Chapter1. THE GENESIS OF FOURIER ANALYSIS

陈斯杰

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摘要

This book begins the talk of Fourier Analysis with two physical problem: vibrating string and heat conduction. In this review, I am going to illustrate these two examples in detail by supplementing the background knowledge.

1 The Vibrating String

1.1 Simple harmonic motion 简谐运动

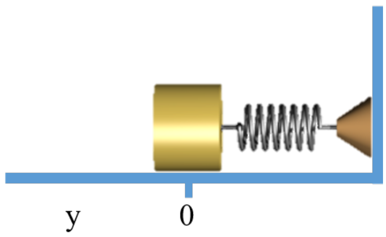

图 1: 简谐运动谐振子

Newton's law produces a 2nd order ODE

$$F = -ky(t) \Rightarrow -ky(t) = my''(t) \tag{1}$$

Let $c = \sqrt{\frac{k}{m}}$, we get a neater form:

$$y''(t) + c^2 y(t) = 0 \tag{2}$$

Equation(2) is a linear homogeneous 2nd-order differential equation(二阶常系数线性方程). For a general case $y'' + py' + qy = 0$, we first solve the characteristic equation $\lambda^2 + p\lambda + q = 0$ and get the characteristic roots: λ_1, λ_2.

Characteristic Roots	Linear Independent Sol. Pair	General Sol.
$\lambda_1, \lambda_2 \in \mathbb{R}$ *and* $\lambda_1 \neq \lambda_2$	$y_1 = e^{\lambda_1 x}, y_2 = e^{\lambda_2 x}$	$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$
$\lambda_1, \lambda_2 \in \mathbb{R}$ *and* $\lambda_1 = \lambda_2$	$y_1 = e^{\lambda_1 x}, y_2 = x e^{\lambda_1 x}$	$y = (C_1 + C_2 x) e^{\lambda_1 x}$
$\lambda_1, \lambda_2 = \alpha \pm i\beta$	$y_1 = e^{\alpha x} \cos \beta x, y_2 = e^{\alpha x} \sin \beta x$	$y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$

The corresponding characteristic equation of (2) is $\lambda^2 + c^2 = 0$, and the roots are $\lambda = \pm ic$; therefore, the general solution of (2) is

$$y(t) = a \cos ct + b \sin ct \tag{3}$$

and it's 1st order derivative is

$$y'(t) = -ac \sin ct + bc \cos ct \tag{4}$$

To get the particular solution, we need 2 initial condition: $y(0)$ *and* $y'(0)$.
$y(0) = a,\ y'(0) = bc$, thereby getting the particular solution:

$$y(t) = y'(0) \cos ct + \frac{y'(0)}{c} \sin ct \tag{5}$$

1.2 Standing Wave 驻波

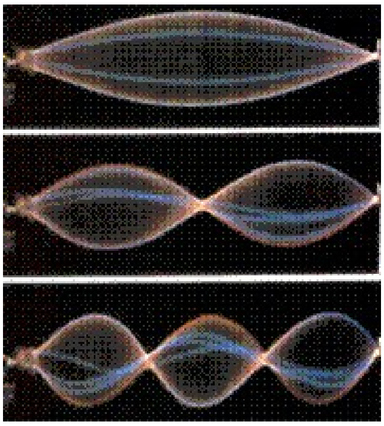

图 2: 驻波

驻波是两个振幅、波长、频率都相同的正弦波，相向而行形成的。驻波的波形并不前进，波形上的每个质点皆作Simple Harmonic Motion(简谐运动)。
We use $u(x,t)$ to describe the standing wave. The nature of standing waves suggest the mathematical idea of "separation of variables".
As standing waves do not travel, every mass point just oscillates around the balance position at

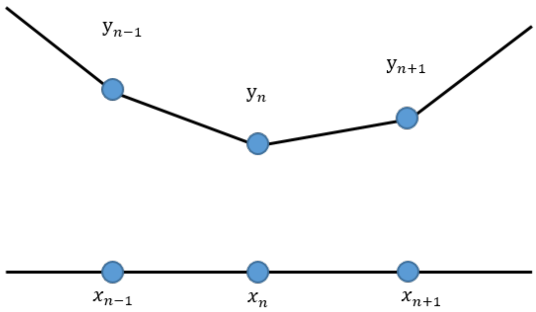

图 3: A Vibrating String

different amplitudes. So the equation of standing wave movement can be expressed as:

$$u(x,t) = \phi(x)\psi(t) \tag{6}$$

where:
$\phi(x)$ represents the amplitute of the mass points at location x.
$\psi(t)$ is a oscillating factor, which makes every mass point oscillates within the amplitute $\phi(x)$.

拨动两端固定并拉紧的弦，机械波经过两固定端的反射，反射的波叠加可形成驻波。

下面推导弦上的驻波方程：

将固定弦分为N段，弦上的每一段的端点坐标为(x_n, y_n)

$$\begin{cases} x_n = nh = n\dfrac{L}{N} \\ y_n(t) = u(x_n, t) \end{cases} \tag{7}$$

每一段的受力分析如下:

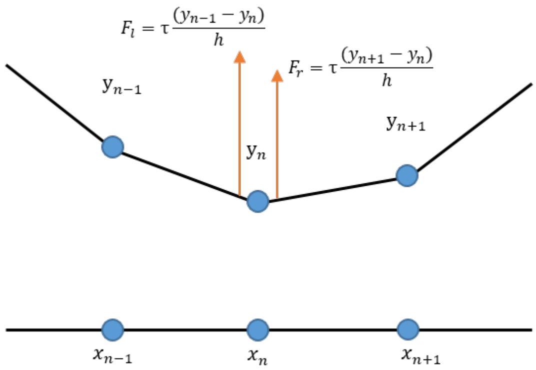

图 4: A Vibrating String

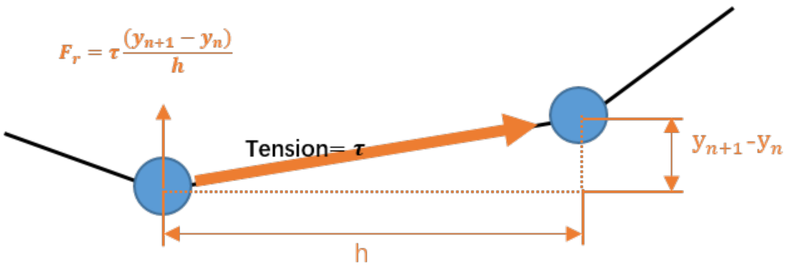

图 5: Force Analysis