Note Title

Time-dependent Non-equilibrium phenomena
(Close to equilibrium)
→ linear response theory.

Example: Harmonic Oscillator (classical)

$$\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2}kx^2 \qquad \omega_0 = \pm\sqrt{\frac{k}{m}}$$

damping term (e.g. viscous friction) $\quad f_{vis} = -\alpha v$

$\alpha = 6\pi\eta a$ ← particle size

↑ viscosity

$$\ddot{x} + \omega_0^2 x + \gamma\dot{x} = f/m$$

← driven h.o.

↑ dissipation (from damping) $\qquad \gamma = \frac{\alpha}{m}$

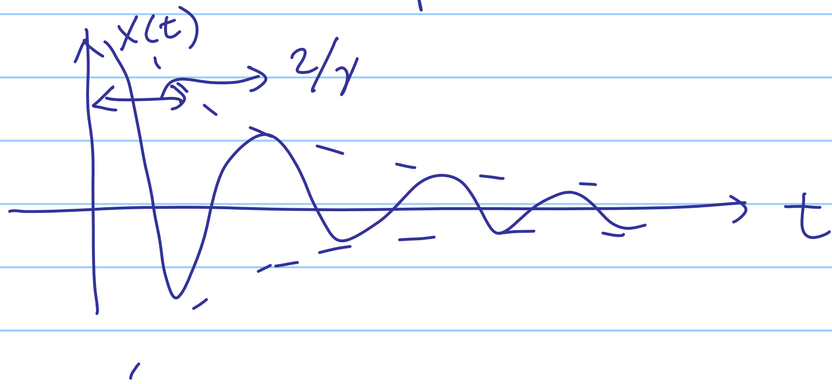
$\boxed{f=0}$ $\quad x \sim e^{-i\omega t}$

$$\Rightarrow \quad -\omega^2 + \omega_0^2 - i\omega\gamma = 0$$

$$\omega = -i\frac{\gamma}{2} \pm \sqrt{-\frac{\gamma^2}{4} + \omega_0^2} \equiv \pm\omega_1 - i\frac{\gamma}{2}$$

$$\omega_1 \equiv \sqrt{\omega_0^2 - \gamma^2/4}$$

For $\omega_0^2 > \gamma^2/4$: underdamped. $\quad x \sim e^{-\frac{\gamma}{2}t}\, e^{\pm i\omega_1 t}$



For $\omega_0^2 < \gamma^2/4$: overdamped.

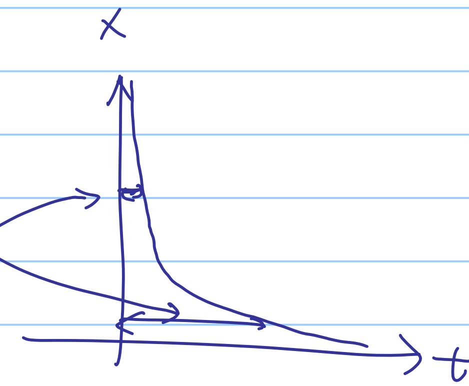
$$x \sim e^{-t/\tau}$$

$$\omega = -i\frac{\gamma}{2}\left[1 \mp \sqrt{1-\frac{4\omega_0^2}{\gamma^2}}\right] = -i\tau^{-1}$$

$$\tau^{-1} = \frac{\gamma}{2}\left[\qquad\qquad\right]$$

$$= \frac{\gamma}{2}\left[1 \mp \left(1-\frac{2\omega_0^2}{\gamma^2}\right)\right]$$

$$= \begin{cases} \dfrac{\omega_0^2}{\gamma} & \text{(slow)} \\ \gamma & \text{(fast)} \end{cases}$$



$\downarrow$ can be ignored for $\omega_0^2 \ll \gamma^2$

Response function

$$f = f(\omega)e^{-i\omega t} \Longrightarrow x = x(\omega)e^{-i\omega t}$$

$$\chi(\omega) \equiv \frac{x(\omega)}{f(\omega)} = \frac{1}{m}\frac{1}{-\omega^2+\omega_0^2-i\gamma\omega}$$

① modes: poles in $\chi(\omega)$: $\omega = \pm\omega_1 - i\frac{\gamma}{2}$

② $\omega = 0$; $\lim_{\omega\to 0}\chi(\omega) = \frac{1}{m}\cdot\frac{1}{\omega_0^2} = \frac{1}{k} = \chi_0$

spring constant

physically $f = kx$ $\quad \frac{x}{f} = \frac{1}{k}$

③ high frequency: $\chi(\omega) \approx -\frac{1}{m\omega^2}$

$$\underline{\chi} = \underline{\chi}' + i\underline{\chi}''$$

Imaginary Part (χ'')

$$\chi''(\omega) = \frac{1}{m} \frac{\omega\gamma}{(\omega^2-\omega_0^2)^2+(\gamma\omega)^2} = \cdots\cdots$$

$$= \frac{1}{2m\omega_1}\left[\underbrace{\frac{\gamma/2}{(\omega-\omega_1)^2+(\gamma/2)^2}}_{\text{Lorentzian}} - \frac{\gamma/2}{(\omega+\omega_1)^2+(\gamma/2)^2}\right]$$

$$\overset{\gamma\to 0}{=} \frac{\pi}{2m\omega_0}\left[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right]$$

① χ'' is real and odd.

② Lorentzian peaks @ $\omega = \pm\omega_1$

width $= \gamma/2$

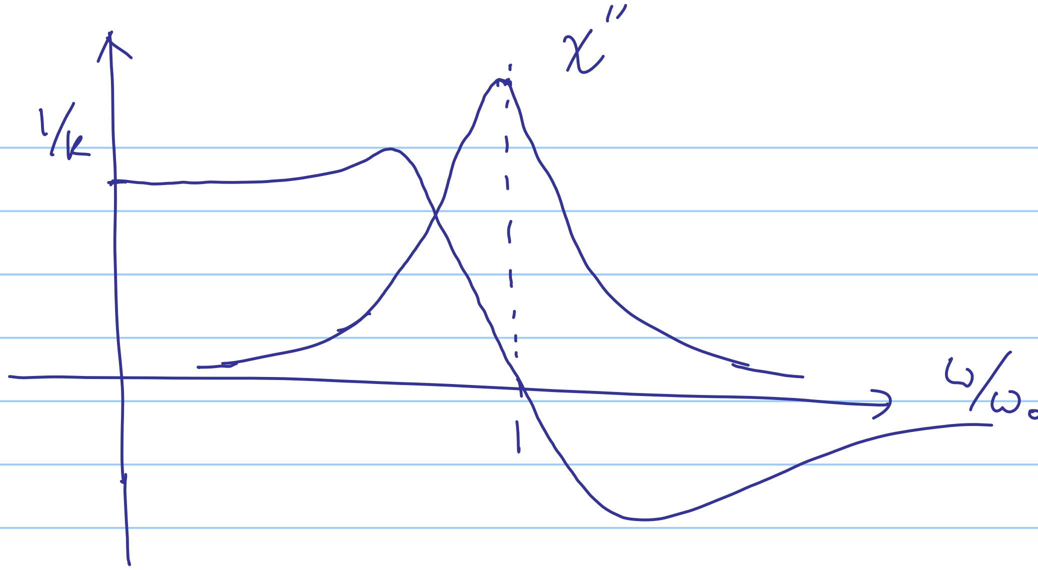
③ When damping is zero ($\gamma = 0$) : δ - fn.

Real part:

$$\chi'(\omega) = \frac{1}{m} \frac{\omega_0^2-\omega^2}{(\omega-\omega_0)^2+\omega^2\gamma^2}$$

① $\chi'(\omega)$ is positive for $\omega < \omega_0$

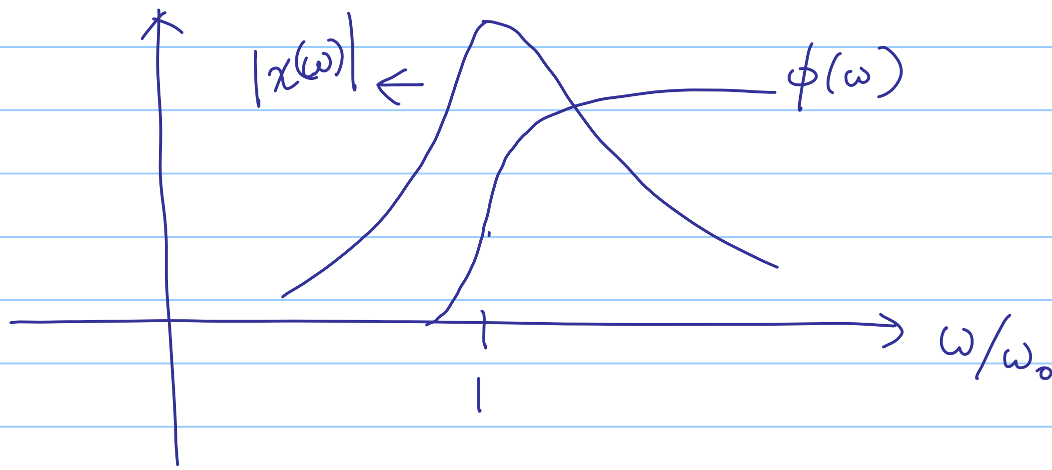
$\chi'(\omega) \longrightarrow \frac{1}{m\omega_0^2} = \frac{1}{k}$ for $\omega \to 0$



$$\chi(\omega) = |\chi(\omega)|\, e^{-i\phi(\omega)} \leftarrow \text{phase angle}$$

$$|\chi(\omega)| = \frac{1}{m} \frac{1}{\left[(\omega-\omega_0)^2 + \omega^2\gamma^2\right]^{1/2}}$$

$$\frac{\chi''}{\chi'} = \tan\phi = \frac{\omega\gamma}{\omega_0^2 - \omega^2}$$



If $f = \mathrm{Re}\left[f_0 e^{-i\omega t}\right] \rightarrow x = \chi(\omega) f = |\chi(\omega)|\, e^{-i(\omega t - \phi)}$

$\rightarrow x(t) = f_0 |\chi(\omega)| \cos(\omega t - \phi)$

<u>Steady state time-dependence</u>

In steady-state: external force is doing work.

$$\delta W = f(t) \cdot \dot{x}(t) \cdot \delta t$$

The power dissipation over one period $\left(= \frac{2\pi}{\omega}\right)$ is

$$P = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} f(t)\, \dot{x}(t)\, dt$$

$$= \frac{\omega}{2\pi} \int_0^{2\pi/\omega} dt \cdot f_0 \cos\omega t \left[-\omega f_0 |\chi(\omega)| \sin(\omega t - \phi) \right]$$

$$= -\frac{\omega^2}{2\pi} f_0^2 |\chi(\omega)| \underbrace{\int_0^{2\pi/\omega} \cos\omega t \sin(\omega t - \phi)\, dt}$$

$\because$

$$= \frac{\omega}{2} f_0^2 \underbrace{|\chi(\omega)| \sin[\phi(\omega)]} \longrightarrow \chi''(\omega)$$

$$\chi = |\chi| \cos\phi + i |\chi| \sin\phi$$

$$= \frac{1}{2} f_0^2 \cdot \omega \chi''(\omega)$$