Revêtements Étales and the Fundamental Group (SGA I)

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Translation Notes

The original can be found on the arXiv.

CHAPTER 1

Étale morphisms

To simplify the exposition we assume that all preschemes in the following are locally Noetherian (at least, starting from section 2). 1

1. Basics of differential calculus

Let X be a prescheme on Y, and $\Delta_{X/Y}$ the diagonal morphism $X \to X \times_Y X$. This is an immersion, and thus a closed immersion of X into an open subset V of $X \times_Y X$. Let $\mathscr{I}_X$ be the ideal of the closed sub-prescheme corresponding to the diagonal in V (N.B. if one really wishes to do things intrinsically, without assuming that X is separated over Y — a misleading hypothesis — then one should consider the set-theoretic inverse image of $\mathcal{O}_{X\times X}$ in X and denote by $\mathscr{I}_X$ the augmentation ideal in the above...). The sheaf $\mathscr{I}_X/\mathscr{I}_X^2$ can be thought of as a quasi-coherent sheaf on X, which we denote by $\Omega^1_{X/Y}$. It is of finite type if $X \to Y$ is of finite type. It behaves well with respect to a change of base $Y' \to Y$. We also introduce the sheaves $\mathcal{O}_{X\times_Y X}/\mathscr{I}_X^{n+1} = \mathscr{P}^n_{X/Y}$, which are sheaves of *rings* on X, giving us preschemes denoted by $\Delta^n_{X/Y}$ and called the *n-th infinitesimal neighbourhood of* X/Y. The polysyllogism is entirely trivial, even if rather long[1]; it seems wise to not discuss it until we use it for something helpful, with smooth morphisms.

2. Quasi-finite morphisms

Proposition 2.1. *Let $A \to B$ be a local homomorphism (N.B. all rings are now Noetherian) and $\mathfrak{m}$ the maximal ideal of A. Then the following conditions are equivalent:*

(i) *$B/\mathfrak{m}B$ is of finite dimension over $k = A/\mathfrak{m}$.*
(ii) *$\mathfrak{m}B$ is [...] and $B/\mathfrak{r}(B) = k(B)$ is an extension of $k = k(A)$.*
(iii) *The completion $\hat{B}$ of B is finite over the completion $\hat{A}$ of A.*

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We then say that B *is quasi-finite* over A. A morphism $f\colon X \to Y$ is said to be quasi-finite in x (or the Y-prescheme f is said to be quasi-finite in x) if $\mathcal{O}_x$ is quasi-finite over $\mathcal{O}_{f(x)}$. [...]A morphism is said to be quasi-finite if it is quasi-finite in each point[2].

Corollary 2.2. *If A is complete then quasi-finiteness is equivalent to finiteness.*

We could give the usual polysyllogism (i), (ii), (iii), (iv), (v) for quasi-finite morphisms, but that doesn't seem necessary here.

[1] cf. EGA IV 16.3.
[2] In EGA II 6.2.3 we further suppose that f is of finite type.

3. Unramified morphisms

PROPOSITION 3.1. *Let $f\colon X \to Y$ be a morphism of finite type, $x \in X$, and $y = f(x)$. Then the following conditions are equivalent:*

(i) *$\mathcal{O}_x/\mathfrak{m}_y\mathcal{O}_x$ is a finite separable extension of $k(y)$.*
(ii) *$\Omega^1_{X/Y}$ is null in x.*
(iii) *The diagonal morphism $\Delta_{X/Y}$ is an open immersion on a neighbourhood of x.*

For the implication (i) $\implies$ (ii), we are brought by Nakayama to the case where $Y = \operatorname{Spec}(k)$ and $X = \operatorname{Spec}(k')$, where it is well known and otherwise trivial by the definition of separable; (ii) $\implies$ (iii) comes from a nice and easy characterisation of open immersions, using Krull; (iii) $\implies$ (i) follows as well from reducing to the case where $Y = \operatorname{Spec}(k)$ and the diagonal morphism is everywhere an open immersion. One must then prove that X is finite [...]and this leads us to consider the case where k is algebraically closed. But then every closed point of X is isolated (since it is identical to the inverse image of the diagonal by the morphism $X \to X \times_k X$ defined by x), whence X is finite. We can thus suppose that X reduces to a point, of the ring A, and so $A \otimes_k A \to A$ is an isomorphism, hence $A = k$. $\square$

DEFINITION 3.2. (a) *We then say that f is* unramified *in x, or that X is unramified in x on Y.*
(b) *Let $A \to B$ be a local homomorphism. We say that it is* unramified*, or that B is a local* unramified *algebra on A, if $B/mathfrakr(A)B$ is a finite separable extension of $A/\mathfrak{r}(A)$, i.e. if $\mathfrak{r}(A)B = \mathfrak{r}(B)$ and $k(B)$ is a separable extension of $k(A)$*[3].

3

REMARK. The fact that B is unramified over A [...]. Unramified implies quasi-finite.

COROLLARY 3.3. *The set of points where f is unramified is open.*

COROLLARY 3.4. *Let X' and X be two preschemes of finite type over Y, and $g\colon X' \to X$ a Y-morphism. If X is unramified over Y then the graph morphism $\Gamma_g\colon X' \to X \times_Y X$ is an open immersion.*

In effect, this is the inverse image of the diagonal morphism $X \to X \times_Y X$ by

$$g \times_Y \mathrm{id}_{X'}\colon X' \times_Y X \to X \times_Y X.$$

One can also introduce the annihilator ideal $\mathfrak{d}_{X/Y}$ of $\Omega^1_{X/Y}$, called the different ideal of X/Y; it defines a closed sub-prescheme of X which, set theoretically, is the set of point where X/Y is ramified, i.e. not unramified.

PROPOSITION 3.5. (i) *An immersion is ramified.*
(ii) *The composition of two ramified morphisms is also ramified.*
(iii) *Base extension of a ramified morphisms is also ramified.*

We don't care so much about (ii) or (iii) (the second seems more entertaining to me). We can also [...]by giving a few punctual comments; it is no more general than in appearance (except in the case of definition b) and also boring. We obtain, as per usual, the corollaries:

[3]Cf. regrets in III 1.2.

COROLLARY 3.6. (iv) *The cartesian product of two unramified morphisms is unramified.*

(v) *If gf is unramified then so too is f.*

(vi) *If f is unramified then so too is f_{red}.*

PROPOSITION 3.7. *Let $A \to B$ be a local homomorphism and suppose that the residue extension $k(B)/k(A)$ is trivial with $k(A)$ algebraically closed. In order for B/A to be unramified it is necessary and sufficient that $\hat{B}$ be (as an $\hat{A}$-algebra) a quotient of $\hat{A}$.*

REMARK.

- In the case where we don't suppose that the residue extension is trivial, we can bring ourselves to the case where it is by taking a suitable finite flat extension on A which destroys the aforementioned extension.
- Consider the example where A is the local ring of an ordinary double point of a curve, and B a point of the normalised curve: then $A \subset B$, B is unramified over A with trivial residue extension, and $\hat{A} \to \hat{B}$ is **4**
surjective but *not injective*. We are thus going to strengthen the notion of unramifiedness.

4. Étale morphisms. Étale covers.

CHAPTER 6

Fibred Categories and Descent

0. Introduction

Contrary to what was said in the introduction to the previous chapter, it was impossible to make the descent in the category of preschemes, even in special cases, without having developed the language of descent in the general categories beforehand with sufficient care.

The notion of "descent" provides the general framework for all processes of "re-gluing" objects, and therefore of "gluing" of categories. The most classic case of re-gluing relates to the data of a topological space X and of a cover of X by open sets X_i; if we consider for all i a space fibre (say) E_i above X_i, and for every pair (i,j) an isomorphism f_{ji} of $E_i|X_{ij}$ on $E_j|X_{ij}$ (where we put $X_{ij} = X_i \cap Xj$), satisfying a well-known condition of transitivity (which we write in a short way $f_{kj}f_{ji} = f_{ki}$), we know that there is a fibrous space E on X, defined as isomorphism with the condition that we have isomorphisms $f_i \colon E|X_i \xrightarrow{\simeq} E_i$, satisfying the relations $f_{ji} = f_j f_i^{-1}$ (with the usual writing abuse). Let X' be the sum space of X_i, which is therefore a fibrous space above X (i.e. provided with a continuous mapping $X' to X$ of $\mathscr{E}'$ on $X'' = X' \times_X X'$, the collation condition can then be written as an identity between isomorphisms of fibrous spaces $\mathscr{E}_1'''$ and $\mathscr{E}_3'''$ on the triple fiber product $X''' = X' \times_X X' \times_X X'$ (where E_i''' designates the inverse image of E' on X''' by the canonical projection of index i). The construction of E, from E' and f, is a typical case of "descent". Moreover, starting from a fibered space E on X, one says that X is "locally trivial", of fiber F, if there exists an open covering (X_i) of X such that the $E|X_i$ are isomorphic to $F \times X_i$, or what amounts to the same, such as the inverse image E' of E on $X' = \coprod_i Xi$ is isomorphic to $X' \times F$.

Thus, the notion of "gluing" objects like the "location" of a property, are related to the study of certain types of "basic changes" $X' \to X$. In algebraic geometry, many other types of basic change, and in particular morphisms $X' \to X$ Thus, the notion of "gluing" objects like that of "localization" of a property and, are related to the study of certain types of "basic changes" $X' \to X$. In algebraic geometry, many other types of change of base, including the flattened $X' \to X$ morphisms must be considered as corresponding to a "localization" process with respect to the preschemes, or other objects, "above" X. This type of localization is used just as much as the simple topological localization (which is a special case besides). The same is true (to a lesser extent) in Analytical Geometry.

Most demonstrations, reducing to verifications, are omitted or simply sketched out; in this case we specify the less obvious diagrams that are introduced into a demonstration.

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Call, to abbreviate, pseudofunctor of $\mathscr{E}^{op}$ in Cat (it should be said, pseudo-normalized functor), a set of data a), b), c) as above, satisfying the conditions A ') and B). In the previous section, we have associated a pseudofunctor $\mathscr{E}^{op} \to \mathsf{Cat}$ with a split fibration normalized on $\mathscr{E}$, here we will indicate the inverse construct. We will leave the reader to verify most of the details, as well as the fact that these constructions are "inverse" to each other. Specifically, we should consider the pseudofunctors $\mathscr{E}^{op} \to \mathsf{Cat}$ as the objects of a new category, and show that our constructions provide equivalences, quasi-inverses one from the other, between the latter and the category of cloven categories above $\mathscr{E}$, defined in the preceding section.

We put

$$\mathscr{F}_\circ = \coprod_{S \in \mathrm{Ob}(\mathscr{E})} \mathrm{Ob}(\mathscr{F}(S)),$$

together sum of sets $\mathrm{Ob}(\mathscr{F}(S))$ (N.B. we will note here $\mathscr{F}(S)$ and not $\mathscr{F}_S$ the value in the object S of $\mathscr{E}$ of the pseudofunctor gives, to avoid confusion of notation thereafter). So we have an obvious mapping:

$$p_\circ \colon \mathscr{F}_\circ \to \mathrm{Ob}\mathscr{E}.$$

Let

$$\overline{\xi} = (S, \xi),\ \overline{\eta} = (T, \eta) \qquad (\text{where } \xi \in \mathrm{Ob}\mathscr{F}(S),\ \eta \in \mathrm{Ob}\mathscr{F}(T))$$

two elements of $\mathscr{F}_\circ$, and $f \in \mathrm{Hom}(T, S)$, we let

$$h_f(\overline{\eta}, \overline{\xi}) = \mathrm{Hom}_{\mathscr{F}(T)}(\eta, f^*(\xi)).$$

If we also have a morphism $g \colon U \to T$ in $\mathscr{E}$, and a $\zeta \in \mathrm{Ob}\mathscr{F}(U)$, we define a function, denoted $(u, v) \mapsto u \circ v$:

$$h_f(\overline{\eta}, \overline{\xi}) \times h_g(\overline{\zeta}, \overline{\eta}) \to h_{fg}(\overline{\zeta}, \overline{\xi}),$$

i.e. a function

$$Hom_{\mathscr{F}(T)}(\eta, f^*(\xi)) \times \mathrm{Hom}_{\mathscr{F}(U)}(\zeta, g^*(\eta)) \to \mathrm{Hom}_{\mathscr{F}(U)}(\zeta, (fg)^*(\xi)),$$

for the formula

$$u \circ v = c_{f,g}(\xi) \cdot g^*(U) \cdot v,$$

i.e. $u \circ v$ is the composite of the sequence

$$\zeta \xrightarrow{u} g^*(\eta) \xrightarrow{g^*(u)} g^* f^*(\xi) \xrightarrow{c_{f,g}(\xi)} (fg)^*(\xi).$$

On the other hand

$$h(\overline{\eta},\overline{\xi}) = \coprod_{f\in \mathrm{Hom}(T,S)} h_f(\overline{\eta},\overline{\xi}),$$

and the previous couplings define couplings

$$h(\overline{\eta},\overline{\xi}) \times h(\overline{\zeta},\overline{\eta}) \to h(\overline{\zeta},\overline{\xi}),$$

while the definition of $h(\overline{\eta},\overline{\xi})$ implies an obvious function:

$$p_{\overline{\eta},\overline{\xi}} \colon h(\overline{\eta},\overline{\xi}) \to \mathrm{Hom}(T,S).$$

That said, we check the following points:

(1) The composition between elements of $h(\overline{\eta},\overline{\xi})$ is *associative*.

(2) For all $\overline{\xi} = (\xi, S)$ in $\mathscr{F}_\circ$, consider the identity element of

$$h_{id_S}(\overline{\xi},\overline{\xi}) = \mathrm{Hom}_{\mathscr{F}(S)}(id_S^*(\xi),\xi) = \mathrm{Hom}_{\mathscr{F}(S)}(\xi,\xi),$$

and its image in $h(\overline{\xi},\overline{\xi})$. This object is a unit on the left and right for the composition between elements of $h(\overline{\eta},\overline{\xi})$.

This already shows that we get a category $\mathscr{F}$, by posing

$$\mathrm{Ob}\mathscr{F} = \mathscr{F}_\circ, \quad \mathrm{Fl}\mathscr{F} = \coprod_{\overline{\xi},\overline{\eta}\in\mathscr{F}_\circ} h(\overline{\eta},\overline{\xi}).$$

(N.B. we can not simply take for $\mathrm{Fl}\mathscr{F}$ the union of the sets $h(\overline{\eta},\overline{\xi})$, because these are not necessarily disjoint.) Furthermore:

(3) The functions $p_\circ \colon \mathrm{Ob}\mathscr{F} \to \mathrm{Ob}\mathscr{E}$ and $p_1 = (p_{\overline{\eta},\overline{\xi}} \colon \mathrm{Fl}\mathscr{F} \to \mathrm{Fl}\mathscr{E}$ define a *functor* $p \colon \mathscr{F} \to \mathscr{E}$. In this way, F becomes a category on E, moreover the obvious function $h_f(\overline{\eta},\overline{\xi}) \to \mathrm{Hom}(\overline{\eta},\overline{\xi})$ induces a *bijection*

$$h_f(\overline{\eta},\overline{\xi}) \xrightarrow{\sim} \mathrm{Hom}_f(\overline{\eta},\overline{\xi}).$$

(4) The obvious functions

$$\mathrm{Ob}\mathscr{F}(S) \to \mathscr{F}_\circ = \mathrm{Ob}\mathscr{F}, \quad \mathrm{Fl}\mathscr{F}(S) \to \mathrm{Fl}\mathscr{F},$$

or the second is defined by the obvious functions

$$\mathrm{Hom}_{\mathscr{F}(S)}(\xi,\xi') = h_{id_S}(\overline{\xi},\overline{\xi}') \to \mathrm{Hom}(\overline{\xi},\overline{\xi}')$$

define an *isomorphism*

$$i_S \colon \mathscr{F}(S) \xrightarrow{\sim} \mathscr{F}_S.$$

(5) For every object $\overline{\xi} = (S,\xi)$ of $\mathscr{F}$, and every morphism $f \colon T \to S$ of $\mathscr{E}$, let us consider the element $\overline{\eta} = (T,\eta)$ of $\mathscr{F}_T$, with $\eta = f^*(\xi)$, and the element $\alpha_f(\xi)$ of $\mathrm{Hom}(\overline{\eta},\overline{\xi})$, image of $id_{f^*(\xi)}$ by the morphism $\mathrm{Hom}_{\mathscr{F}(T)}(f^*(\xi), f^*(\xi)) = h_f(\overline{\eta},\overline{\xi}) \to \mathrm{Hom}_f(\overline{\eta},\overline{\xi})$. *This element is cartesian, and that's the identity in ξ if $f = id_S$*, in other words, the whole $\alpha_f(\xi)$ defines a normalized cleavage of F over E. Moreover, by construction, we have commutativity in the functor diagram

$$\begin{tikzcd}
\mathscr{F}(S) \arrow[r, "f^*"] \arrow[d, "i_S"'] & \mathscr{F}(T) \arrow[d, "i_T"] \\
\mathscr{F}_S \arrow[r, "f^*_{\mathscr{F}}"'] & \mathscr{F}_T
\end{tikzcd}$$

where $f^*_{\mathscr{F}}$ is the inverse image functor by f, relative to the cleavage considered on $\mathscr{F}$. Finally:

(6) the homomorphisms $c_{f,g}$ given by the pseudofunctor are transformed, by the isomorphisms i_S, in functorial homomorphisms $c_{f,g}$ associated with the cleavage of $\mathscr{F}$.

We confine ourselves to giving the verification of 1) (which is, if possible, less trivial than the others). It is enough to prove the associativity of the composition between the objects of sets of the form $h_f(\overline{\eta}, \overline{\xi})$. So let's consider in $\mathscr{E}$ morphisms

$$S \xleftarrow{f} T \xleftarrow{g} U \xleftarrow{h} V$$

and the objects

$$\xi, \eta, \zeta, \tau$$

in $\mathscr{F}(S), \mathscr{F}(T), \mathscr{F}(U), \mathscr{F}(V)$, finally some elements

$$\begin{aligned} u &\in h_f(\overline{\eta}, \overline{\xi}) = \mathrm{Hom}_{\mathscr{F}(T)}(\eta, f^*(\xi)) \\ v &\in h_g(\overline{\zeta}, \overline{\eta}) = \mathrm{Hom}_{\mathscr{F}(U)}(\zeta, g^*(\eta)) \\ w &\in h_h(\overline{\tau}, \overline{\zeta}) = \mathrm{Hom}_{\mathscr{F}(V)}(\tau, h^*(\zeta)). \end{aligned}$$

We want to prove the formula

$$(u \circ v) \circ w = u \circ (v \circ w),$$

which is an equality in $\mathrm{Hom}_{\mathscr{F}(V)}(\tau, (fgh)^*(\xi))$. By virtue of the definitions both members of this equality are obtained by composition following the superior contour and bottom of the diagram below:

$$\begin{tikzcd}
\tau \arrow[r, "w"] \arrow[rrd, "v \circ w"'] & h^*(\zeta) \arrow[r, "h^*(v)"] \arrow[rrr, bend left, "h^*(u \circ v)"] & h^*g^*(\eta) \arrow[r, "h^*g^*(u)"] \arrow[d, "c_{g,h}(\eta)"'] & h^*g^*f^*(\xi) \arrow[r, "h^*(c_{f,g}(\xi))"] \arrow[d, "c_{g,h}(f^*(\xi))"] & h^*(fg)^*(\xi) \arrow[d, "c_{fg,h}(\xi)"] \\
& & (gh)^*(\eta) \arrow[r, "(gh)^*(\eta)"'] & (gh)^*f^*(\xi) \arrow[r, "c_{f,gh}(\xi)"'] & (fgh)^*(\xi)
\end{tikzcd}$$

Now the middle square is commutative because $c_{g,h}$ is a functorial homomorphism, and the square on the right is commutative under condition B) for a pseudofunctor. Hence the desired result.

Of course, it remains to be precise, when the pseudofunctor envisaged comes from already a normalized cloven category $\mathscr{F}'$ over $\mathscr{E}$, how do we get an natural isomorphism between $\mathscr{F}'$ and $\mathscr{F}$. We leave the details to the reader.

We also allow the reader to interpret, in terms of pseudofunctors, the notion of inverse image of a split category $\mathscr{F}$ over $\mathscr{E}$ by a change of base functor $\mathscr{E}' \to \mathscr{E}$.