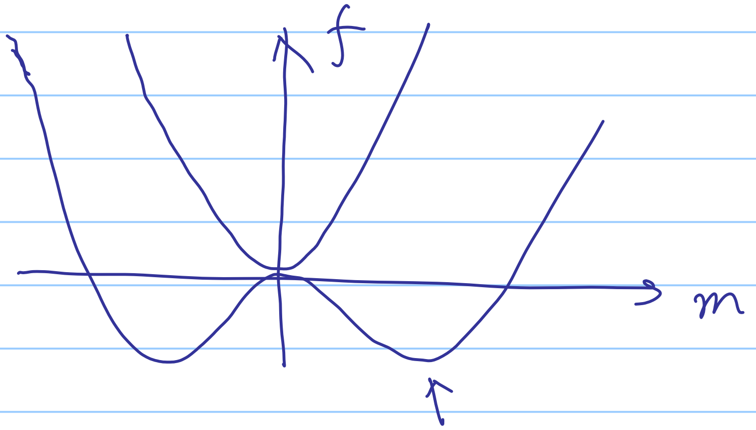
Continuous symmetry

Note Title 05/03/2012

- Broken symmetry $\longleftrightarrow$ phase transition
  We have seen an example of broken discrete symmetry (Ising model)



non-zero order parameter
@ two free energy minima

Now, let's consider continuous symmetry.
example: O(n) model $\vec{m} = (m_1, m_2, m_3, \cdots, m_n)$

$n=3$: Heisenberg model
$n=2$: XY-model.

Now, Landau free energy in general is

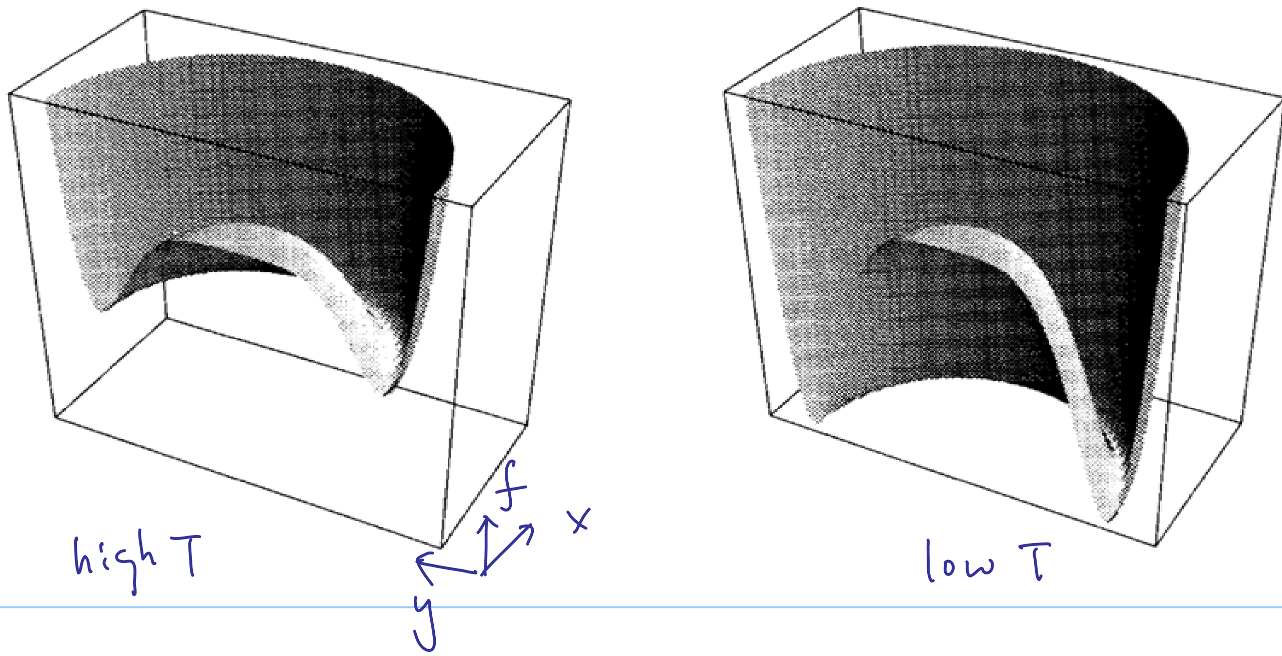
$$f = \int d^d\vec{r} \left[\frac{1}{2} r |\vec{m}|^2 + \frac{1}{4} u |\vec{m}|^4 - \vec{h} \cdot \vec{m} + \frac{1}{2} (\vec{\nabla} \vec{m})^2 \right]$$

where $(\nabla m)^2 = \sum_{i=1}^{d} \sum_{\alpha=1}^{n} \left(\frac{\partial m}{\partial x_i} \right)^2$

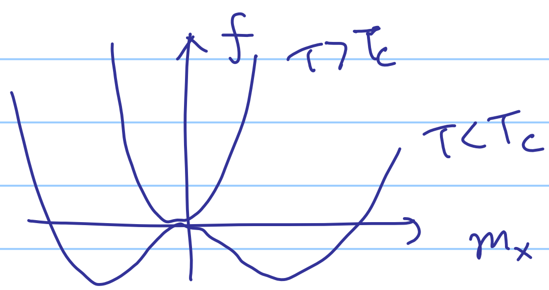
If we ignore the fluctuation part

$$f = \frac{1}{2} r m^2 + \frac{1}{4} u m^4 + \cdot$$

$n=2$ example



The cross-section will look exactly like the Ising case



for $h=0$ → phase transition @ $T=T_c$ ($r=0$)

But for $T<T_c$: infinite degeneracy

Susceptibility

$\vec{h} = h\vec{n}$ where $\vec{n}$ is a unit vector (e.g. $\vec{n} = (0,0,1)$.)

$m_z = m$

$m_x, m_y, \cdots = 0$

* Generalized susceptibility $\chi_{\alpha\beta} = \dfrac{\partial m_\alpha}{\partial h_\beta}$ —— ①

* Generalized correl. fn.

$C_{\parallel}(\vec{r}) = \langle m_z m_z \rangle - \langle m_z \rangle \langle m_z \rangle$: Longitudinal correl. fn.

$C_{\perp}(\vec{r}) = \langle m_x m_x \rangle - \langle m_x \rangle \langle m_x \rangle$: Transverse " "

$\langle m_y m_y \rangle - \langle m_y \rangle \langle m_y \rangle$

⋮

* Static Susceptibility Sum rule :

$S_{\parallel}(q=0) = k_B T \chi_{\parallel}$

$S_{\perp}(q=0) = k_B T \chi_{\perp}$

* $\dfrac{\partial f}{\partial m_\alpha} = h_\alpha$: Eq. of state (from thermodynamic) —②

$$\chi_{\alpha\beta} = -\frac{\partial^2 f}{\partial h_\alpha \partial h_\beta} = -\frac{\partial}{\partial h_\alpha}\left(\frac{h_\beta}{h}\frac{\partial f}{\partial h}\right)$$

....

$$= -\frac{h_\alpha h_\beta}{h}\frac{\partial^2 f}{\partial h^2} - \frac{1}{h}\frac{\partial f}{\partial h}\left(\delta_{\alpha\beta} - \frac{h_\alpha h_\beta}{h^2}\right)$$

For $\vec{n} = (0,0,1)$

$$\chi_{zz} = -\frac{\partial^2 f}{\partial h^2} = \chi_{\parallel}$$

$$\chi_{\alpha\beta} = -\frac{1}{h}\frac{\partial f}{\partial h}\delta_{\alpha\beta} = \frac{m}{h} = \chi_{\perp}$$

$T > T_c$: $\chi_\perp = 0$

$T < T_c$: $m \neq 0$, for $h \to 0 \implies \chi_\perp = \infty$

Two types of fluctuation $\left[\begin{array}{l}\text{longitudinal} \\ \text{transverse} \to \text{very large}\end{array}\right.$

Consider fluctuations explicitly

For $T < T_c$: MFT gives a solution for nonzero m.

but $\vec{m}$ direction is not defined.

Choose $\vec{n}$ as the direction of $\vec{m}$

$$\vec{m} \to m(\hat{n} + \phi_1 \hat{n} + \vec{\phi}_\perp) \qquad \phi_1 \to (0, 0, \phi_1)$$

$\hat{n}$ is the unit vector along $\vec{m}$, $\qquad \phi_\perp \to (\phi_x, \phi_y, 0)$

ϕ_1 is fluctuation along $\hat{n}$,

$\vec{\phi}_\perp$ is " $\perp$ to $\hat{n}$

$$|\vec{m}|^2 = m^2(1 + 2\phi_1 + \phi_1^2 + \phi_\perp^2)$$

$$|\vec{m}|^4 = m^4(1 + 4\phi_1 + 6\phi_1^2 + 4\phi_1^3 + \cdots + \phi_\perp^4 + 2\phi_\perp^2 + \cdots)$$

Using this $f = \frac{1}{2}r|\vec{m}|^2 + \frac{1}{4}u|\vec{m}|^4 = -\frac{1}{4}\frac{r^2}{u} - rm^2\phi_1^2$

The gradient term

$$\frac{1}{2}(\nabla\vec{m})^2 = \frac{m^2}{2}(\nabla\phi_1)^2 + \frac{m^2}{2}(\nabla\phi_\perp)^2$$

$$\Delta f = \frac{m^2}{2}\int d^d\vec{r}\left[(\nabla\phi_1)^2 + (\nabla\phi_\perp)^2 + 2r\phi_1^2\right] + \cdots$$

$$= (\text{longitudinal part}) + \underbrace{\frac{1}{2}\int m^2(\nabla\phi_\perp)^2 d^d\vec{r}}_{\text{transverse part}}$$

free energy due to nonzero fluctuation

Any fluctuation $\nabla\phi_\perp$ raises Δf

and restoring force $\Rightarrow$ rigidity m^2