Conics Assignment

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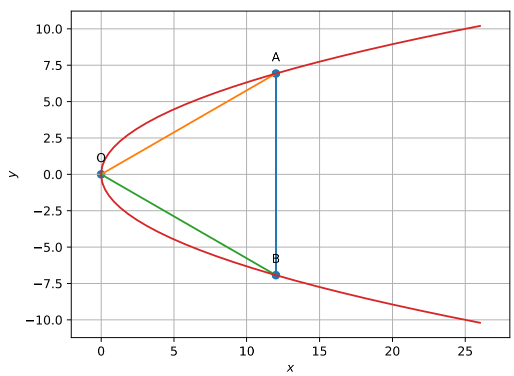
Problem Statement **- An equilateral triangle is inscribed in the parabola $y^2 = 4ax$,where one vertex is at the vertex of the parabola. Find the length of the side of the triangle.**

Solution

Given, the axis of parabola is horizotal.
Given,one vertex of $\triangle$OAB is at vertex of parabola.



$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{1}$$

where,

$$\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2}$$

Equation Parabola is $y^2 = 4ax$
From $\triangle$OAB

$$\|\mathbf{A}\| = \|\mathbf{B}\| = \|\mathbf{A} - \mathbf{B}\| \tag{3}$$

$$\|\mathbf{A}\|^2 = \|\mathbf{B}\|^2 = \|\mathbf{A} - \mathbf{B}\|^2 \tag{4}$$

$$\|\mathbf{A}\|^2 + \|\mathbf{B}\|^2 - 2\mathbf{A}^T\mathbf{B} = \|\mathbf{A}\|^2 = \|\mathbf{B}\|^2 \tag{5}$$

$$\frac{\mathbf{A}^T\mathbf{B}}{\|\mathbf{A}\|^2} = \frac{\mathbf{A}^T\mathbf{B}}{\|\mathbf{B}^2\|} = \frac{1}{2} \tag{6}$$

$\triangle$OAB is a equilateral triangle
α=60^0
The side length of equilateral triangle,OA=OB=AB=r
Let

$$\mathbf{A} = \begin{pmatrix} r\cos\theta_1 \\ r\sin\theta_1 \end{pmatrix} \tag{7}$$

$$\mathbf{B} = \begin{pmatrix} r\cos\theta_2 \\ r\sin\theta_2 \end{pmatrix} \tag{8}$$

$$\mathbf{A}^T\mathbf{B} = \frac{\|\mathbf{A}\|^2}{2} \tag{9}$$

$$r^2\cos(\theta_1 - \theta_2) = \frac{r^2}{2} \tag{10}$$

$$\theta_1 - \theta_2 = \cos^{-1}\frac{1}{2} \tag{11}$$

Given $\mathbf{A}$ satisfy the eq1

$$\mathbf{A}^T\mathbf{V}\mathbf{A} + 2\mathbf{u}^T\mathbf{A} + f = 0 \tag{12}$$

$$\mathbf{A}^T\mathbf{V}\mathbf{A} + 2\mathbf{u}^T\mathbf{A} = 0 \tag{13}$$

$$\begin{pmatrix} r\cos\theta_1 & r\sin\theta_1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} r\cos\theta_1 \\ r\sin\theta_1 \end{pmatrix} + 2\begin{pmatrix} -2a & 0 \end{pmatrix}\begin{pmatrix} r\cos\theta_1 \\ r\sin\theta_1 \end{pmatrix} = 0 \tag{14}$$

$$\begin{pmatrix} r\cos\theta_1 & r\sin\theta_1 \end{pmatrix}\begin{pmatrix} 0 \\ r\sin\theta_1 \end{pmatrix} + 2(-2ar\cos\theta_1) = 0 \tag{15}$$

$$r^2\sin^2\theta_1 = 4ar\cos\theta_1 \tag{16}$$

$$r = \frac{4a\cos\theta_1}{\sin^2\theta_1} \tag{17}$$

Similarly $\mathbf{B}$ satisfy the eq1

$$r = \frac{4a\cos\theta_2}{\sin^2\theta_2} \tag{18}$$

Form eq17 and eq18
Yeilding

$$\cos(\theta_1 + \theta_2) = 1 \tag{19}$$

$$\theta_1 + \theta_2 = \cos^{-1}1 \tag{20}$$

Add eq11 and eq20

$$\theta_1 = \frac{\cos^{-1}\frac{1}{2} + \cos^{-1}1}{2} \tag{21}$$

Subtract eq20 from eq11

$$\theta_2 = \frac{-\cos^{-1}\frac{1}{2} + \cos^{-1}1}{2} \tag{22}$$

Construction

The input parameters are V,u,f
$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} -2a \\ 0 \end{pmatrix}, f = 0$

Symbol	**Value**	**Description**
a	1	
α	60^0	$\angle A = \angle B = \angle O$
r	Solving eq18	OA=OB=AB
O	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	center of parabola and Point O
A	$\begin{pmatrix} r\cos\theta_1 \\ r\sin\theta_1 \end{pmatrix}$	Point A
B	$\begin{pmatrix} r\cos\theta_2 \\ r\sin\theta_2 \end{pmatrix}$	Point B