Chapter1. THE GENESIS OF FOURIER ANALYSIS

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摘要

This book begins the talk of Fourier Analysis with two physical problem: vibrating string and heat conduction. In this review, I am going to illustrate these two examples in detail by supplementing the background knowledge.

1 The Vibrating String

1.1 Simple harmonic motion 简谐运动

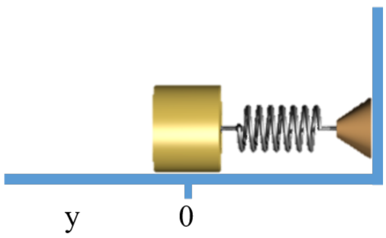

Figure 1: 简谐运动谐振子

Newton's law produces a 2nd order ODE

$$F = -ky(t) \Rightarrow -ky(t) = my''(t) \tag{1}$$

Let $c = \sqrt{\frac{k}{m}}$, we get a neater form:

$$y''(t) + c^2 y(t) = 0 \tag{2}$$

Equation(2) is a linear homogeneous 2nd-order differential equation(二阶常系数线性方程). For a general case $y'' + py' + qy = 0$, we first solve the characteristic equation $\lambda^2 + p\lambda + q = 0$ and get the characteristic roots: λ_1, λ_2.

Characteristic Roots	Linear Independent Sol. Pair	General Sol.
$\lambda_1, \lambda_2 \in \mathbb{R}$ *and* $\lambda_1 \neq \lambda_2$	$y_1 = e^{\lambda_1 x}, y_2 = e^{\lambda_2 x}$	$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$
$\lambda_1, \lambda_2 \in \mathbb{R}$ *and* $\lambda_1 = \lambda_2$	$y_1 = e^{\lambda_1 x}, y_2 = x e^{\lambda_1 x}$	$y = (C_1 + C_2 x) e^{\lambda_1 x}$
$\lambda_1, \lambda_2 = \alpha \pm i\beta$	$y_1 = e^{\alpha x}\cos\beta x, y_2 = e^{\alpha x}\sin\beta x$	$y = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x)$

The corresponding characteristic equation of (2) is $\lambda^2 + c^2 = 0$, and the roots are $\lambda = \pm ic$; therefore, the general solution of (2) is

$$y(t) = a\cos ct + b\sin ct \tag{3}$$

and it's 1st order derivative is

$$y'(t) = -ac\sin ct + bc\cos ct \tag{4}$$

To get the particular solution, we need 2 initial condition: $y(0)$ *and* $y'(0)$.
$y(0) = a,\ y'(0) = bc$, thereby getting the particular solution:

$$y(t) = y'(0)\cos ct + \frac{y'(0)}{c}\sin ct \tag{5}$$

1.2 Wave Equation on String

下面推导弦上的波动方程：
We subdivided the string into a large number N of mass points, distributed uniformly along the x-axis. (See Figure 2)
Each particle's position is (x_n, y_n) , where:

$$\begin{cases} x_n = nh = n\dfrac{L}{N} \\ y_n(t) = u(x_n, t) \end{cases} \tag{6}$$

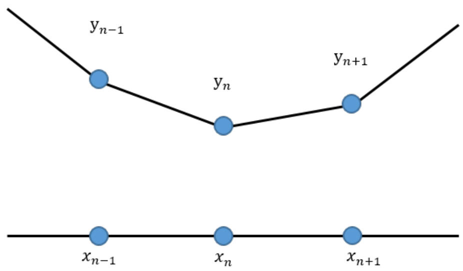

Figure 2: A Vibrating String

There are two forces acted upon the particle by two neighbour particles; i.e. the sum of forces on (x_n, y_n) is $F_l + F_r$. (See Figure 3)

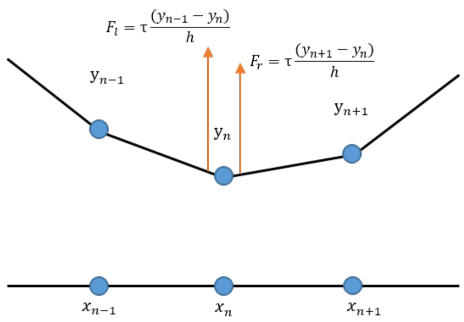

Figure 3: Forces on the string's particle

Now we are going to illustrate why F_r equals $\tau\frac{(y_{n+1}-y_n)}{h}$. As the particle only oscillates in the y-direction, the horizontal net force is zero.

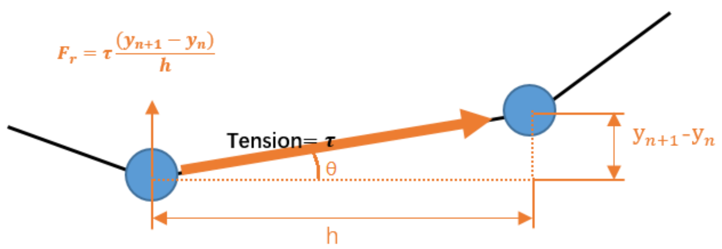

Figure 4: Force Analysis

The y-direction force is the y-component of the tension τ between the particle and the next particle. (Figure 4)
F_r equals $\tau \sin\theta$ (θ is the angle between the tension and the horizontal line), Considering θ is very small, we substitute $\sin\theta$ with $\tan\theta$.
Hence, F_r approximately equals to $(\tau\tan\theta) = (\tau\frac{(y_{n+1}-y_n)}{h})$
The linear density of the string is ρ, so each particle's mass is ρh According to Newton's law $F = ma$, we derive the following equation.

$$\begin{aligned}
\rho h y_n''(t) &= F_l + F_r \\
&= \tau\frac{y_{n+1}(t) - y_n(t) + y_{n-1}(t) - y_n(t)}{h} \\
&= \tau\frac{u(x_n + h, t) - u(x_n, t) + u(x_n - h, t) - u(x_n, t)}{h} \\
&= \tau\frac{u(x_n + h, t) + u(x_n - h, t) - 2u(x_n, t)}{h}
\end{aligned}$$

Let's take the limit of this equation's right side with respect to h,

$$\rho h y_n''(t) = \lim_{h\to 0} \tau \frac{u(x_n+h,t)+u(x_n-h,t)-2u(x_n,t)}{h}$$

and move h to the right side, and use $u(x,t)$ notation to substitute $y''(t)$

$$\rho \frac{\partial^2 u}{\partial t^2} = \lim_{h\to 0} \tau \frac{u(x_n+h,t)+u(x_n-h,t)-2u(x_n,t)}{h^2} \tag{7}$$

The right side is right the definition of 2nd order partial derivatives of u with respect to x, so in short:

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2},\ with\ c = \sqrt{\tau/\rho} \tag{8}$$

This relation is known as 1-D wave equation, $c > 0$ is called the **velocity** of the motion.
We can examine c's physical meaning using Dimensional Analysis (量纲分析).
The Unit of c is $(N \bullet \frac{kg}{m})^{0.5} = (kg \bullet m/s^2 \bullet \frac{kg}{m})^{0.5} = m/s$

1.3 Solving Wave Equation

For simplicity, we assume that $c = 1$ and length of string $L = \pi$, so Equation (9) becomes

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}\ (0 \leqslant x \leqslant \pi,\ t \geqslant 0) \tag{9}$$

1.3.1 using travelling waves

我们用$F(x)$来表示某一时刻的波形，$F(x+vt)$则表示t个时间后的波形（假设波向右移动）。

1.3.2 using the superposition of standing waves

Standing Wave 驻波 驻波是两个振幅、波长、频率都相同的正弦波，相向而行形成的。驻波的波形并不前进，波形上的每个质点皆作Simple Harmonic Motion(简谐运动)。拨动两端固定并拉紧的弦，机械波经过两固定端的反射，反射的波叠加可形成驻波。

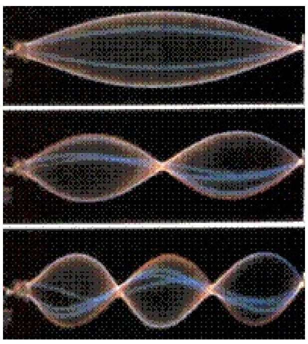

Figure 5: 驻波

We use $u(x,t)$ to describe the standing wave. The nature of standing waves suggest the mathematical idea of "separation of variables" (分离变量的思想).
As standing waves do not travel, every mass point just oscillates around the balance position at different amplitudes. So the equation of standing wave movement can be expressed as:

$$u(x,t) = \phi(x)\psi(t) \tag{10}$$

where:

$\phi(x)$ represents the amplitute of the mass points at location x.
$\psi(t)$ is a oscillating factor, which makes every mass point
oscillates within the amplitute $\phi(x)$.