Revêtements Étales and the Fundamental Group (SGA I)

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Contents

Translation Notes

The original can be found on the arXiv.

CHAPTER 1

Étale morphisms

1

To simplify the exposition we assume that all preschemes in the following are locally Noetherian (at least, starting from section 2).

1. Basics of differential calculus

Let X be a prescheme on Y, and $\Delta_{X/Y}$ the diagonal morphism $X \to X \times_Y X$. This is an immersion, and thus a closed immersion of X into an open subset V of $X \times_Y X$. Let $\mathscr{I}_X$ be the ideal of the closed sub-prescheme corresponding to the diagonal in V (N.B. if one really wishes to do things intrinsically, without assuming that X is separated over Y — a misleading hypothesis — then one should consider the set-theoretic inverse image of $\mathcal{O}_{X\times X}$ in X and denote by $\mathscr{I}_X$ the augmentation ideal in the above…). The sheaf $\mathscr{I}_X/\mathscr{I}_X^2$ can be thought of as a quasi-coherent sheaf on X, which we denote by $\Omega^1_{X/Y}$. It is of finite type if $X \to Y$ is of finite type. It behaves well with respect to a change of base $Y' \to Y$. We also introduce the sheaves $\mathcal{O}_{X\times_Y X}/\mathscr{I}_X^{n+1} = \mathscr{P}^n_{X/Y}$, which are sheaves of *rings* on X, giving us preschemes denoted by $\Delta^n_{X/Y}$ and called the *n-th infinitesimal neighbourhood of* X/Y. The polysyllogism is entirely trivial, even if rather long[1]; it seems wise to not discuss it until we use it for something helpful, with smooth morphisms.

2. Quasi-finite morphisms

PROPOSITION 2.1. *Let $A \to B$ be a local homomorphism (N.B. all rings are now Noetherian) and $\mathfrak{m}$ the maximal ideal of A. Then the following conditions are equivalent:*

(i) *$B/\mathfrak{m}B$ is of finite dimension over $k = A/\mathfrak{m}$.*
(ii) *$\mathfrak{m}B$ is [. . .] and $B/\mathfrak{r}(B) = k(B)$ is an extension of $k = k(A)$.*
(iii) *The completion $\hat{B}$ of B is finite over the completion $\hat{A}$ of A.*

2

We then say that B *is quasi-finite* over A. A morphism $f\colon X \to Y$ is said to be quasi-finite at x (or the Y-prescheme f is said to be quasi-finite at x) if $\mathcal{O}_x$ is quasi-finite over $\mathcal{O}_{f(x)}$. [. . .]A morphism is said to be quasi-finite if it is quasi-finite in each point[2].

COROLLARY 2.2. *If A is complete then quasi-finiteness is equivalent to finiteness.*

We could give the usual polysyllogism (i), (ii), (iii), (iv), (v) for quasi-finite morphisms, but that doesn't seem necessary here.

[1]cf. EGA IV 16.3.
[2]In EGA II 6.2.3 we further suppose that f is of finite type.

3. Unramified morphisms

Proposition 3.1. *Let $f\colon X \to Y$ be a morphism of finite type, $x \in X$, and $y = f(x)$. Then the following conditions are equivalent:*

(i) *$\mathcal{O}_x/\mathfrak{m}_y\mathcal{O}_x$ is a finite separable extension of $k(y)$.*
(ii) *$\Omega^1_{X/Y}$ is null at x.*
(iii) *The diagonal morphism $\Delta_{X/Y}$ is an open immersion on a neighbourhood of x.*

For the implication (i) $\Longrightarrow$ (ii), we are brought by Nakayama to the case where $Y = \mathrm{Spec}(k)$ and $X = \mathrm{Spec}(k')$, where it is well known and otherwise trivial by the definition of separable; (ii) $\Longrightarrow$ (iii) comes from a nice and easy characterisation of open immersions, using Krull; (iii) $\Longrightarrow$ (i) follows as well from reducing to the case where $Y = \mathrm{Spec}(k)$ and the diagonal morphism is everywhere an open immersion. One must then prove that X is finite [...]and this leads us to consider the case where k is algebraically closed. But then every closed point of X is isolated (since it is identical to the inverse image of the diagonal by the morphism $X \to X \times_k X$ defined by x), whence X is finite. We can thus suppose that X reduces to a point, of the ring A, and so $A \otimes_k A \to A$ is an isomorphism, hence $A = k$. □

Definition 3.2. (a) *We then say that f is* unramified *at x, or that X is unramified at x on Y.*
(b) *Let $A \to B$ be a local homomorphism. We say that it is* unramified*, or that B is a local* unramified *algebra on A, if $B/mathfrakr(A)B$ is a finite separable extension of $A/\mathfrak{r}(A)$, i.e. if $\mathfrak{r}(A)B = \mathfrak{r}(B)$ and $k(B)$ is a separable extension of*
3 *$k(A)$*[3].

Remark. The fact that B is unramified over A [...]. Unramified implies quasi-finite.

Corollary 3.3. *The set of points where f is unramified is open.*

Corollary 3.4. *Let X' and X be two preschemes of finite type over Y, and $g\colon X' \to X$ a Y-morphism. If X is unramified over Y then the graph morphism $\Gamma_g\colon X' \to X \times_Y X$ is an open immersion.*

In effect, this is the inverse image of the diagonal morphism $X \to X \times_Y X$ by

$$g \times_Y \mathrm{id}_{X'}\colon X' \times_Y X \to X \times_Y X.$$

One can also introduce the annihilator ideal $\mathfrak{d}_{X/Y}$ of $\Omega^1_{X/Y}$, called the different ideal of X/Y; it defines a closed sub-prescheme of X which, set theoretically, is the set of point where X/Y is ramified, i.e. not unramified.

Proposition 3.5. (i) *An immersion is ramified.*
(ii) *The composition of two ramified morphisms is also ramified.*
(iii) *Base extension of a ramified morphisms is also ramified.*

We don't care so much about (ii) or (iii) (the second seems more entertaining to me). We can also [...]by giving a few punctual comments; it is no more general than in appearance (except in the case of definition b) and also boring. We obtain, as per usual, the corollaries:

[3]Cf. regrets in III 1.2.

Corollary 3.6. (iv) *The cartesian product of two unramified morphisms is unramified.*

(v) *If gf is unramified then so too is f.*

(vi) *If f is unramified then so too is f_{red}.*

Proposition 3.7. *Let $A \to B$ be a local homomorphism and suppose that the residue extension $k(B)/k(A)$ is trivial with $k(A)$ algebraically closed. In order for B/A to be unramified it is necessary and sufficient that $\hat{B}$ be (as an $\hat{A}$-algebra) a quotient of $\hat{A}$.*

Remark.
- In the case where we don't suppose that the residue extension is trivial, we can bring ourselves to the case where it is by taking a suitable finite flat extension on A which destroys the aforementioned extension.
- Consider the example where A is the local ring of an ordinary double point of a curve, and B a point of the normalised curve: then $A \subset B$, B is unramified over A with trivial residue extension, and $\hat{A} \to \hat{B}$ is surjective but *not injective*. We are thus going to strengthen the notion of unramifiedness. **4**

4. Étale morphisms. Étale covers.

We are going to suppose all that will be necessary concerning flat morphisms; these facts will be proved later, if there is time[4].

Definition 4.1. a) *Let $f\colon X \to Y$ be a morphism of finite type. We say that f is* étale *at x if f is both flat and unramified at x. We say that f is étale if it is étale at all points. We say that X is étale at x over Y, or that it is a Y-prescheme étale at x etc.*

b) *Let $f\colon A \to B$ be a local homomorphism. We say that f is étale, or that B is étale over A, if B is flat and unramified over A[5].*

Proposition 4.2. *For B/A to be étale, it is necessary and sufficient that $\hat{B}/\hat{A}$ be étale.* **5**

In effect, this is true individually for both "unramified" and "flat".

Corollary 4.3. *Let $f\colon X \to Y$ be of finite type, and $x \in X$. The property of f being étale at x depends only on the local homomorphism $\mathcal{O}_{f(x)} \to \mathcal{O}_x$, and in fact only on the corresponding homomorphism for the completions.*

Corollary 4.4. *Suppose that the residue extension $k(A) \to k(B)$ is trivial, or that $k(A)$ is algebraically closed. Then B/A is étale if and only if $\hat{A} \to \hat{B}$ is an isomorphism.*

We combine the platitude and 3.7.

Proposition 4.5. *Let $f\colon X \to Y$ be a morphism of finite type. Then the set of points where f is étale is open.*

Again, this is in fact true individually for both "unramified" and "flat".

This proposition shows that we can forget about the punctual comments in the study of morphisms of finite type that are somewhere étale.

[4] Cf. Exp. IV.

[5] Cf. regrets in III 1.2.

Proposition 4.6. (i) *An open immersion is étale.*
(ii) *The composition of two étale morphisms is étale.*
(iii) *The base extension of an étale morphism is étale.*

In effect, (i) is trivial, and for (ii) and (iii) it suffices to note that it is true for "unramified" and "flat". As a matter of fact, there are also corresponding comments to make about local homomorphisms (without the finiteness condition), which in any case should figure in the multiplodoque (starting with the case of unramified).

Grothendieck's *multiplodoque d'algèbre homologique* was the final version of his *Tohoku paper* — see (2.1) in 'Life and work of Alexander Grothendieck' by Ching-Li Chan and Frans Oort for more information. [trans.]

Corollary 4.7. *A cartesian product of two étale morphisms is étale.*

Corollary 4.8. *Let X and X' be of finite type over Y, and $g\colon X \to X'$ a Y-morphism. If X' is unramified over Y and X is étale over Y, then g is étale.*

In effect, g is the composition of the graph morphism $\Gamma_g\colon X \to X \times_Y X'$ (which is an open immersion by 3.4) and the projection morphism, which is étale since it is just a "change of base" by $X' \to Y$ of the étale morphism $X \to Y$.

Definition 4.9. *We say that a cover of Y is étale (resp. unramified) if it is a Y-scheme X that is finite over Y and étale (resp. unramified) over Y.*

The first condition means that X is defined by a coherent sheaf of algebras $\mathscr{B}$ over Y. The second means that $\mathscr{B}$ is locally free over Y (resp. absolutely nothing) *and* further that, for all $y \in Y$, the fibre $\mathscr{B}(y) = \mathscr{B}_y \otimes_{\mathcal{O}_y} k(y)$ is a separable algebra (i.e. a finite composition of finite separable extensions) over $k(y)$.

Proposition 4.10. *Let X be a flat cover of Y of degree n (the definition of this term deserved to figure in 4.9) defined by a locally free coherent sheaf $\mathscr{B}$ of algebras. We define, as usual, the trace homomorphism $\mathscr{B} \to \mathscr{A}$ (that is a homomorphism of $\mathscr{A}$-modules, where $\mathscr{A} = \mathcal{O}_Y$). For X to be étale it is necessary and sufficient that the corresponding bilinear form $\operatorname{tr}_{\mathscr{B}/\mathscr{A}} xy$ defines an isomorphism of $\mathscr{B}$ over $\mathscr{B}$, or, equivalently, that the* discriminant section

$$d_{X/Y} = d_{\mathscr{B}/\mathscr{A}} \in \Gamma(Y, \wedge^n \check{\mathscr{B}} \otimes_{\mathscr{A}} \wedge^n \check{\mathscr{B}})$$

is invertible, or that the discriminant ideal defined by this section is the unit ring.

In effect, we can reduce to the case where $Y = \operatorname{Spec}(k)$, and then it is a well-known criterion of separability, and trivial by passing to the algebraic closure of k.

6 Remark. We will have a less trivial statement to make later on, when we do not suppose a priori that X is flat over Y, but instead require some normality hypothesis.

CHAPTER 6

Fibred Categories and Descent

0. Introduction 145

Contrary to what was said in the introduction to the previous chapter, it proved impossible to do descent in the category of preschemes, even in special cases, without having already carefully developed the language of descent in general categories.

The notion of "descent" provides the general framework for all processes of "gluing" objects, and therefore of "gluing" of categories. The most classic case of gluing relates to the data of a topological space X and a cover of X by open sets X_i; if we consider for all i a fibre bundle (say) E_i above X_i, and for every pair (i, j) an isomorphism f_{ji} of $E_i|X_{ij}$ on $E_j|X_{ij}$ (where we put $X_{ij} = X_i \cap Xj$), satisfying a well-known condition of transitivity (which we write in a short way $f_{kj} f_{ji} = f_{ki}$), we know that there is a fibre bundle E on X, defined up to isomorphism with the condition that we have isomorphisms $f_i \colon E|X_i \xrightarrow{\simeq} E_i$, satisfying the relations $f_{ji} = f_j f_i^{-1}$ (with the usual writing abuse). Let X' be the space given by the sum of the X_i, which is therefore a fibre space over X (i.e. equipped with a continuous mapping $X' to X$). We can then more concisely think of the data of the E_i as a fibre bundle E' over X', and the data of the f_{ji} as an isomorphism between the two inverse images (by the two canonical projections) E''_1 and E''_2 of E' over $X'' = X' \times_X X'$, whence the gluing condition can be written as an identity between isomorphisms of fibre bundles $\mathscr{E}'''_1$ and $\mathscr{E}'''_2$ on the triple fibre product $X''' = X' \times_X X' \times_X X'$ (where E'''_i denotes the inverse image of E' on X''' by the canonical projection of index i). The construction of E, from E' and f, is a typical case of "descent". Moreover, starting from a fibre bundle E on X, one says that X 146
is "locally trivial", of fibre F, if there exists an open covering (X_i) of X such that the $E|X_i$ are isomorphic to $F \times X_i$, or, equivalently, such that the inverse image E' of E over $X' = \coprod_i X_i$ is isomorphic to $X' \times F$.

Thus the notion of "gluing" objects, like the "localisation" of a property, is related to the study of certain types of "base changes" $X' \to X$. In algebraic geometry many other types of base change, and in particular faithfully flat morphisms $X' \to X$, should be considered as corresponding to a process of "localisation" relative to preschemes, or other objects, "over" X. This type of localization is used just as much as the simple topological localization (which is a special case of this more general idea). The same is true (to a lesser extent) in Analytic Geometry.

Most proofs, reducing to verifications, are omitted or simply sketched out; if so then we specify the less-obvious diagrams that are needed in the proof.

1. Universe, Categories, Equivalence of Categories

To avoid certain logical difficulties, we introduce here the notion of a *Universe*, which is a set "big enough" that we do not leave it by the usual operations of set theory; an "Axiom of Universes" guarantees that every object is in some universe. For details, see a book in preparation by C. Chevalley and the conférencier[1]. Thus the abbreviation Ens designates, not the category of all sets (a notion that does not make sense), but the category of all sets found in a given Universe (which we omit from the notation). Similarly, Cat will indicate the category of categories in the Universe in question, and the "morphisms" from an object X of Cat to another Y are by definition the *functors* from X to Y.

The *conférencier* is the master of ceremonies, or chairman, of a conference. In this case (so it would appear) it was P. Gabriel. [trans.]

147 If $\mathscr{C}$ is a category, we denote by $\mathrm{Ob}(\mathscr{C})$ the *set of objects* of $\mathscr{C}$ and by $\mathrm{Fl}(\mathscr{C})$ the *set of arrows* of $\mathscr{C}$ (or morphisms of $\mathscr{C}$). We will then write $X \in \mathrm{Ob}(\mathscr{C})$, avoiding the current abuse of notation $X \in \mathscr{C}$. If $\mathscr{C}$ and $\mathscr{C}'$ are two categories then a *functor* from $\mathscr{C}$ to $\mathscr{C}'$ will always be what is usually called a *covariant functor* from $\mathscr{C}$ to $\mathscr{C}'$; its data implies that of the target category and the source category, $\mathscr{C}$ and $\mathscr{C}'$. The functors from $\mathscr{C}$ to $\mathscr{C}'$ form a set, denoted $\mathrm{Hom}(\mathscr{C}, \mathscr{C}')$, which is the set of objects of a category denoted $\mathbf{Hom}(\mathscr{C}, \mathscr{C}')$. By definition, a *contravariant functor* from $\mathscr{C}$ to $\mathscr{C}'$ is a functor from the *opposite category* $\mathscr{C}^{\mathrm{op}}$ of $\mathscr{C}$ to $\mathscr{C}'$.

We also take as given the notion of a *projective limit* and *inductive limit* of a functor $F\colon \mathscr{I} \to \mathscr{C}$, and in particular the most common special cases of this notion: Cartesian products and fibre products, dual notions of direct sums and amalgamated sums, and the usual formal properties of these operations.

For example, in the category Cat introduced earlier, the projective limits (relative to categories found in the chosen Universe) exist; the set of objects (respectively the set of arrows) of the projective limit category $\mathscr{C}$ of the $\mathscr{C}_i$ is obtained by taking the projective limit of sets of objects (resp. arrows) of the categories $\mathscr{C}_i$. The best known case is that of the product of a family of categories. We will constantly use the fiber product of two categories over a third.

For everything concerning categories and functors, while waiting for the book already in preparation, see [**1**] (which is necessarily very incomplete, even concerning the generalities outlined here).

We take this opportunity to explain the concept of equivalence of categories, which is not satisfactorily exposed in [**1**]. A functor $F\colon \mathscr{C} \to \mathscr{C}'$ is said to be *faithful* (resp. *fully faithful*) if, for any pair of objects S, T of $\mathscr{C}$, the function $u \mapsto F(u)$ from $\mathrm{Hom}(S,T)$ to $\mathrm{Hom}(F(S), F(T))$ is injective (respectively bijective). We say
148 that F is an *equivalence* of categories if F is fully faithful, and if any object S' of $\mathscr{C}'$ is isomorphic to an object of the form $F(S)$. We show that it is the same to say that there exists a functor G from $\mathscr{C}'$ to $\mathscr{C}$ that is quasi-inverse of F, i.e. such that GF is isomorphic to $id_{\mathscr{C}}$. When this is so, the data of a functor $G\colon \mathscr{C}' \to \mathscr{C}$ and an isomorphism $\varphi\colon GF \to id_{\mathscr{C}'}$ equals the data, for all $S' \in \mathrm{Ob}(\mathscr{C}')$, of a pair (S, u) consisting of an object S of $\mathscr{C}$ and an isomorphism $u\colon F(S) \to S'$, i.e. $(G(S), \varphi(S))$. (With this notation, there exists a unique functor $C' \to C$ having the given function $S \mapsto G(S)$ as function-objects, and such that the function $S \mapsto \varphi(S)$ is a homomorphism of functors $FG \to id_{\mathscr{C}'}$). Finally, if G is a quasi-inverse functor of F, and if one chooses isomorphisms $\varphi\colon FG \xrightarrow{\sim} id_{\mathscr{C}'}$ and $\psi\colon GF \xrightarrow{sim} id_{\mathscr{C}}$, then the two compatibility conditions on φ and ψ stated in [**1**, I.1.2] are in fact equivalent to each other, and for every isomorphism φ chosen, there exists a unique isomorphism such that these conditions are satisfied.

[1]The finalised authors are C. Chevalley and P. Gabriel. The book should come out in the year 3000. Whilst waiting, cf. as well SGA 4 I.

2. Categories over another

3. Change of base in categories over E

4. Fibred categories; Equivalence of E-categories

5. Cartesian morphisms, inverse images, Cartesian functors

6. Fibred categories and prefibred categories. Products and change of base within.

7. Cloven categories over E

8. Cloven Fibration defined from a pseudofunctor

As an abbreviation, we say *pseudofunctor* of $\mathscr{E}^{op}$ in Cat (really we should say *normalized* pseudo-functor) to mean a set of data a), b), c) as above, satisfying the conditions A') and B). In the previous section, we associated a pseudofunctor $\mathscr{E}^{op} \to \mathsf{Cat}$ to a normalized cloven fibration on $\mathscr{E}$, and here we will describe the inverse construction. We will leave to the reader the verification of most of the details, as well as the fact that these constructions are "inverse" to each other. Specifically, we should consider the pseudofunctors $\mathscr{E}^{op} \to \mathsf{Cat}$ as the objects of a new category, and show that our constructions provide equivalences, quasi-inverse to each other, between the above and the category of cloven categories above $\mathscr{E}$, as defined in the preceding section.

We write

$$\mathscr{F}_\circ = \coprod_{S \in \mathrm{Ob}(\mathscr{E})} \mathrm{Ob}(\mathscr{F}(S)),$$

the sum of sets $\mathrm{Ob}(\mathscr{F}(S))$ (N.B. we will write $\mathscr{F}(S)$ and not $\mathscr{F}_S$ to mean the the value in the object S of $\mathscr{E}$ of the given pseudofunctor to avoid confusion of notation thereafter). We then have an obvious map:

$$p_\circ \colon \mathscr{F}_\circ \to \mathrm{Ob}\mathscr{E}.$$

Let

$$\overline{\xi} = (S, \xi),\, \overline{\eta} = (T, \eta) \qquad (\text{where } \xi \in \mathrm{Ob}\mathscr{F}(S),\, \eta \in \mathrm{Ob}\mathscr{F}(T))$$

be two elements of $\mathscr{F}_\circ$, and let $f \in \mathrm{Hom}(T, S)$. We then define

$$h_f(\overline{\eta}, \overline{\xi}) = \mathrm{Hom}_{\mathscr{F}(T)}(\eta, f^*(\xi)).$$

If we also have a morphism $g \colon U \to T$ in $\mathscr{E}$, and some $\zeta \in \mathrm{Ob}\mathscr{F}(U)$, we define a function, denoted $(u, v) \mapsto u \circ v$: **176**

$$h_f(\overline{\eta}, \overline{\xi}) \times h_g(\overline{\zeta}, \overline{\eta}) \to h_{fg}(\overline{\zeta}, \overline{\xi}),$$

i.e. a function

$$Hom_{\mathscr{F}(T)}(\eta, f^*(\xi)) \times \mathrm{Hom}_{\mathscr{F}(U)}(\zeta, g^*(\eta)) \to \mathrm{Hom}_{\mathscr{F}(U)}(\zeta, (fg)^*(\xi)),$$

given by the formula

$$u \circ v = c_{f,g}(\xi) \cdot g^*(U) \cdot v,$$

i.e. $u \circ v$ is the composition of the sequence

$$\zeta \xrightarrow{u} g^*(\eta) \xrightarrow{g^*(u)} g^* f^*(\xi) \xrightarrow{c_{f,g}(\xi)} (fg)^*(\xi).$$

Further define

$$h(\overline{\eta},\overline{\xi}) = \coprod_{f\in \mathrm{Hom}(T,S)} h_f(\overline{\eta},\overline{\xi}),$$

and the previous couplings define couplings

$$h(\overline{\eta},\overline{\xi}) \times h(\overline{\zeta},\overline{\eta}) \to h(\overline{\zeta},\overline{\xi}),$$

and the definition of $h(\overline{\eta},\overline{\xi})$ gives an obvious function:

$$p_{\overline{\eta},\overline{\xi}} \colon h(\overline{\eta},\overline{\xi}) \to \mathrm{Hom}(T,S).$$

177 With these definitions, we verify the following properties:

(1) The composition of elements of $h(\overline{\eta},\overline{\xi})$ is *associative.*
(2) For all $\overline{\xi} = (\xi, S)$ in $\mathscr{F}_\circ$, consider the identity element of

$$h_{id_S}(\overline{\xi},\overline{\xi}) = \mathrm{Hom}_{\mathscr{F}(S)}(id_S^*(\xi),\xi) = \mathrm{Hom}_{\mathscr{F}(S)}(\xi,\xi),$$

and its image in $h(\overline{\xi},\overline{\xi})$. This object is a left and right unit for the composition of elements of $h(\overline{\eta},\overline{\xi})$.

Already, this shows that we get a category $\mathscr{F}$ by setting

$$\mathrm{Ob}\mathscr{F} = \mathscr{F}_\circ, \quad \mathrm{Fl}\mathscr{F} = \coprod_{\overline{\xi},\overline{\eta}\in\mathscr{F}_\circ} h(\overline{\eta},\overline{\xi}).$$

(N.B. we can not simply take for $\mathrm{Fl}\mathscr{F}$ the union of the sets $h(\overline{\eta},\overline{\xi})$, because these are not necessarily disjoint.). Furthermore:

(3) The functions $p_\circ \colon \mathrm{Ob}\mathscr{F} \to \mathrm{Ob}\mathscr{E}$ and $p_1 = (p_{\overline{\eta},\overline{\xi}} \colon \mathrm{Fl}\mathscr{F} \to \mathrm{Fl}\mathscr{E}$ define a *functor* $p\colon \mathscr{F} \to \mathscr{E}$. In this way, $\mathscr{F}$ becomes a category over $\mathscr{E}$, and moreover the obvious function $h_f(\overline{\eta},\overline{\xi}) \to \mathrm{Hom}(\overline{\eta},\overline{\xi})$ induces a *bijection*

$$h_f(\overline{\eta},\overline{\xi}) \xrightarrow{\sim} \mathrm{Hom}_f(\overline{\eta},\overline{\xi}).$$

(4) The obvious functions

$$\mathrm{Ob}\mathscr{F}(S) \to \mathscr{F}_\circ = \mathrm{Ob}\mathscr{F}, \quad \mathrm{Fl}\mathscr{F}(S) \to \mathrm{Fl}\mathscr{F},$$

where the second is defined by the obvious maps

$$\mathrm{Hom}_{\mathscr{F}(S)}(\xi,\xi') = h_{id_S}(\overline{\xi},\overline{\xi}') \to \mathrm{Hom}(\overline{\xi},\overline{\xi}'),$$

define an *isomorphism*

178

$$i_S \colon \mathscr{F}(S) \xrightarrow{\sim} \mathscr{F}_S.$$

(5) For every object $\overline{\xi} = (S,\xi)$ of $\mathscr{F}$, and every morphism $f\colon T \to S$ of $\mathscr{E}$, consider the element $\overline{\eta} = (T,\eta)$ of $\mathscr{F}_T$, with $\eta = f^*(\xi)$, and $\alpha_f(\xi)$ of $\mathrm{Hom}(\overline{\eta},\overline{\xi})$, the image of $id_{f^*(\xi)}$ under the morphism $\mathrm{Hom}_{\mathscr{F}(T)}(f^*(\xi), f^*(\xi)) = h_f(\overline{\eta},\overline{\xi}) \to \mathrm{Hom}_f(\overline{\eta},\overline{\xi})$. *This element is cartesian, and it is the identity in ξ if $f = id_S$.* In other words, the set of all $\alpha_f(\xi)$ defines a normalized cleavage of $\mathscr{F}$ over $\mathscr{E}$. Further, by construction, we have commutativity in the diagram of functors

$$\begin{tikzcd}
\mathscr{F}(S) \arrow[r, "f^*"] \arrow[d, "i_S"'] & \mathscr{F}(T) \arrow[d, "i_T"] \\
\mathscr{F}_S \arrow[r, "f^*_{\mathscr{F}}"'] & \mathscr{F}_T
\end{tikzcd}$$

where $f^*_{\mathscr{F}}$ is the inverse image functor by f, relative to the cleavage considered on $\mathscr{F}$. Finally:

(6) the homomorphisms $c_{f,g}$ given by the pseudofunctor are transformed, by the isomorphisms i_S, to functorial homomorphisms $c_{f,g}$ associated with the cleavage of $\mathscr{F}$.

We restrict ourselves to giving the verification of 1) (which is, if possible, less trivial than the others). It is enough to prove the associativity of the composition between the objects of the sets of the form $h_f(\overline{\eta}, \overline{\xi})$. So consider the morphisms in $\mathscr{E}$

$$S \xleftarrow{f} T \xleftarrow{g} U \xleftarrow{h} V$$

and the objects

$$\xi, \eta, \zeta, \tau$$

in $\mathscr{F}(S), \mathscr{F}(T), \mathscr{F}(U), \mathscr{F}(V)$, as well as some elements

$$\begin{aligned}
u &\in h_f(\overline{\eta}, \overline{\xi}) = \mathrm{Hom}_{\mathscr{F}(T)}(\eta, f^*(\xi)) \\
v &\in h_g(\overline{\zeta}, \overline{\eta}) = \mathrm{Hom}_{\mathscr{F}(U)}(\zeta, g^*(\eta)) \\
w &\in h_h(\overline{\tau}, \overline{\zeta}) = \mathrm{Hom}_{\mathscr{F}(V)}(\tau, h^*(\zeta)).
\end{aligned}$$

179

We want to prove the formula

$$(u \circ v) \circ w = u \circ (v \circ w),$$

which is an equality in $\mathrm{Hom}_{\mathscr{F}(V)}(\tau, (fgh)^*(\xi))$. By virtue of the definitions both members of this equality are obtained by composition following the top and bottom arrows of the diagram below:

$$\begin{tikzcd}
\tau \arrow[r, "w"] \arrow[drr, "v \circ w"'] & h^*(\zeta) \arrow[r, "h^*(v)"] \arrow[rrr, bend left, "h^*(u \circ v)"] & h^*g^*(\eta) \arrow[r, "h^*g^*(u)"] \arrow[d, "c_{g,h}(\eta)"'] & h^*g^*f^*(\xi) \arrow[r, "h^*(c_{f,g}(\xi))"] \arrow[d, "c_{g,h}(f^*(\xi))"] & h^*(fg)^*(\xi) \arrow[d, "c_{fg,h}(\xi)"] \\
& & (gh)^*(\eta) \arrow[r, "(gh)^*(\eta)"'] & (gh)^*f^*(\xi) \arrow[r, "c_{f,gh}(\xi)"'] & (fgh)^*(\xi)
\end{tikzcd}$$

Now the middle square is commutative because $c_{g,h}$ is a functorial homomorphism, and the square on the right is commutative under condition B) for a pseudofunctor, whence the desired result.

Of course, it remains to make precise, in the case where the pseudofunctor already comes from a normalized cloven category $\mathscr{F}'$ over $\mathscr{E}$, how we get n natural isomorphism between $\mathscr{F}'$ and $\mathscr{F}$. We leave the details to the reader.

We also allow the reader to interpret, in terms of pseudofunctors, the notion of inverse image of a cloven category $\mathscr{F}$ over $\mathscr{E}$ by a change of base functor $\mathscr{E}' \to \mathscr{E}$.