Revêtements Étales and the Fundamental Group (SGA I)

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Translation Notes

The original can be found on the arXiv. I don't know French, I just used Google translate and some knowledge of the subject matter.

CHAPTER 1

Étale morphisms

To simplify the exposition we assume that all preschemes in the following are locally Noetherian (at least, starting from section 2).

1. Basics of differential calculus

Let X be a prescheme on Y, and $\Delta_{X/Y}$ the diagonal morphism $X \to X \times_Y X$. This is an immersion, and thus a closed immersion of X into an open subset V of $X \times_Y X$. Let $\mathscr{I}_X$ be the ideal of the closed sub-prescheme corresponding to the diagonal in V (N.B. if one really wishes to do things intrinsically, without assuming that X is separated over Y — a misleading hypothesis — then one should consider the set-theoretic inverse image of $\mathcal{O}_{X\times X}$ in X and denote by $\mathscr{I}_X$ the augmentation ideal in the above…). The sheaf $\mathscr{I}_X/\mathscr{I}_X^2$ can be thought of as a quasi-coherent sheaf on X, which we denote by $\Omega^1_{X/Y}$. It is of finite type if $X \to Y$ is of finite type. It behaves well with respect to a change of base $Y' \to Y$. We also introduce the sheaves $\mathcal{O}_{X\times_Y X}/\mathscr{I}_X^{n+1} = \mathscr{P}^n_{X/Y}$, which are sheaves of *rings* on X, giving us preschemes denoted by $\Delta^n_{X/Y}$ and called the *n-th infinitesimal neighbourhood of* X/Y. The polysyllogism is entirely trivial, even if rather long[1]; it seems wise to not discuss it until we use it for something helpful, with smooth morphisms.

2. Quasi-finite morphisms

PROPOSITION 2.1. *Let $A \to B$ be a local homomorphism (N.B. all rings are now Noetherian) and $\mathfrak{m}$ the maximal ideal of A. Then the following conditions are equivalent:*

(i) *$B/\mathfrak{m}B$ is of finite dimension over $k = A/\mathfrak{m}$.*
(ii) *$\mathfrak{m}B$ is […] and $B/\mathfrak{r}(B) = k(B)$ is an extension of $k = k(A)$.*
(iii) *The completion $\hat{B}$ of B is finite over the completion $\hat{A}$ of A.*

We then say that B *is quasi-finite* over A. A morphism $f\colon X \to Y$ is said to be quasi-finite in x (or the Y-prescheme f is said to be quasi-finite in x) if $\mathcal{O}_x$ is quasi-finite over $\mathcal{O}_{f(x)}$. […]A morphism is said to be quasi-finite if it is quasi-finite in each point[2].

COROLLARY 2.2. *If A is complete then quasi-finiteness is equivalent to finiteness.*

We could give the usual polysyllogism (i), (ii), (iii), (iv), (v) for quasi-finite morphisms, but that doesn't seem necessary here.

[1] cf. EGA IV 16.3.
[2] In EGA II 6.2.3 we further suppose that f is of finite type.

3. Unramified morphisms

Proposition 3.1. *Let $f\colon X \to Y$ be a morphism of finite type, $x \in X$, and $y = f(x)$. Then the following conditions are equivalent:*

(i) *$\mathcal{O}_x/\mathfrak{m}_y\mathcal{O}_x$ is a finite separable extension of $k(y)$.*

(ii) *$\Omega^1_{X/Y}$ is null in x.*

(iii) *The diagonal morphism $\Delta_{X/Y}$ is an open immersion on a neighbourhood of x.*

Proof. For the implication (i) $\Longrightarrow$ (ii), we are brought by Nakayama to the case where $Y = \mathrm{Spec}(k)$ and $X = \mathrm{Spec}(k')$, where it is well known and otherwise trivial by the definition of separable; (ii) $\Longrightarrow$ (iii) comes from a nice and easy characterisation of open immersions, using Krull; (iii) $\Longrightarrow$ (i) follows as well from reducing to the case where $Y = \mathrm{Spec}(k)$ and the diagonal morphism is everywhere an open immersion. One must then prove that X is finite [...]and this leads us to consider the case where k is algebraically closed. But then every closed point of X is isolated (since it is identical to the inverse image of the diagonal by the morphism $X \to X \times_k X$ defined by x), whence X is finite. We can thus suppose that X reduces to a point, of the ring A, and so $A \otimes_k A \to A$ is an isomorphism, hence $A = k$. □

Definition 3.2. (a) *We then say that f is* unramified *in x, or that X is unramified in x on Y.*

(b) *Let $A \to B$ be a local homomorphism. We say that it is* unramified*, or that B is a local* unramified *algebra on A, if $B/mathfrakr(A)B$ is a finite separable extension of $A/\mathfrak{r}(A)$, i.e. if $\mathfrak{r}(A)B = \mathfrak{r}(B)$ and $k(B)$ is a separable extension of $k(A)$*[3].

Remark. The fact that B is unramified over A [...]. Unramified implies quasi-finite.

Corollary 3.3. *The set of points where f is unramified is open.*

[3]Cf. regrets in III 1.2.