SDS 385 Exercise Set

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The Laplacian matrix has the following form (where $d(i)$ is the degree of vertex i):

$$\mathcal{L}_{i,j} = \begin{cases} d(i) & \text{for } i = j \\ -1 & \text{for } i \text{ adjacent to } j \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

If we let D_i denote the ith column of D, then the i,jth element of $D^T D$ is equal to $D_i \cdot D_j$. We now seek to show that $D_i \cdot D_j$ is equivalent to the i,jth element of $\mathcal{L}$ as described in Equation (1).

Proof. From the description in the text, D_i has nnz (number of nonzero elements) equal to the degree of vertex i. Let $nzi(D_i)$ be the nonzero indicies of D_i, that is, if $j \in nzi(D_i)$, then D_{ij} (the j-th element of D_i) is equal to either plus or minus one. Then

$$D_i \cdot D_i = \sum_{j \in nzi(D_i)} D_{ij}^2 = \sum_{j \in nzi(D_i)} 1 = \text{nnz}(D_i) = d(i)$$

So $D^T D_{i,j} = d(i)$ when $i = j$, which matches Equation (1).

Now suppose $i \neq j$. If vertex i and vertex j do not share any edges, then there does not exist a k such that $D_{ik} \neq 0$ and $D_{jk} \neq 0$. Then $D_i \cdot D_j$ is necessarily zero, in agreement with Equation (1)

Now suppose that the vertices share one (and exactly one) edge. Suppose this edge is edge k. Then if $i < j$, $D_{ik} = 1$ and $D_{ji} = -1$, or vice versa if $i > k$. In either case, $D_i^T D_j = -1$. $\square$

Since they are elementwise equal, $\mathcal{L} = D^T D$, and so $x^T \mathcal{L} x = x^T D^T D x = \|Dx\|_2^2$.

This minimization problem can now be expressed as

$$\min_{x \in \mathbb{R}^n} \frac{1}{2}(y - x)^T (y - x) + \frac{\lambda}{2}(Dx)^T (Dx)$$

After some algebraic manipulation, we find that this is equivalent to

$$\min_{x \in \mathbb{R}^n} \frac{1}{2} x^T (\lambda D^T D + I^T I) x - y^T x + \frac{y^T y}{2}$$

Since this is a quadratic form, the minimum is found by solving

$$(\lambda D^T D + I)\hat{x} = y$$

so we choose $b = y$ and $C = (\lambda D^T D + I)$