Mean Field Theory (Ch. 4, C+L)

Note Title 07/02/2012

Phase transition $\begin{cases} 2^{nd} \text{ order (continuous transition)} \\ 1^{st} \text{ order (discontinuous transition)} \end{cases}$

Ising Model:

$$\mathcal{H} = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j \qquad \sigma_i : \pm 1$$

f.m. nearest neighbors

Order parameter: $m = \langle \sigma \rangle = \dfrac{N_\uparrow - N_\downarrow}{N}$ $\longrightarrow$ $N_\uparrow - N_\downarrow = mN$, $N_\uparrow + N_\downarrow = N$

(microcanonical ensemble)

Entropy: $S = \ln \left(\dfrac{N!}{N_\uparrow ! N_\downarrow !} \right)$ $\quad (k_B = 1)$

$$S = \ln N! - \ln N_\uparrow ! - \ln N_\downarrow !$$

$\longleftarrow$ $\ln N! = N \ln N - N$

$= \cdots\cdots$

$$= N \ln 2 - \frac{N}{2} \left[(1+m) \ln (1+m) + (1-m) \ln (1-m) \right]$$

$N_\uparrow = \frac{N}{2}(1+m)$

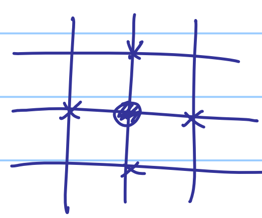
$N_\downarrow = \frac{N}{2}(1-m)$

$$\frac{S}{N} = \ln 2 - \frac{1}{2} \left[\qquad \text{''} \qquad \right]$$

For energy: $Z = \mathrm{Tr}\, e^{-\beta \mathcal{H}}$: difficult to do but can be done

1D: transfer matrix.

not possible in general



Mean Field Approximation

Bragg + Williams (1934)

Replace σ_i with $\langle \sigma_i \rangle = m$

$$E = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j = -\frac{1}{2} J N z m^2$$

m^2 # of N.N.

$$F = E - TS$$

$$f = \frac{F}{N} = -\frac{1}{2} J z m^2 + \frac{1}{2} T \left[(1+m)\ln(1+m) + (1-m)\ln(1-m) \right] - T\ln 2$$

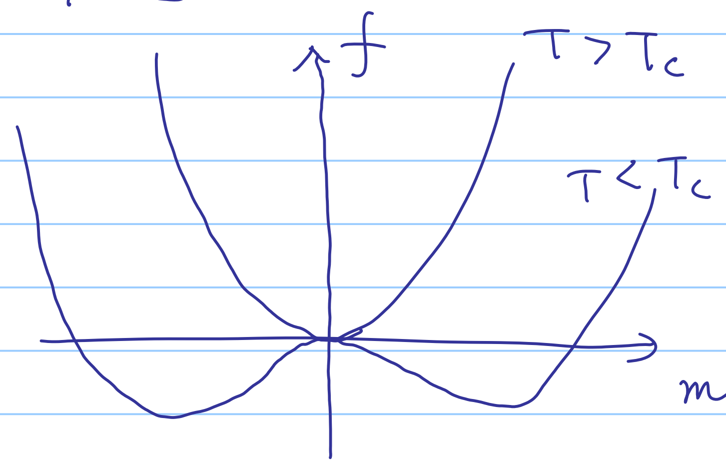
For $m \ll 1$ (near T_c) $\qquad \ln(1+m) = m - \frac{m^2}{2} + \frac{m^3}{3} - \frac{m^4}{4} + \cdots$

$$\frac{S}{N} = \cdots = \ln 2 - \frac{m^2}{2} - \frac{m^4}{12} + \cdots$$

$$f = -\frac{1}{2} J z m^2 + \frac{T}{2} m^2 + \frac{T}{12} m^4 - T\ln 2 + \cdots$$

$$= \frac{1}{2} \underset{\uparrow}{(T - Jz)} m^2 + \frac{T}{12} m^4 - T\ln 2$$

$$T_c \equiv Jz$$



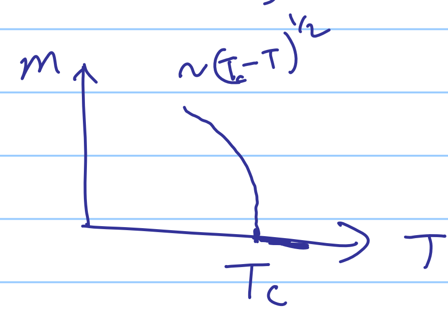
$$\frac{\partial f}{\partial m} = (T - T_c) m + \frac{T}{3} m^3$$

$$= m \left[(T - T_c) + \frac{T}{3} m^2 \right] = 0$$

gives slope = 0 point.

$T < T_c \qquad m^2 = \frac{3(T_c - T)}{T} \simeq \frac{3(T_c - T)}{T_c} \qquad (T \sim T_c)$

$$m = \left(\frac{3}{T_c}\right)^{1/2} (T_c - T)^{1/2}$$



Order parameter exponent

$$m \propto (T_c - T)^{\beta} \qquad \boxed{\beta = 1/2}$$

critical exponents; $\alpha, \beta, \gamma, \nu, \eta, \delta, z$

ν: $\xi \sim |T - T_c|^{\nu}$

γ: $\chi \sim |T - T_c|^{\gamma}$

Specific Heat

$$C_V = -T \frac{\partial^2 F}{\partial T^2} \qquad \frac{C_V}{N} = -T \frac{\partial^2 f}{\partial T^2}$$

$$f = \frac{1}{2}(T - T_c)\, m^2 + \frac{T}{12} m^4 - T \ln 2$$

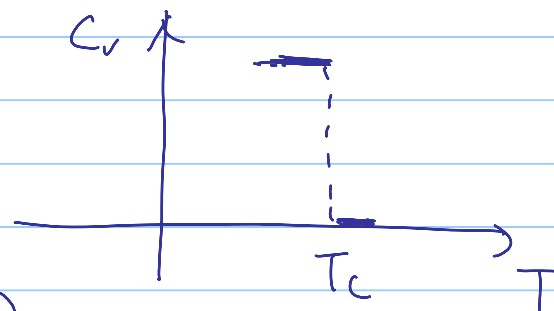
$$m = \begin{cases} \left(\frac{3}{T_c}\right)^{1/2} (T_c - T)^{1/2} & T < T_c \\ 0 & T > T_c \end{cases}$$

$$f = \frac{1}{2}(T - T_c) \cdot \frac{3(T_c - T)}{T_c} + \frac{T}{12} \frac{9(T_c - T)^2}{T_c^2} - T \ln 2$$

$$\simeq -\frac{3}{4T_c}(T - T_c)^2 - T \ln 2$$

$$C_V = \begin{cases} \frac{3}{2} N k_B \ \text{(const)} & T < T_c \\ 0 & T > T_c \end{cases}$$

(Near T_c)



$$C_V \sim (T - T_c)^\alpha \qquad \boxed{\alpha = 0} : \text{M.F.T.}$$