Question 1

A fundamental relation must meet the requirement:

$$S(\lambda U, \lambda V, \lambda N_1, ...) = \lambda S(U, V, N_1, ...)$$

For a:

$$\begin{aligned} S &= C\left(\frac{NU}{V}\right)^{\frac{2}{3}} \\ S(\lambda U, \lambda V, \lambda N) &= C\left(\frac{\lambda^2 NU}{\lambda V}\right)^{\frac{2}{3}} \\ &= C\lambda^{\frac{2}{3}}\left(\frac{NU}{V}\right)^{\frac{2}{3}} \end{aligned}$$

S is not a fundamental relation.

For b:

$$\begin{aligned} S &= C\frac{V^3}{NU} \\ S(\lambda U, \lambda V, \lambda N) &= C\frac{(\lambda V)^3}{\lambda N \lambda U} \\ &= C\lambda\frac{V^3}{NU} \end{aligned}$$

S is a fundamental relation.

Question 2

Given

$$\begin{aligned} T &= \frac{\partial U}{\partial S}_{V,N_1...N_r} \\ P &= -\frac{\partial U}{\partial V}_{S,N_1...N_r} \end{aligned}$$

We have

$$\begin{aligned} \frac{P}{T} &= -\frac{\partial S}{\partial V}_{V,N_1...N_r} \\ &= -\frac{\partial S(U, V, N_1...N_r)}{\partial V} \end{aligned}$$

Check that the above relation is a zero-order homogeneous function

$$\begin{aligned}\frac{\partial S(\lambda U, \lambda V, \lambda N_1 ... \lambda N_r)}{\partial V}_{\lambda U, \lambda V, \lambda N_1 ... \lambda N_r} &= \frac{\partial S(\lambda U, \lambda V, \lambda N_1 ... \lambda N_r)}{\partial \lambda V}\frac{\partial \lambda V}{\partial V} \\ &= \lambda\frac{\partial S(\lambda U, \lambda V, \lambda N_1 ... \lambda N_r)}{\partial \lambda V}\end{aligned}$$

Recalling

$$\begin{aligned}S(\lambda U, \lambda V, \lambda N_1 ... \lambda N_r) &= \lambda S(U, V, N_1 ... N_r) \\ \lambda\frac{\partial S}{\partial V} &= \lambda\frac{\partial S(\lambda U, \lambda V, \lambda N_1 ... \lambda N_r)}{\partial \lambda V}\end{aligned}$$

Thus $\frac{P}{T}$ is an intensive property.

Question 3

Begin with the entropy formulation

$$\begin{aligned}dS &= \frac{1}{T}dU + \frac{P}{V}dV - \sum_{i=1}^{r}\frac{\mu_i}{T}dN_i \\ &= \frac{\partial S}{\partial U}dU + \frac{\partial S}{\partial V}dV + \frac{\partial S}{\partial N_1}dN_1 + \frac{\partial S}{\partial N_2}dN_2\end{aligned}$$

From the relation of S

$$\begin{aligned}\frac{\partial S}{\partial U} &= \frac{3R(N_1+N_2)}{2U} \\ \frac{\partial S}{\partial V} &= \frac{R(N_1+N_2)}{V} \\ \frac{\partial S}{\partial N_i} &= C + R\ln\frac{VU^{\frac{3}{2}}}{(N_1+N_2)^{\frac{5}{2}}} - R\ln\frac{N_i}{N_1+N_2} - \frac{5}{2}R\end{aligned}$$

By matching the differential terms, we find

$$\begin{aligned}\frac{\partial S}{\partial U} &= \frac{1}{T} \\ U &= \frac{3RT(N_1+N_2)}{2} \\ \frac{\partial S}{\partial V} &= \frac{P}{V} \\ P &= R(N_1+N_2) \\ \frac{\partial S}{\partial N_1} &= -\frac{\mu_1}{T} \\ \mu_1 &= -T(C + R\ln\frac{V(\frac{3RT}{2})^{\frac{3}{2}}}{(N_1+N_2)} - R\ln\frac{N_1}{N_1+N_2} - \frac{5}{2}R)\end{aligned}$$