NLA Quiz-4

~~Householder's QR factorization~~ :

Upper heisenberg form :- It is a sq matrix having all elements equal to 0 below first subdiagonal.

Q $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 2 & 1 & 5 & 0 \\ 4 & 2 & 1 & 0 \end{bmatrix}$ Q find upper Heisenberg matrix.

⇒ Theory :- a) $HAH^T = H_s$ { H = householder matrix, H_s = Hessenberg matrix

b) $H = \begin{bmatrix} 1 & 0 \cdots \\ 0 & \\ \vdots & \hat{H} \end{bmatrix}$

c) $\hat{H} = I - 2\frac{UU^T}{U^TU}$

d) $U = X + \text{sign}(x1) \cdot \|X\| \cdot e^{(1)}$

e) X = first column excluding first row. = d

$e^{(1)}$ $[1\ 0\ 0 \cdots]^T$ { same dim as X }

Soln :- [I] $d = [4\ 2\ 4]^T = X$

$\beta = -\|d\|_2 = -\sqrt{4^2+2^2+4^2} = -6$

$\alpha = \frac{\sqrt{2}}{\|d - \beta e^{(1)}\|_2}$

here, $d - \beta e^{(1)} = \begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix} - \begin{bmatrix} -6 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ 2 \\ 4 \end{bmatrix}$

$\|d - \beta e^{(1)}\|_2 = \sqrt{100+4+16} = \sqrt{120}$ } $\alpha = \frac{1}{\sqrt{60}}$

Then, $v = \alpha(d - \beta e^{(1)})$

$= \frac{1}{\sqrt{60}} \begin{bmatrix} 10 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1.745 \end{bmatrix}$

Then, $U = I - vv^* = I - \begin{bmatrix} 1.667 & 0.334 & 0.666 \\ 0.334 & 0.066 & 0.133 \\ 0.6668 & 0.1334 & 0.266 \end{bmatrix}$

$U = \begin{bmatrix} -0.667 & -0.334 & -0.668 \\ -0.334 & 0.933 & -0.133 \\ -0.668 & -0.133 & 0.733 \end{bmatrix}$

$UAU^* = \begin{bmatrix} I & 0 \\ 0 & U \end{bmatrix} A \begin{bmatrix} I & 0 \\ 0 & U^* \end{bmatrix}$

$= \left[\begin{bmatrix} 1 & -5 \\ -6 & 8.563 \end{bmatrix} \begin{bmatrix} 1.6 & 1.199 \\ 3.623 & 0.757 \end{bmatrix} \right.$
$\left. \begin{bmatrix} 0 & 1.379 \\ 0 & 5.759 \end{bmatrix} \begin{bmatrix} 3.009 & -1.382 \\ -2.382 & -1.564 \end{bmatrix} \right]$

[II] $d = [1.379 \quad 5.759]^T$

$\beta = -\|d\|_2^2 = -5.921$

$\|d - \beta e^{(1)}\|_2 = [1.379 \quad 5.759] - [-5.921 \quad 0] = [7.3 \quad 5.759]$
$= 9.298$

$\alpha = \frac{\sqrt{2}}{\|d - \beta e^{(1)}\|_2} = 0.152$

Then, $v = 0.152\,(a - \beta e^{(1)})$

$= \begin{bmatrix} 1.1096 & 0.875 \end{bmatrix}$

Then,

$$U = I - vv^* = I - \begin{bmatrix} 1.231 & 0.971 \\ 0.971 & 0.766 \end{bmatrix}$$

$$= \begin{bmatrix} -0.231 & 0.971 \\ 0.971 & 0.234 \end{bmatrix}$$

Then,

$$H = UAU^* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -0.231 & 0.971 \\ 0 & 0 & -0.231 & 0.971 \\ 0 & 0 & 0.971 & 0.234 \end{bmatrix}$$

[book pg. 317.] Ans