Inversion and Rejection

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1 Simulate a Beta(2,2) random variable

A $\mathrm{Beta}(2,2)$ random variable has density

$$f(x) \propto x\,(1-x) \qquad \text{for all} \quad x \in [0,1].$$

Use rejection sampling to simulate a $\mathrm{Beta}(2,2)$ random variable.

2 Density $f(x) \propto x^4$

Consider the probability density $f(x)$ on the interval $[0,1]$ defined by

$$f(x) \propto x^4 \qquad \text{for all} \quad x \in [0,1].$$

Use the inversion method to simulate a random variable with density $f(x)$.

3 Cauchy distribution

The Cauchy distribution has density

$$f(x) = \frac{1}{\pi}\frac{1}{1+x^2} \qquad \text{for all} \quad x \in \mathbb{R}.$$

Use the inversion method to simulate a random variable with density $f(x)$.

4 Expected number of simulations for rejection sampling

Consider the rejection sampling algorithm to simulate a random variable with density $f(x)$ when using an envelope function $E(x)$. Assume that $\alpha = \int E(x)\,dx$.

1. Prove that $\alpha \geq 1$.
2. Let N be the number of simulations needed to generate a random variable distributed according to $f(x)$ when using the rejection sampling algorithm with envelope function $E(x)$. Compute the expected number of simulations $\mathrm{E}[N]$.

5 Sine distribution

Generate a random variable with density $f(x) \propto (1 + \sin(6x))$ on the interval $[0, \pi]$.

6 Weibull distribution

The Weibull distribution is often used to model the lifetime of a device or system. In particular, it is often used in reliability engineering. The Weibull distribution with shape parameter $\alpha > 0$ and scale parameter $\lambda > 0$ takes values in $[0, +\infty)$ and is often defined through its cumulative distribution function (CDF)

$$F(x) = 1 - \exp\left(-\left(\frac{x}{\lambda}\right)^{\alpha}\right).$$

1. Explain how you would simulate a random variable with Weibull distribution using the inversion method.
2. Draw a histogram of 10^4 random variables with Weibull distribution with shape parameter $\alpha = 2$ and scale parameter $\lambda = 1$. Restrict the range of the histogram to $[0, 5]$.

7 Gaussian from a Cauchy

How would you simulate a standard Gaussian random variable if the only available random number generator is a standard Cauchy random variable? *Hint:* Use rejection sampling.

8 Averaging Cauchy random variables

A Cauchy random variable has density $f(x) = \frac{1}{\pi}\frac{1}{1+x^2}$. Furthermore, its mean is not well-defined because the integral $\int x\, f(x)\, dx$ does not converge. Suppose one averages n independent Cauchy random variables $X_1, \ldots, X_n$ to obtain the random variable $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. What do you think is the distribution of $\bar{X}_n$? Use simulations to support your answer.

9 Box-Muller algorithm (Non-examinable)

The Box-Muller algorithm is a method to generate two independent standard normal random variables. The algorithm is as follows:

1. Generate two independent random variables U_1 and U_2 uniformly distributed on $[0, 1]$.
2. Compute $Z_1 = \sqrt{-2\log(U_1)}\cos(2\pi U_2)$ and $Z_2 = \sqrt{-2\log(U_1)}\sin(2\pi U_2)$.

Prove that Z_1 and Z_2 are independent standard normal random variables.