Circle Assignment

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***Problem Statement* - XY and EF are two parallel tangents to a circle , x with centre O and another tangent AB with point of contact C intersecting XY at A and EF at B. Prove that $\angle$AOB = 90°.**

Solution

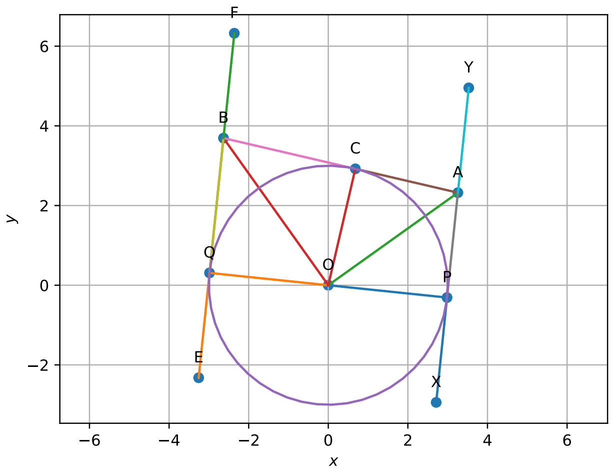

Figure 1: Tangents from A to circle through P, C and Tangent from B to circle through Q

Solution

In order to find the intersection points C and P of tangents from A, the origin is O. The equation of the circle in the new frame is

$$\boldsymbol{x}^T\boldsymbol{x} = r^2 \tag{1}$$

$$\boldsymbol{x} = \mathbf{A} + \lambda(\mathbf{m}) \tag{2}$$

Sub eq2 in eq1

$$(\mathbf{A} + \lambda(\mathbf{m}))^\top(\mathbf{A} + \lambda(\mathbf{m})) = r^2 \tag{3}$$

Simplify eq3

$$\lambda^2 \left\|\boldsymbol{m}\right\|^2 + 2\lambda\mathbf{A}^\top\mathbf{m} + \left\|\boldsymbol{A}\right\|^2 = r^2 \tag{4}$$

Solving eq4

$$\lambda = \frac{-\mathbf{A}^\top\mathbf{m}}{\left\|\boldsymbol{m}\right\|^2} \tag{5}$$

$$(\mathbf{A}^\top\mathbf{m})^2 = (\left\|\boldsymbol{A}\right\| - r^2)(\left\|\boldsymbol{m}\right\|^2) \tag{6}$$

consider

$$\mathbf{m} = \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{7}$$

From eq6

$$\mathbf{A}^\top\mathbf{m}\mathbf{m}^\top\mathbf{A} = \left\|\boldsymbol{m}\right\|^2 (\left\|\boldsymbol{A}\right\|^2 - r^2) \tag{8}$$

$$\mathbf{A}^\top \begin{pmatrix} 1 & m \\ m & m^2 \end{pmatrix} \mathbf{A} = (1+m^2)(\left\|\boldsymbol{A}\right\| - r^2) \tag{9}$$

consider

$$\mathbf{A} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = r_1 \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \tag{10}$$

sub eq10 in eq9

$$\begin{pmatrix} a_1 & a_2 \end{pmatrix} \begin{pmatrix} 1 & m \\ m & m^2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = (1+m^2)(\left\|\boldsymbol{A}\right\|^2 - r^2) \tag{11}$$

$$a_1^2 + 2ma_1a_2 + m^2a_2^2 = (1+m^2)(\left\|\boldsymbol{A}\right\|^2 - r^2) \tag{12}$$

$$m = \frac{-a_1a_2 \pm \sqrt{r^2\left\|\mathbf{A}\right\|^2 - r^4}}{r^2 - a_1^2} \tag{13}$$

Solving eq13,we get two values m_1, m_2.using λ, m_1, m_2 find two Points intersecting circle at **P** and **C** from eq2

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ m_1 \end{pmatrix} \tag{14}$$

$$\mathbf{m_2} = \begin{pmatrix} 1 \\ m_2 \end{pmatrix} \tag{15}$$

$$\mathbf{P} = \mathbf{A} + \lambda\mathbf{m_1} \tag{16}$$

$$\mathbf{C} = \mathbf{A} + \lambda\mathbf{m_2} \tag{17}$$

here $\boldsymbol{O}$ is mid point of $\boldsymbol{P}$ and $\boldsymbol{Q}$

$$\mathbf{Q} = 2(\mathbf{O}) - \mathbf{P} \tag{18}$$

assume

$$\mathbf{D} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{19}$$

$$\mathbf{j} = \mathbf{D}(\mathbf{P} - \mathbf{A}) \tag{20}$$

$$\mathbf{k} = \mathbf{D}(\mathbf{A} - \mathbf{C}) \tag{21}$$

$$\mathbf{C} = \begin{pmatrix} \mathbf{j}^\top \\ \mathbf{k}^\top \end{pmatrix} \begin{pmatrix} \mathbf{Q} & \mathbf{A} \end{pmatrix} \tag{22}$$

$$\mathbf{B} = \begin{pmatrix} \mathbf{j}^\top \\ \mathbf{k}^\top \end{pmatrix}^{-1} \mathbf{C} \tag{23}$$

Using Parallelogram Law Vector Addition

$$\mathbf{A} - \mathbf{O} = (\mathbf{P} - \mathbf{O}) + (\mathbf{C} - \mathbf{O}) \tag{24}$$

$$\mathbf{B}-\mathbf{O}=(\mathbf{Q}-\mathbf{O})+(\mathbf{C}-\mathbf{O}) \tag{25}$$
$$\mathbf{B}-\mathbf{O}=-(\mathbf{O}-\mathbf{Q})+(\mathbf{C}-\mathbf{O}) \tag{26}$$

$$(\mathbf{A}-\mathbf{O})^\top(\mathbf{B}-\mathbf{O})=((\mathbf{P}-\mathbf{O})+(\mathbf{C}-\mathbf{O}))^\top(-(\mathbf{O}-\mathbf{Q})+(\mathbf{C}-\mathbf{O})) \tag{27}$$

We Know

$$\mathbf{P}-\mathbf{O}=\mathbf{O}-\mathbf{Q} \tag{28}$$

From eq27 and eq28

$$(\mathbf{O}-\mathbf{A})^\top(\mathbf{O}-\mathbf{B})=0 \tag{29}$$

Finally
Dot Product of two Vectors is Zero
then,Angle between those two Vectors is 90°

$$\angle \boldsymbol{AOB}=\mathbf{90°} \tag{30}$$

Construction

Symbol	Value	Description
r_1	4	OA
θ	120	
r	3	Radius
$\mathbf{A}$	$r_1\begin{pmatrix} cos\theta \\ sin\theta \end{pmatrix}$	Point A
$\mathbf{O}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	Point O
$\mathbf{A_O}$	$\mathbf{A}-\mathbf{O}$	$\mathbf{A}$ when origin shifted to $\mathbf{O}$
m_1, m_2	evaluate eq13	solution of eq13
$\mathbf{C}$	$\mathbf{A}+\lambda\mathbf{m_2}$	Point C
$\mathbf{P}$	$\mathbf{A}+\lambda\mathbf{m_1}$	Point P
$\mathbf{Q}$	$2(\mathbf{O})-\mathbf{P}$	Point Q
$\mathbf{B}$	$\begin{pmatrix} j^\top \\ k^\top \end{pmatrix}\mathbf{c}$	Point B