Line Assignment

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Problem Statement **- The side AB of a parallelogram ABCD is produced to any point P. A line through A and parallel to CP meets CB produced at Q and then parallelogram PBQR is completed.Show that**

1. **ar(ABCD) = ar(PBQR)**

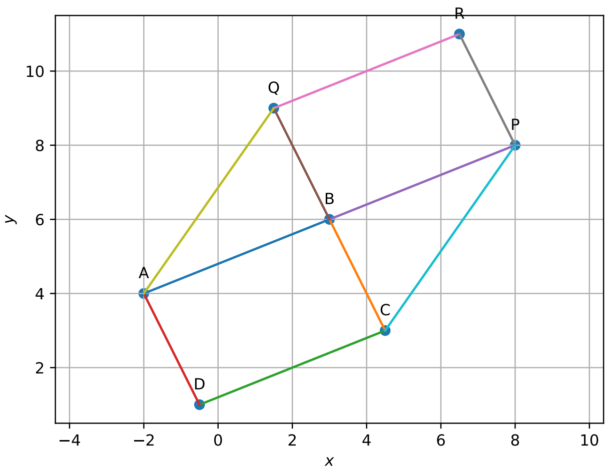

Figure 1: parallelogram PBQR is produced using parallelogram ABCD

Solution

METHOD 1

Let

$$\boldsymbol{B}-\boldsymbol{A}=\boldsymbol{m} \tag{1}$$

$$\boldsymbol{D}-\boldsymbol{A}=\boldsymbol{n} \tag{2}$$

n written as two perpendicular components one along $\boldsymbol{m}$ and one perpendicular to $\boldsymbol{m}$

$$\boldsymbol{n}=\boldsymbol{m}\parallel_A+\boldsymbol{m}\perp_A \tag{3}$$

$$\boldsymbol{m}\parallel_A=\frac{\boldsymbol{n}^T\boldsymbol{m}}{||\boldsymbol{m}||^2}\boldsymbol{m} \tag{4}$$

$$\boldsymbol{m}\perp_A=\boldsymbol{n}-\boldsymbol{m}\parallel_A \tag{5}$$

$$\boldsymbol{m}\perp_A=\boldsymbol{n}-\frac{\boldsymbol{n}^T\boldsymbol{m}}{||\boldsymbol{m}||^2}\boldsymbol{m} \tag{6}$$

$$||\boldsymbol{m}\perp_A||^2=||\boldsymbol{n}||^2-\frac{\boldsymbol{n}^2T\boldsymbol{m}^2}{||\boldsymbol{m}||^2} \tag{7}$$

$$AreaofparallelogramABCD=\sqrt{base^2height^2} \tag{8}$$

$$AreaofparallelogramABCD=\sqrt{||\boldsymbol{m}\perp_A||^2||\boldsymbol{m}||^2} \tag{9}$$

$$AreaofparallelogramABCD=\sqrt{(||\boldsymbol{n}||^2-\frac{\boldsymbol{n}^2T\boldsymbol{m}^2}{||\boldsymbol{m}||^2})||\boldsymbol{m}||^2} \tag{10}$$

Let
Q lies on BC extended

$$\boldsymbol{P}=\boldsymbol{B}+x\boldsymbol{m} \tag{11}$$

$$\boldsymbol{Q}=\boldsymbol{B}+k\boldsymbol{n} \tag{12}$$

Q line passes through A and parallel to CP

$$\boldsymbol{Q}=\boldsymbol{A}+u\boldsymbol{o} \tag{13}$$

where

$$\boldsymbol{o}=\boldsymbol{P}-\boldsymbol{C} \tag{14}$$

equating 13 and 14

$$\boldsymbol{B}+k\boldsymbol{n}=\boldsymbol{A}+u\boldsymbol{o} \tag{15}$$

$$B-A=u\boldsymbol{o}-k\boldsymbol{n} \tag{16}$$

$$\textbf{B-A}=\begin{pmatrix}\boldsymbol{o} & -\boldsymbol{n}\end{pmatrix}\begin{pmatrix}u\\k\end{pmatrix} \tag{17}$$

$$\begin{pmatrix}u\\k\end{pmatrix}=\begin{pmatrix}\boldsymbol{o} & -\boldsymbol{n}\end{pmatrix}^{-1}\textbf{(B-A)} \tag{18}$$

OR

$$k=e2^T\begin{pmatrix}\boldsymbol{o} & -\boldsymbol{n}\end{pmatrix}^{-1}\textbf{(B-A)} \tag{19}$$

for parallelogram **PQRS**,
Base $=||\boldsymbol{P}\textbf{-}\boldsymbol{B}||$

$$Base=x||\boldsymbol{m}||(from-eq12) \tag{20}$$

To find height, write $\boldsymbol{Q}-\boldsymbol{B}$ as sum of two perpendicular components

$$\boldsymbol{Q}-\boldsymbol{B}=k\boldsymbol{n} \tag{21}$$

$$\boldsymbol{Q}-\boldsymbol{B}=\boldsymbol{m}\parallel_P+\boldsymbol{m}\perp_P \tag{22}$$

$$\boldsymbol{m}\parallel_P=k\frac{\boldsymbol{n}^T\boldsymbol{m}}{||\boldsymbol{m}||^2}\boldsymbol{m} \tag{23}$$

$$\boldsymbol{m}\perp_P=k(\boldsymbol{n}-\frac{\boldsymbol{n}^T\boldsymbol{m}}{||\boldsymbol{m}||^2}\boldsymbol{m}) \tag{24}$$

compare eq 6 and eq 24

$$\boldsymbol{m}\perp_P=k(\boldsymbol{m}\perp_A) \tag{25}$$

From eq20

$$Base\mathbf{PB}=|\boldsymbol{P}-\boldsymbol{B}=x||\boldsymbol{m}||=x(||\boldsymbol{A}-\boldsymbol{B}||)=Base\mathbf{AB} \tag{26}$$

From eq25 and eq26
Area of parallelogram=$|\boldsymbol{m} \perp_P|$(Base**PB**)
=k($|\boldsymbol{m} \perp_A|$)x(Base**AB**)

$$= kx(area(\mathbf{ABCD})) \tag{27}$$

Let
Point B be $\begin{pmatrix} 0 & 0 \end{pmatrix}$, A be $\begin{pmatrix} i \\ 0 \end{pmatrix}$ and D= $\begin{pmatrix} j \\ k \end{pmatrix}$. The figure can be relocated to other origin and rotated by transformation Py+C without any changes.So,results valid with assume points A,B and D is valid.

$$\boldsymbol{o} = \boldsymbol{P} - \boldsymbol{C} = \boldsymbol{B} + x(\textbf{B-A})\textbf{-((B-A)+(D-A)+A)} \tag{28}$$

$$\boldsymbol{o} = x\boldsymbol{B} + (1-x)\boldsymbol{A} - \boldsymbol{D} \tag{29}$$

$$\boldsymbol{o} = \begin{pmatrix} (1-x)i - j \\ -k \end{pmatrix} \tag{30}$$

$$-\boldsymbol{n} = -(\boldsymbol{o} - \boldsymbol{A}) = \begin{pmatrix} i-j \\ -k \end{pmatrix} \tag{31}$$

$$(\boldsymbol{o} - \boldsymbol{A}) = \begin{pmatrix} (1-x)i - j & i-j \\ -k & -k \end{pmatrix} \tag{32}$$

$$(\boldsymbol{o} - \boldsymbol{A})^-1 = \begin{pmatrix} -k & j-i \\ k & (1-x)i-j \end{pmatrix} (\frac{1}{det((\boldsymbol{o} - \boldsymbol{n}))}) \tag{33}$$

$det(\boldsymbol{o} - \boldsymbol{n}) = (x-1)ki + kj - kj + ki$
$det(\boldsymbol{o} - \boldsymbol{n} = xki$
substuting eq33 in eq19

$$k = (\frac{1}{xki}) \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} -k & j-i \\ k & (1-x)i-j \end{pmatrix} \begin{pmatrix} -i \\ 0 \end{pmatrix} \tag{34}$$

$$k = (\frac{1}{xki}) \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} ki \\ -ki \end{pmatrix} \tag{35}$$

$$k = (\frac{1}{xki})(-ki) = \frac{-1}{x} \tag{36}$$

sub eq36 in eq27
area(**PBQR**)=($\frac{-1}{x}$)$(x)aera(\mathbf{ABCD})$
Finally
$|aera(\mathbf{PBQR})| = |aera(\mathbf{ABCD})|$

METHOD 2

GIVEN **ABCB** is parallelogram
AB$\|$**CB** and **BC**$\|$**DA**
P is extension of AB
Q is extension of CB
PBQR parrallelogram is produced
CP$\|$**AQ**
From the above
We Know

$$\textbf{Q-B} = \lambda_1(\textbf{D-A}) \tag{37}$$

$$\textbf{P-B} = \lambda_2(\textbf{C-D}) \tag{38}$$

$$\textbf{B-A} = \textbf{A-D} \tag{39}$$

$$\textbf{B-C} = \textbf{A-D} \tag{40}$$

$$\textbf{R-P} = \textbf{Q-B} \tag{41}$$

$$\textbf{R-Q} = \textbf{P-B} \tag{42}$$

$$\textbf{Q-A} = \lambda(\textbf{P-C}) \tag{43}$$

Where $\lambda = (\lambda_1) * (\lambda_2)$
using A,B,C,D,P,Q,R Position Vectors
We find
From eq37,$\lambda_1 = 1$
From eq38,$\lambda_2 = 1$
Finally

$$\lambda = (\lambda_1) * (\lambda_2) = 1 \tag{44}$$

We Know
Area of parallelogram **ABCD**=**AB** $\times$ **AD**
$\angle(AB, AD) = \theta$

$$\theta = \arccos(\frac{\mathbf{AB} \cdot \mathbf{AD}}{|AB| * |AD|}) \tag{45}$$

Using Position Vectors
$\theta = 17.64$
Area of parallelogram **ABCD**=$|\mathbf{AB}||\mathbf{AD}|\sin\theta$
using Position Vectors

$$Areaofparallelogram\mathbf{ABCD} = 5.47\textbf{square units} \tag{46}$$

Area of parallelogram **PBQR**=**QB** $\times$ **BP**
$\angle(QB, BP) = \alpha$

$$\alpha = \arccos(\frac{\mathbf{QB} \cdot \mathbf{BP}}{|QB| * |BP|}) \tag{47}$$

Using Position Vectors
$\alpha = 17.64$
Area of parallelogram **PBQR**=$|\mathbf{QB}||\mathbf{BP}|\sin\alpha$
using Position Vectors

$$Areaofparallelogram\mathbf{PBQR} = 5.47\textbf{square units} \tag{48}$$

From eq44,eq46 and eq48
Finally
Area of parallelogram **ABCD**=Area of parallelogram **PBQR**

Construction

The input parameters are the values i,j,k,x.

Symbol	**Value**	**Description**
i	-2	
j	-0.5	
k	1	
x	-1	
A	$\begin{pmatrix} i \\ 4 \end{pmatrix}$	Position Vector A
B	$\begin{pmatrix} 3 \\ 6 \end{pmatrix}$	Position Vector B
D	$\begin{pmatrix} j \\ k \end{pmatrix}$	Position Vector D
C	D+B-A	Position Vector C
m	B-C	Diretion Vector of AB
n	D-A	Diretion Vector of DA
P	B-x*m	Position Vector P
Q	B+x*n	Position Vector Q
R	P+Q-B	Position Vector R