0$ corresponds to exact resonance.

The decay rate (γ) is, in general, the sum of three contributions—namely, the one caused by atomic collisions, the one relating to Doppler broadening, and the one relating to spontaneous radiative decay. In a gaseous assembly of the atoms, all three mechanisms operate to varying

degrees in causing the broadening of the spectral line. The production of antibunched radiation in resonance fluorescence, however, is conditional on the scattering being caused by individual atoms so that the field at any given instant does not appear as the sum of contributions from a large number of atoms. This means that the collisional and Doppler mechanisms have a negligible effect in determining γ, which is therefore solely a consequence of spontaneous radiative decay.

In an actual setup meant for the production of antibunched light, a laser beam is made to shine on a region through which a stream of the scattering atoms is passed in a perpendicular direction at such a rate that at any given instant the incident radiation interacts with only one or a few of the scattering atoms. The scattered radiation is observed in a direction perpendicular to both the laser beam and the atomic stream, and is divided into two parts by a beam splitter. The two beams produced by the beam splitter are then made to activate two photodetectors, with a variable time delay introduced between the two. One then uses the photocount correlation recorded by the detectors in inferring the second-order degree of coherence.

Fig. 8.25 depicts the variation of $g^{(2)}(\tau)$ with τ for zero detuning ($\Omega = 0$) and for two different values of the Rabi frequency Ω_R, one corresponding to a weak incident field ($\Omega_R \ll \gamma$) and the other corresponding to a strong field ($\Omega_R \gg \gamma$). Though the nature of the variation differs in the two cases, both graphs are indicative of antibunching since $g^{(2)}(\tau) > g^{(2)}(0)$ for all τ in each of them. In addition, the radiation has sub-Poissonian statistics since $g^{(2)}(0) = 0$.

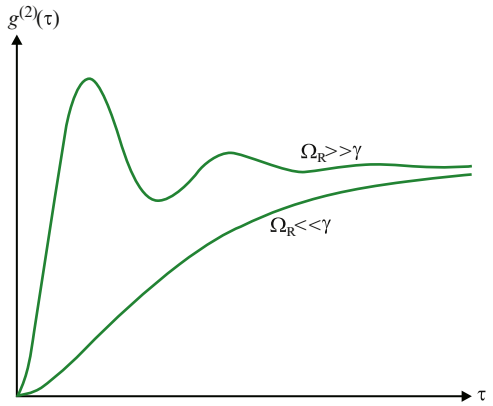

Fig. 8.25
Illustrating the feature of antibunching in resonance fluorescence (schematic). The variation of the second-order degree of coherence $g^{(2)}(\tau)$ with the time delay τ is depicted for zero detuning and for two different strengths of the incident field—namely, one corresponding to $\Omega_R \gg \gamma$ and the other corresponding to $\Omega_R \ll \gamma$. In both graphs, $g^{(2)}(\tau) > g^{(2)}(0)$ for all τ. (Adapted from C. Gerry and P. Knight, *Introductory Quantum Optics,* Cambridge University Press, Cambridge, 2005; Fig. 7.14.)

The basic reason why the radiation produced in resonance fluorescence is characterized by the feature of antibunching is rather simple. Since a time lag of $\tau_0 \sim \gamma^{-1}$ occurs between the excitation of an atom and the emission of radiation from it, an interval slightly larger than τ_0 has to elapse between successive emission events, and a spacing close to zero is quite improbable. Since $g^{(2)}(\tau)$ is essentially the joint probability of detection of emitted photons with a spacing τ between them, one expects $g^{(2)}(\tau) > g^{(2)}(0)$ for $\tau \sim \tau_0$.

8.18 Two-Photon Interference

The single-beam and two-beam photon pair states introduced in Section 8.9.5 can be used to demonstrate a nonclassical interference effect referred to as *two-photon interference* (also known as *Hong-Ou-Mandel interference*, first demonstrated by the three physicists). While the experiment is generally performed with two-photon (ie, squeezed) pair states, we outline here an explanation of the phenomenon in terms of the single photon pair states.

We first consider the single-beam photon pair state of Eq. (8.224) made to be incident at one of the two input ports of a beam splitter, with the other input port carrying the vacuum state. If we number the input and output ports of the beam splitter as in Fig. 8.12 and assume that the input beam is fed to port '1,' the field state is given by

$$|(2_1)_\beta\rangle = \frac{1}{\sqrt{2}} \int dt dt' \tilde{\beta}(t,t') \hat{a}_1^\dagger(t) \hat{a}_1^\dagger(t') |0\rangle, \tag{8.383}$$

where the notation is a slight modification of that adopted in Section 8.9.5, with the beam splitter ports indicated by subscript numerals, and where the time domain operators $\hat{a}_1^\dagger(t)$ and $\hat{a}_1(t)$ have been used instead of the frequency domain ones, the time domain spectral function $\tilde{\beta}(t,t')$ being the Fourier transform of $\beta(\omega,\omega')$. In writing formula (8.383), we have left the vacuum port implied. The above input state is obtained by parametric down conversion in a nonlinear crystal.

The annihilation and creation operators at the various ports are related by the beam splitter matrix

$$B = \begin{pmatrix} R & T' \\ T & R' \end{pmatrix} \tag{8.384}$$

in Eq. (8.236) (with a slightly different notation for the matrix elements), since each frequency component of the continuous-mode field is reflected and refracted at the beam splitter in a manner analogous to a single-mode field (we ignore the frequency dependence of the beam splitter matrix elements for simplicity).

Using the beam splitter matrix, one can express the state (8.383) in terms of the creation operators for the two output ports, where the relevant operator transformation is

$$\hat{a}_1^\dagger(t) = R\hat{a}_3^\dagger(t) + T\hat{a}_4^\dagger(t). \tag{8.385}$$

For a single continuous-mode field, the spectral function satisfies the symmetry property

$$\tilde{\beta}(t,t') = \tilde{\beta}(t',t). \tag{8.386}$$

With use of this symmetry, the state under consideration is seen to be

$$|(2_1)_\beta\rangle = R^2|(2_3)_\beta\rangle + \sqrt{2}RT|(1_3 1_4)_\beta\rangle + T^2|(2_4)_\beta\rangle, \tag{8.387}$$

where the middle term denotes a two-beam photon pair state as in Eq. (8.230) and where, once again, there is a slight modification in the notation used, with the number of photons in each of the relevant ports having been indicated.

The experimentally determined quantity is the joint probability distribution of the photons in the two output ports '3' and '4,' as revealed by joint photocount records of two detectors placed against the two ports.

> This is a photocount correlation measurement of the Hanbury Brown-Twiss type, and the probability distribution mentioned above relates to the *interbeam* and *intrabeam* second-order degrees of coherence.

Looking at formula (8.387), one finds that the respective probabilities are

$$P(2_3, 0_4) = |R|^4, \quad P(0_3, 2_4) = |T|^4, \quad P(1_3, 1_4) = 2|R|^2|T|^2, \tag{8.388}$$

where the notation is self-explanatory. With a single-beam input state as in Eq. (8.383), an experimental measurement of the probabilities is seen to corroborate the above theoretical values. One observes that the probability distribution is independent of the spectral function β (ie, independent of the form of the two-photon wave packet) and is simply a classical probability distribution between two alternatives (corresponding to the two output ports), with respective probabilities $|R|^2$ and $|T|^2$.

With a two-beam input, on the other hand, there appears a nonclassical second-order interference effect whereby the joint probability of one photon in each of the two output ports is reduced and can even be made to assume the value zero. Thus for the state

$$|(1_1 1_2)_\beta\rangle = \frac{1}{\sqrt{2}} \int dt dt' \tilde{\beta}(t,t') \hat{a}_1^\dagger(t) \hat{a}_2^\dagger(t') |0\rangle, \tag{8.389}$$

with one beam at each input port, the beam splitter transformation once again gives the joint output port probability distribution, which is

$$P(2_3, 0_4) = P(0_3, 2_4) = |R|^2|T|^2(1 + |J|^2), \quad P(1_3, 1_4) = 1 - 2|R|^2|T^2|(1 + |J|^2). \tag{8.390a}$$

In these expressions, $|J|^2$ is the *overlap integral* between the two input beams, given by

$$|J|^2 = \int dt dt' \tilde{\beta}^*(t,t')) \tilde{\beta}(t',t). \tag{8.390b}$$

Expressions (8.390a) can be compared with the corresponding classical ones where the photons incident in the two arms are reflected and transmitted independently:

$$P_{\rm cl}(2_3, 0_4) = P_{\rm cl}(0_3, 2_4) = |R|^2|T|^2, \quad P_{\rm cl}(1_3, 1_4) = 1 - 2|R|^2|T^2|. \tag{8.391}$$

It is thus apparent that there is a quantum mechanical 'two-photon interference' effect whereby particles tend to appear at the same port with a greater probability than the classical rule implies. The probability for the two photons to appear at distinct ports is reduced correspondingly.

For instance, if the two photons in the pair state are independent (ie, are not entangled), then the spectral function factorizes as

$$\tilde{\beta}(t, t') = \tilde{\xi}_1(t)\tilde{\xi}_2(t'), \tag{8.392}$$

where $\tilde{\xi}_1$ and $\tilde{\xi}_2$ denote the spectral functions for the wave packets. In the case of Gaussian wave packets of identical shape, the overlap integral assumes the simple form

$$J = \exp\left(-\frac{1}{2}\Delta^2(t_{01} - t_{02})\right), \tag{8.393}$$

where Δ characterizes the width of either wave packet and t_{01} and t_{02} denote the time instants at which the centers of the two wave packets arrive at the respective input points. For $t_{01} = t_{02}$ (ie, for the case where the two wave packets arrive simultaneously at the beam splitter), the interference effect is maximum since $|J|^2 = 1$, and one then has for a 50:50 beam splitter

$$P(1_3, 1_4) = 0. \tag{8.394}$$

The joint count probability can be recorded as a function of the position of the beam splitter as one makes the latter change in small steps by shifting the beam splitter in the path of either of the incident beams. Fig. 8.26 depicts the variation of $P(1_3, 1_4)$ as a function of the beam splitter position where the Hong-Ou-Mandel 'dip' is seen to make its appearance, signaling the second-order interference effect that has no classical explanation.

8.19 Homodyne Detection

In direct detection with a photodetector one measures the intensity (or the photon flux) of an optical field at the location of the detector. The *homodyne detector*, on the other hand, uses an indirect detection procedure where the coherent field of a *local oscillator* is mixed with the signal (ie, the field to be detected and measured) so as to segregate any one of the quadrature components of the latter, depending on the *phase* of the local oscillator. This method is especially useful for the detection of quadrature squeezed light since the two quadrature components of a squeezed optical field have distinct fluctuation properties. The term 'homodyne' signifies that the frequency of the local oscillator is to be the same as that of the signal, or the same as the central frequency of the latter, assuming the signal to be a narrowband one. In practice, both are often derived from the same source.

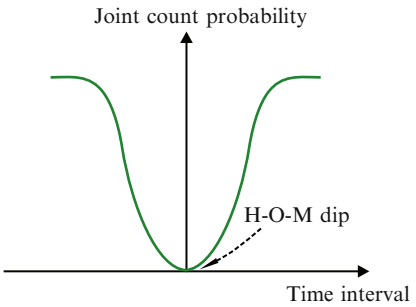

Fig. 8.26
Two-photon interference. The variation of the joint photocount probability $P(1_3, 1_4)$ with the time interval $t_{01} - t_{02}$, which one varies by changing the position of the beam splitter in the path of either of the input beams. The dip reaches down to $P(1_3, 1_4) = 0$ for a 50:50 beam splitter and for uncorrelated input photons. *H-O-M*: Hong-Ou-Mandel.

Fig. 8.27 depicts the basic arrangement for a *balanced* homodyne detector. The signal field and the local oscillator field are fed to the two input ports of a 50:50 beam splitter B. The output beams are fed to photodetectors D_1 and D_2, each having a high quantum efficiency. The circuit C gives as output the difference in the number of counts registered by D_1 and D_2 in any specified time interval T.

Let the annihilation operators for the signal and local oscillator fields be $\hat{a}(t)$ and $\hat{a}_L(t)$, respectively. The corresponding operators for the two output fields of the beam splitter are, say, $\hat{b}_1(t)$ and $\hat{b}_2(t)$, these being related to the input operators by the beam splitter matrix B in Eq. (8.384), which we assume to be a symmetric one, with

$$R = R' = \frac{1}{\sqrt{2}}, \quad T = T' = \frac{\mathrm{i}}{\sqrt{2}}. \tag{8.395}$$

The photon flux operators for the fields incident on D_1 and D_2 are $\hat{b}_1^\dagger \hat{b}_1$ and $\hat{b}_2^\dagger \hat{b}_2$ (see Section 8.9.1). The difference between the two, when expressed in terms of $\hat{a}^\dagger$, $\hat{a}_L^\dagger$, $\hat{a}$, and $\hat{a}_L$ by making use of the beam splitter matrix B, is given by

$$\hat{N}_{\mathrm{diff}}(t) = \mathrm{i}(\hat{a}^\dagger(t)\hat{a}_L(t) - \hat{a}_L^\dagger(t)\hat{a}(t)). \tag{8.396}$$

This is the quantity that the output of the arrangement shown in Fig. 8.27 is supposed to measure (in an actual experiment, one often measures the integral of this quantity over some specified time interval). However, the detectors D_1 and D_2 have quantum efficiencies less than unity and the actual output is obtained from the equivalent beam splitter model of each detector, as indicated in Section 8.14.1. The mean value of $\hat{N}_{\mathrm{diff}}$ that is actually obtained is thus

$$\langle \hat{N}_{\mathrm{diff}}(t) \rangle = \mathrm{i}\eta_Q \langle \hat{a}^\dagger(t)\hat{a}_L(t) - \hat{a}_L^\dagger(t)\hat{a}(t) \rangle, \tag{8.397}$$

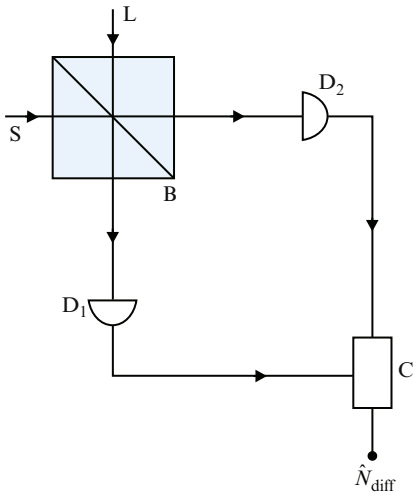

Fig. 8.27
The basic arrangement in a balanced homodyne detector. L and S are the local oscillator and signal beams fed to the input ports of the 50:50 beam splitter B. The output beams are fed to the photodetectors D_1 and D_2. The pulses registered by the detectors are fed to the circuit C, which registers the difference in the photodetector counts. For each value of the local oscillator phase, the setup determines the mean and the variance of the difference between the photon fluxes incident on D_1 and D_2; these are proportional to the mean and the variance of a phase-dependent quadrature of the signal field.

where each of the detectors is assumed to have quantum efficiency η_{Q}. In the following we assume, for the sake of simplicity, that $\eta_{\text{Q}} = 1$ and that the detectors measure the true fluxes.

In a homodyne setup, a strong coherent beam is chosen as the local oscillator field, for which $a_{\text{L}}(t)$ can be replaced with

$$\hat{a}_{\text{L}}(t) \to \alpha \mathrm{e}^{-\mathrm{i}\omega t + \mathrm{i}\theta}; \tag{8.398}$$

that is, with a classical field of real amplitude, say, α (the square root of the local oscillator intensity), and phase θ that is defined with reference to a specified choice of the origin of time for the signal and the local oscillator fields. The phase can be varied in small steps from one run of an experiment to another, where in each run the mean and the variance of $\hat{N}_{\text{diff}}$ are measured.

Let us assume for simplicity that the signal field is a single-mode one, for which

$$\hat{a}(t) = \hat{a}\mathrm{e}^{-\mathrm{i}\omega t}, \quad \hat{a} = \hat{X}_1 + \mathrm{i}\hat{X}_2, \tag{8.399}$$

where $\hat{X}_1$ and $\hat{X}_2$ are the two quadrature operators for the signal field. For such a single-mode field, the time dependence is removed from $\hat{N}_{\text{diff}}$, and one obtains

$$\langle \hat{N}_{\text{diff}} \rangle = 2\alpha \langle X(\theta) \rangle, \tag{8.400a}$$

where

$$\hat{X}(\theta) \equiv \frac{\mathrm{i}}{2}(\hat{a}^{\dagger}\mathrm{e}^{\mathrm{i}\theta} - \hat{a}\mathrm{e}^{-\mathrm{i}\theta}) \tag{8.400b}$$

is a phase-dependent quadrature that reduces to $\hat{X}_1$ and $\hat{X}_2$ for $\theta = \frac{3\pi}{2}$ and $\theta = 0$, respectively.

On working out the variance of $\hat{N}_{\mathrm{diff}}$, in a similar manner, one obtains

$$(\Delta\hat{N}_{\mathrm{diff}})^2 = 4\alpha^2(\Delta\hat{X}(\theta))^2. \tag{8.400c}$$

Thus for any specified phase of the local oscillator, the setup measures the mean and the variance of the phase-dependent quadrature $\hat{X}(\theta)$. Hence, by varying θ in small steps from 0 to 2π, one obtains the mean and the variance of $\hat{X}_1$ and $\hat{X}_2$ for the two values mentioned above.

One first calibrates the homodyne detector by making the measurements in the absence of the signal, in which case the signal port of the beam splitter receives the vacuum field, and one obtains the readings as caused by the vacuum fluctuations (in this case, $\langle\hat{N}_{\mathrm{diff}}\rangle = 0$ for a perfectly balanced homodyne detector), which determines the shot noise level of the setup.

Supposing now that a beam of quadrature squeezed light is used as a signal and the phase θ of the local oscillator is made to vary in small steps through a range 2π, when one will find two values of θ, for one of which the variance will have a minimum value less than the shot noise level, while for the other it will have a maximum value greater than the shot noise. These correspond respectively to the deamplified and the amplified quadratures of the squeezed light. The determination of the minimum and maximum variances then constitutes the detection and characterization of the squeezed light.

8.20 Cavity Quantum Electrodynamics

8.20.1 CQED: Introduction

CQED is now a large subject in which great advances have been made in recent years that are based on rapid technological progress in a number of related areas. The basic idea in CQED is to place an atom, a molecule, or a quantum dot (a miniature semiconductor piece with 3D electron confinement resulting in discrete energy levels; a dot can have a large dipolar interaction with an electromagnetic field) in an optical cavity, or a *resonator*, and to explore the interaction between the atom and the radiation inside the cavity. This makes possible a deep study of fundamental features of atom-field interaction and an exploration of new frontiers of application, including the production of entangled states of atoms and photons, and the opening up of great possibilities in quantum information science.

In this section I will give a very brief and sketchy outline of a few basic concepts in CQED, on the basis of which you can engage in a more serious study of the subject from more competent

and authentic texts. As in many other places in this book, I will skip the derivations leading to the statements I will be making, instead concentrating on presenting the overall picture these statements pertain to. I can only hope that this will be of some value to you.

The subject of CQED can be partitioned into two more or less distinct areas, one relating to the so-called *weak coupling* regime and the other to the *strong coupling* regime, where the former was principally explored in the early days of the development of the subject, while the latter is of more recent vintage.

As regards experimental realizations of the atom-field system, one approach has been to use resonant microwave cavities with Rydberg atoms interacting with the microwave fields, where the term 'Rydberg atom' means an atom (commonly one with a single valence electron) excited to a state close to ionization for which the transition frequency between successive levels lies in the microwave region of the electromagnetic spectrum. Microwave cavities with superconducting walls (perfect reflectors) housing Rydberg atoms have opened up a wide field of investigations in CQED. Technological developments have made it possible to explore the optical range of the spectrum as well, with the help of Fabry-Pérot-type optical resonators of miniature size.

8.20.2 Atom in a Cavity: The Relevant Parameters

A CQED setup is characterized by a number of parameters whose values are to be appropriate for the specific features of the atom-field interaction that it seeks to explore. For instance, the two energy levels of the atom between which transitions are to be induced by the field inside the cavity determine the transition frequency and the resonant mode of the field that the cavity is to support. The latter is determined by the linear dimension and the shape of the cavity, where, in practice, the cavity dimension has to conform to the available technology, and often the choice of the atomic states, the field mode(s), and the cavity size are determined in a mutually consistent way.

The atom is characterized by a certain rate of *spontaneous decay* (Γ) from the excited state $|e\rangle$ to the ground state $|g\rangle$, where these are the states between which the transitions are induced by the cavity field (the state $|g\rangle$ is only *effectively* the ground state for the relevant electronic transition, and is not necessarily the ground state of the atom as a whole). The spontaneous decay rate (or the associated mean lifetime) is determined by the atomic levels chosen. For instance, a Rydberg atom with a large principal quantum number n has a low decay rate and a long lifetime compared with one with a small value of n. The basic theory relating to the spontaneous decay rate is the one outlined in Section 8.12.4, in which the vacuum-induced transition rate W^{vacuum} is what we now refer to as the *natural decay rate*. In the following, as in the Jaynes-Cummings Hamiltonian in Section 8.12.3, the atomic transition frequency is denoted by ω_0.

Another important parameter is the *strength* (g) of the interaction between the atom and the cavity field, which is again determined by the atomic states ($|e\rangle$, $|g\rangle$) involved in the transition under consideration, which is commonly a *dipolar* one, where the strength is proportional to the dipole matrix element between the states. Large-n Rydberg atoms are commonly characterized by large values of the dipole matrix element as compared with low-n ones. Quantum dots are of especial relevance in CQED because of their large dipole matrix elements, the latter being essentially due to their relatively large size.

The design of the cavity determines its *Q factor*, which tells us how selective it is in supporting field modes of various different propagation vectors and frequencies. A high-Q cavity can sustain standing wave modes in it belonging to only a very narrow range of frequencies. When energy is fed into the cavity at a steady rate either from the atom placed in it or from external sources, a buildup of energy occurs within the cavity in an initial transient phase, where the energy is distributed between various different frequencies (ω). The buildup is most pronounced for the *resonant* frequency ω_c, while, in comparison, modes with other frequencies are suppressed by destructive interference effects. Higher harmonics of the fundamental mode ω_c are also possible but these are mostly not of much relevance in CQED setups.

Along with the buildup process, there also occurs a *leakage* of photons from within the cavity to its exterior owing to imperfect reflection at the walls, due to scattering, and, in the case of Fabry-Pérot-type cavities, due to diffraction at the mirrors. The preferential buildup of energy in the resonant mode is limited by enhanced leakage at higher energy densities, while other modes are affected to a relatively lesser extent by leakage. Eventually, a steady state is reached characterized by a distribution of the energy density ($U(\omega)$) among various different frequencies (ω) as in the graph in Fig. 8.28. The Q factor of the cavity—a quantitative measure of its selectivity—is defined as

$$Q = \frac{\omega_c}{\Delta\omega}, \tag{8.401a}$$

where $\Delta\omega$ is the width of the graph at half the maximum.

The Q factor is related to the *photon decay rate* (κ) of the cavity, defined as the inverse of the photon lifetime, where the latter is the time in which the energy density falls to the fraction e^{-1} of its steady-state value (the energy decay is exponential in nature) if the energy supply to the cavity is stopped. An alternative expression for the Q factor in terms of the photon decay rate is

$$Q = \frac{\omega_c}{\kappa}. \tag{8.401b}$$

The design of the cavity also determines the extent to which the atom in the cavity can 'see' the outer world. Fig. 8.29 schematically depicts an optical cavity formed of two parallel

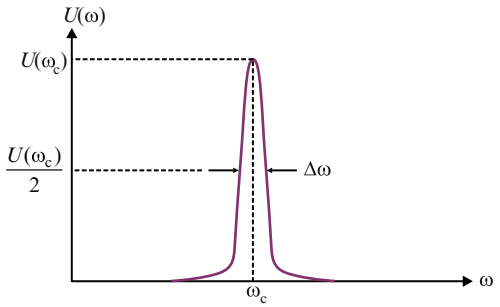

Fig. 8.28
Depicting the energy spectrum within a resonant cavity. The energy density $U(\omega)$ (energy per unit volume and frequency interval) in the steady state is plotted as a function of ω. The graph has a sharp peak, with a maximum at the resonant frequency (ω_c). The width ($\Delta\omega$) at half the maximum determines the Q factor of the cavity and equals the leakage rate κ.

mirrors M_1 and M_2. The separation L between the mirrors is related to the frequency ω_c of the fundamental resonant mode as $\omega_c = \frac{\pi c}{L}$ (see formula (8.157)). The dotted lines define the solid angle by which the outer world is 'visible' to the atom. This solid angle is one of the factors that determines the photon decay rate κ since it controls the rate at which photons leak from the cavity by diffraction. It also determines the *effective* spontaneous decay rate Γ' of the atom since the latter no longer radiates into open space. I mentioned in Section 8.12.4 that the spontaneous decay rate (Γ) of an atom placed in open space will be termed its *natural decay rate* in this book. By comparison, Γ' will be termed the *effective natural decay rate*. The difference between Γ' and Γ is pronounced in the case of microwave cavities with superconducting walls, where we have $\Gamma' \approx 0$, while in the case of a Fabry-Pérot cavity, Γ' is of the same order as Γ.

For an atom in a cavity, it is Γ' rather than Γ that is to be taken as the natural decay rate. The *cavity decay rate* (Γ_{cav}) of the atom placed in the cavity, on the other hand, is determined by its coupling with the cavity field and the rate at which the field energy is dissipated to the outer world, the latter being determined by Γ' and κ. The two parameters Γ' and κ characterize the coupling of the atom-cavity system to the field exterior to the cavity, where the latter may be regarded as a large *reservoir* responsible for the dissipation of energy from the atom-cavity system. Of the two, the former constitutes the direct residual coupling of the atom to the reservoir, while the latter represents the cavity-reservoir coupling.

The basic processes determining the parameters indicated above, characterizing the atom, the cavity field, and the atom-field interaction are depicted symbolically in Fig. 8.29.

These processes occurring in the cavity are of two categories. The atom-field interaction is a reversible process with energy moving back and forth between the atom and the cavity field.

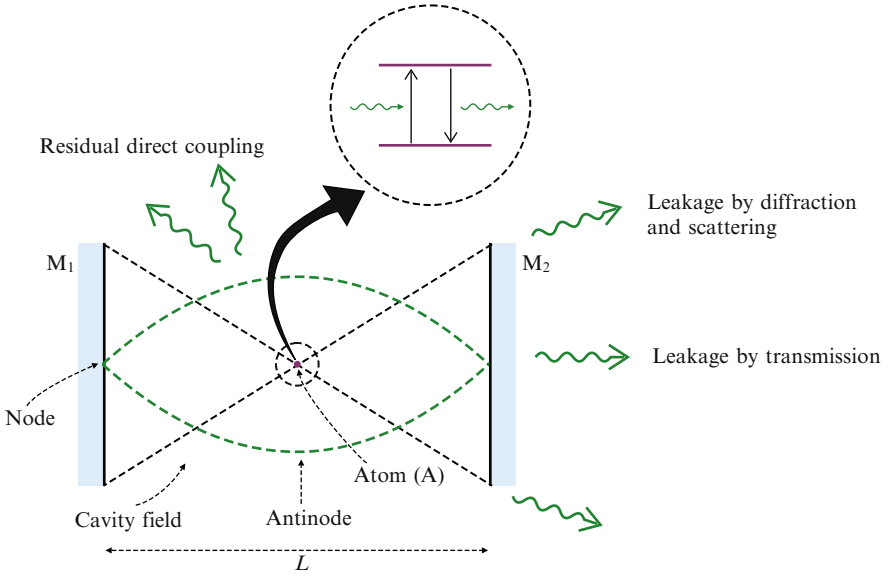

Fig. 8.29

Schematic illustration of a resonant optical cavity. M_1 and M_2 are parallel mirrors with a high reflectance. The separation L determines the resonant frequency ω_c of the cavity field ($\omega_c = \frac{c\pi}{L}$). The atom A placed within the cavity interacts resonantly with the cavity field, exchanging energy with the latter. The dotted lines indicate the solid angle through which the residual direct interaction of the atom with the external environment occurs, where the latter determines its effective natural decay rate (Γ'). The cavity field decays by leakage (caused by transmission and diffraction) to the external environment at rate κ. The resonant mode supported by the cavity has a field node at the mirrors and an antinode in the middle. The field-induced transition between the two relevant atomic levels is shown, pointed out by the thick arrow.

This interaction is described by a unitary evolution operator resulting from a Hamiltonian which, to a good degree of approximation, can be taken to be the Jaynes-Cummings Hamiltonian in Section 8.12.3. On the other hand, the spontaneous decay of the atom and the leakage of photons from the cavity are irreversible processes by which the atom-field system is coupled to the external reservoir and is gradually depleted of energy.

The *weak coupling* regime corresponds to a situation where the irreversible processes described in terms of the parameters Γ' and κ dominate over the reversible atom-field interaction controlled by the parameter g ($g \ll \Gamma', \kappa$). On the other hand, the *strong coupling regime* corresponds to a large value of g relative to Γ' and κ ($g \gg \Gamma', \kappa$). In the weak coupling case, the spontaneous decay rate and the vacuum-induced shift (or *Lamb shift*) of energy levels are modified by the atom-field interaction. In the strong coupling regime, on the other hand, a number of new phenomena including those relating to entangled states of atoms and photons make their appearance, opening up immense possibilities in the applied field of quantum information science.

Incidentally, the strength of the atom-field interaction in the cavity is determined by the cavity itself since, in a typical CQED setup, there is no external field with which the interaction occurs. Instead, the vacuum fluctuations at the resonant frequency and the response of the atom to these fluctuations result in the buildup of the cavity field that causes the direct and the reverse atomic transitions. The strength g is thus determined ultimately by the atomic dipole transition matrix element and the cavity volume and shape, these being parameters intrinsic to the atom and the cavity.

8.20.3 The Weak Coupling Regime

8.20.3.1 The cavity-controlled decay rate

Let us, for simplicity, assume the cavity to be completely closed, which implies that there is no residual direct coupling of the atom to the environmental modes (ie, $\Gamma' = 0$), and the spontaneous decay of the atom from its excited state to the ground state cannot occur by means of direct coupling to the propagating modes of the environment. The atom, however, can still interact with the modes of the cavity field. Though the latter are in the nature of standing waves, these standing wave modes are, in turn, coupled to the propagating modes of the environment, which provides an indirect mechanism for the spontaneous decay of the atomic excited state. This spontaneous decay rate within the cavity differs from the spontaneous decay rate of the atom when it is placed in open space (we have agreed to call it the *natural spontaneous decay rate*) and constitutes a CQED effect where the field environment affects a characteristic feature of the atom-field coupling.

The cavity-controlled modification of the spontaneous decay rate can be worked out by the use of Fermi's golden rule as we did in deriving the expression for the natural decay rate in Section 8.12.4. Recall that the golden rule expresses the end result of first-order perturbation theory (see Section 8.12.2.4) pertaining to the weak interaction of two quantum systems where, in the present instance, the interaction is between the atom and the cavity field. The cavity field, in turn, behaves like a damped harmonic oscillator because of its dissipative interaction with the field external to the cavity.

Referring to formula (8.310), recall that the total rate of decay W from the excited state of the atom to the ground state is made up of a field-induced part and a vacuum part (Eq. 8.311a), where the expression for the latter is given by

$$\Gamma = W^{\text{vacuum}} = \frac{\pi p^2 \omega_0}{3\epsilon_0 \hbar V} \rho(\omega_0) \tag{8.402}$$

(check this out, refer to the paragraphs following Eq. 8.310), where the notation is as in Section 8.12.4. Here $\rho(\omega_0)$ stands for the density of propagating field modes in open space. For an atom placed in a cavity, we have to replace this with the density of modes within the cavity, which we denote by the symbol ρ_{cav}. Further, we have to modify the factor $\frac{1}{3}$ in

expression (8.402) in arriving at the cavity decay rate since it arose by virtue of the isotropy of the open-space vacuum field. The cavity field, on the other hand, has a preferred direction because of the geometry of the cavity wall with reference to the location of the atom. Assuming, for simplicity, that the dipole matrix element is oriented along the field direction, one obtains

$$\Gamma_{\text{cav}} = \frac{\pi p^2 \omega_0}{\epsilon_0 \hbar V} \rho_{\text{cav}}(\omega_0), \tag{8.403}$$

where $\rho_{\text{cav}}(\omega_0)$ stands for the density of modes of the cavity field. Let us assume, for generality, that there may exist a detuning between the resonant frequency of the cavity ω_c and the atomic transition frequency ω_0. The density of modes function $\rho_{\text{cav}}(\omega)$ will then be sharply peaked at $\omega = \omega_c$ and will have a width $\Delta\omega = \kappa$. The cavity decay rate will then depend on the extent to which ω_0 deviates from ω_c.

The function ρ_{cav} is determined by the statistical spread in the frequencies of the incoherent components making up the cavity field and is analogous to the line shape function in an atomic transition. Assuming for concreteness that the graph of $\rho_{\text{cav}}(\omega)$ has a Lorentzian shape,

$$\rho_{\text{cav}}(\omega) = \frac{1}{2\pi} \frac{\kappa}{(\omega - \omega_c)^2 + \left(\frac{\kappa}{2}\right)^2}, \tag{8.404}$$

one can work out the cavity-controlled decay rate Γ_{cav} for any given mismatch between the transition frequency ω_0 and the cavity resonant frequency ω_c. At exact resonance ($\omega_0 = \omega_c$), assuming that the atom is placed at a field antinode ($\sin(kz_0) = 1$ in the notation of Section 8.12.3.1), one obtains

$$\Gamma_{\text{cav}} = \frac{3\rho_{\text{cav}}(\omega_0)}{\rho(\omega_0)} \Gamma = \left(\frac{3\lambda_0^3 Q}{4\pi^2 V} \right) \Gamma, \tag{8.405a}$$

where Γ stands for the natural decay rate and ρ stands for the density of modes in open space. In the case of a mismatched cavity with $|\omega_0 - \omega_c| \gg \kappa$, on the other hand,

$$\Gamma_{\text{cav}} = \left(\frac{3\lambda_0^3}{16\pi^2 V Q} \right) \frac{\omega_0 \omega_c}{(\omega_0 - \omega_c)^2} \Gamma. \tag{8.405b}$$

In these expressions, λ_0 stands for the free-space wavelength corresponding to the transition frequency ω_0.

One observes from Eq. (8.405a) that the free-space decay rate *is enhanced* in a high-Q cavity of small volume at resonance ($\omega_0 = \omega_c$) by the factor $\frac{3\lambda_0^3 Q}{4\pi^2 V}$ (referred to as the *Purcell factor* because it was Purcell who first pointed out the possibility of such enhancement). Conversely, as formula (8.405b) shows, the decay rate is *suppressed* if there occurs a mismatch between

the atomic transition frequency and the cavity resonant frequency, which is how the atom can be made to have a long lifetime in a cavity. The physical reason why the decay rate is suppressed is that the cavity does not support the mode into which the atom radiates.

> The decay of the cavity field by the loss of photons to the electromagnetic field exterior to the cavity makes the cavity field behave like a damped simple harmonic oscillator, for which κ plays the role of the damping constant. The cavity bandwidth $\Delta\omega$ is then the width of the resonance curve of the damped oscillator, which increases in proportion to the damping constant. The quantum damped oscillator resembles its classical analogue in the shape of the resonance curve—namely the Lorentzian curve of Eq. (8.404).

8.20.3.2 Vacuum energy shift in a cavity

In the quantum mechanical calculation of the energy levels of an atom, the latter is assumed to be an isolated system with no electromagnetic interactions. This, however, is not strictly true since there is always the interaction with the *vacuum field*. As we know, this causes the spontaneous transition from an excited state to the ground state of the atom. A related effect of the vacuum field is to cause a *shift* of the atomic energy levels, known as the *Lamb shift* (Lamb was the first to point out this vacuum effect and predicted a split between the $^2S_{\frac{1}{2}}$ and $^2P_{\frac{1}{2}}$ levels of hydrogen, and Lamb and Retherford experimentally confirmed the effect by using microwave techniques).

In the case of an atom in a closed cavity, the natural transition rate is modified, being determined now by the cavity features, as we saw in Section 8.20.3.1. At the same time, there occurs a vacuum shift of energy levels analogous to the Lamb shift, again determined by cavity features, as we will now see.

For this we invoke the results of the Jaynes-Cummings model from Section 8.12.3, assuming that the atom characterized by the transition frequency ω_0 is interacting with the cavity field of frequency ω_c (ie, substitute ω_c for ω in the Jaynes-Cummings model), the detuning between the two being $\Omega = \omega_c - \omega_0$.

We consider the weak coupling limit (g small) and, at the same time, the limit of small values of $\frac{g}{\Omega}$, which can be achieved without Ω being so large as to vitiate the rotating wave approximation assumption. Referring to Eq. (8.304b), we find that the dressed energies are given by

$$E_{1,2} \approx \left(n + \frac{1}{2}\right)\hbar\omega_c \pm \frac{\hbar}{2}\Omega\left(1 + \frac{2g^2(n+1)}{\Omega^2}\right), \tag{8.406}$$

where these correspond to perturbed energies of the bare states $|a\rangle \otimes |n+1\rangle$ and $|b\rangle \otimes |n\rangle$, which were denoted by $|\tilde{a}\rangle$ and $|\tilde{b}\rangle$ in Section 8.12.3.4. To indicate the photon numbers in these bare states, we now use the notation $|\tilde{a}_{n+1}\rangle$ and $|\tilde{b}_n\rangle$.

Considering in this notation the bare states $|\tilde{a}_0\rangle$ and $|\tilde{b}_0\rangle$, for each of which the cavity field is in the bare vacuum state, we find that the shifts in the energies of these due the perturbation caused by the coupling between the atom and the field are, respectively,

$$\delta E(\tilde{a}_0) = 0, \quad \delta E(\tilde{b}_0) = -\hbar\frac{g^2}{\Omega} = \hbar\frac{g^2}{\omega_0 - \omega_c}. \tag{8.407}$$

This results in the shift, depicted in Fig. 8.30, in the frequency ω_0, caused by vacuum fluctuations in the cavity, and describes the effect we were looking for. Along with this, there occurs a shift when the bare field in the cavity is in an n-photon state ($n = 1, 2, \ldots$), this being analogous to the field-induced energy shift observed in atomic transitions in open space (AC Stark effect).

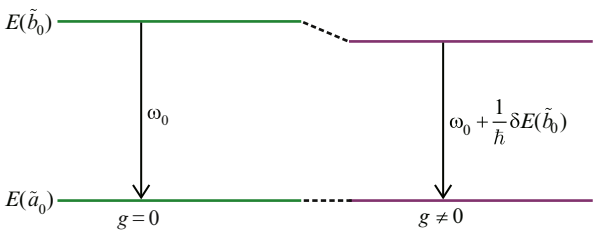

Fig. 8.30

Depicting the vacuum energy shift of the atomic level $|b\rangle$ due to a weak atom-field coupling in a cavity. The detuning Ω is assumed to be small, though $\frac{\Omega}{g}$ is relatively large. The ground state $|a\rangle$ is not affected by the vacuum field. The bare energies $E(\tilde{a}_0)$ and $E(\tilde{b}_0)$ are shown on the left, while the modified energies are shown on the right. The transition frequency ω_0 is shifted by $-\frac{g^2}{\Omega}$. A positive value of $\Omega = \omega_c - \omega_0$ is assumed in the diagram.

8.20.3.3 Modification of the cavity refractive index

In explaining the vacuum energy shift of atomic levels in the cavity, we considered a fixed state of the bare cavity field—namely, the bare vacuum (more generally, a bare field state with a fixed nonzero photon number can be considered to explain field-induced changes in atomic energy levels). Conversely, one can consider a fixed bare atomic state and investigate the modification of the photon spectrum in the cavity. In the absence of interaction ($g = 0$), the fixed atomic state $|a\rangle$ gives rise to joint states of the form $|a\rangle \otimes |n\rangle$ ($n = 0, 1, 2, \ldots$) of the composite system made of the atom and the field, for which the photon spectrum consists of the frequency ω_c corresponding to transitions between successive levels. Similarly, for the fixed atomic state $|b\rangle$, one has the states $|b\rangle \otimes |n\rangle$, and the photon spectrum is again made up of the single frequency ω_c. The two sets of energy levels and the associated photon spectrum are depicted in Fig. 8.31.

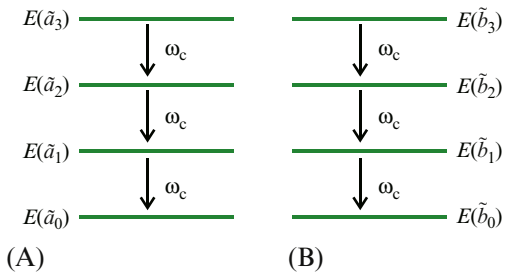

Fig. 8.31

Depicting the photon spectrum in a cavity with zero coupling ($g = 0$). (A) The atom in the bare state $|a\rangle$; energy levels $E(\tilde{a}_n)$ for composite states $|\tilde{a}_n\rangle = |a\rangle \otimes |n\rangle$ ($n = 0, 1, 2, \ldots$) are shown, forming an equispaced ladder with transition frequency ω_c between successive states. (B) Corresponding energy levels $E(\tilde{b}_n)$ when the atom is in the bare state $|b\rangle$; the energy ladder is again equispaced with transition frequency ω_c. In either case the photon spectrum consists of the single frequency ω_c.

For a small nonzero coupling, on the other hand, the bare states $|\tilde{a}_n\rangle \equiv |a\rangle \otimes |n\rangle$ ($n = 0, 1, 2, \ldots$) are modified to energies

$$E_1^{(n)} \approx \left(n + \frac{1}{2}\right)\hbar\omega_c + \frac{\hbar}{2}\Omega\left(1 + \frac{2g^2(n+1)}{\Omega^2}\right), \tag{8.408}$$

as shown in Fig. 8.32A. The photon spectrum, corresponding to transitions between successive energy levels, then consists of the modified frequency

$$\omega_c' = \omega_c + \frac{\hbar g^2}{\Omega}. \tag{8.409a}$$

Analogously, the bare states $|\tilde{b}_n\rangle \equiv |b\rangle \otimes |n\rangle$ ($n = 0, 1, 2, \ldots$) give rise to the modified energy levels $E_2^{(n)}$ shown in Fig. 8.32B, in which case the photon spectrum is made up of the single frequency

$$\omega_c'' = \omega_c - \frac{\hbar g^2}{\Omega}. \tag{8.409b}$$

This modification of the spectrum from frequency ω_c to ω_c' or ω_c'' (depending on the bare atomic state chosen) is equivalent to a modification of the refractive index of the cavity space by the single atom placed in it. With no atom placed in the cavity, one has

$$\omega_c = \frac{c\pi}{L} \tag{8.410a}$$

for an evacuated Fabry-Pérot-type cavity (which we consider here for the sake of illustration), while with the atom placed in the cavity, the frequency can be expressed as

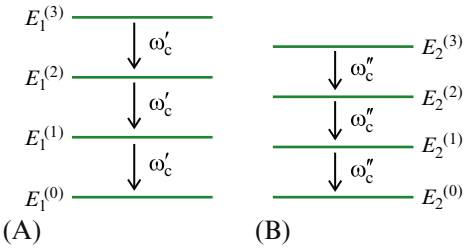

Fig. 8.32

Depicting the photon spectrum in a cavity with a small nonzero coupling g. (A) Energy levels $E_1^{(n)}$ for perturbed states produced from $|\tilde{a}_n\rangle$ $(n = 0, 1, 2, \ldots)$ by the weak atom-field interaction; the transition frequency between successive levels is ω_c' given by Eq. (8.409a). (B) The levels $E_2^{(n)}$ produced from the bare states $|\tilde{b}_n\rangle$ $(n = 0, 1, 2, \ldots)$ by perturbation; the spacing between the levels now corresponds to the frequency ω_c'' given by Eq. (8.409b). The modification of the photon spectrum compared with the uncoupled case shown in Fig. 8.31 results in a change in the cavity refractive index given by Eq. (8.410c).

$$\omega_c + \delta\omega_c = \frac{c\pi}{(1 + \delta n)L}, \tag{8.410b}$$

where

$$\delta n = \mp\frac{\hbar g^2}{\Omega\omega_c} \tag{8.410c}$$

depending on whether the atomic state in question is $|a\rangle$ or $|b\rangle$. This indicates a modification of the effective *refractive index* in the cavity from 1 to $1 + \delta n$.

This modification of the refractive index is dispersive in nature (ie, depends on the frequency ω_c) and is used in *nondemolition*-type photon number measurements and in the production of *cat states* in the cavity.

8.20.4 Open Systems in Quantum Optics

8.20.4.1 Open systems: The master equation

The atom and the cavity field form a composite quantum system. The latter, in turn, is an *open* system that interacts with the environment external to the cavity, the atom by means of its residual direct coupling to the environment modes and the cavity field by means of photon 'leakage' (ie by photons escaping to the environment by means of absorption, transmission, scattering, and diffraction). The environment can be regarded as an infinitely large reservoir to which the atom and the cavity field are coupled, where, in the joint evolution of the atom-cavity system and the reservoir, one can, at any given instant, average over the possible states of the reservoir so as to arrive at the *reduced* density matrix (see Section 8.3.12) describing the state of the atom-cavity system.

This reduction gives the time evolution of the atom-cavity system without regard to what happens to the state of the environment. Because of the fact that the environment is an infinitely large system, it is not possible to monitor its instantaneous state. Since one is interested only in describing what happens to the atom and the cavity, one regards these as forming an open system, on which the effect of the environment is described in the sense of an average.

By way of a general formulation of this approach, we consider a system S interacting weakly with a reservoir R (which makes S an *open* system), the two together forming a closed system, described by the Hamiltonian

$$\hat{H} = \hat{H}_{\mathrm{S}} + \hat{H}_{\mathrm{R}} + \hat{V}, \tag{8.411}$$

where $\hat{H}_{\mathrm{S}}$ and $\hat{H}_{\mathrm{R}}$ stand for the Hamiltonian operators for S and R considered without reference each other, and $\hat{V}$ describes the interaction between the two.

> In writing Eq. (8.411), I have simplified things a bit since $\hat{H}$ and $\hat{V}$ stand for operators in the state space of the composite system made up of S and R as they should, while $\hat{H}_{\mathrm{S}}$ and $\hat{H}_{\mathrm{R}}$ are operators in the state spaces of S and R. More precise pairs of expressions to use in Eq. (8.411) are, respectively, $\hat{H}_{\mathrm{S}} \otimes \hat{I}_{\mathrm{R}}$ and $\hat{I}_{\mathrm{S}} \otimes \hat{H}_{\mathrm{R}}$, where $\hat{I}_{\mathrm{S}}$ and $\hat{I}_{\mathrm{R}}$ are unit operators in the state spaces of S and R. The simplified expressions I use need not cause confusion if interpreted appropriately in each case.

Under a certain set of assumptions about the systems S and R and about the interaction Hamiltonian $\hat{V}$, one can describe the effect of R on S in the form of a first-order differential equation satisfied by the reduced density operator $\hat{\rho}_{\mathrm{S}}(t) = \mathrm{Tr}_{\mathrm{R}}\hat{\rho}(t)$ (ie, the partial trace, taken over the states of R, of the density operator $\hat{\rho}$ for the composite system made up of S and R) such that, knowing the reduced density operator at an initial time $t = 0$ one can, in principle, construct the same at any subsequent time t. Such an equation is referred to as the *master equation* for the open system S.

The time evolution of the state of the composite system made up of S and R is given by

$$\mathrm{i}\hbar\frac{d}{dt}\hat{\rho}(t) = [\hat{H}, \hat{\rho}], \tag{8.412}$$

from which, by taking the partial trace over the states of R, one obtains

$$\mathrm{i}\hbar\frac{d}{dt}\hat{\rho}_{\mathrm{S}}(t) = \mathrm{Tr}_{\mathrm{R}}[\hat{H}, \hat{\rho}]. \tag{8.413}$$

In general, the right-hand side of this equation cannot be reduced to a form involving $\hat{\rho}_{\mathrm{S}}(t)$ and operators pertaining to the system S without specific reference to the reservoir R. In other words, the evolution of the reduced state of S cannot be described in the desired closed form of a master equation unless the systems S and R, and the interaction Hamiltonian $\hat{V}$, meet an appropriate set of restrictive requirements.

Going to the basics of the problem, one can formulate in quite general terms what these requirements should be and check that these are indeed met with in any given situation of practical relevance. Once this has been done, one can set up the master equation and can then work out how the system under consideration evolves in time. In particular, quantum optics setups mostly do conform to the requirements referred to above, and their dynamics can be described, to a good degree of approximation, in terms of master equations. The mathematical steps necessary to effect the reduction from Eq. (8.413) to the master equation are said to constitute the *microscopic derivation* of the latter.

Briefly, the basic assumption from which the microscopic derivation proceeds is that of a *small system S coupled weakly to a large system*—namely, the reservoir R. Commonly, R is assumed to be a system in equilibrium, described by a time-invariant density operator $\hat{\rho}_{\mathrm{R}}^{(0)}$. Assuming that the composite system made up of S and R is initially in the product state

$$\hat{\rho}(0) = \hat{\rho}_{\mathrm{S}}(0) \otimes \hat{\rho}_{\mathrm{R}}^{(0)}, \tag{8.414}$$

one can use the basic assumption mentioned above in effecting a number of convenient approximations. On a very short timescale, the coupling between S and R leads to an entanglement between the two, and builds up *correlations* in the reservoir R. These correlations, however, are quickly smothered, typically over a characteristic time τ_{R} (the *correlation time* of the reservoir), and R continues to remain close to the equilibrium state. One can thus assume that the coupling to R causes a non-negligible change in the reduced state of S but the state of R itself is not changed appreciably. One can then assume that, in an approximate sense, the state of the composite system at time t is of the form $\hat{\rho}_{\mathrm{S}}(t) \otimes \hat{\rho}_{\mathrm{R}}^{(0)}$.

One other approximation involved in the microscopic derivation is based on the assumption of a weak coupling between S and R, as a result of which the evolution equation can be expanded in a perturbation series in which one can retain only the terms up to the second order in the interaction Hamiltonian V. The coupling, moreover, is assumed to be *linear* in two specific sets of operators pertaining to S and R—namely, ones that act as *raising* and *lowering* operators for either system in its energy basis. A third approximation is referred to as that of a *Markovian evolution* of the state of S wherein the rate of change of the reduced state $\hat{\rho}_{\mathrm{S}}$ at time t is assumed not to depend on the reduced state at times earlier than t. Strictly speaking, the evolution of $\hat{\rho}_{\mathrm{S}}$ depends on the history of the evolution because of the cumulative effect of the correlations that build up between S and R. However, the fact that the reservoir returns to its equilibrium state within a very short time implies that the correlations do not last over a longer timescale.

Finally, the microscopic derivation uses the *rotating wave approximation* in a spirit analogous to that adopted in the context of the atom-field interaction discussed in Sections 8.12.2.3 and 8.12.3. This requires that the characteristic time τ_{S} over which the internal dynamics of the system S, governed by its Hamiltonian $\hat{H}_{\mathrm{S}}$, causes its state to change appreciably be small compared with the *relaxation time* τ_{relax} (ie, the characteristic time over which the

system-reservoir interaction causes the reduced state of S to change appreciably). If the rotating wave approximation is not made, then the evolution equation for the reduced density operator contains terms oscillating with various Bohr frequencies, where a Bohr frequency signifies a difference of the form $\omega' - \omega$, with $\hbar\omega$ and $\hbar\omega'$ being typical eigenvalues of $\hat{H}_{\mathrm{S}}$. The rotating wave approximation amounts to averaging out these oscillations on the assumption that these are fast compared with the relaxation of $\hat{\rho}_{\mathrm{S}}$.

On making all these approximations, one arrives at the master equation for the system S coupled to the reservoir R. The master equation appears in a certain standard form, referred to as the *Lindblad form*. The form of the latter can be deduced from general considerations without reference to specific details of the systems S and R and without reference to the interaction Hamiltonian $\hat{V}$ (ie, without going through the steps of the microscopic derivation), on the basis of a set of assumptions regarding the *dynamical map* in the state space of S (see Section 8.20.4.2 below).

It may be mentioned in this context that evolution equations more general than ones in the standard form can be set up for the reduced density operator, wherefrom useful information can be extracted regarding the dynamics of the system S, and some of these are generally referred to as 'master equations.' For instance, one can arrive at a master equation having a more general form if one drops the rotating wave approximation. While the standard form of the master equation (also referred to as the *Markovian master equation*) is seen to be useful in the context of a large number of quantum optics setups, equations of these more general forms are also of relevance in a limited number of situations.

In summary, one arrives, by means of a microscopic derivation, at a master equation in the standard form under the assumption of a weak coupling between S and R, provided the relevant timescales ($\tau_{\mathrm{R}},\ \tau_{\mathrm{S}},\ \tau_{\mathrm{relax}}$, see earlier) satisfy the inequalities

$$\tau_{\mathrm{R}} \ll \tau_{\mathrm{relax}}, \quad \tau_{\mathrm{S}} \ll \tau_{\mathrm{relax}}. \tag{8.415}$$

I will now outline what the standard form of the master equation is.

8.20.4.2 Master equation: The standard form

In a related and complementary approach to the master equation, one does not ask questions about the specific system under consideration and, instead, attempts to work out the general form of the master equation, *assuming* that such an equation does describe the evolution of the reduced state of the system. For the master equation to be a useful description of the system dynamics, it has to satisfy a number of requirements. To see what these requirements are, let us consider the time-dependent *map* W_t, commonly referred to as the *dynamical map*, that transforms any given initial reduced state $\hat{\rho}_{\mathrm{S}}(0)$ to the state $\hat{\rho}_{\mathrm{S}}(t)$, at any later time t, where the latter is the state resulting from the time evolution as implied by the master equation,

$$W_t : \hat{\rho}_{\mathrm{S}}(0) \to \hat{\rho}_{\mathrm{S}}(t). \tag{8.416}$$

For $\hat{\rho}_{\rm S}(t)$ to be a valid density operator, the mapping must be a *trace-preserving, convex linear, and completely positive* one. Moreover, it is required to represent a *dynamical semigroup* if the master equation is to represent the time evolution in a consistent manner.

> Here I opt not to explain what these requirements mean since such an explanation will constitute a distraction from what I want to tell you in these sections of the book. While, on the face of it, these requirements seem to be in the nature of mathematical ones, they do have a physical basis. In particular, the assumption of the dynamical semigroup corresponds to the requirement of Markovian time evolution of the system under consideration. I repeat that while the assumption of Markovian evolution is a convenient and realistic one in a large number of contexts, it is not a *necessary* requirement of the theory.

Given these requirements on the time-dependent mapping W_t, one can infer what the general form of the master equation is to be. Briefly, the master equation can be expressed in the form

$$\frac{d}{dt}\hat{\rho}_{\rm S}(t) = \frac{1}{{\rm i}\hbar}[\hat{H}', \hat{\rho}_{\rm S}] + \sum_{k=1}^{N} \gamma_k \left(\hat{L}_k \hat{\rho}_{\rm S} \hat{L}_k^\dagger - \frac{1}{2}\hat{L}_k^\dagger \hat{L}_k \hat{\rho}_{\rm S} - \frac{1}{2}\hat{\rho}_{\rm S}\hat{L}_k^\dagger \hat{L}_k \right). \tag{8.417}$$

In the expression on the right-hand side, all the operators pertain to the system S under consideration (ie, act in the state space of S), while, at the same time, carrying the imprint of its coupling to the reservoir R. In the first term of this expression, $\hat{H}'$ is a Hermitian operator that may differ from the Hamiltonian of S. While this term represents a unitary and reversible time evolution of the reduced state of S, the next term, involving the N operators $\hat{L}_k$ $(k = 1, 2, \ldots, N)$ (referred to as 'Lindblad operators' or *jump operators*), represents an irreversible and dissipative evolution, and is at times referred to as the *dissipator*. The number N is an integer less than or equal to $d^2 - 1$, where d is the dimension of the state space of the system S (we assume that the state space is finite-dimensional for simplicity). Finally, γ_k $(k = 1, 2, \ldots, N)$ are a set of positive constants characterizing the dissipator. The assumptions relating to the nature of the mapping W_t mentioned above do not determine the operators $\hat{H}'$ and $\hat{L}_k$, or the constants γ_k, but only determine the form of the evolution equation as stated above, this being the *standard*, or *Lindblad*, form of the master equation.

To determine the operators $\hat{H}'$ and $\hat{L}_k$ $(k = 1, 2, \ldots, N)$, one has to look at the details of the systems S and R and also at the interaction Hamiltonian V, and go through the *microscopic* derivation of the master equation. As indicated in Section 8.20.4.1, one has to make a number of approximations so as to obtain an equation of the Lindblad form. In the case of a considerable number of quantum optics setups, these approximations have a good measure of validity.

8.20.4.3 Example: Decay of the two-level atom

The two-level atom coupled to a heat bath at any given temperature T is among the most well-studied objects in the theory of open systems. Here the two-level atom, with the ground state $|a\rangle$ and the excited state $|b\rangle$, constitutes the system S, described by the Hamiltonian

$$\hat{H}_{\mathrm{S}} = -\frac{1}{2}\hbar\omega_0\sigma_3, \tag{8.418a}$$

where ω_0 is the atomic transition frequency and σ_3 is the Pauli matrix given in Eq. (8.288a). Such a system is also referred to as a *qubit* in quantum information theory. The reservoir is assumed to be made up of electromagnetic field modes in thermal equilibrium at a temperature T. This corresponds to an infinitely large assembly of bosons, where the bosons are independent harmonic oscillators, with the oscillator for the mode labeled α (recall from Eq. (8.6) that the index α is made up of the relevant propagation vector $\mathbf{k}$ and the polarization index s) having a frequency ω_α (recall further that ω_α does not depend on the polarization index) and annihilation and creation operators $\hat{b}_\alpha$ and $\hat{b}_\alpha^\dagger$. The frequency spectrum ($\{\omega_\alpha\}$) of the reservoir modes is assumed to be an effectively continuous one. The Hamiltonian of the reservoir is given by

$$\hat{H}_{\mathrm{R}} = \sum_\alpha \hbar\omega_\alpha \hat{b}_\alpha^\dagger \hat{b}_\alpha, \tag{8.418b}$$

where the vacuum energies of the oscillators have been ignored since these are not of relevance in the present context. Finally, the atom is assumed to be coupled to the reservoir by the dipole interaction:

$$\hat{V} = -\hat{\mathbf{d}} \cdot \hat{\mathbf{E}}, \tag{8.418c}$$

where $\hat{\mathbf{d}}$ is the vector dipole moment operator of the atom and $\hat{\mathbf{E}}$ is the electric field operator, which appears as a sum of polarized plane wave modes of the form (8.166a) in open space (recall, however, that expression (8.166a) has been written for the special case of a mode with its wave vector along the z-axis).

The reservoir made up of the field modes is assumed to be in thermal equilibrium at some specified temperature T, in which case the reservoir density operator $\hat{\rho}_{\mathrm{R}}^{(0)}$ is a direct product of density operators for all the individual modes, with the density operator for the mode labeled α being given by

$$\hat{\rho}_\alpha^{(0)} = (1 - e^{-\beta\hbar\omega_\alpha})^{-1} \exp(-\beta\hbar\omega_\alpha \hat{b}_\alpha^\dagger \hat{b}_\alpha) \quad \left(\beta = \frac{1}{k_{\mathrm{B}}T}\right) \tag{8.419}$$

(compare this with formula (8.172a), which is written in a different notation). One can now go through the steps of the microscopic derivation, making appropriate assumptions indicated in Section 8.20.4.1, and arrive at the Markovian master equation, which I state here for the special case $T = 0$ (ie, for the case where the two-level atom interacts with the *electromagnetic field vacuum*, which acts as the reservoir):

$$\frac{d\hat{\rho}_{\mathrm{S}}}{dt} = \frac{1}{\mathrm{i}\hbar}[\hat{H}', \hat{\rho}_{\mathrm{S}}] - \frac{\gamma}{2}\left(\sigma_-\sigma_+\hat{\rho}_{\mathrm{S}} + \hat{\rho}_{\mathrm{S}}\sigma_-\sigma_+ - 2\sigma_+\hat{\rho}_{\mathrm{S}}\sigma_-\right). \tag{8.420}$$

This formula tells us that there is just a single Lindblad operator $\hat{L} = \sigma_+$ (see the definitions of σ_+, σ_-, and σ_3 in Eq. 8.288a) in the present case and, correspondingly, a single constant γ,

which turns out to be the *spontaneous decay rate* (Γ) of the atom (ie, the Einstein A coefficient) given by the first relation in Eq. (8.312). The operator $\hat{H}'$ occurring in this master equation differs from the Hamiltonian $\hat{H}_S$ of the isolated atom in that it involves an additional term relating to the *vacuum shift* of the energy of the excited state $|b\rangle$ of the atom, also referred to as the 'Lamb shift.' This, however, is not of interest to us in the present context, and will not be considered in detail.

The master equation lets us write down the rates of change of the matrix elements of the reduced density operator in the energy basis of the system under consideration (ie, in the energy basis of the two-level atom in the present context). For instance, one has, from Eq. (8.420)

$$\frac{d}{dt}\langle b|\hat{\rho}_S|b\rangle = -\Gamma\langle b|\hat{\rho}_S|b\rangle, \tag{8.421}$$

which shows that Γ is indeed the spontaneous rate of decay from the excited state $|b\rangle$ to the ground state $|a\rangle$ (reason this out).

While formula (8.420) has been written for $T = 0$ for simplicity, one can also obtain the master equation for a nonzero value of T, when it is modified by the inclusion of field-induced terms in the dissipator in addition to the vacuum terms. This can be inferred from the full form of the master equation for the Jaynes-Cummings model given in Section 8.20.4.5.

The two-level atom coupled to a reservoir is sometimes referred to as a *damped spin*, since the two-level atom is equivalent to a spin-half particle.

8.20.4.4 Example: Decay of the cavity field

As the next example of an open system in quantum optics, consider a closed cavity with a single resonant mode of the electromagnetic field in it, equivalent to a harmonic oscillator characterized by a frequency ω_0 and by annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$. The cavity field (system S) is coupled to the environment exterior to the cavity, which we again assume to be made up of an infinite number of modes of the external electromagnetic field, where these modes form a system (the reservoir R) in equilibrium at some specified temperature T. In this case $\hat{H}_S$ is the harmonic oscillator Hamiltonian

$$\hat{H}_S = \hbar\omega_0\hat{a}^\dagger\hat{a}, \tag{8.422}$$

where the vacuum energy is ignored as having no consequence in the present context. The reservoir Hamiltonian ($\hat{H}_R$) is given by formula (8.418b), while the interaction Hamiltonian is taken to be of the form

$$\hat{V} = \sum_\alpha g_\alpha(\hat{a}\hat{b}_\alpha^\dagger + \hat{a}^\dagger\hat{b}_\alpha), \tag{8.423}$$

where $\{g_\alpha\}$ denotes a set of coupling strengths, and where possible terms such as $\hat{a}\hat{b}_\alpha$ and $\hat{a}^\dagger\hat{b}_\alpha^\dagger$ are not included in view of the rotating wave approximation. The cavity field coupled to the reservoir is referred to as the *damped harmonic oscillator* ('damped oscillator' in brief), which appears in numerous different contexts in physics.

With $\hat{H}_S$, $\hat{H}_R$, and $\hat{V}$ defined as above, one can go through the microscopic derivation to arrive at the master equation for the reduced density operator of the cavity field. Once again we confine ourselves to the case of the reservoir at $T = 0$ for simplicity, when the dissipator involves only one Lindblad operator $\hat{L} = \hat{a}$, and the master equation appears as

$$\frac{d\hat{\rho}_S}{dt} = \frac{1}{i\hbar}[\hat{H}', \hat{\rho}_S] - \frac{\kappa}{2}\left(\hat{a}^\dagger\hat{a}\hat{\rho}_S + \hat{\rho}_S\hat{a}^\dagger\hat{a} - 2\hat{a}\hat{\rho}_S\hat{a}^\dagger\right). \tag{8.424}$$

Here $\hat{H}'$ is again a renormalized Hamiltonian which differs from the Hamiltonian $\hat{H}_S$ of the isolated oscillator in that it involves a frequency shift $\omega_0 \to \omega_0'$ analogous to the frequency shift of a classical damped oscillator. We, however, concentrate on the dissipative dynamics of the oscillator described by the second term on the right-hand side (ie, by the dissipator), which contains the constant κ. Assuming that the frequencies of the field modes making up the reservoir form an effectively continuous set and that the coupling constants g_α do not depend on the polarization index in α, one obtains a smooth function $g(\omega)$, in terms of which the constant κ is

$$\kappa = 2\pi|g(\omega_0)|^2\rho(\omega_o), \tag{8.425}$$

where $\rho(\omega_0)$ is the density of modes of the reservoir field at the cavity frequency ω_0.

> One obtains the same result by invoking the Fermi golden rule for the cavity-reservoir interacting system.

Analogous to the case of the damped spin, κ represents the cavity decay rate. Thus, considering the diagonal matrix element ($\langle 0|\hat{\rho}_S|0\rangle$) of the reduced density operator between cavity vacuum states, one obtains the rate of change of this matrix element from Eq. (8.424) as

$$\frac{d}{dt}\langle 0|\hat{\rho}_S|0\rangle = \kappa\langle 1|\hat{\rho}_S|1\rangle, \tag{8.426}$$

which tells us that there occurs a decay from the one-photon state to the vacuum state (by the release of one photon to the external field) at a rate κ. This decay rate determines the bandwidth $\Delta\omega$ and the Q value of the cavity field, as indicated in Section 8.20.3.1.

The master equation obtained from the microscopic derivation for the case of the reservoir modes forming a system at equilibrium at a nonzero temperature T includes additional temperature-dependent terms representing the external field–induced dynamics of the cavity mode, again in analogy to the damped-spin case. The complete form of the master equation

for the damped oscillator can be inferred from the master equation for the Jaynes-Cummings model given in Section 8.20.4.5.

8.20.4.5 Master equation for the atom-cavity system

With all this background, we finally consider the composite system made up of a two-level atom *and* a single resonant cavity mode as the system of interest (S) coupled to the reservoir (R), made up of the infinite number of modes of the electromagnetic field external to the cavity. The atom-cavity system will be assumed to be described by the Jaynes-Cummings Hamiltonian given by Eq. (8.293a),

$$\hat{H}_{\mathrm{S}} = -\frac{1}{2}\hbar\omega_0\sigma_3 + \hbar\omega\hat{a}^\dagger\hat{a} + \mathrm{i}\hbar g(\sigma_-\hat{a} - \sigma_+\hat{a}^\dagger). \tag{8.427}$$

The reservoir is again described by the Hamiltonian $\hat{H}_{\mathrm{R}}$ given by Eq. (8.418b), while the interaction Hamiltonian is assumed to be made up of contributions corresponding to the right-hand sides of Eqs. (8.418c) and (8.423). In this, the dipole interaction term in Eq. (8.418c) is to be taken with a certain weight factor, indicating that the two-level atom does not interact with the reservoir field in open space but does so from within the cavity, as a result of which there occurs a scaled-down residual interaction, which for a completely closed cavity may even be reduced to zero. As mentioned later, this will be reflected in a reduced value of the parameter γ (see Eq. 8.420) compared with its value (Γ) in the case of the atom being placed in open space.

With the problem defined in this way, one can attempt to work through a microscopic derivation so as to arrive at a Markovian master equation describing the dissipative dynamics of the atom-cavity system. This, however, is fraught with problems of a technical nature since the interaction $\hat{V}$ described above does not lend itself to being reduced to a form appropriate for the microscopic derivation to work. A notable feature of the problem that goes against the conditions necessary for the validity of a Markovian master equation is the existence of the set of frequencies $\Omega_n = 2g\sqrt{n}$ ($n = 1, 2, \ldots$) (see Eq. 8.297b) characterizing the internal dynamics of the atom-cavity system S.

One approach commonly adopted for the system under consideration is to ignore the residual direct coupling of the two-level atom with the reservoir field, which, in any case, is suppressed in a closed cavity. In other words, one assumes that the dominant mode of dissipation is by photon leakage from the cavity field to the reservoir field, and then writes down phenomenologically a master equation where the dissipator looks formally the same as that for the damped harmonic oscillator (see Eq. 8.424, which is written for the special case of a reservoir at $T = 0$). A microscopic derivation gives a master equation in a nonstandard form that differs, in general, from such a phenomenological master equation but a comparison of

the two shows that the predictions from the two equations agree over a considerably wide range of situations involving the damped Jaynes-Cummings model.

A more general phenomenological model of the Lindblad form is obtained by *adding together the dissipators* for the damped spin and the damped harmonic oscillator. I write this down below for a nonzero temperature T characterizing the reservoir. To begin with, the master equation is of the Lindblad form with *four* Lindblad operators $\hat{L}_k$ $(k = 1, 2, 3, 4)$:

$$\frac{d}{dt}\hat{\rho}_{\rm S}(t) = \frac{1}{{\rm i}\hbar}[\hat{H}', \hat{\rho}_{\rm S}] + \sum_{k=1}^{4} \gamma_k \left(\hat{L}_k \hat{\rho}_{\rm S} \hat{L}_k^\dagger - \frac{1}{2}\hat{L}_k^\dagger \hat{L}_k \hat{\rho}_{\rm S} - \frac{1}{2}\hat{\rho}_{\rm S} \hat{L}_k^\dagger \hat{L}_k \right), \tag{8.428a}$$

where

$$L_1 = \sigma_+, \quad L_2 = \sigma_-, \quad L_3 = \hat{a}, \quad L_4 = \hat{a}^\dagger, \tag{8.428b}$$

and where the constants γ_k are

$$\gamma_1 = (n_1(T) + 1)\Gamma', \quad \gamma_2 = n_1(T)\Gamma', \quad \gamma_3 = (n_2(T) + 1)\kappa, \quad \gamma_4 = n_2(T)\kappa. \tag{8.428c}$$

In these expressions, $n_1(T)$ stands for the mean number of thermal photons of frequency ω_0 (the atomic transition frequency) in the reservoir field at the specified temperature T, while $n_2(T)$ is the mean number of photons of frequency ω (the cavity resonant frequency):

$$n_1(T) = \frac{1}{{\rm e}^{\beta\hbar\omega_0} - 1}, \quad n_2(T) = \frac{1}{{\rm e}^{\beta\hbar\omega} - 1} \quad \left(\beta = \frac{1}{k_{\rm B} T} \right), \tag{8.428d}$$

where $k_{\rm B}$ stands for the Boltzmann constant. With reference to the expressions for the constants γ_k given earlier, the constant κ is the rate of photon leakage from the cavity field to the reservoir field, and is given by expression (8.425) (with the replacement $\omega_0 \to \omega$), while Γ' does not have a corresponding microscopic interpretation, and appears as a phenomenological constant representing the spontaneous atomic decay rate due to the residual direct coupling of the two-level atom to the reservoir field. Finally, in expression (8.428a), $\hat{H}'$ stands for a renormalized system Hamiltonian where $H_{\rm S}$ is modified by the inclusion of terms representing the vacuum shifts (ie, the Lamb shifts) and the field-induced shifts (ie, the Stark shifts) in the atomic excited level energy and in the cavity frequency. Here I opt not to go into the details of these energy shifts.

As mentioned earlier, the dissipator in the phenomenological master equation (8.428a) is a sum of two terms, where these correspond to the atom-reservoir coupling and the cavity-reservoir coupling, each considered independently of the other. At zero temperature the two dissipators reduce to the ones appearing in formulae (8.420) and (8.424), pertaining to the damped spin and damped oscillator, respectively. I repeat that for a closed cavity one may put $\Gamma' = 0$, in which case the dissipator for the atom-cavity system reduces to that for the damped oscillator, and the master equation, written out in full, appears as

$$\frac{d\hat{\rho}_S}{dt} = \frac{1}{i\hbar}[\hat{H}', \hat{\rho}_S] - \frac{\kappa}{2}(1 + n(T))\left(\hat{a}^\dagger\hat{a}\hat{\rho}_S + \hat{\rho}_S\hat{a}^\dagger\hat{a} - 2\hat{a}\hat{\rho}_S\hat{a}^\dagger\right) - \frac{\kappa}{2}n(T)\left(\hat{a}\hat{a}^\dagger\hat{\rho}_S + \hat{\rho}_S\hat{a}\hat{a}^\dagger - 2\hat{a}^\dagger\hat{\rho}_S\hat{a}\right), \tag{8.429}$$

where the notation differs slightly compared with that in Eq. (8.428c) in that the symbol $n(T)$ has been used in the place of $n_2(T)$, representing the mean number of reservoir photons at the cavity frequency ω.

The master equation given by Eq. (8.429) leads to a number of consequences that can be checked against observations on atom-cavity systems, when it is found that, even though it is of a phenomenological nature, it yields valid inferences in a considerably wide range of situations, especially in those involving closed cavities with microwave cavity fields.

As a simple illustration of the application of the master equation, we consider the case of an atom, initially in the excited state, placed in a cavity in the vacuum state, where the reservoir field is at $T = 0$. This means that $n(T) = 0$ in Eq. (8.429) and that the atom-cavity system remains confined to the subspace of states made up of the bare states $|\tilde{b}_0\rangle$, $|\tilde{a}_1\rangle$, and $|\tilde{a}_0\rangle$ (in the notation of Section 8.20.3.2; the last of these three is referred to as the 'zero quantum state,' the other two being 'one quantum' states). The coupling to the reservoir now gives rise to a reduced density operator (in the following we write this as $\hat{\rho}$ for simplicity, omitting the subscript 'S') whose matrix elements in the space of these three states form a closed system of equations resulting from Eq. (8.429).

The basic processes involved in the space of these three states are, first, the coherent coupling between $|\tilde{b}_0\rangle$ and $|\tilde{a}_1\rangle$ by the Jaynes-Cummings Hamiltonian and, second, the decay of $|\tilde{a}_1\rangle$ into $|\tilde{a}_0\rangle$. While there are, in all, nine possible matrix elements in the subspace under consideration, there are only four nontrivial components of the master equation, which we write as (we ignore the vacuum frequency shift of the cavity field and assume that the atom-cavity system is at resonance, ie, $\omega = \omega_0$)

$$\begin{aligned}
\frac{d\rho_{11}}{dt} &= g(\rho_{12} + \rho_{21}),\\
\frac{d\rho_{22}}{dt} &= -g(\rho_{12} + \rho_{21}) - \kappa\rho_{22},\\
\frac{d(\rho_{12} + \rho_{21})}{dt} &= -2g(\rho_{11} - \rho_{22}) - \frac{\kappa}{2}(\rho_{12} + \rho_{21}),\\
\frac{d\rho_{33}}{dt} &= \kappa\rho_{22}
\end{aligned} \tag{8.430}$$

(check this out), where the subscripts '1,' '2,' and '3' correspond to states $|\tilde{b}_0\rangle$, $|\tilde{a}_1\rangle$, and $|\tilde{a}_0\rangle$, respectively.

In writing these equations, we have chosen a representation obtained by applying on the Schrödinger picture states and operators a unitary transformation corresponding to the

evolution operator relating to the free evolution in the Jaynes-Cummings Hamiltonian—that is, the evolution that results from Eq. (8.429) by putting $\kappa = 0$ and $g = 0$. Making such a transformation, one avoids unnecessary phase factors in the solutions for the reduced density matrix elements describing the atom-cavity system.

These equations involve only three independent dynamical variables,

$$x_1 \equiv \rho_{11}, \quad x_2 \equiv \rho_{22}, \quad x_3 \equiv \rho_{12} + \rho_{21}, \tag{8.431}$$

since the evolution of ρ_{33} is determined by that of ρ_{22}, while ρ_{33} does not enter into the first three of the four equations above. This leaves us with three independent linear differential equations:

$$\frac{dX}{dt} = AX, \tag{8.432a}$$

where

$$X \equiv \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad A \equiv \begin{pmatrix} 0 & 0 & g \\ 0 & -\kappa & -g \\ -2g & 2g & -\frac{\kappa}{2} \end{pmatrix}. \tag{8.432b}$$

The eigenvalues of the matrix A are

$$\lambda_0 = -\frac{\kappa}{2}, \quad \lambda_\pm = -\frac{\kappa}{2} \pm \sqrt{\frac{\kappa^2}{4} - 4g^2}, \tag{8.433}$$

where these expressions bring out, in the present context, the distinction between the weak coupling and strong coupling regimes as

$$\begin{aligned} &\text{weak coupling: } |g| < \frac{\kappa}{4}, \\ &\text{strong coupling: } |g| > \frac{\kappa}{4}. \end{aligned} \tag{8.434}$$

In the weak coupling regime, all three eigenvalues are negative, indicating an overdamped dynamics, where the slowest decay mode of ρ_{11} (recall that the initial state is assumed to be $|i\rangle = |\tilde{b}_0\rangle$), which determines the effective decay constant, is $|\lambda_+| \approx \frac{4g^2}{\kappa}$ (for $|g| \ll \kappa$). Recalling the way the coupling strength g was arrived at in Eq. (8.12.3.1) and assuming that the dipole matrix element is oriented along the cavity axis, with the atom placed at a field antinode ($\sin(kz_0) = 1$), one arrives at the formula (8.405a) obtained earlier for the cavity-controlled decay rate at resonance (check this out).

In the strong coupling regime, on the other hand, the coherent Rabi oscillations appear as a signature of the reduced dynamics, and an oscillatory decay occurs.

8.20.5 The Strong Coupling Regime

Assuming once again that the atom-cavity reduced dynamics is confined to the subspace of states $|\tilde{b}_0\rangle$, $|\tilde{a}_1\rangle$, and $|\tilde{a}_0\rangle$, the strong coupling regime corresponds to $|g| > \frac{\kappa}{4}$ (Eq. 8.434), for which one finds from Eq. (8.433) that two of the eigenvalues ($\lambda_\pm$) of the matrix A have nonzero imaginary parts, which implies an oscillatory decay since the real parts of the eigenvalues are all negative. The strong coupling between the atom and the cavity sets up the Rabi oscillations caused by the reversible exchange of a single photon between the two, while the relatively weak coupling to the reservoir leads to a slow damping of the oscillations. Eventually, the photon leaks out of the cavity, and the system tends asymptotically to the state $|\tilde{a}_0\rangle$.

Fig. 8.33 depicts schematically the overdamped and oscillatory decay processes in the weak and strong coupling regimes as compared with the spontaneous decay of an atom in free space. The two-level atom is initially assumed to be in the excited state ($\rho_{11}(0) = 1$), and the probability of the atom being in the excited state at time t—that is, $\rho_{11}(t)$—is plotted as a function of t (in the spontaneous decay case $\rho_{11}(t)$ is defined appropriately). The enhanced damping at resonance in the weak coupling regime compared with the open-space spontaneous decay is evident from the curves marked 'A' and 'B.'

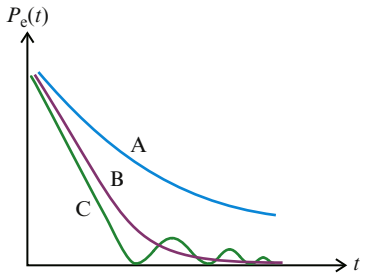

Fig. 8.33

Depicting the overdamped and oscillatory decay processes for a two-level atom in a cavity coupled to a reservoir field. The reservoir is assumed to be at temperature $T = 0$. The matrix element $\rho_{11}(t)$ of the reduced density operator is plotted against time t, for the initial condition $\rho_{11}(0) = 1$, where $\rho_{11}(t)$ represents the probability ($P_e(t)$) of the atom to be in the excited state at time t. Curve A depicts the probability for the case of spontaneous decay in open space, while curve B corresponds to the overdamped case in the weak coupling regime, where the Purcell effect involving an enhanced decay rate is apparent. Curve C shows the oscillatory decay in the strong coupling regime, where the Rabi oscillations occur but are damped by the reservoir.

The dynamics of the system can be inferred qualitatively from what we saw in the Jaynes-Cummings model in Section 8.12.3. As we found in Section 8.12.3.4, as a result of the interaction between the atom and the cavity field, the bare states $|\tilde{a}_1\rangle$ and $|\tilde{b}_0\rangle$ do not constitute stationary states of the Jaynes-Cummings Hamiltonian. Instead, the stationary states

correspond to the dressed states $|E_1\rangle$ and $|E_2\rangle$ of the form (8.303a) (with α an β related by Eq. 8.304a), with energies given by Eq. (8.304b). If the system is now assumed to be coupled to the reservoir, both dressed states decay to the ground state $|\tilde{a}_0\rangle$.

The energy splitting between the dressed states is revealed in the strong coupling regime by a *doublet structure* of the emission spectrum (and also of the absorption spectrum) of the atom-cavity system. In the weak coupling case the splitting is masked by the damping, and the spectrum consists of only one peak at the resonant frequency of the cavity, with a width determined by the damping rate. In the strong coupling case, on the other hand, the splitting is enhanced with increasing value of $|g|$ relative to the damping parameter κ. This is shown in Fig. 8.34.

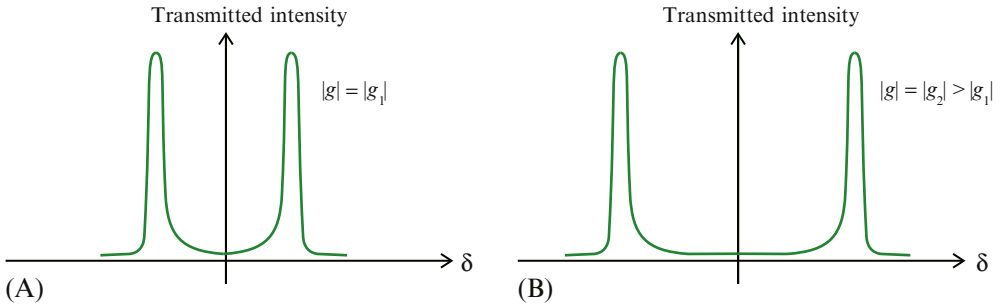

Fig. 8.34
Depicting the doublet structure of the spectrum of the atom-cavity system in the strong coupling regime. A laser beam of frequency $\omega = \omega_0 + \delta$ (the 'probe' beam) is passed through the cavity, through which a low-intensity stream of atoms is passed in a perpendicular direction. Here ω_0 stands for the atomic transition frequency, with the cavity field locked to this frequency, while δ represents the detuning of the probe field from ω_0. The transmitted intensity of the probe beam is plotted as a function of δ, and shows a doublet structure. The splittings for the coupling parameters g_1 and g_2 are depicted in (A) and (B), respectively, for a fixed value of κ, with $|g_2| > |g_1|$. The splitting increases with the strength of coupling.

8.21 Quantum Optics and Quantum Information

8.21.1 Information Processing: Classical and Quantum

Theoretical and technological progress in quantum optics and in related areas has opened up a great and vast expanse for further progress in the form of theoretical developments in and practical applications of *quantum information* theory.

Quantum information theory shares a certain common set of foundational concepts with its classical counterpart, where the latter has culminated in the development of classical

computers, but is distinguished by a large number of radically new concepts and potential applications based on exotic possibilities inherent in quantum theory itself.

As in the classical case, quantum information processing systems use sets of basic units carrying and encoding information but the constitution and functioning of these units, the *qubits*, differ fundamentally from the classical units—namely, the *bits*. While a bit has just two possible states (namely, '0' and '1'), a qubit can have a nondenumerably infinite number of possible states, all of which can be built up from two basic states, commonly denoted $|0\rangle$ and $|1\rangle$. A single qubit can thus encode an infinite number of classical bits. What is more, the set of joint states of more than one qubit incorporates *correlated* information of a nature fundamentally different from what is possible with classical bits, where such nonclassical correlations appear in the form of *entangled* states of the qubits.

In classical information processing the joint states of bits are changed and manipulated by means of classical logic *gates*. Quantum information processing is also achieved by the operation of gates. As in the classical case, the states of qubits making up a quantum *register* can be changed step by step by the action of a number of *quantum gates* that constitute a *universal* set, where there can be more than one possible set of universal gates. A set of universal gates is made up of ones that act on a single qubit and of others that work on the joint states of more than one qubit. Given a set of initial data 'written' on the register of qubits, a step-by-step operation of the gates in a predetermined succession constitutes a *program*, where a program can be regarded as the realization of an *algorithm*. The program takes the register of qubits to some *final* state, yielding data that may constitute the solution to some given problem, provided the algorithm is an appropriate one.

While this is the basic scheme of quantum *computation* that can, in principle, achieve results far beyond the limits of classical computation, other equally remarkable information processing jobs are possible by means of qubits and quantum gates where classical and quantum communications *channels* are used for the transfer of classical and quantum information. An example of quantum information processing of a nonclassical nature is the secure transfer of data in encoded form, referred to as *quantum cryptography*.

8.21.2 Realization of Qubits and Quantum Gates

In this section I briefly outline a few quantum optics schemes for realization of qubits and quantum gates, the fundamental building blocks of quantum information systems.

8.21.2.1 Quantum information hardware: Introduction

The big concern of quantum information science is the physical realization of the qubits and quantum gates—the building blocks of any quantum information system—this being the

essential prerequisite for transforming the possible into the real. It is here that quantum optics has a big role to play.

Several alternative schemes have been explored for the physical realization of quantum information setups. One of these uses nuclear magnetic resonance (NMR), where the qubits are nuclear spins in the molecules in a liquid medium containing a large number of similar molecules, and the function of the gates is performed by AC magnetic fields acting on the nuclear spins. While the field of NMR is distinct from quantum optics, the latter uses a number of basic ideas pertaining to the former, notable among which is that of Rabi oscillations induced by electromagnetic pulses (microwave and laser pulses in quantum optics systems). Another scheme uses arrays of quantum dots where electrons confined in the dots act as the qubits and the individual and joint states of these are controlled by means of magnetic fields and the quantum dot gate voltages.

Three other schemes belong to the field of quantum optics. Of these, one is based on atoms interacting with resonant or near-resonant high-Q cavity fields. A second scheme uses the interaction of internal excitations of *trapped ions* with their vibrational modes, while a third scheme involves the manipulation of polarization states of photons.

8.21.2.2 Quantum optics hardware: The ion trap

Ion traps constitute a promising means of realizing qubits and quantum gates. A combination of AC and DC fields is used to trap ions either singly or in a linear array, where a pair of internal electronic states of an ion are used as the basic qubit states. Among the various possible ion traps, the linear *Paul trap* is a useful and convenient one for the purpose of producing a linear array of trapped ions.

The trap consists of four parallel electrode rods arranged with their lengths along the *axis* of the trap, which we take to be the x-axis of a coordinate system. In a cross section by the y-z plane the rods appear at the corners of a square, with a pair of rods at the opposite ends of a diagonal connected to an AC supply, the other pair of rods being at AC ground. The resulting AC quadrupole field configuration provides a confining potential in the transverse plane for an ion placed inside the trap. Axial confinement is ensured by a pair of electrodes placed on the axis at the two ends of the trap, where both electrodes carry a repelling DC voltage for the trapped ions. Fig. 8.35 illustrates the idea of the ion trap, in which Fig. 8.35A depicts the positions (A, B, C, and D) of the electrode rods in a transverse (y-z) section, with the ion I located at the center, while Fig. 8.35B depicts a linear array of trapped ions, along with the pair of axial trapping electrodes.

An ion placed in the trap is held with almost zero thermal motion by means of *laser cooling*. A tunable laser beam with frequency close to the transition frequency between a pair of ionic states is used for the cooling. For an ion with a thermal motion toward the laser source, the *Doppler-shifted* laser frequency matches the transition frequency, and the resulting resonance

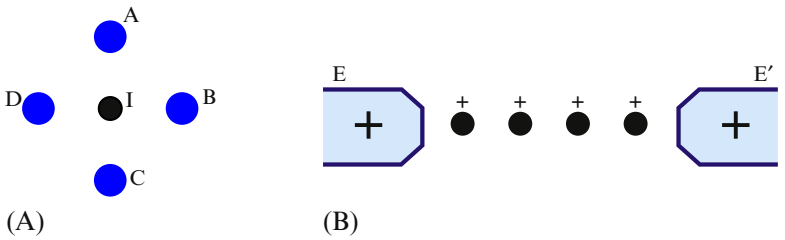

Fig. 8.35
Illustrating the linear Paul trap. (A) A transverse section showing the positions of the AC electrode rods A, B, C, and D, with a trapped ion I at the center. An AC electric field is applied between A and C, while B and D are at AC ground. The resulting quadrupole field configuration provides the trapping potential for the ion I in the transverse direction. The axis of the trap is perpendicular to the plane of the diagram. (B) A linear array of trapped ions along the axis of the trap, E and E′ are electrodes, each carrying a repelling DC voltage for axial confinement. The ions are deprived of their thermal motion by means of laser cooling.

absorption slows down the ion. In the subsequent emission process, a photon is released in a direction that varies randomly from one emission event to another, and there occurs, on average, zero change in the ion momentum. Thus there is a net slowing down in each absorption-emission cycle, equivalent to the action of a retarding force on the ion.

A pair of long-lived internal states of an ion are used as the basic qubit states $|0\rangle$ and $|1\rangle$. A pair of laser beams produced by a beam splitter from a single beam induce transitions between the qubit states of each ion, thereby implementing single-qubit quantum gates. Multiple-qubit gates are based on the coupling between the internal qubit states with vibrational states of the ions, where the latter arise in the form of phonon modes of the ion string. Laser pulses are used to impart momentum to the ion chain, wherein the absorption and emission of photons by an ion changes the phonon mode of the string by a process analogous to the Mo̎ssbauer effect in a crystal. Successive transitions induced in two ions in the array couple the two qubits in an entangled state, where the coupling is mediated by the phonon modes. This constitutes the basic process whereby multiple-qubit gates are realized.

8.21.2.3 Quantum optics hardware: The high-Q cavity

The high-Q cavity is an ideal device for the physical realization of the Jaynes-Cummings model and for the production of entangled states involving the cavity mode and atoms that can be made to pass through the cavity with specified velocities. The atoms (and also the cavity modes) may be made to function as qubits, and appropriately switched microwave pulses can be used to cause transitions between atomic states.

The microwave pulses are applied in the cavity at the entrance and exit zones of the atoms, referred to as the ‘Ramsey zones’. Atoms are prepared in circular Rydberg states before they

enter the cavity, these states being long-lived ones, having large dipole matrix elements for transitions between selected states. In a typical setup, shown in Fig. 8.36, three Rydberg states labeled $|a\rangle$, $|b\rangle$, and $|s\rangle$ are selected, with the states $|a\rangle$ and $|b\rangle$ earmarked as the basic qubit states. The cavity field is prepared with photon number 0 or 1 in the resonant mode, though other values of the photon number can also be selected.

> A circular Rydberg state is a state with a large value of the principal quantum number n, and with the maximum possible value of the azimuthal quantum number l for the given value of n.

As an atom is made to pass through the cavity, a microwave pulse switched in the first Ramsey zone R_1 causes a transition to a specified basic qubit state, following which an entangled state of the atom and the cavity field is produced by means of the atom-cavity interaction described by the Jaynes-Cummings Hamiltonian. The precise state of the atom-cavity system as the atom exits the cavity can be controlled by the tuning of its velocity of traversal. The entanglement persists even after the atom exits the cavity.

While a Ramsey pulse can be used as a single-qubit gate, the entire process involving the Ramsey pulse(s) and the cavity interaction can be regarded as a two-qubit gate, where the atom and the cavity field constitute the two qubits. By use of appropriate pulses in the two Ramsey zones and the atom-cavity interaction, universal quantum gates can be realized. On the other hand, by sending successive atoms through the cavity and again by an appropriate use of the Ramsey pulses and the cavity interaction, one can produce desired entangled states of the atoms, and once again various multiple-qubit gates, involving the atoms as the qubits, can be realized.

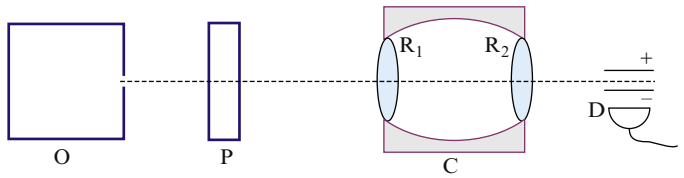

Fig. 8.36

Depicting the high-Q cavity as a quantum information processor. A stream of atoms is generated in the oven O and prepared in a specified quantum state with a specified velocity in zone P. C is the cavity, with Ramsey zones R_1 and R_2 near the entrance and exit of the cavity. Atoms from zone P are made to traverse the cavity in succession, with microwave pulses of appropriate frequency and duration acting on them in the Ramsey zones. On exiting the cavity, the atoms are detected by the ionization detector D.

The high-Q cavity is not considered to be a prime candidate for the quantum computer, though it holds great promise as a versatile device in quantum information processing since it can be made to produce a multitude of entangled states involving several atoms. It is also a useful

device for the study of *decoherence* of entangled states. In particular, it can be used in the production of *cat states* of the cavity field and in investigations of the rates of decoherence of these cat states.

1. Decoherence is the process where the state of a quantum system is degraded by means of its interaction with the environment, where the latter in general is a system with a large number of independent degrees of freedom. This 'degradation' is a complex process whereby the system under consideration becomes progressively entangled with the environment and the *entropy* of the reduced state of the system increases. If the system itself is a composite one made up of two or more subsystems (say, two qubits), then the entanglement between the subsystems is erased by the process of decoherence. The decoherence of multiple-qubit states is a highly deleterious process in quantum computation and quantum information processing.
2. A cat state is a superposed state of a system whose state space has a large dimension and which is quickly degraded into a mixed state by the process of decoherence. In particular, superposed states of macroscopic systems qualify as cat states and are almost never observed in practice because of the process of decoherence. Fine-tuned experiments have made possible the experimental realization of such cat states in sufficiently large quantum systems. The superposition of two coherent states in a cavity, each with a large value of the mean photon number, can be regarded as a cat state. Under decoherence, each of the coherent states continues to persist for a relatively long time, but their phase correlation is lost within a short time, thereby demonstrating the fragility of cat states. However, CQED techniques can be made use of to create cat states that are, relatively speaking, long lived so as to be observed experimentally.

Schemes have been proposed for the building of a CQED-based quantum computer where the above-described means of producing entangled atomic states are to be made use of. In particular, an array of atoms placed in a cavity may be used in a manner analogous to a linear array of trapped ions, where the coupling between different ions in the array by means of phonon modes is replaced with an analogous coupling by means of the cavity field.

8.21.2.4 Quantum optics hardware: Photonic qubits

A photon with any specified propagation vector can have two orthogonal states of polarization, in terms of which any arbitrary state of polarization can be expressed in the form of a linear combination. In this, a photon can be regarded as a qubit with the two polarization states as the basic qubit states. The polarization state of a single qubit can be easily manipulated by means of polarizers and retardation plates. The basic qubit states can be encoded in terms of photon paths by means of a polarizing beam splitter that transmits one polarization component and reflects the other component.

Single-qubit gates are easily realized by means of the linear optical devices mentioned above. Quantum gates whose action results in the entanglement of more than one qubit are, however,

not easy to realize. For instance, a strongly nonlinear medium can make possible the realization of universal quantum gates acting on several qubits at a time. Such strongly nonlinear media are yet to be developed, though considerable progress has been made in this regard. An alternative proposal for a photonic quantum computer uses the concept of *cluster state* computation, where nonlinear media are not required and only linear optical devices are used.

Photonic quantum computation (or, more generally, quantum information processing) is considered to be a promising area of investigation since the process of decoherence of photonic states is, in general, a relatively slow one.