Analysis of nonlinear Langevin Equation

Chinmay Sahu

Communications, Signal Processing, Networking Lab
Department of Electrical and Computer Engineering
Clarkson Univeristy,Potsdam,13699

December 7, 2017

Abstract

In this project, a nonlinear Langevin equation has been considered to analyze the non-linearity of vibrating system under two different methods, i.e. numerical and exact methods for given initial conditions.The results of both methods are simulated in Matlab and compared to draw qualitative conclusions. The effect of non-linear stiffness constant (ϵ) on the response of the system has been studied for sinusoid function and white noise with given power spectrum. Finally the variance of the system with different ϵ values under the influence of white noise is studied and concluded.

1 Introduction

The theory of Brownian motion is perhaps the simplest approximate way to treat the dynamics of non equilibrium systems [1]. The fundamental equation is called the Langevin equation [2]; it contain both frictional forces and random forces. The fluctuation-dissipation theorem relates these forces to each other [3].

2 Langevin Equation

In this project, the a nonlinear Langevin equation has been considered. The equation can be considered a vibrating system with a mass, spring and damper. The equation considered here is given by

$$\ddot{x} + 0.02\dot{x} + x + \epsilon x^3 = f(t), \quad x(0) = 0, \quad \dot{x}(0) = 0 \tag{1}$$

where ϵ is non linear stiffness coefficient. *f(t)* can be sinusoid input or white noise.

3 Results and Discussion

First, the system of equation is solved using ode45 solver in Matlab with sinusoid input and ϵ=0. The exact solution of the system of equation is identified with help of dsolve function in Matlab, which is given in equation 2. From the figure 1, The comparison of exact and numerical solution can be observed. It is clear that both the solutions are fitting together.

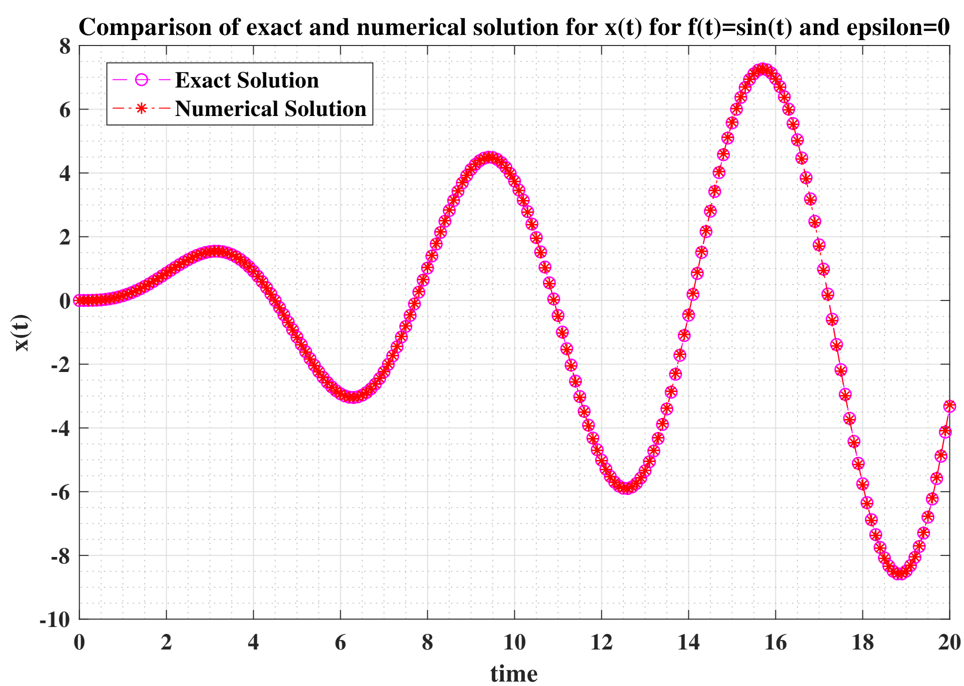

Figure 1: comparison of exact and numerical solution with sine input when ϵ=0 .

$$x(t) = 50cos(t)e^{\frac{-t}{100}} - 50cos(t) + 0.5e^{\frac{-t}{100}}sin(t) \tag{2}$$

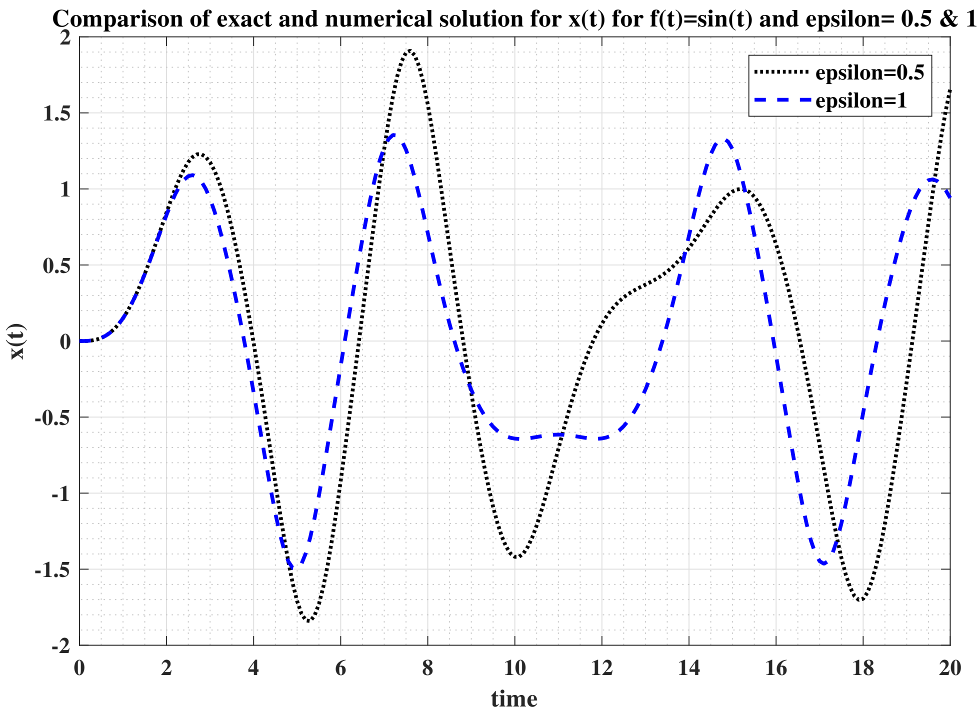

Figure 2: comparison of exact and numerical solution with sine input when ϵ=0.5 and 1.0.

Then effect of increasing stiffness constant ϵ on non-linearity is evaluated for different values of ϵ with sine input. The simulation results for ϵ=0.5 and 1 have been shown in figure 1. As the stiffness ϵ increases from 0 to 0.5, there is visible drop in magnitude of x(t). As ϵ further increases to 1, the magnitude drops further. The velocity values change in smaller scale with small values of ϵ and It is in accordance with theoretical results. Next, a white noise with power spectrum (S_0)=55.44 $\frac{cm^2}{s^3}$ is simulated and shown in figure 3. With this white noise process as input, the non-linear langevin equation is solved numerically with help of Matlab solver ode45 for stiffness constant ϵ=0 and 0.1. The simulated results are plotted in the figure 4. It can be clearly inferred that as the stiffness increases from 0 to 0.1, the amplitude and velocity decreases. Next the exact variance of x(t) is evaluated for white noise input when ϵ=0 using the equation 3. Then variance is calculated numerically for 200 samples and plotted in figure 5 for ϵ=0 and 0.1. Similarly the exact variance is plotted in figure 6. It is observed as the ϵ increases, the amplitude and oscillations in variance reduces. However, exact variance magnitude seems to really high, which does not exactly concur with results obtained from numerical solution.

$$\sigma_x^2 = \frac{\pi S_0}{\zeta\omega_0^3}\{1 - \frac{1}{\omega_d^2}\exp^{-2\zeta\omega_0 t}[\omega_d^2 + 2(\zeta\omega_0)^2 sin^2\omega_d t + \omega_d\omega_0\zeta sin2\omega_d t]\} \tag{3}$$

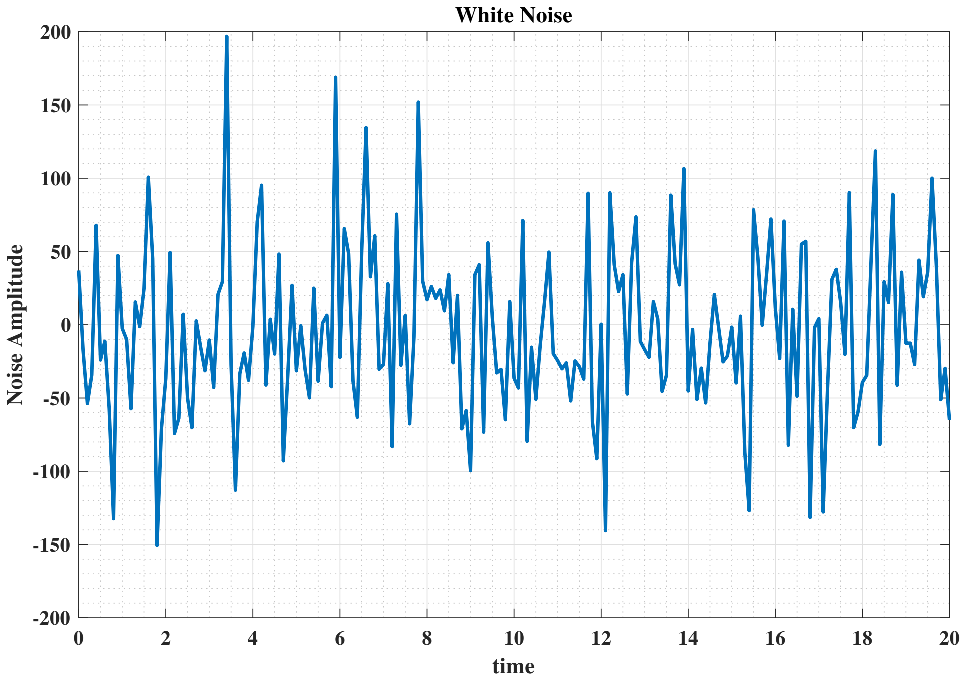

Figure 3: x(t) with White Noise as input with S_0=55.44 $\frac{cm^2}{s^3}$.

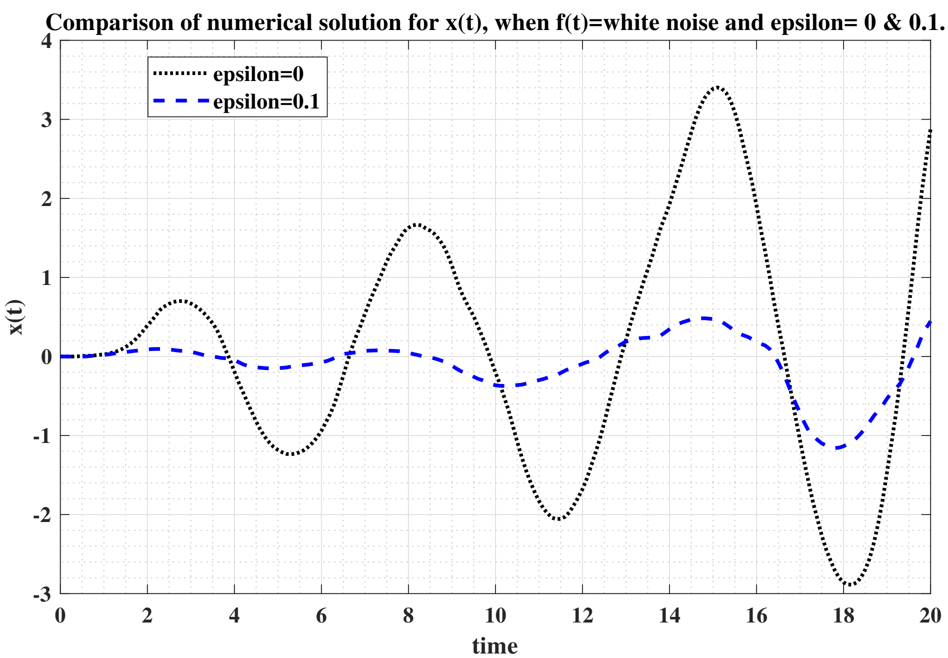

Figure 4: White Noise with S_0=55.44 $\frac{cm^2}{s^3}$.

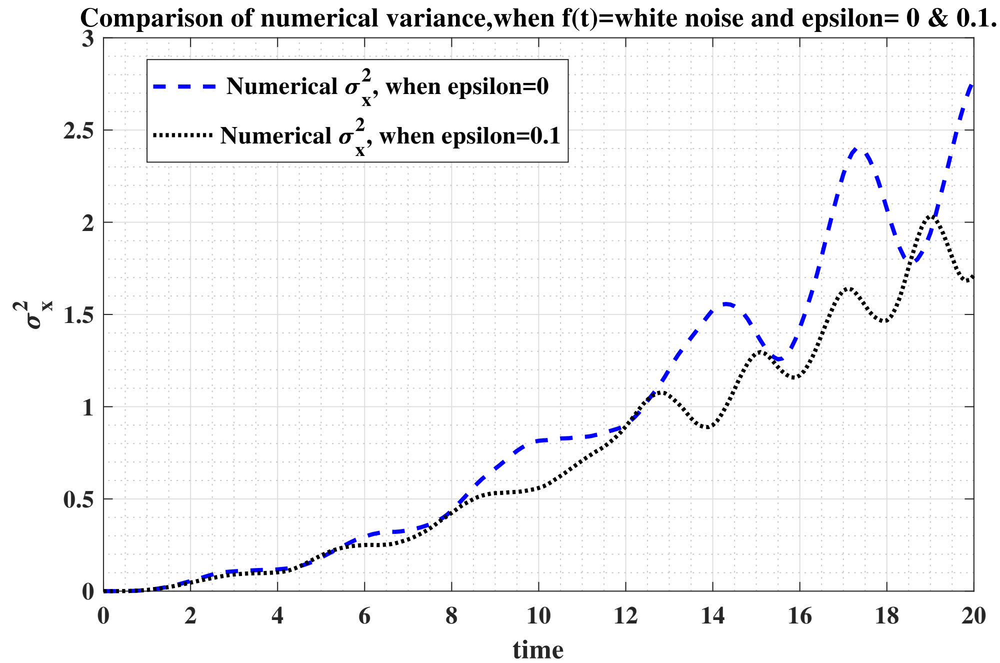

Figure 5: comparison of numerical σ_x^2 with white noise as input, when ϵ=0 and 0.1.

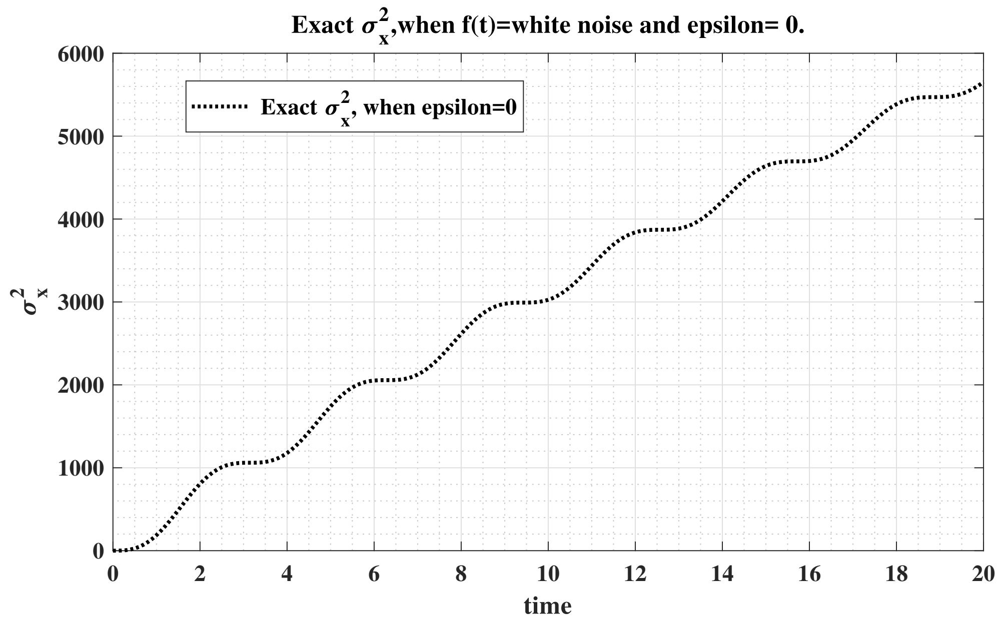

Figure 6: Exact σ_x^2 with white noise as input, when ϵ=0.

4 Conclusions

The non-linear langevin equation is solved numerically for different values of stiffness constant ϵ with sine input and white noise process as input. As ϵ increases, the amplitude of solution decreases and so also the velocity.

Variance is evaluated for ϵ=0 and 0.1 numerically. The exact variance is also evaluated for ϵ=0 and compared with numerical solution. The results are analyzed qualitatively and conclusions are drawn.

References

[1] Coffey, William T and Kalmykov, Yuri P, The Langevin equation: with applications to stochastic problems in physics, chemistry and electrical engineering, World Scientific (2004).

[2] Ullersma P, An exactly solvable model for Brownian motion: I. Derivation of the Langevin equation, Physica ,Elsevier, Vol. 4, pp.27-55(1996).

[3] Adelman, SA and Doll, JD, Generalized Langevin equation approach for atom/solid-surface scattering: General formulation for classical scattering off harmonic solids, The Journal of chemical physics, Vol. 64-6, pp.2375-2388(1976).

Appendix

Matlab Code For Numerical Solution

```
clc
clear
tspan = 0:0.1:20; % time duration for which we solve ODE
x0=[0;0]; %initial conditions
% Command to invoke this function
[t,x]=ode45(@Call_ODE,tspan,x0);

 %Numerical Solution
function xdot= Call_ODE(t,x)
step=20/200;
S0=55.44;  % power density
WhiteNoise=(sqrt((2*pi*S0)/step))*randn(1,1);

ft=WhiteNoise; % input
% ft=sin(t);  %input
epsilon=0; %0.1,0.5,1 epsilon changes here
xdot=[x(2);ft-0.02*x(2)-x(1)-epsilon*x(1)^3];
end
```

Matlab Code For Exact Solution

```
% % Exact Solution
syms y
X_exact = simplify(dsolve('D2X + 0.02*DX + X + 0*X^3 - sin(y) = ...
    0','X(0)=0, DX(0)=0','y'));
pretty(X_exact); %solve langevin equation gives exact solution
ezplot(X_exact,[0, 20]) %plot figure
```

Matlab Code For Variance Calculation

```
tspan = 0:0.1:20;
x0=[0;0];
x_e_for_var=zeros(201);
% Command to invoke this function
% [t,x]=ode45(@Test,tspan,x0);
for j=1:201
[t,x]=ode45(@Test,tspan,x0);
x_e_for_var(:,j)=x(:,1); % collects x(t)
j %shows code progress
end
var_x_num=zeros(1,201);
for j=1:201
    var_x_num(j)=var(x_e_for_var(j,:)); % variance calulation
```

```
end

% Exact Variance calculation
Var_x=[];
 for ti=0:0.1:20
 w0=1;
 zeta=0.01;
 wd=w0*sqrt(1-zeta^2);
 Var_xx=((pi*55.44)/(zeta*w0^3))*(1-((1/wd^2)*exp(-2*zeta*w0*ti)*(wd^2+(2*(zeta*w0)^2*si
 Var_x=[Var_x;Var_xx]; % variance calculation and plot
 end

%Numerical Solution
function xdot= Test(t,x)
step=20/200;
S0=55.44;  % power density
WhiteNoise=(sqrt((2*pi*S0)/step))*randn(1,1);

ft=WhiteNoise;
% ft=sin(t);
epsilon=0; %0.1,0.5,1
xdot=[x(2);ft-0.02*x(2)-x(1)-epsilon*x(1)^3];
end
```

Matlab Code For White Noise

```
clc
clear
T=20;
m=200;
delT=T/m;
time=0:delT:T; %time
S0=55.44;  % power density
WhiteNoise=(sqrt((2*pi*S0)/delT))*randn(m+1,1);

plot(time,WhiteNoise,'LineWidth',2);
title('White Noise','FontName','Times New ...
    Roman','FontSize',12,'FontWeight','bold');
```