Basic Monte Carlo Methods

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Version: 0.2

1 Buffon's Needle

Estimate π by simulation of Buffon's needle experiment. The experiment consists of throwing a needle of length L on a floor with parallel lines separated by a distance L. When the needle length equals the line spacing, the probability that the needle intersects a line is $2/\pi$.
Simulate N throws of the needle and produce an estimate of π, as well as a 95% confidence interval.

2 Estimating π

Estimate π by simulation of the following experiment. Consider the unit square $[0,1] \times [0,1]$ and the circle centered at $(1/2, 1/2)$ with radius $1/2$. The area of the circle is $\pi/4$.
Estimate the area of the circle by simulating N points uniformly distributed in the square. The ratio of points in the circle to the total number of points is an estimate of $\pi/4$.
Produce an estimate of π and a 95% confidence interval.

3 $Q = \int_{-1}^{1} e^{-x^2/2} \, dx$

Estimate $Q = \int_{-1}^{1} e^{-x^2/2} \, dx$ by simulation and produce a 95% confidence interval.
Express the exact value of Q in terms of cumulative distribution function of the standard normal distribution.

4 $Q = \int_{-\infty}^{\infty} e^{-x^2/2} dx$

For this exercise, we propose to estimate $Q = \int_{-\infty}^{\infty} e^{-x^2/2} dx$ by simulation. For this purpose, we propose to use a change of variable to transform the integral into a finite one.
Consider the function $T(x) = 1/(1 + e^{-x})$.

1. Show that the inverse function of T is $T^{-1}(z) = \log(z/(1-z))$.

2. Using the change of variable $z = T(x)$, show that for any integrable function $\varphi : \mathbb{R} \to \mathbb{R}$ we have that

$$\int_{-\infty}^{\infty} \varphi(x)\, dx = \int_0^1 \varphi\left(\log \frac{z}{1-z}\right) \left(\frac{1}{z} + \frac{1}{1-z}\right) dz.$$

3. Estimate Q by simulation and produce a 95% confidence interval.

4. What is the exact value of Q?

5 Monte Carlo estimation of moments

Consider the density on $[0, 1] \times [0, 1]$ with density $f(x, y)$ given by

$$f(x, y) \propto \exp(xy + \sin(3\,x + y)).$$

Compute the mean $\mu \in \mathbb{R}^2$ as well as the covariance matrix $\Sigma \in \mathbb{R}^{2\times 2}$ of the distribution.

6 Aces in a deck of cards

Consider a well shuffled deck of 52 cards. The entire deck is split equally (26 cards each) between two players. Estimate the probability that the first player gets all four aces.

7 Increasing random uniforms

Estimate the probability that $U_1 < U_2 < U_3 < U_4$ where U_1, U_2, U_3, U_4 are independent uniforms on $[0, 1]$.
Can you compute the exact value of this probability?

8 Number of rolls to see all faces of a die

Estimate the expected number of times one needs to roll a die to see all six faces at least once.

9 Three birthdays on the same day

Estimate the probability that in a group of $n = 100$ people, at least 3 people have their birthday on the same day.

10 Total sum game

Two players alternately roll a die. After each roll, the result is added to a shared running total (starting at 0). The player whose roll causes this combined total to exceed 11 loses. Do you prefer to be the first or the second player?

11 Random walk escape

A particle starts at position 0 on the integers and moves $+1$ or -1 with equal probability at each step.

1. Estimate the expected number of steps to reach either -5 or $+10$ for the first time.
2. Estimate the probability of reaching $+10$ before -5.
3. Can you compute the exact value of this probability?

12 Gambler's ruin

A gambler starts with \$20 and bets \$1 on each round of a fair game (win or lose with probability $1/2$ each).

1. Estimate the probability of reaching \$50 before going bankrupt.
2. How does this probability change if the win probability is 0.49 instead of 0.5?
3. Can you compute the exact probability in the fair case?

13 The secretary problem (optimal stopping)

You interview $n = 20$ candidates sequentially for a position. After each interview, you must immediately accept or reject the candidate. Once rejected, a candidate cannot be recalled. You can only compare candidates you have already seen (i.e., you know their relative ranking among those interviewed so far). Consider the following strategy: reject the first k candidates regardless of their quality, then accept the first subsequent candidate who is better than all previously seen candidates.

1. For a fixed k, estimate by simulation the probability of selecting the best overall candidate.
2. Find the value of k that maximizes this probability.
3. What is the optimal fraction k/n as n becomes large?