Trilateration Index - Masters Thesis

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Contents

Using Trilateration Distances as an Alternate Coordinate System

Abstract

We present an alternative method for pre-processing and storing point data (such as Geospatial data) to improve query performance for some distance related queries such as "find all points in a set P within distance d of query point q". This pre-processing can be used to improve existing algorithms for Nearest-Neighbor (NN) searches, and lend themselves to new algorithms which offer some performance improvements and tradeoffs vs. existing algorithms.

While our results show that we aren't able to reach performance of well-tuned modern graph-based NN algorithms such as FAISS and HNSW on euclidean datasets, our approach has the benefit of being easily implemented in existing SQL databases, where we see a roughly 10x performance improvement to other in-database queries. Our approach is particularly beneficial on geospatial data where a single distance calculation may be significantly more expensive than in euclidean distance calculations.

Our construction includes storing the distances from fixed points (typically three, as in trilateration) as an alternative to Latitude and Longitude. This effectively creates a coordinate system where the coordinates are the trilateration distances. We explore this alternative coordinate system and the theoretical, technical, and practical implications of using it. Trilateration is a common concept, widely used in GPS and closely related to Triangulation used in cartography, surveying, and orienteering, although algorithmic use of these concepts for NN-style problems are scarce. Trilateration (or, more generally, "n-point lateration") is applicable to 2D, 3D, and higher dimension systems with minimal adaptation.

Initial results are promising for some use cases. Nearest-neighbor logic is both simplified compared to space-partitioning (R/kd/ball-Tree) or graph-based (FAISS or HNSW) approaches, and performant in some use cases.

Further, we discuss the problem of "Network Adequacy" common to medical and communications businesses, to analyze questions such as "are at least 90% of patients living within 50 miles of a covered emergency room". This is in fact the class of question that led to the creation of our pre-processing and algorithms, and is a generalization of a class of Nearest-Neighbor problems.

We show constructions of the Trilateration Index in Python and SQL to show flexibility and to allow benchmarking against existing solutions in both domains.

While we focus primarily on geospatial data, potential applications to this approach extend to any distance-measured n-dimensional metric space, and we touch on those (briefly, here, to constrain our scope). For example, we consider applying the technique to Levenshtein distance, MINST datasets via cosine-similarity, and even facial recognition or taxicab systems where distances do not follow the triangle inequality.

1 The Trilateration Geospatial Index

1.1 Trilateration Index – General Definition

Given an n-dimensional metric space (M, d) (the universe of points M in the space and a distance function d which respects the triangle inequality), a typical point X in the coordinate system will be described by coordinates $x_1, x_2, \ldots, x_n$, which, typically, represents the decomposition of a vector V from an "origin" point $O : 0, 0, \ldots, 0$ to X into orthogonal vectors $0, x1, 0, x2, \ldots, 0, xn$ along each of the n dimensional axes of the space.

The Trilateration of such a point requires $n+1$ fixed points F_p (p from 1 to $n+1$), no three of which occupy the same $(n-1)$-dimensional hyperplane. The Trilateration Coordinate for the point X is then: $t_1, t_2, \ldots, t_{n+1}$ where t_i is the distance (according to d) from X to F_i (in units applicable to the system).

1.2 Underlying Mathematical Concepts

The benefit of storing geographic points as a set of trilateration distances rather than latitude and longitude boils down to the simplification of comparing distances between points by shortcutting complex distance queries using simple subtractions. We discuss the math behind the geospatial queries, to exhibit their complexity, and set some theoretical bounds on quick distance calculations using the trilateration index.

1.2.1 High Cost of Geospatial Calculations

Calculating the distance between two points around the globe with precision is required for Satellite Communications and Geospatial Positioning Systems (GPS), as well as for ground based surveying and generally all applications requiring precise (sub-meter) measurements accounting for the curvature of the earth.(ASPRS 2015)

1.2.1.1 Haversine

One of the simplest distance calculations between two points on the earth's surface – namely the Haversine Formula (Gade 2010), which dates to the early 1800s– works by assuming the earth is a sphere. The calculation for the distance between two points on the earth, using this formula goes as:

Given the radius of a spherical representation of the earth as $r = 6356.752km$ and the coordinates of two points (latitude, longitude) given by (ϕ_1, λ_1) and (ϕ_2, λ_2), the distance d between those points along the surface of the earth is:

$$d = 2r \sin^{-1}\left(\sqrt{\sin^2\left(\frac{\phi_2 - \phi_1}{2}\right) + \cos(\phi_1)\cos(\phi_2)\sin^2\left(\frac{\lambda_2 - \lambda_1}{2}\right)}\right)$$

Obviously this is somewhat computationally complex, comprising five trigonometric functions, two subtractions and a square root. While it is a closed form solution, it causes an error over long distances of up to 0.3%, which can mean distances are off by up to 3 meters over distances of 1000 kilometers. From the equator to the north pole, which on a sphere is defined as precisely 10,000 km, the actual distance is off by over 2 km, which is a sizeable error for even the most robust applications.

1.2.1.2 Vincenty and Karney's Improvements (Geodesics)

The shortcomings of the spherical calculation was thoroughly discussed by Walter Lambert in 1942.(Lambert 1942) However it wasn't until 1975 that an iterative computational approach came about to give more accurate distance measurements with a model of the earth more consistent with reality. By considering the earth as an ellipsoid, rather than a sphere, the distance calculations are more complex, but far more precise. Vincenty was able to create an iterative approach accurate down to the millimeter level on an ideal elliptical earth; far more accurate than the Haversine calculations(Vincenty 1975). This algorithm, however, was a series which failed to converge for points at near opposite sides of the earth. Karney was able to improve upon this in 2013 to fix these antipodal non-convergences, and the resulting formulae are now widely available in geospatial software libraries where precision is required (commonly referred to as "Geodesic" distances. (Karney 2013)

To get an idea of the relative complexity, we ran some basic timings using widely available python libraries that perform both calculations. The Haversine is about 22 times faster than Karney's iterative approach. For comparison, we include Euclidean functions, which are of course computationally simple, although their usefulness on curved surfaces are minimal:

Table 1: Timings (seconds) of 5000 Calls to Distance Functions

title	time	ratio
Geodesic	1.2110690	575.54843
Haversine	0.0553729	26.31542
Euclidean	0.0021042	1.00000

1.3 Simple distance functions

Before jumping into Network Adequacy and Nearest Neighbor algorithms let's look at the core usage of the trilateration data structure and its use in simple distance functions.

What we mean by 'simple distance functions' is one of the following primitive functions common to SQL or map related software libraries:

- $D(p, q)$: returns the distance between points p and q
- $Within(d, q, P)$: returns the set of all points in P within distance d of query point q
- $AnyWithin(d, q, P)$: returns a boolean result - True if $Within(d, q, P)$ is non-empty; False otherwise

1.3.1 Distance Function

How can we use the Trilateration Index (TI) to improve the performance of a single distance function $D(p, q)$? In the simplest case, we cannot… the construction of the TI structures requires three distance functions to be calculated each for p and q (to the three fixed reference points).

However, for large datasets with fixed points where many distances need to be calculated between them, particularly if the distance function itself is computationally intensive (such as geospatial distances on an accurate ellipsoid model of earth) (Lambert 1942), we can use the TI structure to create approximate distances, and provide upper and lower bounds on exact values.

For example, let's take our sample data:

1.4 Network Adequacy

The trilateration index was originally designed to improve efficiency of the "network adequacy" problem for health care. Network adequacy is a common legal requirement for medicare or insurance companies with constraints such as:

- 90% of members must live within 50 miles of a covered emergency room
- 80% of female members over the age of 13 must live within 25 miles of a covered OB/GYN
- 80% of members under the age of 16 must live within 25 miles of a covered pediatrician
- etc.

Similar requirements, legal or otherwise, show up in cellular network and satellite communication technology (numbers are illustrative):

- Maximize the number of people living within 10 miles of a 5G cell tower
- 100% of all major highways should be within 5 miles of a 4G cell tower
- There must be at least 2 satellites within 200 km of a point 450 km directly above every ground station for satellite network connectivity at any given time
- There must be at least 1 satellite with access to a ground station within 50 km of a point 450 km directly above as many households as possible at any given time

The nearest-neighbor problem was called the "Post-Office Problem" in early incarnations, and the system of post offices lends itself to a similar construction: * Ensure that all US Postal addresses are within range of a post office

and so forth.

It is worth noting that the phrase "Network Adequacy" appears in studies of electric grids bearing a meaning that is NOT related to these distance algorithms. (Mahdavi and Mahdavi 2011; Ahmadi et al. 2019)

Note that these are all illustrative examples; the real "Medicare Advantage Network Adequacy Criteria Guidance" document for example, is a 75 page document.

1.4.1 Formalization of Network Adequacy

We formalize the concept of "Network Adequacy" mathematically:

1.4.1.1 Network Adequacy Definition

Given a non-empty set of points P and a non-empty set of query points Q in a metric space M (where $P \cap Q$ comprises the 'network'), the network is 'adequate' for a distance d and a distance function $D(a, b)$ describing the distance between points a and b for $a \in M$ and $b \in M$ if for every point q (where $q \in Q$) $\exists$ at least one point p ($p \in P$) $\ni D(p, q) <= d$. Otherwise the network is 'inadequate'.

1.4.1.2 Network Adequacy Threshold Definition

We can generalize this slightly more by describing a network as 'adequate with threshold T' by introducing a percent T ($0 <= T <= 1$) such that the same network is adequate if for at least $T * |Q|$ (or T percent of points in Q) there exists at least one point $p \ni D(p, q) <= d$.

In this case, if $T == 1$ we have the original case. If $T == 0$ we have a trivial case where the network is always adequate (even if Q and/or P are empty, which is generally disallowed).

1.4.2 Existing solutions

We can find no literature where this topic is solved in a particular algorighmic way. There are numerous discussions in health care about satisfying network adequacy, but more as policy or health care topics. (Wishner and Marks 2017),(Mahdavi and Mahdavi 2011)

In general, it appears that most practical solutions are done in SQL databases which are commonly the source of member and provider data for health care datasets. Still, there is little published here; this information is anecdotal based on the author's personal direct knowledge and informal research.

Satellite and cellular network discussions of this problem appear to be proprietary.

Where we can find references to actual applications, the implemented solutions tend to be iterative, exhaustive implementations of existing Nearest-Neighbor algorithms. That is, for each point q, find the nearest point p

and if $D(p, q) < d$ count it as conforming, otherwise count it as non-conforming. We then calculate the ratio of $r = \frac{conforming}{|Q|}$ if $r >= T$.

If we set $m = |Q|$ and $n = |P|$, and if we use a Nearest-Neighbor algorithm with $O(n\ log\ n)$ then the time complexity for Network Adequacy becomes $O(mn\ log\ n)$.

In the worst-case, we cannot improve on this mathematically, but we can introduce what we believe are novel algorithms, based on the Trilateration Index, which execute efficiently compared to this iterative approach, by deeply reducing the search space and number of times the distance functions must be called in typical real-world cases. Also, trivially, it is unnecessary to find the nearest point p, merely prove that at least one such point exists (or none exists) for a given q.

1.4.3 Experimental Results for Network Adequacy

TODO:

1.5 Nearest Neighbor

1.6 Random 20-dimension euclidean distance:

```
writing output to results/random-xs-20-euclidean.png
  0:                                   TrilatExpand        1.000      401.899
  1:                                  Trilateration        1.000       25.971
  2:                            TrilaterationApprox        0.585       71.236
  3:                           TrilaterationExpand2        1.000     1411.711
  4:                         BallTree(leaf_size=40)        1.000     3040.686
  5:                         BallTree(leaf_size=10)        1.000     3974.894
  6:                       BallTree(leaf_size=1000)        1.000     2224.275
  7:                        BallTree(leaf_size=100)        1.000     2561.925
  8:                        BallTree(leaf_size=400)        1.000     2265.526
  9:                        BallTree(leaf_size=200)        1.000     2352.948
 10:                         BallTree(leaf_size=20)        1.000     3532.328
 11:                                   BruteForce()        1.000      670.163
```

1.7 Glove 25-dimension Angular distance

```
writing output to results/glove-25-angular.png
  0:               FaissIVF(n_list=2048, n_probe=1)        0.416     1938.482
  1:              FaissIVF(n_list=4096, n_probe=50)        0.965     1098.394
  2:               FaissIVF(n_list=64, n_probe=200)        1.000       35.085
  3:                 FaissIVF(n_list=64, n_probe=1)        0.664     1123.371
  4:            FaissIVF(n_list=1024, n_probe=200)        1.000      164.781
  5:                 FaissIVF(n_list=32, n_probe=1)        0.704      714.849
  6:            FaissIVF(n_list=4096, n_probe=100)        0.988      733.391
  7:               FaissIVF(n_list=8192, n_probe=1)        0.334     1703.636
  8:               FaissIVF(n_list=32, n_probe=200)        1.000       35.108
  9:               FaissIVF(n_list=512, n_probe=50)        0.995      305.273
 10:              FaissIVF(n_list=128, n_probe=100)        1.000       43.439
 11:                FaissIVF(n_list=32, n_probe=10)        0.995      108.414
 12:            FaissIVF(n_list=2048, n_probe=200)        0.999      295.694
 13:              FaissIVF(n_list=8192, n_probe=50)        0.946     1201.676
 14:                FaissIVF(n_list=256, n_probe=1)        0.554     1742.011
```

15:	FaissIVF(n_list=512, n_probe=100)	0.999	166.827
16:	FaissIVF(n_list=64, n_probe=100)	1.000	35.081
17:	FaissIVF(n_list=4096, n_probe=200)	0.997	441.810
18:	FaissIVF(n_list=256, n_probe=200)	1.000	44.869
19:	FaissIVF(n_list=8192, n_probe=100)	0.978	932.471
20:	FaissIVF(n_list=1024, n_probe=10)	0.896	1358.564
21:	FaissIVF(n_list=8192, n_probe=5)	0.658	1709.966
22:	FaissIVF(n_list=128, n_probe=10)	0.971	354.258
23:	FaissIVF(n_list=8192, n_probe=10)	0.776	1569.960
24:	FaissIVF(n_list=64, n_probe=5)	0.948	363.314
25:	FaissIVF(n_list=128, n_probe=50)	1.000	83.435
26:	FaissIVF(n_list=128, n_probe=1)	0.607	1517.637
27:	FaissIVF(n_list=1024, n_probe=1)	0.467	1894.326
28:	FaissIVF(n_list=32, n_probe=5)	0.970	203.792
29:	FaissIVF(n_list=256, n_probe=50)	0.998	169.421
30:	FaissIVF(n_list=256, n_probe=5)	0.884	997.154
31:	FaissIVF(n_list=512, n_probe=200)	1.000	87.226
32:	FaissIVF(n_list=512, n_probe=10)	0.925	990.031
33:	FaissIVF(n_list=1024, n_probe=50)	0.990	495.363
34:	FaissIVF(n_list=1024, n_probe=5)	0.805	1648.008
35:	FaissIVF(n_list=2048, n_probe=50)	0.979	826.074
36:	FaissIVF(n_list=256, n_probe=10)	0.952	605.637
37:	FaissIVF(n_list=2048, n_probe=100)	0.994	465.453
38:	FaissIVF(n_list=4096, n_probe=1)	0.375	1878.121
39:	FaissIVF(n_list=256, n_probe=100)	1.000	88.562
40:	FaissIVF(n_list=128, n_probe=5)	0.918	591.155
41:	FaissIVF(n_list=32, n_probe=100)	1.000	35.103
42:	FaissIVF(n_list=4096, n_probe=5)	0.708	1772.210
43:	FaissIVF(n_list=512, n_probe=5)	0.845	1353.636
44:	FaissIVF(n_list=8192, n_probe=200)	0.993	615.649
45:	FaissIVF(n_list=4096, n_probe=10)	0.819	1693.549
46:	FaissIVF(n_list=1024, n_probe=100)	0.997	300.927
47:	FaissIVF(n_list=128, n_probe=200)	1.000	34.741
48:	FaissIVF(n_list=64, n_probe=50)	1.000	44.832
49:	FaissIVF(n_list=64, n_probe=10)	0.986	205.624
50:	FaissIVF(n_list=2048, n_probe=10)	0.858	1608.728
51:	FaissIVF(n_list=2048, n_probe=5)	0.757	1812.419
52:	FaissIVF(n_list=32, n_probe=50)	1.000	35.020
53:	FaissIVF(n_list=512, n_probe=1)	0.507	1873.798
54:	TrilaterationExpand2	1.000	6.125
55:	BallTree(leaf_size=200)	1.000	8.084
56:	BruteForce()	1.000	3.765

1.8 Review of Current Literature

In general, we split our current literature review into two major sections: 1. Geospatial considerations and 2. Nearest Neighbor Algorithms, and a third minor section for ancillary references outside those two areas:

1.8.1 Geospatial References

1.8.1.1 Haversine

1.8.1.2 Vincenty's Formula

Vincenty's Formula is a common non-spherical numeric solution to Earth-shaped ellipsoidal distance calculations [https://arxiv.org/pdf/1109.4448.pdf] however it "fails to converge for nearly antipodal points" #### Karney's Formula - https://arxiv.org/abs/1109.4448 Improvements upon Vincenty's Formula have already been implemented in Python (https://pypi.org/project/geographiclib/), which we use in our Python implementations.

1.9 2-D Bounded Example

Consider a 2-dimensional grid – a flattened map, a video game map, or any mathematical $x - y$ coordinate grid with boundaries. WOLOG in this example consider the two-dimensional Euclidean space $M = \mathbb{R}^{\nvDash}$ and bounded by $x, y \ \epsilon \ \{0..100\}$. Also, let us use the standard Euclidean distance function for d. This is, trivially, a valid metric space.

Since the space has dimension $n = 2$, we need 3 fixed points F_p. While the Geospatial example on Earth has a specific prescription for the fixed points, an arbitrary space does not. We therefore prescribe the following construction for bounded spaces:

Construct a circle (hypersphere for other dimensions) with the largest area inscribable in the space. In this example, that will be the circle centered at $(50, 50)$ with radius $r = 50$.

Select the point at which the circle touches the boundary at the first dimension (for spaces with uneven boundary ratios, select the point at which the circle touches the earliest boundary x_i). Such a point is guaranteed to exist since the circle is largest (if it does not, then the circle can be expanded since there is space between every point on the circle and an axis, and it is not a largest possible circle).

From this point, create a regular $n + 1$-gon (triangle here) which touches the circle at $n + 1$ points. These are the points we will use as F_p. They are, by construction, not all co-linear (or in general do not all exist on the same n-dimensional hyperplane) satisfying our requirement [proof].

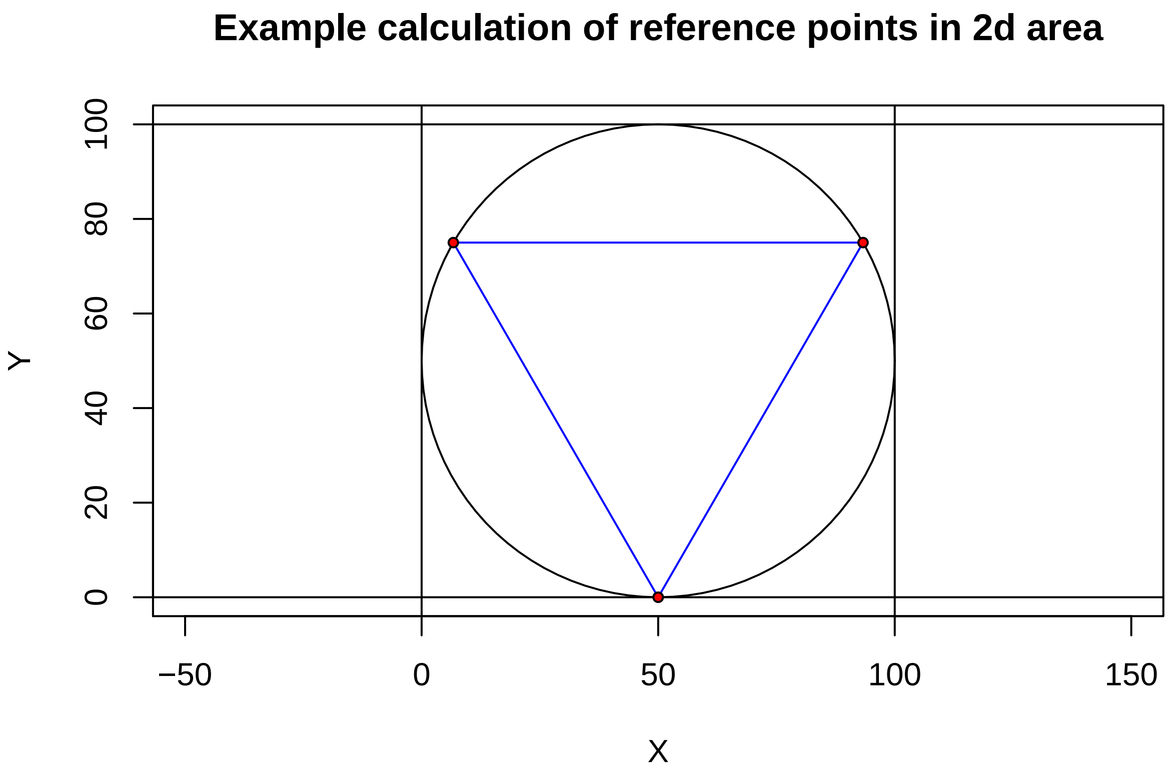
The point $y = 0$, $x = 50$ is the first point of the equilateral triangle. The slope of the triangle's line is $tan(\frac{pi}{3})$, so setting the equation of the circle:

$(x - 50)^2 + (y - 50)^2 = 50^2$ equal to the lines: $y = tan(\frac{\pi}{3})(x - 50)$ gives $x = 25(2 + \sqrt{3})$ on the right and $y = tan(\frac{-\pi}{3})(x - 50)$ gives $x = -25(\sqrt{3} - 2)$ on the left, and of course the original $(0, 50)$ point. Applying x to our earlier equations for y we get a final set of three points:

$$F_1 = (x = 50, y = 0)$$

$$F_2 = (x = 25(2 + \sqrt{3}), y = tan(\frac{\pi}{3})((25(2 + \sqrt{3})) - 50)$$

$$F_3 = (x = -25(\sqrt{3} - 2), y = tan(\frac{-\pi}{3})((-25(\sqrt{3} - 2)) - 50)$$



Remember, any three non-colinear points will do, but this construction spaces them fairly evenly throughout the space, which may be beneficial later* [Add section (reference) with discussions of precision and examples where reference points are very near one another].

The trilateration of any given point X in the space, now, is given by:

$$T(X) = d(F_1, X), d(F_2, X), d(F_3, X)$$

That is, the set of (three) distances d from X to F_1, F_2, and F_3 respectively.

1.9.0.1 10 Random Points

As a quick example of the trilateration calculations, we use a basic collection of 10 data points:

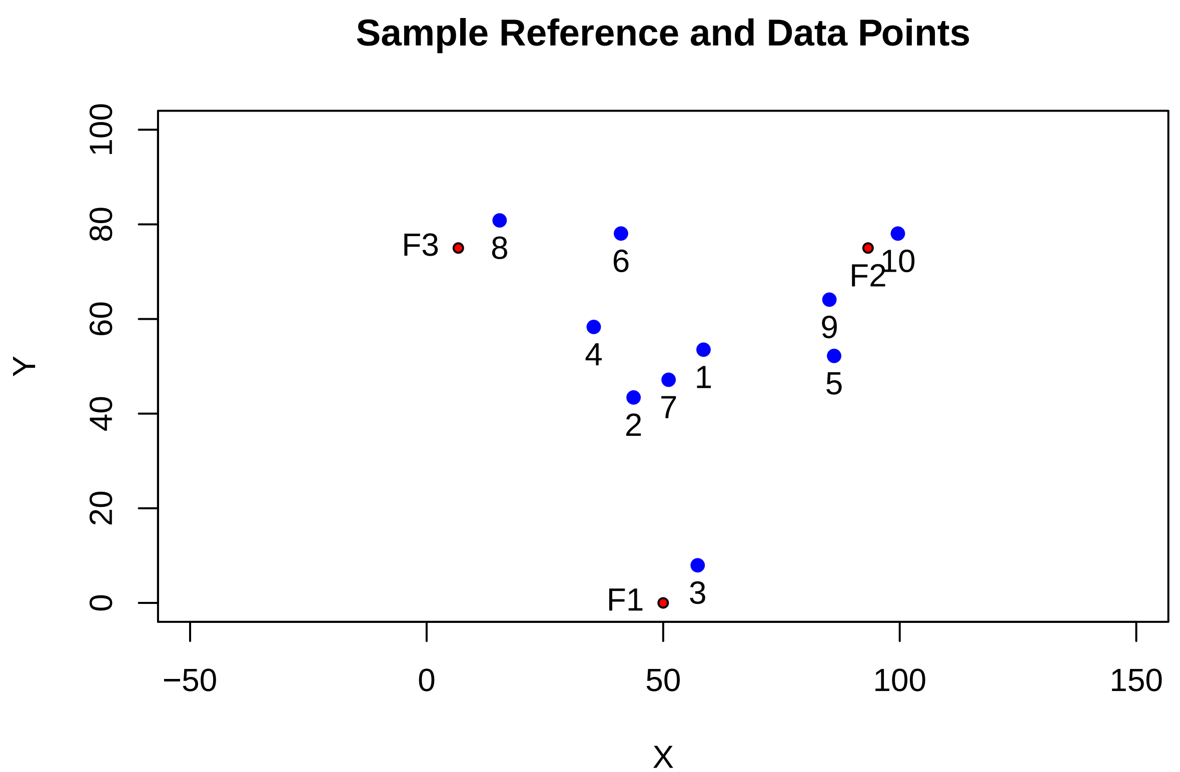
```
##           x         y
## 1  58.52396 53.516719
## 2  43.73940 43.418684
## 3  57.28944  7.950161
## 4  35.32139 58.321179
## 5  86.12714 52.201894
## 6  41.08036 78.065907
## 7  51.14533 47.157734
## 8  15.42852 80.836340
## 9  85.13531 64.090063
## 10 99.60833 78.055071
```

The trilateration of those points, that is, the three points $d_1, d_2, d_3 = d(F_1, X), d(F_2, X), d(F_3, X)$ are (next to the respective x_n):

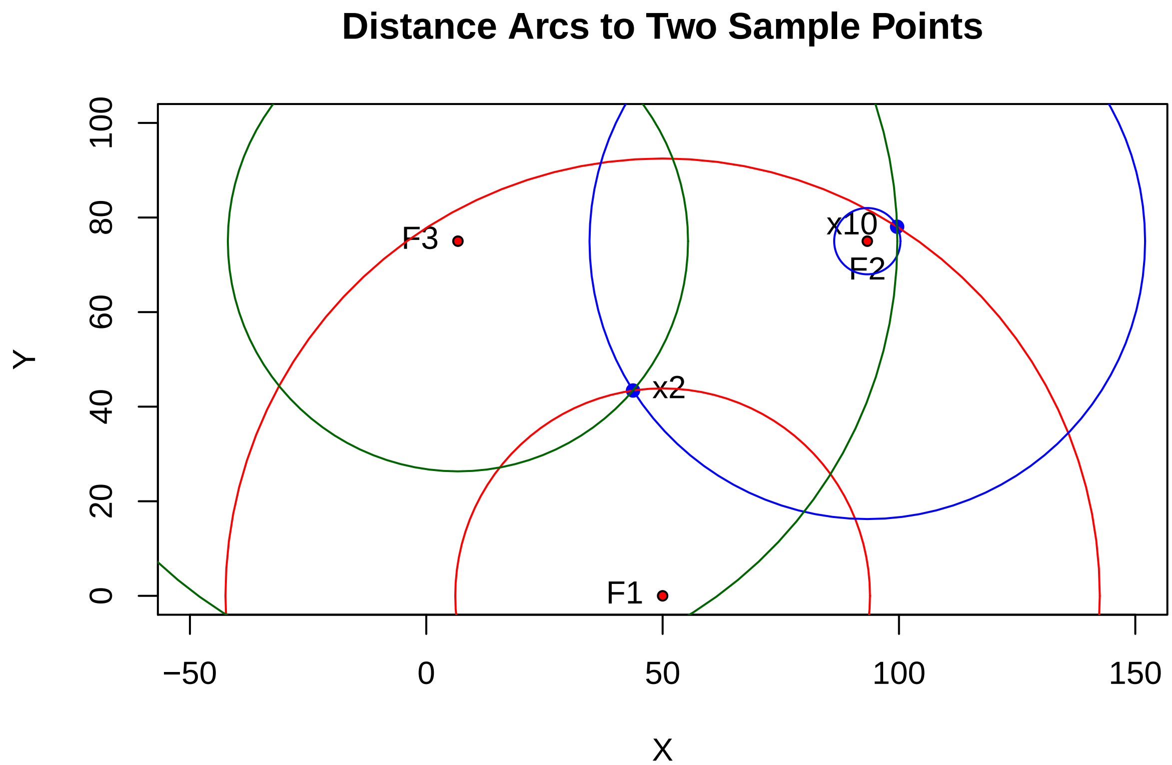
```
##           x         y       d1        d2       d3
## 1  58.52396 53.516719 54.19130 40.877779 56.10157
## 2  43.73940 43.418684 43.86772 58.768687 48.67639
## 3  57.28944  7.950161 10.78615 76.108693 83.99465
## 4  35.32139 58.321179 60.14002 60.331164 33.12763
## 5  86.12714 52.201894 63.48392 23.900246 82.63550
## 6  41.08036 78.065907 78.57382 52.310835 34.51806
## 7  51.14533 47.157734 47.17164 50.520446 52.44704
## 8  15.42852 80.836340 87.91872 78.091149 10.50105
## 9  85.13531 64.090063 73.08916 13.627535 79.19169
## 10 99.60833 78.055071 92.48557  7.008032 92.95982
```

Note that we do not need to continue to store the original latitude and longitude. We can convert the three d_n distances back to Latitude and Longitude within some ϵ based on the available precision. Geospatial coordinates in Latitude and Longitude with six digits of precision are accurate to within < 1 *meter*, and 8 digits is accuract to within < 1 *centimeter*, although this varies based on the latitude and longitude itself; latitudes closer to the equator are less accurate than those at the poles. The distance values d_x are more predictable, since they measure distances directly. While the units in this sample are arbitrary, $F(x)$ in a real geospatial example could be in kilometers, so three decimal digits would precisely relate to 1 *meter*, and so on. This is one reason that we will later examine using the trilateration values as an outright replacement for Longitude and Latitide, and this feature is important when considering storage requirements for this data in large real-world database applications.

For now, continuing with the example, those 10 points are shown here in blue with the three reference points F_1, F_2, F_3 in red:



To help understand the above values, the following chart shows the distances for points x_2 and x_10 above. Specifically, the distances d_1 from point F_1 are shown as arcs in red, the distances d_2 from point F_2 in blue, and d_3 from point F_3 in green.



1.9.0.2 Use of trilateration as an index for nearest-neighbor

One of our expected benefits of this approach is an improvement in algorithms like nearest-neighbor search.

1.9.1 Geospatial Example

Applying this to real sample points; let the following be the initial reference points on the globe:

Point 1: $90.000000, 0.000000$ (The geographic north pole)

Point 2: $38.260000, -85.760000$ (Louisville, KY on the Ohio River)

Point 3: $-19.22000, 159.93000$ (Sandy Island, New Caledonia)

Optional Point 4: $-9.42000, 46.33000$ (Aldabra)

Optional Point 5: $-48.87000, -123.39000$ (Point Nemo)

Note that the reference points are defined precisely, as exact latitude and longitude to stated decimals (all remaining decimal points are 0). This is to avoid confusion, and why the derivation of the points is immaterial (Point Nemo, for example is actually at a nearby location requiring more than two digits of precision).

Only three points are required for trilateration (literally; thus the "tri" prefix of the term), but we include 5 points to explore the pros and cons of n-fold geodistance indexing for higher values of n.

1.10 Theoretical Discussion

1.10.1 Theoretical benefits:

Precision: Queries are not constrained by precision choices dictated by the index, as can be the case in Grid Indexes and similar R-tree indexes. R-tree indexes improve upon naïve Grid Indexes in this area, by allowing the data to dictate the size of individual grid elements, and even Grid Indexes are normally tunable to specific data requirements. Still, this involves analysis of the data ahead of time for optimal sizing, and causes resistance to changes in the data.

Distributed Computing: Trilateration distances can be used as hash values, compatible with distributed computing (I.e. MongoDB shards or Teradata AMP Hashes).

Geohashing: Trilateration distances can be used as the basis for Geohashes, which improve somewhat on Latitude/Longitude geohashes in that distances between similar geohashes are more consistent in their proximity.

Bounding Bands: The intersection of Bounding Bands create effective metaphors to bounding boxes, without having to artificially nest or constrain them, nor build them in advance.

Readily Indexed (B-Tree compatible): Trilateration distances can be stored in traditional B-Tree indexes, rather than R-tree indexes, which can improve the sorting, merging, updating, and other functions performed on the data.

Fault Tolerant: This coordinate system is somewhat self-checking, in that many sets of coordinates that are individually within the correct bounds, cannot be real, and can therefore be identified as data quality issues. For example, a point cannot be 5 kilometers from the north pole (fixed point F1) and 5 kilometers from Louisville, KY (fixed point F2) at the same time. A point stored with those distances could be easily identified as invalid.

Theoretical shortcomings:

Index Build Cost: Up front calculation of each trilateration is expensive, when translating from standard coordinates. Each point requires three (at least) distance calculations from fixed points and the sorting of the resulting three lists of distances. This results in `O(n*logn)` just to set up the index.

*This could be mitigated by upgrading sensor devices and pushing the calculations back to the data acquisition step, in much the way that Latitude and Longitude are now trivial to calculate in practice by use of GPS devices. Also, we briefly discuss how GPS direct measurements (prior to converstion to Lat/Long) may be useful in constructing trilateration values.

Storage: The storing of three distances (32- or 64- bits per distance) is potentially a sizeable percent increase in storage requirement from storing only Latitude/Longitude and some R-Tree or similar index structure.

*Note that if the distances are stored instead of the Lat/Long, rather than in addition to them, storage need not increase.

Projection-Bound: The up-front distance calculations means that transforming from one spatial reference system (I.e. map projection – geodetic – get references to be specific) to another requires costly recalculations bearing no benefit from the calculation. For example a distance on a spherical projection of the earth between a given lat/long combination will be different than the distance calculated on the earth according to the standard WGS84 calculations).

*This said, we expect in most real-world situations, cross-geodetic comparisons are rare.

Difficult Bounding Band Intersection: Bounding Bands intersect in odd shapes, which, particularly on ellipsoids, but even on 2D grids, are difficult to describe mathematically. Bounding boxes on the other hand, while they distort on ellipsoids, are still easily understandable as rectangles.

Figure 1 - An example problem with radio towers R1, R2, and R3, and various receivers. Dashed lines represent the bounding bands with +/- a small distance from a given receiver (center black circle)

Figure 2 - A close look at the intersection of three bounding bands limiting an index search around a point with a search radius giving the circle in A. Note that the area B is an intersection of two of the three bands. Area C is the intersection of all three.

2 Appendixy stuff

Alternate Database Indexes

References (I need to finish reading and digesting these):

http://ieeexplore.ieee.org/document/5437947/

https://prezi.com/chnisgybkshy/gps-trilateration/

https://www.maa.org/sites/default/files/pdf/cms_upload/Thompson07734.pdf

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