APPENDIX D

A Touch of Probability

Probability theory plays a very important role in data communication and networking because this theory is the best way of quantizing uncertainty and the field of data communication is full of uncertainty. For example, when we send a frame, we are not sure how much of it will arrive at the destination uncorrupted. Also, when a station tries to access the network, we are not certain how successful it will be.

This appendix is just a review of basic concepts of probability theory that are needed to understand some topics discussed in this book.

D.1 DEFINITION

Although many definitions have been written for probability, we use the classical one, which is close to our purpose.

The probability of an event A is a number, P [A], that is interpreted as

$$\mathbf{P\,[A] = N_A / N}$$

where N is the total number of possible outcomes (also referred to as the sample space) and N_A is the number of possible outcomes related to event A.

Example D.1

We flip a coin. What is the probability of a head?

Solution

The total number of outcomes is 2 (*head* or *tail*). The number of possible outcomes related to this event is 1 (only *head*). Therefore, we have

$$\mathbf{P\,[head] = N_{head} / N = 1 / 2}$$

Example D.2

We roll a die. What is the probability of getting a 5?

Solution

The total number of outcomes is 6 (1, 2, 3, 4, 5, 6). The number of possible outcomes related to this event is 1 (only one 5). Therefore, we have

$$\mathbf{P\,[5] = N_5 / N = 1 / 6}$$

Example D.3

We flip two coins. What is the probability of getting two heads?

Solution

The total number of outcomes is 4 (head-head, head-tail, tail-head, or tail-tail). The number of possible outcomes related to this event is 1 (head-head). Therefore, we have

$$\mathbf{P\ [head\text{-}head] = N_{head\text{-}head} / N = 1 / 4}$$

D.2 AXIOMS AND PROPERTIES

To be able to use probability theory, we need axioms and properties.

D.2.1 Axioms

To find the probabilities of events, we accept some axioms. Axioms cannot be proved, but they are assumed. The following three axioms are fundamental to probability theory.

Axiom 1

This axiom states that the probability of an event is a nonnegative value:

$$\mathbf{P\ [A] \geq 0}$$

Axiom 2

This axiom states that the probability of the sample space is 1. In other words, the probability that one of the possible outcomes occurs is 1:

$$\mathbf{P\ [S] = 1}$$

Axiom 3

This axiom states that if A1, A2, A3,...... are disjoint events (occurrence of one, does not change the probability of the occurrence of the others), then

$$\mathbf{P\ [A1\ or\ \ A2\ or\ A3\ or.........] = P[A1] + P[A2] + P[A3] + ...}$$

D.2.2 Properties

Accepting the above axioms, a list of properties can be proven. The following is the minimum number of properties we need in order to understand the rest of this discussion (we leave the proofs to the books on probability):

Property 1

If A is an event and A’ is the complement of that event, then we have

$$\mathbf{P\,[A] = 1 - P[A']}$$

For example, if the probability of getting a 2 in rolling a die is 1/6, the probability of not getting a 2 is 1 – 1/6 or 5/6.

Property 2

We always have an outcome or

$$\mathbf{P\,[no\ outcome] = 0}$$

In other words, if we roll a die, the probability that none of the numbers shows is 0; that is, a number always shows.

Property 3

If an event is a subset of another event, the probability of the first event is less than or equal to the probability of the second event.

$$\mathbf{If\ A\ is\ a\ subset\ of\ B,\ then\ P[A] \leq P[B]}$$

For example, the probability of getting a 2 or 3 in a roll of dice, P [2 or 3], is less than the probability of getting 2, 3, or 4, P [2 or 3 or 4].

Property 4

The probability of an event is always between 0 and 1.

$$\mathbf{0 \leq P[A] \leq 1}$$

Property 5

If A, B, C,... are independent events, then

$$\mathbf{P\,[A\ and\ B\ and\ C\ and...] = P\,[A] \times P\,[B] \times P\,[C] \times ...}$$

If the events are independent (occurrence of the one does not change the probability of the occurrence of the others), then the probability of all events happening together is the product of their probabilities.

D.3 REPEATED TRIALS

So far, we have concentrated on the probability of events in a single trial such as flipping one coin. We are also interested in the probability of events when there is more than one trial. For example, what is the probability of getting one *head* if we flip a coin 10 times? What is the probability of getting five *heads* if we flip the coin 20 times.

Example D.4

Let us find the probability of getting exactly one *head* when we flip a biased coin three times. Assume that probability of getting a head is p, which means that the probability of not getting a head (getting a tail) is $1 - p$.

Solution

We can get one head either in the first trial, the second trial, or the third trial. However, if a head comes up in one of the trials, it must not come up in the other two trials. We can therefore say that

P [only one head in three trials] =
P [a head in first trial] × P [a tail in second trial] × P [a tail in third trial] +
P [a head in second trial] × P [a tail in first trial] × P [a tail in third trial] +
P [a head in third trial] × P [a tail in first trial] × P [a tail in second trial]

Using the probabilities, we get

P [only one head in three trials] =
$$p \times (1 - p) \times (1 - p) + p \times (1 - p) \times (1 - p) + p \times (1 - p) \times (1 - p)$$

Or we can say that

P [only one head in three trials] $= 3\,p \times (1 - p)^2$

Example D.5

We send a small frame of 3 bits. If the probability of each bit changing in transmission is 0.10 and each bit is independent, what is the probability that exactly one bit changes?

Solution

This problem is similar to Example 1. We can assume that sending a bit corresponds to flipping a biased coin; a bit can reach the destination with or without change. Therefore, we have

P [exactly one bit is changed] $= 3\,p \times (1 - p)^2 = 3\,(0.1)\,(1 - 0.1)^2 = 0.243$

D.3.1 Bernoulli Trials

Bernoulli found the probability of k successful occurrences of an event in n trials. Assuming the probability of success is p and the probability of failure is q (or $1 - p$), then

$$\textbf{P [A successful event occurs } k \textbf{ times in } n \textbf{ trials]} = \mathbf{C}\ (n, k)\ p^k\ q^{n-k}$$

where C (n, k) is the combination of n objects k at a time; its value is

$$\mathbf{C}\ (n, k) = (n!)\ /\ [k!\ (n - k)!]$$

Examples 1 and 2 are cases of Bernoulli trials with $n = 3$ and $k = 1$.

Example D.6

We send a small frame of 10 bits. If the probability that a bit changes in transmission is 10 percent (0.1) and each bit is independent, what is the probability that exactly three bits change?

Solution

Using the result of Bernoulli trials, we can find the probability of three bits changing as

$$\textbf{P [exactly three bits changed]} = \mathbf{C\ (10, 3)\ p^3 \times (1 - p)^7 = 0.057}$$

or 5.7 percent, which is much less than the probability of one bit being changed.

Example D.7

In Example D.6, what is the probability of no bits changing?

Solution

We can find the probability of no changes

$$\textbf{P [no changes]} = \mathbf{C\ (10, 0)\ p^0 \times (1 - p)^{10} = 0.346}$$

or 34.6 percent. Note that C (10, 0) here is just 1.

Example D.8

Let us assume that we have a CSMA/CD network with n stations. For any time slot, the probability that a station has a frame to send is p. Now the question is what is the probability that a time slot is used successfully (with no collision). This probability is important for calculating the efficiency of the network.

Solution

For a successful time slot, one and only one station has a frame to send; all others do not. However, this station can be any of the n stations. This problem is similar to flipping a coin and getting exactly one *head* in n trials. Think of each station as a trial and a successful slot as a head. The probability can be calculated by adding the probability that the first station has a frame to send with the probability that the others do not, p $(1 - p)^{n-1}$, or the second station has a frame to send with the probability that the others do not, p $(1 - p)^{n-1}$, and so on. We can use the Bernoulli formula to calculate one success out of n trials.

$$\textbf{P [successful slot]} = \mathbf{C}\ (n, 1)\ p\ \ (1 - p)^{n-1} = n\ p\ (1 - p)^{n-1}$$