Green's Functions

Green's functions are defined as solutions of inhomogeneous differential equations of the type ($\tilde{L}$ is a linear differential operator):

$$\tilde{L}G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}')$$

$$(z - \tilde{L(\mathbf{r})})G(\mathbf{r}, \mathbf{r}'; z) = \delta(\mathbf{r} - \mathbf{r}')$$

$$\tilde{L(\mathbf{r})}\phi_n(\mathbf{r}) = \lambda_n \phi_n(\mathbf{r})$$

The corresponding matrix definition of a Green's function is:

$$(A - \lambda B)G = I$$

$$G_\lambda = (A - \lambda B)^{-1}$$

Thereom 1. *Fundamental Theorem of Green's Functions*

Let $G(\mathbf{r}, \mathbf{r}')$ be a function which: Satisfies the differential equation

$$\tilde{L}G(\mathbf{r}, \mathbf{r}') = 0$$

everywhere in (a,b) except at the point $\mathbf{r} = \mathbf{r}'$. Satisfies a the given homogeneous boundary conditions. Is continuous for fixed $\mathbf{r}'$, even at $\mathbf{r} = \mathbf{r}'$. Has continuous first and second derivative everywhere in (a,b), except at the point $\mathbf{r} = \mathbf{r}'$, where it has a jump discontinuity:

$$\frac{d}{dx}G(\mathbf{r}, \mathbf{r}')|_{\mathbf{r}'+}^{\mathbf{r}'+} = \frac{-1}{p(\mathbf{r}')}$$

$$\delta(\mathbf{r}-\mathbf{r}')L(\mathbf{r}) = \langle\mathbf{r}|\, L\, |\mathbf{r}'\rangle$$

$$G(\mathbf{r},\mathbf{r}';z) = \langle\mathbf{r}|\, G(z)\, |\mathbf{r}'\rangle$$

$$\delta(\mathbf{r}-\mathbf{r}') = \langle\mathbf{r}|\, |\mathbf{r}'\rangle$$

The poles of an appropriate analytic continuation of G in the complex E-plane can be interpreted as the energy (the real pole) and the inverse life-time (the imaginary part).