Maths for Intelligent Systems

Topic 2:

Linear Algebra II

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Determinants

If $ax + by = p$ and

$cx + dy = q$ Then it can be represented as

$$Ax = b$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad x = \begin{bmatrix} x \\ y \end{bmatrix} \; and \; b = \begin{bmatrix} p \\ q \end{bmatrix}$$

If $(a\,d - b\,c \neq 0)$, then the system of linear equations has a unique solution.

The number$(a\,d - b\,c)$, which determines uniqueness of solution is associated with the matrix and is called the determinant of A and denoted by |A| or Δ or det.A.

Note:

1. Only square matrices have determinants
2. Expanding a determinant along any row or column gives same value.

YouTube Video: https://www.youtube.com/watch?v=Ip3X9LOh2dk (Determinant)

Properties:

- det(A)=det(A^T)
- If two rows (or columns) of A are equal, then det(A)=0.
- If a row (or column) of A consists entirely of 0, then det(A)=0.
- If B result from the matrix A by interchanging two rows (or columns) of A, then det(B)= - det(A)

Example: Evaluate the determinant $\Delta = \begin{vmatrix} 1 & 2 & 4 \\ -1 & 3 & 3 \\ 4 & 1 & 2 \end{vmatrix}$

So expanding along First Row (R1), we get

$$\Delta = \mathbf{1}\begin{vmatrix} \mathbf{3} & \mathbf{3} \\ \mathbf{1} & \mathbf{2} \end{vmatrix} - \mathbf{2}\begin{vmatrix} \mathbf{-1} & \mathbf{3} \\ \mathbf{4} & \mathbf{2} \end{vmatrix} + \mathbf{4}\begin{vmatrix} \mathbf{-1} & \mathbf{3} \\ \mathbf{4} & \mathbf{1} \end{vmatrix}$$

$$= \mathbf{1}(\mathbf{3} \times \mathbf{2} - \mathbf{3} \times \mathbf{1}) - \mathbf{2}((\mathbf{-1}) \times \mathbf{2} - \mathbf{3} \times \mathbf{4}) + \mathbf{4}((\mathbf{-1}) \times \mathbf{1} - \mathbf{3} \times \mathbf{4})$$

$$= \mathbf{3} + \mathbf{28} - \mathbf{52}$$

$$= \mathbf{-21}$$

Inverse of Matrix

Let A and B be $n\ x\ n$ matrices then A and B are *inverses* of each other, then

$AB = BA = I_n$

Inverse of Matrix A is denoted by A^{-1}.

Properties of Inverses of Matrix:

1. $(A^{-1})^{-1} = A$

2. $(AB)^{-1} = B^{-1}A^{-1}$

3. $(A^T)^{-1} = (A^{-1})^T$

4. If A and B are matrices with
   $AB = I_n$ then, A and B are inverses of each other.

We can calculate the Inverse of Matrix by:

Step 1: calculating the Matrix of Minors,

Step 2: calculating the Matrix of Cofactors,

Step 3: then the Adjoint, and

Step 4: multiply to Adjoint Matrix by 1/Det(A)

Example: find the Inverse of A using Minors, Cofactors and Adjoint

$$A = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

Step 1: Matrix of Minors

The first step is to create a "Matrix of Minors". This step has the most calculations.

Here are the first two, and last two, calculations of the "Matrix of Minors"

$$\begin{bmatrix} \bullet & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} \quad 0\times1 - (-2)\times1 = 2$$

$$\begin{bmatrix} 3 & \bullet & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} \quad 2\times1 - (-2)\times0 = 2$$

$$\cdots$$

$$\begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & \bullet & 1 \end{bmatrix} \quad 3\times-2 - 2\times2 = -10$$

$$\begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & \bullet \end{bmatrix} \quad 3\times0 - 0\times2 = 0$$

And here are the Minors for each element of the matrix:

$$\begin{bmatrix} 2 & 2 & 2 \\ -2 & 3 & 3 \\ 0 & -10 & 0 \end{bmatrix}$$

Matrix of Minors

Step 2: Matrix of Cofactors

$$\begin{bmatrix} 2 & 2 & 2 \\ -2 & 3 & 3 \\ 0 & -10 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & -2 & 2 \\ +2 & 3 & -3 \\ 0 & +10 & 0 \end{bmatrix}$$

Matrix of Minors *Matrix of CoFactors*

Step 3: Adjoint

Now "Transpose" all elements of the Cofactor matrix

$$\begin{bmatrix} 2 & 2 & 0 \\ -2 & 3 & 10 \\ 2 & -3 & 0 \end{bmatrix}$$

Step 4: Multiply to Adjoint Matrix by 1/Det(A)

multiply the Adjoint matrix by 1/Det(A):

$$A^{-1} = \frac{1}{10} \underbrace{\begin{bmatrix} 2 & 2 & 0 \\ -2 & 3 & 10 \\ 2 & -3 & 0 \end{bmatrix}}_{\textit{Adjugate}} = \underbrace{\begin{bmatrix} 0.2 & 0.2 & 0 \\ -0.2 & 0.3 & 1 \\ 0.2 & -0.3 & 0 \end{bmatrix}}_{\textit{Inverse}}$$

Solving System of Linear Equations

Solving a Linear System of Equations:

- Elimination of variables / Substitution Method
- Cramer's rule
- Row reduction/ Gaussian elimination / Gauss–Jordan elimination
- Matrix solution

Elimination of variables / Substitution Method:

The simplest method for solving a system of linear equations is to repeatedly eliminate variables. This method can be described as follows:

- In the first equation, solve for one of the variables in terms of the others.
- Substitute this expression into the remaining equations. This yields a system of equations with one fewer equation and one fewer unknown.
- Repeat until the system is reduced to a single linear equation.
- Solve this equation and then back-substitute until the entire solution is found.

For example, consider the following system:

$$x + 3y - 2z = 5$$

$$3x + 5y + 6z = 7$$

$$2x + 4y + 3z = 8$$

Solving the first equation for x gives $x = 5 + 2z - 3y$, and plugging this into the second and third equation yields

$$-4y + 12z = -8$$

$$-2y + 7z = -2$$

Solving the first of these equations for y yields $y = 2 + 3z$, and plugging this into the second equation yields

$$z = 2.$$

We now have:

$$x = 5 + 2z - 3y$$

$$y = 2 + 3z$$

$$z = 2$$

Substituting $z = 2$ into the second equation gives $y = 8$. Substituting $z = 2$ and $y = 8$ into the first equation yields $x = -15$.

Therefore, the solution set is the single point $(x, y, z) = (-15, 8, 2)$.

Cramer's Rule:

Cramer's rule is an explicit formula for the solution of a system of linear equations, with each variable given by a quotient of two determinants. For example, the solution to the system is given by

$$\begin{aligned} x + 3y - 2z &= 5 \\ 3x + 5y + 6z &= 7 \\ 2x + 4y + 3z &= 8 \end{aligned}$$

Is given by,

$$x = \frac{\begin{vmatrix} 5 & 3 & -2 \\ 7 & 5 & 6 \\ 8 & 4 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 3 & -2 \\ 3 & 5 & 6 \\ 2 & 4 & 3 \end{vmatrix}}, y = \frac{\begin{vmatrix} 1 & 5 & -2 \\ 3 & 7 & 6 \\ 2 & 8 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 3 & -2 \\ 3 & 5 & 6 \\ 2 & 4 & 3 \end{vmatrix}}, z = \frac{\begin{vmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \\ 2 & 4 & 8 \end{vmatrix}}{\begin{vmatrix} 1 & 3 & -2 \\ 3 & 5 & 6 \\ 2 & 4 & 3 \end{vmatrix}}$$

Though Cramer's rule is important theoretically, it has little practical value for large matrices, since the computation of large determinants is somewhat awkward.

Row reduction/ Gaussian elimination / Gauss–Jordan elimination

Linear equations

$2x + 3y = 7$

$3x - 2y = 4$

Matrix

$$\begin{array}{rc} \text{Row 1} \rightarrow & \multirow{2}{*}{$\begin{bmatrix} 2 & 3 \\ 3 & -2 \end{bmatrix}_{2 \times 2}$} \\ \text{Row 2} \rightarrow & \\ & \uparrow \quad\quad \uparrow \\ & \text{Column 1} \quad \text{Column 2} \end{array}$$

Augmented Matrix

$$\left[\begin{array}{cc:c} 2 & 3 & 7 \\ 3 & -2 & 4 \end{array}\right]$$

Elementary Row / Column Operations:

1. $R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$
2. $R_i \rightarrow k \cdot R_i$ or $C_i \rightarrow k \cdot C_i$
3. $R_i \rightarrow R_i + kR_j$ or $C_i \rightarrow C_i + kC_j$

Row Echelon Form:

4x+8y-4z=4
3x+8y+5z=-11
-2x+y+12z=-17

$$\left[\begin{array}{ccc:c} 4 & 8 & -4 & 4 \\ 3 & 8 & 5 & -11 \\ -2 & 1 & 12 & -17 \end{array}\right]$$

Method of
Gaussian Elimination

$$\left[\begin{array}{ccc:c} 1 & 2 & -1 & 1 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

x+2y- z =1
y+4z=-7
z=-2

Reduced Row Echelon Form:

4x+8y-4z=4
3x+8y+5z=-11
-2x+y+12z=-17

$$\left[\begin{array}{ccc:c} 4 & 8 & -4 & 4 \\ 3 & 8 & 5 & -11 \\ -2 & 1 & 12 & -17 \end{array}\right]$$

Method of
Gaussian Elimination

$$\left[\begin{array}{ccc:c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

x = -3
y = 1
z=-2

Method of Gaussian Elimination:

$$\left[\begin{array}{ccc:c} 4 & 8 & -4 & 4 \\ 3 & 8 & 5 & -11 \\ -2 & 1 & 12 & -17 \end{array}\right]$$

$R_1 \rightarrow 1/4\ R_1$

$$\left[\begin{array}{ccc:c} 1 & 2 & -1 & 1 \\ 3 & 8 & 5 & -11 \\ -2 & 1 & 12 & -17 \end{array}\right]$$

$R_2 \rightarrow R_2 - 3R_1$ and $R_3 \rightarrow R_3 + 2R_1$

$$\left[\begin{array}{ccc:c} 1 & 2 & -1 & 1 \\ 0 & 2 & 8 & -14 \\ 0 & 5 & 10 & -15 \end{array}\right]$$

$R_2 \rightarrow 1/2\ R_2$

$$\left[\begin{array}{ccc:c} 1 & 2 & -1 & 1 \\ 0 & 1 & 4 & -7 \\ 0 & 5 & 10 & -15 \end{array}\right]$$

$R3 \rightarrow R3 - 5R2$

$$\left[\begin{array}{ccc:c} 1 & 2 & -1 & 1 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & -10 & 20 \end{array}\right]$$

$R_3 \rightarrow -(1/10)R_3$

Row Echelon Form $\rightarrow$

$$\left[\begin{array}{ccc:c} 1 & 2 & -1 & 1 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$R_1 \rightarrow R_1 - 2R_2$

$$\left[\begin{array}{ccc:c} 1 & 0 & -9 & 15 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$R_1 \rightarrow R_1 + 9R_3$ and $R_2 \rightarrow R_2 - 4R_3$

Reduced Row Echelon Form $\rightarrow$

$$\left[\begin{array}{ccc:c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

(Case: No Solution)

$$\begin{aligned} x+y+z&=2 \\ y-3z&=1 \\ 2x+y+5z&=0 \end{aligned} \Rightarrow \left[\begin{array}{ccc:c} 1 & 1 & 1 & 2 \\ 0 & 1 & -3 & 1 \\ 2 & 1 & 5 & 0 \end{array}\right]$$

Method of Gaussian Elimination

$$\left[\begin{array}{ccc:c} 1 & 0 & 4 & 1 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & -3 \end{array}\right]$$

No Solution

Since, $0x+0y+0z=-3$

$0=-3$ **?(Think!)**

(Case: Infinite Solution)

$$\begin{aligned} -3x-5y+36z&=10 \\ -x \quad +7z&=5 \\ x+y-10z&=-4 \end{aligned} \quad \Rightarrow \quad \left[\begin{array}{ccc|c} -3 & -5 & 36 & 10 \\ -1 & 0 & 7 & 5 \\ 1 & 1 & -10 & -4 \end{array}\right]$$

Method of Gaussian Elimination

$$\left[\begin{array}{ccc|c} 1 & 0 & -7 & -5 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Infinite Solution

Since, $0x+0y+0z=0$ $\quad$ $0=0$ **?(Think!)**

Matrix solution:

If $ax + by = p \quad and$

$cx + dy = q$

Then it can be represented as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}$$

$Ax = b$ where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ $and\ b = \begin{bmatrix} p \\ q \end{bmatrix}$

$$A^{-1}Ax = A^{-1}b$$

$$Ix = A^{-1}b$$

$$x = A^{-1}b$$

Determinants

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$cx + dy = q$ Then it can be represented as

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$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad X = \begin{bmatrix} x \\ y \end{bmatrix} \; and \; b = \begin{bmatrix} p \\ q \end{bmatrix}$$

If $(a\,d - b\,c \neq 0)$ then the system of linear equations has a unique solution.

The number$(a\,d - b\,c)$, which determines uniqueness of solution is associated with the matrix and is called the determinant of A and denoted by |A| or Δ or det.A.

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- det(A)=det(A^T)
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Example: Evaluate the determinant $\Delta = \begin{vmatrix} 1 & 2 & 4 \\ -1 & 3 & 3 \\ 4 & 1 & 2 \end{vmatrix}$

So expanding along First Row (R1), we get

$$\Delta = 1\begin{vmatrix} 3 & 3 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} -1 & 3 \\ 4 & 2 \end{vmatrix} + 4\begin{vmatrix} -1 & 3 \\ 4 & 1 \end{vmatrix}$$

$$= 1(3 \times 2 - 3 \times 1) - 2((-1) \times 2 - 3 \times 4) + 4((-1) \times 1 - 3 \times 4)$$

$$= 3 + 28 - 52$$

$$= -21$$

Eigenvalues and Eigenvector:

The eigenvectors of a square matrix are the non-zero vectors which, after being multiplied by the matrix, remain proportional to the original vector, i.e. any vector x that satisfies the equation: $Ax = \lambda x,$

where A is the matrix in question, x is the eigenvector and λ is the eigenvalue.

In order to find the eigenvectors of a matrix we must start by finding the eigenvalues.

$$Ax - \lambda x = 0,$$

then we pull the vector x outside of a set of brackets:

$$(A - \lambda I)x = 0,$$

The only way this can be solved is if $A - \lambda I$ does not have an inverse, therefore we find values of λ such that the determinant of $A - \lambda I$ is zero:

$$|A - \lambda I| = 0$$

Once we have a set of eigenvalues we can substitute them back into the original equation to find the eigenvectors.

Example 1

Find the eigenvalues and eigenvectors of the matrix:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

First we start by finding the eigenvalues,

$$|A - \lambda I| = \left| \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix}$$

Next we derive a formula for the determinant, which must equal zero:

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)(2-\lambda) - 1 = \lambda^2 - 4\lambda + 3$$

$$\lambda^2 - 4\lambda + 3 = 0$$

Now, we need to find the roots of this quadratic equation in λ.

$$\lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1) = 0$$

Therefore, the solutions to this equation are $\lambda = 3 \; and \; \lambda = 1$

These solutions are the eigenvalues of the matrix A.

Now we will now solve for an eigenvector.

First we will solve for $\lambda_1 = 1$:

From equation $Ax = \lambda x$,

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1 \times \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 + x_2 \\ x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$2x_1 + x_2 = x_1 \qquad \rightarrow \qquad x_1 + x_2 = 0$$

$$x_1 + 2x_2 = x_2 \qquad \rightarrow \qquad x_1 + x_2 = 0$$

These equations are not solvable!

$$x_1 = -x_2$$

$$x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

This is one of the eigenvector of the matrix A.

Now will solve for $\lambda_1 = 3$: (From equation $Ax = \lambda x$)

$$\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3 \times \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 + x_2 \\ x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ 3x_2 \end{bmatrix}$$

and multiply out to find a set of simultaneous equations:

$$2x_1 + x_2 = 3x_1 \quad \rightarrow \quad -x_1 + x_2 = 0$$

$$x_1 + 2x_2 = 3x_2 \quad \rightarrow \quad x_1 - x_2 = 0$$

These equations also not solvable!

$$x_1 = x_2$$

$$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

This is one of the eigenvector of the matrix A.

The full solution is: for $\lambda_1 = 1$, $x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and for $\lambda_1 = 3$, $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$