Dynamic Optimization of Cash Flow Management Decisions: A Stochastic Model

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Abstract—**A stochastic method is proposed based on decision analysis and Bayesian updating to monitor the cash flow and make short term decisions when a liquidity squeeze appears possible. The model involves uncertainties about the payment time of outstanding bills sent out to customers and the updating of this information as the manager gets to know his customers through successive payments. This updating is done through the use of conjugate probability distributions that allow closed form analytical computation of the probability density functions for the payment of each client given past experience. The use of exponential utility functions allows simple computation of the benefits of this cash flow monitoring system. This formulation is adapted to the case of small and new businesses with a small number of customers whose buying and paying schedules are critical for the firm. It can be particularly useful for new high technology ventures as part of their strategy to manage short-term financial risk. An illustrative example is used to assess the benefits of such a monitoring system.**

I. Introduction

While profitability measures such as return on investment and gross margin are important measures of the health of a firm, cash flow remains the ultimate test of survivability in both profit and not-for-profit organizations. Profitability is important, but without cash flow a firm cannot meet basic obligations such as payroll, taxes, and payments to suppliers. Also, profitability can reflect accounting policies (e.g., depreciation, depletion, inventory valuation), while cash flow is arguably a more objective test of short term survivability. Finally, a focus on cash flow is essential for determining what is a sustainable growth rate [1].

Cash flow is particularly crucial when a firm has a high rate of growth, few customers, or pays high interest rates [2]. This is a typical situation in the domain of high technology, for example, for a promising start up in electronics or biotechnology. It may also be the case for a small, successful consulting firm, particularly one which specializes in scientific fields that evolve rapidly with unavoidable ups and downs. Cash flow problems often occur in high-growth firms because in order to maintain high growth, management typically must make investments in capital equipment, personnel, and inventory in anticipation of revenues. Thus, the high growth firm generally faces a committed set of obligations in the form of investments which are due in the near term and must be paid for by an often uncertain, future revenue stream. This problem is accentuated in volatile environments where competition and technological changes increase the uncertainty of revenue and expense forecasts [3]. Timing of payments can be a major source of uncertainty in short-term financial decisions [4]. Cash flow problems are also exacerbated in firms with few customers. In this case, cash inflows are particularly volatile because they depend upon the buying and paying habits of a small number of clients. Finally, cash flow problems are also associated with high interest rates. In this situation, the opportunity costs of maintaining high cash balances are high, while at the same time, the costs of borrowing, a major response to negative cash flow, increase.

These three conditions often come together in small, and, especially, new firms such as high technology ventures [5]. As Welsh and White [6] highlight, small and new businesses face resource poverty since they tend to have fewer and more costly sources of financing, and fewer and more volatile sources of revenue. These authors show, in particular, how critical the reliability of the customers' payment patterns can be for the small or new business manager who finds himself facing predicted or unpredicted expenses, and they further emphasize the necessity of cash flow monitoring.

II. The Problem

The problem of cash flow management depends upon the time horizon. Short term cash flow problems (on the order of one to two weeks) may result from delay in payments from customers at a time when the firm's managers must pay bills or meet the payroll. In the longer term, firm managers can find themselves in the classic situation of a liquidity squeeze if, in anticipation of demand, they must draw upon cash reserves to invest in capital equipment, inventory, and the like. In both cases, uncertainties about the size and timing of actual revenues are an inherent part of the problem. Specifically, these uncertainties include the dates of customers' payments in the short term, and the actual level of demand and the timing of revenue growth in the long term.

Several models of cash management have been proposed in the literature. For example, Baumol [7] approached cash forecasting from an inventory viewpoint. The model assumes that the firm disburses cash in a steady stream, and that cash is obtained by either borrowing or withdrawing from an investment. The Miller–Orr [8] model forecasts optimal cash level by assuming that net cash flows fluctuate in a random manner. That is, instead of assuming that cash flows occur at a constant

Manuscript received March 1989; revised December 1989. The review of this paper was processed by Editor D. F. Kocaoglu.

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IEEE Log Number 9035705.

0018-9391/90/0800-0203$01.00 © 1990 IEEE

rate, as the Baumol model assumes, cash flows are assumed to occur following a random pattern. However, these and other inventory approaches [9], [10] suffer from clear limitations. For example, payment schedules and receipts are assumed to be uncontrollable by the firm, and the only decision is the size and time of transfers between cash accounts and a securities investment [11]. Moreover, as Mullins and Homonoff [12] argue, the models, notably Miller–Orr's, do not outperform simple managerial rules-of-thumb.

The problem of cash flow management has also been approached from a linear programming standpoint. For example, Robichek *et al.* [13] attempted to compute total relevant costs (e.g., interest on loans, lost discounts), given a set of constraints (e.g., size of credit line, required compensating balance). While this and other linear programming models of cash management [14] encompass a broader range of decisions than inventory approaches, they are limited by deterministic assumptions and, perhaps because of their mathematical sophistication, have not been widely used in practice [11].

Finally, the problem of cash flow management has been approached by many firms in terms of short-term financial planning [15]. Although specifics vary from firm to firm, the cash management problem is generally decomposed into two crucial features: one is the anticipation of cash shortfalls and the other is the response to anticipated cash problems. A principal means of anticipation of shortfalls is the statement of sources and uses of funds [16]. Simply, firm managers lay out future revenues such as known receivables plus sales forecasts which may or may not be adjusted by the estimated or average collection periods. Time series analyses [11] or accounts receivable prediction techniques [17] may be used as well to predict inflows. These inflows are then matched with future expenses such as known payables, projected expenses based on forecasted sales, and discretionary expenses which are proposed by various departments. Again, time series regression and other historical methods may be used to predict these outflows [11], [18]. This matching yields a continuous picture of the firm's future cash balance from which cash shortfalls can be predicted. In effect, this procedure ignores the uncertainties and transforms the estimate of future cash flow from a stochastic process into a deterministic time function. Uncertainty, if dealt with at all, is treated using sensitivity analysis [15].

The other crucial feature of cash flow management from the short-term financial planning viewpoint is the selection of an appropriate response to a possible liquidity squeeze. In the long term, firm managers may secure lines of bank credit or cash reserves from equity financing if it is anticipated that the firms will not generate adequate cash. Short term actions include delay of payments to suppliers and employees. In effect, firm managers can stretch accounts payable. Other short term tactics are to tap bank lines of credit, borrow against inventory or receivables, use temporary layoffs, accelerate receivables, or liquidate near-cash investments [2]. Finally, firm managers can take their chances and risk bouncing checks or even bankruptcy. While the categories of options are clear, in reality, there are often many possibilities within each option and it is easy to fall into response routines that do not take into account the entire range of options. This is particularly likely in firms which rely heavily upon the daily bank balances as their signalling mechanism.

While many firms use a planning procedure similar to that described above, a substantial number of firms, especially small or new ones, rely on an even simpler cash management system [19]. This system consists of a daily call to the firm's bank to ascertain the current bank balance. The firm's financial manager holds up payment of bills and processing of purchase orders until there is sufficient cash to cover these expenses. Use of an existing bank credit line is the last resort. This conservative procedure sets the problem in deterministic terms, and substantially restricts the range of options. Obviously the result of this system can be the sacrificing of growth opportunities for the safety of an assured positive cash position. It can also be extremely time-consuming.

The purpose of this paper is to develop an approach to cash flow management using the perspective of a general risk management problem and the paradigm of decision analysis [20], [21]. This approach is presented with the small high tech firm in mind but can obviously be extended to other firms with the same characteristics elsewhere. A method of probabilistic analysis of warning systems [22] is used here in order to guide the management of cash flow in a manner consistent with the uncertainties of revenue realization and with the effectiveness of the various possible risk reduction measures. We provide a simple method for the development of a computer-assisted system and an assessment of the value of information provided by such a system. Our risk analysis approach improves upon inventory-based approaches because the range of options available to the decision maker is broader and more realistic. Our approach also enables managers to incorporate systematically the stochastic features of the problem and their own risk preferences, neither of which is possible in inventory or linear programming models or in traditional cash flow approaches in firms. Unlike current methods of forecasting cash (e.g., [11], [17]), our method also permits focus on individual customers and does not presume historical data which new firms, in particular, simply may not have. This lack of historical data is typical of the small high technology firm. Finally, our model is dynamic in that it explicitly updates or learns based upon the payment features of individual customers.

Although our basic approach can be extended to the longer time frame, we have chosen to focus here on the short term. The structure of the problem is illustrated by the following example. Consider the situation of the manager of a small high technology venture, who has only a few customers and has had some, but limited, experience with their payment patterns. For each outstanding bill he or she[1] can assess a probability distribution of the future time of payment. This manager also knows what bills and payroll charges he himself will have to pay at determined future times. Given the cash on hand, he can then determine the probability of running out of cash in the considered period. Finally, every time a customer pays his bill, the manager can proceed to a reassessment of the time that this particular customer takes to pay his bills.

[1]To simplify the exposition, the manager, in the rest of this paper, is referred to as "he."

In this paper, we treat this updating problem in two ways: 1) a reassessment of a point estimate of the mean time to payment, and 2) a Bayesian updating of this mean time [23]. The former method allows updating the mean of the payment time distribution, but does not consider the variation about this mean. The latter method goes one step further and incorporates information about the entire payment time distribution, not just the mean. This distribution information is required to compute expected utility as a criterion for optimizing management control, as we will show in the numerical illustration below.

The probability that a given bill will be paid is then computed for each bill in order to produce the probability of a negative cash position. When there is a nonzero probability of running out of cash, a decision tree is used to specify the appropriate option. Possible actions may include, for example, delaying payments, borrowing from the bank against an existing credit line, or liquidating a securities investment.

Our paper begins with a probabilistic assessment of future cash flow balance, then continues with an analysis of possible actions given the probability and nature of cash flow problems, and concludes with an assessment of the value of information and risk reduction provided by the proposed cash flow monitoring method. Consistent with the general theory of warning systems [22], the study of anticipated cash flow variations and possibility of cash shortage is called a signal model, and the study of decisions given possible signals is called a response model. A numerical example serves to illustrate the application of the method.

III. Probabilistic Analysis of Short-Term Liquidity (Signal Model)

A. The Problem of Short-Term Liquidity

Consider the problem of the manager who finds himself at a given time (t_0) in the following situation: he knows that at a given future time ($t_0 + \Delta t$) he will have to pay some bills. He has billed several customers and knows when these outstanding bills have been sent. If the cash account is sufficient to cover the bills that he has to pay, the manager need not worry, at least about short term cash flow problems. If the amount of cash available is insufficient at time t_0, there is a chance that some of the customers may pay their bills in time, but there is also some chance that the cash available at time $t_0 + \Delta t$ will be insufficient for the manager to pay his own bills. Assume that it takes the time Δt (e.g., a few days) for the manager to take precautionary measures such as bank borrowing. He has to decide at time t_0 what is an appropriate action to avoid potential cash flow problems. At time $t_0 + \Delta t$, he may, for example, delay paying his own bill or call his customers, either to inquire whether or not payment will arrive in time or to ask for payment, or borrow from his bank. Each of these actions carries its own costs (e.g., costs of borrowing, costs of time, and possibly, loss of credit rating or loss of good will from the customers). These decisions, however, have to be made before the uncertainty about customers' payment is resolved for time $t_0 + \Delta t$.

Every day, the problem is reconsidered: Has any payment occurred? Has any bill been sent out? Does any bill come due within the next period Δt? In order to incorporate the time uncertainty of future cash inflows, the manager needs a prior probability distribution of the time elapsed between the day a bill is sent out to each client and the day the payment is received. While a unique distribution can be used for each customer,[2] we assume that the decision maker has adopted a particular probability distribution for each client, not necessarily the same for all clients. There remains, however, some uncertainty about the value of the parameter of the distribution. Continuous updating of this parameter allows the incorporation of new information and results in more accurate estimation of its true value. At each payment, the manager acquires some experience about his customer and can update the information in two ways: he can either proceed to a maximum likelihood reestimation of the mean, which gives him a point estimate; or proceed to a Bayesian updating of the probability distribution that he previously attributed to the mean itself, and thereby consider the full range of relevant uncertainties in his final distribution on decision outcomes. In the model that we propose, both procedures are presented.

As time passes, the manager thus accumulates information about all his customers. He can then compute the probability that, in the future window Δt, no bill is paid, or that one, or two, or n bills are paid (we assume here that there is no partial payment of a bill). From there, he can derive the probability distribution of the excess cash at time $t_0 + \Delta t$ when next bills must be paid by the business. This process, which is both stochastic and dynamic, is represented on the following time axis (Fig. 1).

B. The Signal Model

The goal of the signal model is to compute the probability distribution of the cash balance at time $t_0 + \Delta t$ after payment of the bill due, assuming that this bill is paid (hence the probability of a negative balance). We compute first the probability that a given bill will be paid between t_0 and $t_0 + \Delta t$. We use both a point estimate and Bayesian updating. The former is simpler, but the latter allows one to include in the model the full range of parametric uncertainties. For each customer, we consider the payment time as a random variable. At any point in time the manager has a probability distribution for this random variable which he updates every time a new payment is received from the customer.

The distribution of this random variable can vary among businesses and customers. Bayesian updating can be performed in the general case by brute force computation. It can also be done by closed form analytical computation for some particular distributions. One choice for the distribution of pay-

[2] The identification of the appropriate prior probability distribution $f_T(t)$ can be a challenging problem in itself, especially when historical data for the payment times are not readily available. In the absence of data, a probability distribution that appears to offer a reasonable description of the anticipated payment times can be used to initiate the procedure. To this end, an informal discussion with the client on that issue may provide valuable information. Once data start accumulating (or in cases where such data are available), the validity of the initial distribution can be tested periodically through the use of histograms and standard goodness-of-fit statistical procedures. If the tests indicate that a different probability distribution (or the same distribution but with different parameters) provides a more accurate representation of the data, this new distribution should be used in the analysis for subsequent decisions.

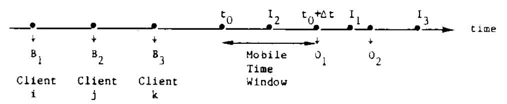

Fig. 1. Typical realization of cash flow over time. B: outstanding bills. I: inflows. O: Outflows (bills to be paid). t_0: current date.

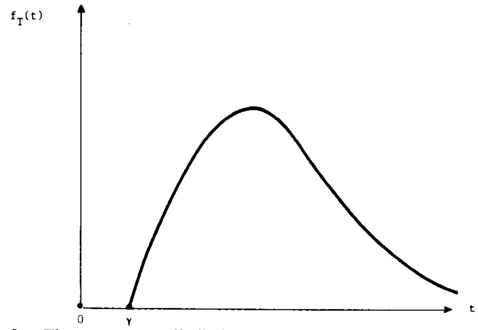

Fig. 2. Three-parameter Weibull probability density function for payment time of each client; shape parameter $\beta = 2$.

ment time for each customer, which is used for illustrative purposes in this paper, is the Weibull distribution with shape parameter $\beta = 2$ (see Appendices I and II for related computations). In actual applications, other distributions can be considered. However, the shape of the Weibull distribution for $\beta = 2$ seems to correspond to a plausible prior distribution of the payment times (see Fig. 2). It allows for a fixed minimum delay between sending the bill and receiving payment, and it leaves open the possibility that payment may never occur. In addition, as it is shown in Appendix II, the Weibull distribution is convenient since Bayesian updating is easy to perform analytically if one wants to keep all information concerning the uncertainty about the mean payment time.

Another choice is a two-parameter family of distributions, such as the Beta distribution, which better distinguishes between predictable and unpredictable customers, as well as between prompt and delinquent ones. Such a distribution allows more freedom in the characterization of the mean and variance of the time to payment. In other cases, a multimodal distribution can be used to represent variations in patterns of payment (e.g., preferred payment times each month). This can be accomplished by choosing, for the corresponding probability distribution of payment time, a linear combination of particular density functions.

Finally, another possibility is to use a discrete distribution by defining the time dimension in terms of business days, or a combination of a finite number of discrete distributions appropriately weighted, each representing the profile of a certain pure form of customer behavior, (e.g., immediate, prompt, semiprompt, payment, etc.). In order to choose the probability distribution that pertains to each client, the manager must examine the pattern of past payments by the client and select the distribution that best fits the existing data. If there is no data available for a client, the manager may choose a distribution that better fits available data for other, similar clients. In general, however, the issue of which probability distribution is the most appropriate and convenient representation of the payment time of a bill is open to further research, and is likely to depend upon the particular circumstances of a given firm.

Given that the manager must pay his first future bill at time $t_0 + \Delta t$, he is interested in the probability distribution of his cash balance at that time knowing 1) his current cash situation, and 2) the probability that his different customers pay their bills in the interim.

Let $C(t)$ denote the cash on hand at time t, 0_1 be the first future bill (outflow) and I_j, $j = 1, \cdots, n$ the amounts of outstanding bills (future inflows) at time t_0. If t_j is the time when bill I_j was sent to the customer, the probability that the payment I_j will be received by time $t_0 + \Delta t$ can be computed from

$$\begin{aligned} p_j = p[t \le t_0 + \Delta t | t \ge t_0] &= \frac{p[t_0 \le t \le t_0 + \Delta t]}{p[t \ge t_0]} \\ &= [F_j(t_0 + \Delta t - t_j) - F_j(t_0 - t_j)]/[1 - F_j(t_0 - t_j)] \end{aligned}$$

where $F_j(\cdot)$ is the cumulative distribution for the payment time of bill I_j. If we consider the payments of bills to be independent events, the cash on hand at time $t_0 + \Delta t$ is given by

$$C(t_0 + \Delta t) = C(t_0) - 0_1 + \sum_{j=1}^{n} I_j x_j$$

where

$$x_j = \begin{cases} 0 & \text{if payment is not received by } t_0 + \Delta t \\ & \quad \text{(probability } 1 - p_j) \\ 1 & \text{if payment is received by } t_0 + \Delta t \\ & \quad \text{(probability } p_j). \end{cases}$$

We can now compute $C(t_0 + \Delta t)$ for all possible combinations of payments or non-payments of the n bills in the time window Δt. We thus obtain a distribution of the cash on hand for the day the first payment from the business is due. If there is a nonzero probability that $C(t_0 + \Delta t) < 0$ (essentially if $C(t_0) < 0_1$), then the manager faces a potential cash flow problem.

The assumption of independence of customers' payment times may not hold in the case of nonstationarities where an external event, like a strike in the postal service, may slow down several payments simultaneously. In those cases the probability distribution of the payment times of different customers should still be conditionally independent but they will depend on the value of a state variable, Z, representing the state of external conditions. If one wants to include this effect in the design of the model, a conditional distribution $f_{T|Z}(t|z)$ should replace $f_T(t)$ in the computation of the probability that an outstanding bill will be received by the time the outflow is due. For example, if receipts are expected to be delayed due to a nonstationarity, the mean of the probability distribution $f_{T|Z}(t|z)$ will be larger than the mean of the distribution under normal conditions. For a given value of Z, the (discrete) probability distribution of the sum S of incoming payments $P(S|z)$ can be derived in the same straightforward way that it is used in the original model. If the uncertainty about Z is captured in the form of a probability density function $f_Z(z)$, then the unconditional distribution of S can be computed from

$$P(S) = \int P(S|z) f_Z(z)\, dz,$$

and the rest of the analysis remains unchanged.

IV. Short Term Decisions and Values of Information (Response Model)

If the signal model indicates a nonzero probability of a cash shortfall, the manager faces the possibility of a short term liquidity problem. Several options (e.g., doing nothing, bank credit, limiting employee hours, stretching payables, accelerating receivables) are possible. To choose the optimal one, we use a decision analytic approach based on the von Neumann axioms of rational choices [20], [21] in which we compute the effect of each alternative on the cash flow (i.e., how costly and how effective it is). The optimal alternative is the one that maximizes expected utility. The value of the information provided by the proposed monitoring system is the additional payment that would make the manager indifferent between adopting the best option available to him on the basis of the new information, and using his current cash management method [24].

This procedure assumes that the overall optimal strategy consists of a sequence of independent optimal decisions. That is, the relevant concern at t_0 is only the next outflow 0_1, and not subsequent outflows (0_2, etc.). This assumption may not hold for anticipated outflows that are so close to each other that any action taken for the first one affects subsequent ones. For example, a phone call to a customer at time t_0 to obtain a payment time affecting outflow 0_1 would also be of value with respect to 0_2. In this case, the myopic policy is a heuristic rather than an optimal solution. A less myopic and more complicated approach would be needed to obtain the exact optimal policy in those situations. Such an approach requires a longer time horizon in the decision tree, and consideration of combinations of contingent decisions based on dependencies among payments.

V. The Signal-Response Model

A. Operation

The model simulates the dynamics of information and cash flow variations in the following way. At the initial time, the business starts with a certain amount of cash. The manager first determines his preferences, assuming, for example, a constant risk attitude across the range of potential outcomes. He also specifies his options, in the event of a potential cash flow problem, and their costs and probabilities. He then initializes the parameters of his prior distribution for the payment time of each customer. He enters the schedule of planned outflows with the amounts involved and an indication of whether or not a particular payment can be delayed and at what cost. He also determines the time horizon (Δt) of his monitoring, i.e., the mobile time window that he considers necessary to make short term decisions if the possibility of a liquidity squeeze appears.

For each new day, the model requires encoding of the bills that are sent out, with the amount, and the customer's identity. If a payment is received from a customer, the model computes the payment time and proceeds to an updating (Bayesian and/or point estimate) of the probability distribution for future payment time. If a scheduled outflow appears in the monitoring window, the probability of a liquidity problem is computed using the signal model from Section II. If this probability is nonzero, the model requires resolution of the decision tree on the basis of the manager's risk attitude for the payment of this particular outflow on this particular day using the response model from Section III. The structure of this tree remains the same day after day. What varies is the probability at time t_0 that the cash on hand is insufficient at the time $t_0 + \Delta t$ when an outflow is scheduled to occur.

B. Implementation Issues

It has already been mentioned that cash flow models are particularly useful in situations in which there is high growth, few customers, and difficult (or expensive) access to additional capital. These three conditions are particularly characteristic of small and, especially, new firms.

In fact, our work with consulting and small high technology firms triggered our approach to cash flow management. We noticed that these firms generally use only one option when faced with cash flow decisions rather than changing options with changing circumstances. These options typically were very safe because managers did not have the analytic tools to address systematically their options. Also, these firms tended to develop payment stereotypes about their customers based on early experiences rather than on the full history of the relationship. We developed our implementation alternatives below based on these experiences and, in particular, our experiences with a small consulting firm.

There are several ways in which our method of cash flow management could be implemented. One way is as a standalone decision support system. The manager (or other employee) could run the model, and then act according to its recommendations. In this scenario, a customer signal model is needed with appropriate distributions for each customer's payment time (e.g., a Weibull or some other distribution), and a mechanism for periodic update of payment times, as described in Section II. The signal model would then need to be interfaced with either a custom or off-the-shelf decision analysis program for the response model.

A more elaborate alternative is to interface our model with the firm's accounts payable system. The accounts payable program would then be triggered when either 1) the signal model computes a zero probability of a cash shortfall, or 2) the response model indicates that payment of the bills is the optimal alternative when a nonzero probability for the cash shortfall exists. In this alternative, alteration of the customer data base of the accounts payable system to include the Bayesian or point estimates of the customer's payment history would provide a smooth way in which to capture the system's ability to learn about customers over time. This systematic learning is a particular advantage of our model over traditional, ad hoc estimates of future cash inflows.

A third possible alternative is an expert system designed to perform the same job on the basis of the heuristics used by

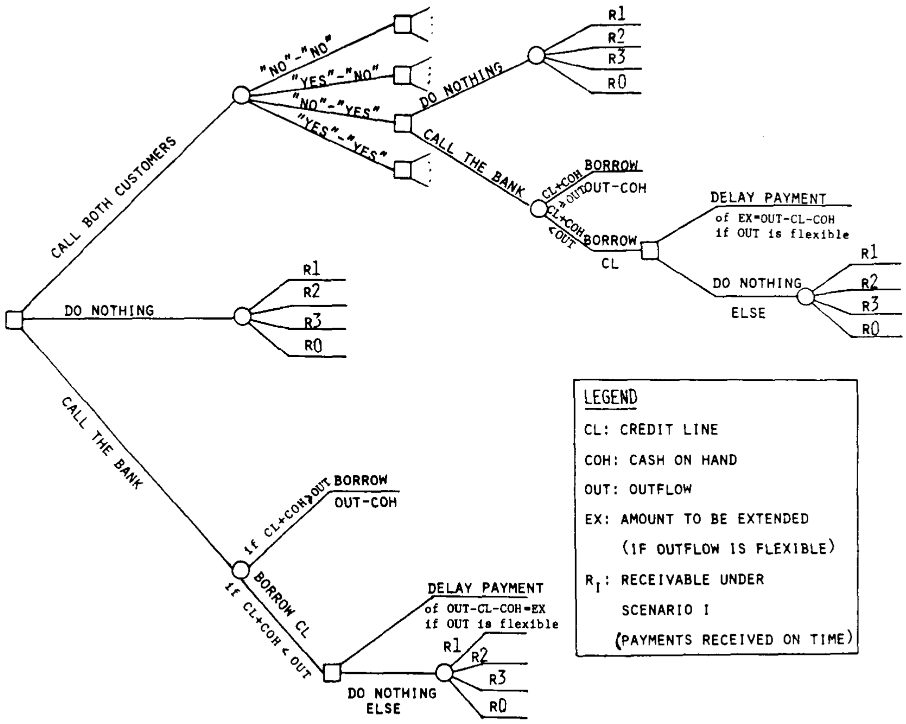

Fig. 3. Decision tree for management of short term cash flow problem.

an experienced finance manager. However, unless the expert system were designed to treat and update random variables in addition to the chaining of simple rules, it will not keep track of uncertainties as logically as decision analytic models. Hybrid systems, however, can be envisioned to combine rules of thumb and probabilistic analysis. They may be particularly suited to longer term cash flow monitoring including more strategic decisions than the tactics of short-term management.

C. An Illustrative Numerical Example

Assume that a small business (such as an engineering consulting or a high technology firm) has two major customers: a private sector business (customer 1) and the government (customer 2). The manager has one bill to pay at time $t_0 + \Delta t$, and a certain amount of cash on hand. If this cash is sufficient to pay his bill, there is no problem. If there is a nonzero probability of insufficient cash at time $t_0 + \Delta t$, we assume that he considers the following alternatives:

Alternative 1: Call his customers and inquire whether or not the check is in the mail (or will be shortly). The cost to the manager of doing so is a fixed cost of $50 for customer 2 (the government) and involves essentially the cost of his time; for customer 1 there is a cost of good will for bothering the customer. We assumed that this cost varies with the time elapsed since the bill was sent (low cost for an old bill, high cost for a recent one). In this example the cost was computed from the formula $100 - (t_0 - t_2 - \gamma)^2/9$ for $t_0 - t_1 - \gamma < 30$, where t_1 is the time that bill 1 was sent to the customer. If $t_0 - t_1 - \gamma \geq 30$, i.e., if the payment has been delayed for over 30 days, this cost was considered negligible. Other formulations of the cost of customer inquiries are also possible. If the customer says that the bill was paid early enough to be received by $t_0 + \Delta t$, we assume a probability 0.95 that the payment will be received on time; otherwise this probability is zero. For the latter answer, the manager can then consider borrowing from his bank on the basis of this additional information.

Alternative 2: Call the bank and activate an existing credit line. We assumed a cost of borrowing, a minimum borrowing time of one month, and a nominal annual interest rate of 15%.

Alternative 3: Delay the payment of the business's own bill by one month at time $t_0 + \Delta t$ if it is feasible. We assumed here also a cost of borrowing and an intangible cost of loss of credit rating.

Alternative 4: Do nothing and face the costs of running out of cash (and possibly, of bouncing a check) at time $t_0 + \Delta t$, with the probability computed according to the signal model as described in Section II.

Fig. 3 represents a decision tree for the choice of the optimal alternative at time t_0 in this simple illustrative case.

The manager is assumed to have a constant risk attitude characterized by a risk aversion coefficient, c. His utility function is therefore exponential, $u(x) = 1 - \exp(-cx)$, and his goal is to maximize his expected utility each time he makes a particular decision.

In this example we assume a Weibull distribution ($\beta = 2$) for payment times and a Gamma distribution for the random variable $U = 1/\delta^2$, where δ is the scale parameter of the

Weibull distribution that is subject to continuous updating. The initial data are as follows.

Customer 1: One bill out; amount: \$200C. The bill has been out for 18 days. The customer's payment history is based on 3 bills paid. The sum of the squared past payment times (λ_1) is 50. The current point estimate of δ_1, the customer 1's scale parameter of the Weibull distribution of the time of payment, is 4.514. The formulas for the computation of λ and the point estimate of δ are given in Appendix I.

Customer 2: One bill out; amount: \$1500. The bill has been out for 30 days. The customer's payment history is based on 2 bills paid. The sum of the squared past payment times (λ_2) is 250. The current point estimate of the scale parameter δ_2 is 11.284.

Cost of borrowing: 15% (annual nominal rate)
Cost of delaying payments: 18% (annual nominal rate)
Cost of running out of
cash by X dollars: $\$10 + X \times 0.0175$
Cash on hand: \$4000.
Time Window: 2 days.
Scheduled outflow: \$6500 due after 2 days. This payment can be delayed.
Minimum payment time: $\gamma = 4$ days.
Cost of calling customers: $\$[100-(14)^2/9]+50 = \$128.22$.
Risk aversion coefficient: $c = 10^{-3}$.

On this particular day, a cash outflow is scheduled within the time window of two days. The probability of receiving the payment from customer j, ($j = 1$ or 2) within $\Delta t = 2$ days is given by

$$p_{ej} = 1 - \exp\{-[2(t_0 - t_j - \gamma)(\Delta t) + (\Delta t)^2]/\hat{\delta}_j^2\}$$

using point estimates, where $t_0 - t_1 = 18$, $t_0 - t_2 = 30$, and by

$$p_{bj} = 1 - \left(\frac{\lambda_j}{\lambda_j + 2(t_0 - t_j - \gamma)(\Delta t) + (\Delta t)^2}\right)^{r_j}$$

using Bayesian updating with $r_1 = 4, r_2 = 3$ (see Appendix I). Given the history of customer's payments, the probability of receiving a payment in time from the first customer is 0.947 (using point estimates) or 0.957 (using Bayesian updating). For the second customer, this probability is respectively 0.572 (point estimate) and 0.659 (Bayesian probability).

Four scenarios can then be envisioned (none, or one, or both customers pay their bills) leading to the following probability distributions for the cash on hand, as estimated at time t_0 for two days later (R_i under scenario i as shown in Fig. 3).

Since there is a nonzero probability of a negative cash position, the signal model warns that there is a potential liquidity problem which requires the response model. In this particular case, given the risk attitude of the decision maker, the response model reveals that his best alternative is to do nothing at this time. This optimal decision happens to be the same for both methods of stochastic updating of information (Table I). This is not always the case, however, because the Bayesian estimate which keeps track of parametric uncertainties, may lead the very risk averse manager to different decisions than the point estimate method.

TABLE I
DAILY UPDATING OF PROBABILITY DISTRIBUTION FOR CASH ON HAND AT SCHEDULED FUTURE PAYMENT TIME

Payments Received	Prospective Amount of Cash	Point Estimate of Probability	Bayesian Updating of Probability
None	$R_1 = \$-2{,}500$	0.0225	0.0145
Customer 2	$R_2 = \$-1{,}000$	0.0301	0.0282
Customer 1	$R_3 = \$-\ 500$	0.4057	0.3260
Both	$R_4 = \$\ 1{,}000$	0.5417	0.6313

Compared to the conservative policy of systematically borrowing if the probability of a negative balance is nonzero, the above recommendation for this particular day leads to a decrease of the certain equivalent (in terms of costs) from \$31.25 to \$9.71 (69% decrease) for the point estimate, and from \$31.25 to \$7.73 (75% decrease) for the Bayesian updating. The computation of the expected utility of the do nothing alternative is shown on Table II. Recall that for the utility function $u(x) = 1 - \exp(-cx)$ the certain equivalent is $CE(u) = -[l_n(1-u)]/C$. For both methods, the manager is clearly happier. That is, the model outperforms the simplistic rule of automatic borrowing. Obviously, in this example, the decision maker is willing to take some risks since the cost of running out of cash is relatively low.

Note that on a different day with a change of data and additional experience about customers' payment delays, the system's recommendation could be quite different. For example, it could be to call the customers or to borrow the balance from the bank. Also, how much better than intuitive business practice the results are depends, as usual, on the skills of the analyst in modeling the firm's situation, and on the skills of the financial manager to process intuitively information and uncertainty in his decisions in the absence of analytical support.

VI. SUMMARY AND CONCLUSION

We have proposed a model of cash flow management based upon a risk analysis approach to warning systems. This method provides a systematic basis to construct a real time cash flow monitoring system, particularly adapted to firms involved in high technology products, including small and new firms and firms facing high growth, few customers, or expensive financing. The proposed approach has several advantages over inventory and linear programming approaches and over the traditional procedures of firms. First, it retains the stochastic nature of the problem and treats explicitly the uncertainties about the time of payments from the clients. Secondly, the system is an intelligent one in that it keeps track of the information and it learns over time. It is not dependent upon the potentially outdated biases of the decision maker. As any decision analytic model, it requires that the analyst spend a considerable amount of time identifying the options that the manager would like to consider to resolve short term cash flow problems, quantifying the manager's risk attitude, and initializing his probabilistic assessment of payment delays. Moreover, the method requires the specification (through knowledge or assumption) of the appropriate probability distribution of each

TABLE II
COMPUTATION OF CERTAIN EQUIVALENTS OF THE DO NOTHING ALTERNATIVE

Payments Received	Cash Balance	Shorage Cost	Utility	Point Estimate of Probability	Bayesian Updating of Probability
None	R_1 = \$-2,500	\$53.75	-0.0552	-0.0225	0.0145
Customer 2	R_2 = \$-1,000	\$27.50	-0.0279	0.0301	0.0282
Customer 1	R_3 = \$ -500	\$18.75	-0.0183	0.4057	0.3260
Both	R_4 = \$ 1,000	0	0	0.5417	0.6313
			Expected Utility:	-0.0098	-0.0078

customer's payment time. However, after initialization, it can be implemented on a microcomputer and used on a daily basis by the individual manager with minimal data input.

VII. Appendix I

Illustration of the Method Using a Weibull Distribution for Payment Time

Let X be the Weibull random variable representing the payment time after sending the bill to a certain client. The general density function for a Weibull distribution is of the form

$$f_X(x) = \frac{\beta}{\delta}\left(\frac{x-\gamma}{\delta}\right)^{\beta-1} \exp\left[-\left(\frac{x-\gamma}{\delta}\right)^{\beta}\right] \quad x \geq \gamma$$
$$= 0 \qquad \text{otherwise,}$$

where β, δ, $\gamma > 0$. The parameter γ is a shift factor that represents the minimum number of days between sending the bill to any client and receiving the payment. The scale parameter δ is the parameter that characterizes the customer's time of payment. (In general, each customer is characterized by a different δ). For a shape parameter $\beta = 2$ the probability density function of X becomes

$$f_X(x) = \frac{2(x-\gamma)}{\delta^2} \exp\left[-\left(\frac{x-\gamma}{\delta}\right)^2\right] \quad x \geq \gamma$$
$$= 0 \qquad \text{otherwise.}$$

The mean of this distribution is $\gamma + (\sqrt{\pi}/2)\delta$. The cumulative distribution is

$$F_X(x) = 1 - \exp\left[-\left(\frac{x-\gamma}{\delta}\right)^2\right] \quad x \geq \gamma$$
$$= 0 \qquad \text{otherwise.}$$

Point Estimate of δ

The average $\bar{X}(n)$ of the n past observations of the payment time of the client is an unbiased estimator of the mean of X. An estimate $\hat{\delta}(n)$ of δ is therefore obtained by the following equations:

$$\bar{X}(n) = \gamma + \frac{\sqrt{\pi}}{2}\hat{\delta}(n)$$
$$\Rightarrow \hat{\delta}(n) = \frac{2(\bar{X}(n)-\gamma)}{\sqrt{\pi}}.$$

After a new piece of information $X(n+1)$, $\hat{\delta}$ is updated as

$$\hat{\delta}(n+1) = \frac{2(\bar{X}(n+1)-\gamma)}{\sqrt{\pi}}$$
$$= \frac{n}{n+1}\hat{\delta}(n) + \frac{2(X(n+1)-\gamma)}{\sqrt{\pi}(n+1)}.$$

Bayesian Updating of $1/\delta^2$

Assume now that in order to make a decision that reflects completely the manager's risk attitude, one decides to consider the full range of uncertainties concerning δ. Define the random variable $U = 1/\delta^2$ for which we can find conjugate periods (Benjamin and Cornell, 1970). In order to be able to do this updating, the prior distribution assumed for U is a Gamma distribution of parameters $r > 0$ and $\lambda > 0$, which is characterized by the following density function:

$$f_U(u) = \frac{\lambda^r u^{r-1} e^{-\lambda u}}{(r-1)!} \quad \text{for } u \geq 0$$
$$= 0 \qquad \text{otherwise.}$$

Let n be the number of existing observations and $\{X_j\}$ be the set of n previous observations for the given client. Then, the parameters of $f_U(u)$ are

$$r = n + 1$$
$$\lambda = \sum_{j=1}^{n}(X_j - \gamma)^2.$$

The implicit assumption in the above equations is that of an initial diffuse state of information characterized by $r = 1$ and $\lambda = 0$ prior to the first observation. Given one additional piece of information (i.e., a new payment has arrived from the client after a time X), the posterior distribution for U after updating ($f'_U(u)$) is also a Gamma distribution of parameters r' and λ', which are simply (see proof in Appendix II):

$$r' = r + 1$$
$$\lambda' = \lambda + (X - \gamma)^2.$$

Probability that a Given Bill will be Paid Between t_0 and $t_0 + \Delta t$

For a specific δ, let t_c be the time at which a bill was sent to the customer, and t the time at which the payment of this bill is received. Then, there are three possible cases:

1) $t_0 - t_c \geq \gamma$.

Then,

$$p[t \leq t_0 + \Delta t | t \geq t_0, \delta] = \frac{p[t_0 \leq t \leq t_0 + \Delta t|\delta]}{p[t \geq t_0|\delta]}$$

$$= \frac{\left[1 - \exp\left[\left(\frac{t_0 - t_c + \Delta t - \gamma}{\delta}\right)^2\right]\right] - \left[1 - \left[\exp\left(\frac{t_0 - t_c - \gamma}{\delta}\right)^2\right]\right]}{\exp\left[-\left(\frac{t_0 - t_c - \gamma}{\delta}\right)^2\right]}$$

$$= 1 - \exp\left[-\left(\frac{t_0 - t_c + \Delta t - \gamma}{\delta}\right)^2 + \left(\frac{t_0 - t_c - \gamma}{\delta}\right)^2\right]$$

$$= 1 - \exp\left[-\frac{2(t_0 - t_c - \gamma)\,\Delta t + (\Delta t)^2}{\delta^2}\right].$$

2) $t_0 - t_c \leq \gamma$, and $t_0 + \Delta t - t_c \geq \gamma$.
Then,

$$p[t \leq t_0 + \Delta t | t \geq t_0, \delta] = p[t \leq t_0 + \Delta t|\delta]$$
$$= 1 - \exp\left[-\frac{(t_0 + \Delta t - t_c - \gamma)^2}{\delta^2}\right].$$

3) $t_0 + \Delta t - t_c \leq \gamma$
Then,

$$p[t \leq t_0 + \Delta t | t \geq t_0, \delta] = 0.$$

Therefore, we can write

$$p[t \leq t_0 + \Delta t | t \geq t_0, \delta] = 1 - \exp[-T/\delta^2]$$

where

$$T = \begin{cases} 0 & \text{for } t_0 - t_c < \gamma - \Delta t \\ (t_0 + \Delta t - t_c - \gamma)^2 & \text{for } \gamma - \Delta t \leq t_0 - t_c \leq \gamma \\ 2(t_0 - t_c - \gamma)\,\Delta t + (\Delta t)^2 & \text{for } t_0 - t_c > \gamma. \end{cases}$$

The probability that the payment arrives in time using a point estimate ($\hat{\delta}$) of δ is thus

$$p_e = 1 - \exp[-T/\hat{\delta}^2].$$

The Bayesian probability that the payment arrives in time using the updated Gamma distribution of $U = 1/\delta^2$ is

$$p_b = \int_0^\infty (1 - e^{-Tu}) f_U(u)\, du$$

$$p_b = 1 - \left(\frac{\lambda}{\lambda + T}\right)^r.$$

VIII. Appendix II

$U = 1/\delta^\beta$ After Bayesian Updating

The prior distribution for $U = 1/\delta^\beta$ is assumed to be Gamma (r, λ):

$$f_U(u) = \frac{\lambda^r u^{r-1} e^{-\lambda u}}{(r-1)!} \quad u \geq 0$$
$$= 0 \qquad \text{otherwise.}$$

with parameters

$$\lambda = \sum_{j=1}^{n} (X_j - \gamma)^\beta$$

$$r = n + 1$$

where n is the number of existing observations. The density function $f_U(u)$ can be written as

$$f_U(u) = N u^{r-1} e^{-\lambda u}, \qquad u \geq 0$$

in which

$$N = \frac{\lambda^r}{(r-1)!}.$$

We want to show that after one additional observation (X), the posterior distribution for U is also Gamma (r', λ'), i.e.,

$$f'_U(u) = \frac{[(\lambda + (X - \gamma)^\beta)]^{r+1}}{r!} u^r \exp[-(\lambda + (X - \gamma)^\beta)u] \quad u \geq 0 \tag{1}$$
$$= 0 \qquad \text{otherwise,}$$

or

$$f'_U(u) = \frac{\lambda'^{r'} u^{r'-1} e^{-\lambda' u}}{(r'-1)!} \quad u \geq 0$$
$$= 0 \qquad \text{otherwise.}$$

The parameters of this distribution are

$$\lambda' = \lambda + (X - \gamma)^\beta$$

$$r' = r + 1.$$

In order for this to be true, we must have that

$$f'_U(u) = N' f(X|u) f_U(u),$$

where the normalizing factor N' depends on X but not upon u. The right side of the above equation can be rewritten as

$$N' f(X|u) f_U(u) = N N' u^{r-1} e^{-\lambda u} \beta u (X - \gamma)^{\beta - 1} \exp[-u(X - \gamma)^\beta]$$
$$= \beta N' N u^r (X - \gamma)^{\beta - 1} \exp[-(\lambda + (X - \gamma)^\beta)u]. \tag{2}$$

From (1) and (2):

$$N' = \frac{[\lambda + (X - \gamma)^{\beta}]^{r+1}}{\beta(X - \gamma)^{\beta - 1} r \lambda^{r}}$$

which is not a function of u. Q.E.D.

ACKNOWLEDGMENT

The authors would like to thank Professor H. Riggs and Professor W. Hausman for guidance and advice in the formulation of this problem, and Professor S. Winterstein for his comments on the mathematical model.

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