2D Solid

Note Title

Consider 2D elastic medium.

Recall: $F = \frac{1}{2} \underbrace{\lambda_{iklm}} u_{ik} u_{lm} \leftarrow u_{ij} \sim \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

$$U = \int d^2r \; \frac{1}{2} \underset{\uparrow \text{const.}}{c} \sum \left(\frac{\partial u_i}{\partial x_j}\right)^2$$

$\left(\frac{\partial u_i}{\partial x_j}\right)\left(\frac{\partial u_i}{\partial x_j}\right)^*$

$$u_i(\vec r) = \sum_{\vec k} e^{i\vec k\cdot\vec r} u_i(\vec k)$$

$$\frac{\partial u_i}{\partial x_j} = \sum_{\vec k} i k_j e^{i\vec k\cdot\vec r} u_i(\vec k)$$

$$= \frac{1}{2} c \sum k_j k_j' \underbrace{\int d^2r \, e^{i(\vec k - \vec k')\cdot\vec r}}_{V\delta_{kk'}} u_i(\vec k) u_i^*(\vec k')$$

$$= \frac{Vc}{2} \sum_{ik} k^2 |u_i(\vec k)|^2$$

Average square displacement

$$\langle u^2 \rangle = \left\langle \int \frac{d^2r}{V} \sum_i u_i(\vec r) u_i^*(\vec r) \right\rangle$$

$$= \left\langle \sum_{ik} |u_{ik}(\vec k)|^2 \right\rangle = \sum_{i\vec k} \langle |u_{ik}(\vec k)|^2 \rangle$$

Restricted to $u(\vec k) = u^*(\vec k)$ ($u(\vec r)$ real)

also for only $\vec k < 1/\lambda$ (ignore short range fluctuation)

$$\langle |u_{ik}(\vec k)|^2 \rangle \underset{\substack{\text{thermal}\\ \text{average}\\ \text{(use } U)}}{=} \cdots = \frac{k_B T}{V c k^2}$$

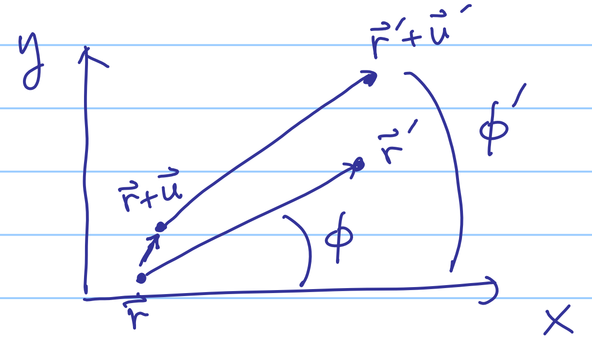
$$\therefore \langle u^2 \rangle = \sum_{ik} \frac{k_B T}{V c k^2} = 2 \int \frac{d^2k}{(2\pi)^2} \frac{k_B T}{c k^2} = \frac{2 k_B T}{c} \int_0^{1/\lambda} \frac{dk}{2\pi k} \longrightarrow \infty$$

small k: long-wavelength limit fluction $\rightarrow$ kills long-range order

Mermin-Wagner theorem: long-range order $\underset{\substack{\uparrow \\ \text{lower critical dimension}}}{d \le 2}$ at $T \to 0$

$\Rightarrow$ Does not mean "melting" of 2D crystal $\rightarrow$ floppy solid.

Orientation Order



$$\tan\phi = \frac{dy}{dx}, \quad \tan\phi' = \frac{dy'}{dx'}$$

$$dx_i' = dx_i + \frac{\partial u_i}{\partial x_k} dx_k$$

$$\tan\phi' = \frac{dy'}{dx'} = \frac{dy + \frac{\partial u_y}{\partial x}dx + \frac{\partial u_y}{\partial y}dy}{dx + \frac{\partial u_x}{\partial x}dx + \frac{\partial u_x}{\partial y}dy}$$

$$\phi - \phi' = \cos\phi\sin\phi\left(\frac{\partial u_y}{\partial y} - \frac{\partial u_x}{\partial x}\right) + \cos^2\phi\left(\frac{\partial u_y}{\partial x}\right) - \sin^2\phi\left(\frac{\partial u_x}{\partial y}\right)$$

$$\delta\phi = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) \quad (\text{averaged over } \phi)$$

Using $u_i = \sum_k e^{i\vec k\cdot\vec r} u_i(\vec k)$

$$\delta\phi = \frac{1}{2}\sum_k \left[ik_x u_y(\vec k) - ik_y u_x(\vec k)\right] e^{i\vec k\cdot\vec r}$$

Spatial average of orientational fluctuations

$$\frac{1}{V}\int d^2r \langle \delta\phi\,\delta\phi\rangle = \frac{1}{4}\sum_{\vec k} k_x^2\langle|u_x(\vec k)|^2\rangle + k_y^2\langle|u_y(\vec k)|^2\rangle$$
$$+ k_x k_y \cancel{\langle (u_x u_y^* + u_y u_x^*)\rangle} \text{ by symmetry}$$

$$= \frac{1}{4}\sum_k \frac{k_B T}{CV\cancel{k^2}} \underbrace{\cancel{(k_x^2 + k_y^2)}}_{k^2}$$

$$= \frac{k_B T}{4C} \sum_k \frac{1}{V} = \frac{k_B T}{4C} \cdot \int_0^{2\pi} d\theta \int_0^{1/\lambda} \frac{dk\,k}{(2\pi)^2}$$

$$= \frac{k_B T}{16\pi \lambda^2 C}$$

Orientational order is destroyed by dislocation unbinding