ST4237 Midterm 2025

1. Estimate the integral

$$\int_{x=0}^{3}\int_{y=0}^{3}\int_{z=0}^{3} e^{\sin(xy+yz^2)}\,dx\,dy\,dz$$

   using the Monte Carlo method.
   [at least one correct decimal expected]

2. Consider three independent random variables $X, Y, Z \sim \mathcal{N}(0,1)$. Estimate the probability

$$\mathbb{P}\big(\,|X/Y|^2 > Z^2\,\big).$$

   [at least one correct decimal expected]

3. Estimate the volume of

$$V = \{(x,y,z) \in \mathbb{R}^3 \,:\, 0 < x < y < z < 2\}.$$

   [at least one correct decimal expected]

4. On average, how many times does one need to roll a fair die until the sum of the rolls strictly exceeds 20.
   [at least one correct decimal expected]

   Remark: the command

   ```
   np.random.randint(low=a, high=b)
   ```

   generates an integer uniformly at random in the range $\{a, a+1, \dots, b-1\}$.
   [at least one correct decimal expected]

5. Consider the probability distribution on $[0,3]$ with density $f(x) \propto \log(1+x^2)$. Estimate the mean of this distribution using the Monte Carlo method.
   [at least one correct decimal expected]

6. Consider the probability distribution on $[0,3]$ with density $f(x) \propto \log(1+x^2)$. Estimate the standard deviation of this distribution using the Monte Carlo method.
   [at least one correct decimal expected]

7. Consider the density on $[0,2]$ with density $f(x) = x^3/Z$ where Z is the normalizing constant. One knows from the lectures that one can simulate from this distribution by setting $X = \Phi(U)$ where $U \sim \mathcal{U}(0,1)$ and Φ is the inverse of the cumulative distribution function of f. Estimate

   $$\mathrm{E}[\Phi(U)\,\Phi(1-U)]$$

   using the Monte Carlo method.
   [at least one correct decimal expected]

8. Consider the density on $[0, 2]$ with density $f(x) = x^3/Z$ where Z is the normalizing constant. What is the probability that $X > 1.8$ if X is distributed according to f?
[at least one correct decimal expected]

9. Estimate the integral

$$\int_{x=-\infty}^{\infty} \log(1+x^2)\, e^{-x^2/2}\, dx$$

using the Monte Carlo method.
[at least one correct decimal expected]

10. Estimate the integral

$$\int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} \frac{\sin(3x^2+2y^2)}{(1+x^2)(1+y^2)}\, dx\, dy$$

using the Monte Carlo method.
[at least one correct decimal expected]

Remark: the Cauchy distribution has density $f(x) = \frac{1}{\pi\,(1+x^2)}$ and can be generated with `np.random.standard_cauchy()`.
[at least one correct decimal expected]

11. To complete a project, ten tasks $T_1, \ldots, T_{10}$ must be completed. The time to complete each task is exponentially distributed with mean 1 hour. The tasks are independent and are all started at the same time and completed concurrently. Estimate the probability that the project is completed in more than 4 hours.

Remark: the exponential distribution with mean $\mu > 0$ has density $f(x) = \frac{1}{\mu}\exp(-x/\mu)$ and can be generated with `np.random.exponential(scale=mu)`.
[at least one correct decimal expected]

12. Consider two independent random variables $X, Y \sim \mathcal{N}(0, 1)$. Compute:

$$\mathbb{E}[\, XY \,].$$

[at least one correct decimal expected]

13. Consider two independent random variables $X, Y \sim \mathcal{N}(0, 1)$. Conditioned on the event that $X + Y > 1$, compute the expectation of their product:

$$\mathbb{E}[\, XY \,|X + Y > 1\,].$$

[at least one correct decimal expected]

14. Consider two independent random variables $X, Y \sim \mathcal{N}(0, 1)$. We would like to estimate accurately the very rare event that $X^2 + Y^2 > 40$. For this purpose, use importance sampling with proposal distributions

$$X_{IS} \sim \mathcal{N}(0, \sigma_{is}^2 = 5^2) \quad \text{and} \quad Y_{IS} \sim \mathcal{N}(0, \sigma_{is}^2 = 5^2).$$

Consider the quantity

$$\alpha = 10^{10} \times \mathbb{P}\big(\, X^2 + Y^2 > 40 \,\big).$$

Estimate α using the Monte Carlo method.

Remark: the normal distribution with mean μ and variance σ^2 has density $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$ and can be generated with

`np.random.normal(loc=mu, scale=sigma)`.

[at least one correct decimal expected]

15. One can approximately simulate from a $\mathcal{N}(0,1)$ distribution by summing 12 independent samples from a $\mathrm{Uniform}([-\frac{1}{2}, \frac{1}{2}])$ distribution (i.e. uniform on the interval $[-\frac{1}{2}, \frac{1}{2}]$). Estimate the quantity

$$\mathbb{P}\left(\sum_{i=1}^{12} U_i > 0.5\right)$$

where the random variables $U_i \sim \mathrm{Uniform}(-\frac{1}{2}, \frac{1}{2})$ are independent.
[at least one correct decimal expected]

16. Estimate the quantity

$$\alpha = 10^{10} \times \mathbb{P}\left(\sum_{i=1}^{12} U_i > 6\right)$$

where the random variables $U_i \sim \mathrm{Uniform}(-\frac{1}{2}, \frac{1}{2})$ are independent.
[at least one correct decimal expected]

17. Consider the distribution on $\{-2 \le x, y, z \le 2\}$ with density

$$f(x,y,z) \propto (x^2 + y^2 + z^2).$$

Use the rejection sampling method to generate samples (X, Y, Z) from this distribution and estimate the quantity

$$\mathbb{E}[\, X^2 + Y^2 + Z^2 \,].$$

[at least one correct decimal expected]

18. A random walk $(X_0, X_1, \ldots, X_{50})$ is defined by $X_0 = 0$ and $X_{t+1} = X_t + Z_t$ where $Z_1, Z_2, \ldots$ are i.i.d. random variables with $\mathbb{P}(Z_t = 1) = \mathbb{P}(Z_t = -1) = 1/2$. We define L has the number of times the random walk is at the level zero,

$$L = \sum_{t=0}^{50} \mathbb{I}_{\{X_t = 0\}}.$$

Estimate the mean of L using the Monte Carlo method.
[at least one correct decimal expected]

19. A random walk $(X_0, X_1, \ldots, X_{50})$ is defined by $X_0 = 0$ and $X_{t+1} = X_t + Z_t$ where $Z_1, Z_2, \ldots$ are i.i.d. random variables with $\mathbb{P}(Z_t = 1) = \mathbb{P}(Z_t = -1) = 1/2$. We define L has the number of times the random walk is at the level zero,

$$L = \sum_{t=0}^{50} \mathbb{I}_{\{X_t = 0\}}.$$

Estimate the standard deviation of L using the Monte Carlo method.
[at least one correct decimal expected]

20. consider 5 points uniformly distributed on the circumference of a circle. Estimate the probability that the 5 points are all contained within a half-circle (i.e. they are all on the same side of a diameter).
[at least one correct decimal expected]