

International Mathematical Olympiad Problems (1971–1973)

Thirteenth International Olympiad, 1971

1971/1. Prove that the assertion is true for $n = 3$ and $n = 5$, and false for every other natural number $n > 2$. If a_1, a_2, \dots, a_n are real numbers, then $(a_1 - a_2)(a_1 - a_3) \cdots (a_1 - a_n) + (a_2 - a_1)(a_2 - a_3) \cdots (a_2 - a_n) + \cdots + (a_n - a_1)(a_n - a_2) \cdots (a_n - a_{n-1}) \geq 0$.

1971/2. Let P_1 be a convex polyhedron with vertices A_1, A_2, \dots, A_9 . For $i = 2, 3, \dots, 9$, let P_i be obtained by translating P_1 so that A_1 moves to A_i . Prove that at least two of the polyhedra have a common interior point.

1971/3. Prove that the set of integers of the form $2^k - 3$ for $k \geq 2$ contains an infinite subset whose members are pairwise relatively prime.

1971/4. All faces of tetrahedron $ABCD$ are acute. Let X, Y, Z, T lie on edges AB, BC, CD, DA respectively. If $\angle DAB + \angle BCD \neq \angle CDA + \angle ABC$, show that no shortest polygonal path exists. If equality holds, show that infinitely many shortest paths exist.

1971/5. For every natural number m , prove that there exists a finite set S in the plane such that for each A in S , exactly m points of S are at unit distance from A .

1971/6. Let $A = (a_{ij})$ be an $n \times n$ matrix of non-negative integers. If $a_{ij} = 0$ implies the sum of the i th row and j th column is at least n , prove that the sum of all entries is at least $n^2/2$.

Fourteenth International Olympiad, 1972

1972/1. From any set of ten distinct two-digit numbers, prove that two disjoint subsets can be chosen with equal sum.

1972/2. Prove that if $n \geq 4$, every cyclic quadrilateral can be dissected into n cyclic quadrilaterals.

1972/3. Let m, n be non-negative integers. Prove that $(2m)!(2n)!/(m!n!(m+n)!)$ is an integer.

1972/4. Find all positive real solutions (x_1, \dots, x_5) satisfying $(x_1^2 - x_3x_5)(x_2^2 - x_3x_5) \leq 0$, $(x_2^2 - x_4x_1)(x_3^2 - x_4x_1) \leq 0$, $(x_3^2 - x_5x_2)(x_4^2 - x_5x_2) \leq 0$, $(x_4^2 - x_1x_3)(x_5^2 - x_1x_3) \leq 0$, $(x_5^2 - x_2x_4)(x_1^2 - x_2x_4) \leq 0$.

1972/5. Let f and g satisfy $f(x+y) + f(x-y) = 2f(x)g(y)$. If f is not identically zero and $|f(x)| \leq 1$, prove that $|g(y)| \leq 1$.

1972/6. Given four distinct parallel planes, prove that a regular tetrahedron exists with one vertex on each plane.

Fifteenth International Olympiad, 1973

1973/1. Let O be the origin and P_1, \dots, P_n lie in a plane on one side of O . Prove that $|OP_1| + \dots + |OP_n| \geq |OP_1 + \dots + OP_n|$.

1973/2. Determine whether a finite set M exists such that for any A, B in M , there exist C, D in M with $AB \parallel CD$ and AB not coincident with CD .

1973/3. Let a, b be real numbers such that $x^4 + ax^3 + bx^2 + ax + 1 = 0$ has a real solution. Find the minimum of $a^2 + b^2$.

1973/4. A soldier must scan an equilateral triangular region with a detector whose range equals half the altitude. Starting from one vertex, find the shortest path covering the region.

1973/5. Let G be the set of non-constant linear functions $f(x) = ax + b$ satisfying: closure under composition, closure under inverse, and each function has a fixed point. Prove that there exists a real number k such that $f(k) = k$ for all f in G .

1973/6. Let $0 < a_i < 1$ and $b_1 + \dots + b_n \geq \ln(a_1 + \dots + a_n)$. Find all real numbers b_1, \dots, b_n satisfying this.