Z-component of magnetization

In order to determine the z-component of magnetization, the given wavefunction is expanded in terms of the conventional eigenfunctions of the p-state:

$$\begin{aligned}\psi_{11} &= \Sigma A_n \phi_n \\ &= f(r)N(A_{11}\frac{(x+iy)}{\sqrt{2}} + A_{10}z + A_{1-1}\frac{(x-iy)}{\sqrt{2}})\end{aligned}$$

Or equivalently in bra-ket notation:

$$\begin{aligned}\mid \psi_{11}\rangle &= \Sigma A_n \mid \phi_n\rangle \\ &= f(r)N(A_{11}\mid \phi_{11}\rangle + A_{10}\mid \phi_{10}\rangle + A_{1-1}\mid \phi_{1-1}\rangle)\end{aligned}$$

Rearranging to solve for A_n ($\langle\phi_n \mid = \mid \phi_n\rangle^\dagger$):

$$A_n = \langle\phi_n \mid \psi_{11}\rangle$$

Substituting in the givens (ignoring the factor $f(r)N$ for the moment and including the $\frac{1}{\sqrt{2}}$ in the expansion coefficient where necessary):

$$A_{11} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -i & 0\end{bmatrix}\begin{bmatrix}1+8i \\ -4+4i \\ 8+i\end{bmatrix} = \frac{1}{\sqrt{2}}(5+12i)$$

$$A_{10} = \begin{bmatrix}0 & 0 & 1\end{bmatrix}\begin{bmatrix}1+8i \\ -4+4i \\ 8+i\end{bmatrix} = (8+i)$$

$$A_{1-1} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & i & 0\end{bmatrix}\begin{bmatrix}1+8i \\ -4+4i \\ 8+i\end{bmatrix} = \frac{1}{\sqrt{2}}(-3+4i)$$

Recalling the possible measurements of the $\tilde{L_z}$:

$$\begin{aligned}\tilde{L_z}\mid \phi_{11}\rangle &= \hbar \mid \phi_{11}\rangle \\ \tilde{L_z}\mid \phi_{10}\rangle &= 0 \mid \phi_{10}\rangle \\ \tilde{L_z}\mid \phi_{1-1}\rangle &= -\hbar \mid \phi_{1-1}\rangle\end{aligned}$$

We can now write the expression of the ensemble average of the expectation values of the z-component of angular momentum:

$$\begin{aligned}
\langle \bar{L_z} \rangle &= \frac{\langle \psi_{11} \mid \tilde{L_z} \mid \psi_{11} \rangle}{\langle \psi_{11} \mid \psi_{11} \rangle} \\
&= \frac{\mid A_{11}A_{11}^* \mid \langle \phi_{11} \mid \tilde{L_z} \mid \phi_{11} \rangle + \mid A_{10}A_{10}^* \mid \langle \phi_{10} \mid \tilde{L_z} \mid \phi_{10} \rangle + \mid A_{1-1}A_{1-1}^* \mid \langle \phi_{1-1} \mid \tilde{L_z} \mid \phi_{1-1} \rangle}{\mid A_{11}A_{11}^* \mid \langle \phi_{11} \mid \phi_{11} \rangle + \mid A_{10}A_{10}^* \mid \langle \phi_{10} \mid \phi_{10} \rangle + \mid A_{1-1}A_{1-1}^* \mid \langle \phi_{1-1} \mid \phi_{1-1} \rangle} \\
&= \frac{169\hbar/2 + 65(0) - 25\hbar/2}{65 + 25/2 + 169/2} \\
&= \frac{144\hbar}{324} \\
&= \frac{4\hbar}{9}
\end{aligned}$$

Therefore, the macroscopic z-component of magnetization is:

$$\begin{aligned}
\vec{\mu_z} &= \frac{\langle \bar{L_z} \rangle QN}{2m_e} \\
&= \frac{(10^{23})4\hbar e}{18m_e} \\
&= 0.271[J/T]
\end{aligned}$$

Justification of quantum number selection

We perform a transformation of coordinates upon the wavefunction:

$$\psi'_{11} = T^{-1}\psi_{11}$$

$$\begin{bmatrix} \psi_{x'} \\ \psi_{y'} \\ \psi_{z'} \end{bmatrix} = \frac{f(r)N}{9} \begin{bmatrix} 1 & -4 & 8 \\ 8 & 4 & 1 \\ -4 & 7 & 4 \end{bmatrix} \begin{bmatrix} 1+8i \\ -4+4i \\ 8+i \end{bmatrix}$$

$$\psi'_{11} = f(r)N \begin{bmatrix} 9 \\ 9i \\ 0 \end{bmatrix}$$

$$\psi'_{11} = 9\sqrt{2}f(r)N\frac{(x+iy)}{\sqrt{2}}$$

Because of the physical laws do not vary upon the application of a coordinate transformation and hence the physical significance of quantum numbers is invariant, we can say, independent of the frame of reference, the given wavefunction represents the p-state $l = 1$ and $m = 1$.