UBC MATH 215/255

Intro

Some Common Equations

$\frac{dy}{dx} = ky$	$y(x) = Ce^{kx}$
$\frac{d^2y}{dx^2} = -k^2y$	$y(x) = C_1\cos(kx) + C_2\sin(kx)$
$\frac{d^2y}{dx^2} = k^2y$	$y(x) = C_1e^{kx} + C_2e^{-kx}$

Definitions

ODE Equations where the derivatives are taken with respect to only one variable.

PDE Equations that depend on partial derivatives of several variables.

Order The highest order derivative found in equation

Linear $a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_0(x)y = b(x)$

Coefficients

$a_n(x), a_{n-1}(x), ...a_0$

Homogeneous

$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_0(x)y = 0$

Autonomous

Equation does not depend on the independent variable

Existence and Uniqueness

Picard's Theorem

If $f(x, y)$ is continuous (as a function of two variables) and $\frac{\partial f}{\partial y}$ exists and is continuous near some (x_0, y_0), then a solution to

$$y' = f(x, y), \qquad y(x_0) = y_0,$$

exists (at least for some small interval of x's) and is unique.

Compiled by Simon Ghyselincks 2021

First Order Equations

Integral Solutions

$$y' = f(x)$$
$$\int y'(x)\,dx = \int f(x)\,dx + C$$
$$y(x) = \int f(x)\,dx + C.$$

For an IVP: $y(x) = \int_{x_0}^{x} f(s)\,ds + y_0$

Separable Equations

$$y' = f(x)g(y) \implies \frac{dy}{g(y)} = f(x)\,dx$$
$$\int \frac{dy}{g(y)} = \int f(x)\,dx + C.$$

Linear Equations and Integrating Factor

$$y' + p(x)y = f(x) \implies r(x)y' + r(x)p(x)y = r(x)f(x)$$
$$\frac{d}{dx}r(x) = r(x)p(x) \implies r(x) = e^{\int p(x)\,dx}$$

General Form:

$$\boxed{y = e^{-\int p(x)\,dx}\left(\int e^{\int p(x)\,dx} f(x)\,dx + C\right)}$$

IVP Form:

$$\boxed{y(x) = e^{-\int_{x_0}^{x} p(s)\,ds}\left(\int_{x_0}^{x} e^{\int_{x_0}^{t} p(s)\,ds} f(t)\,dt + y_0\right)}$$

Substitution

1. $v = f(x, y)$ then find y' in terms of v', v, x
2. Rewrite original equation in terms of v', v, x
3. Solve equation for v and resubstitute $f(x, y) = v$

Bernoulli Equations:

For $y' + p(x)y = q(x)y^n$ use $v = y^{1-n}$

Autonomous Equations

$$\boxed{y' = f(y)}$$

Definitions

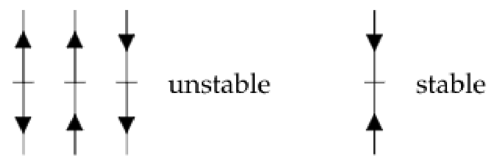
Equilibrium Solution Constant solutions where $y' = 0$

Critical points Points where $f(y) = 0$

Stable A critical point that attracts neighboring positions

Unstable A critical point that is not stable

Phase Diagram:



Euler's Method

$$\boxed{y' = f(x, y),\ \ y(x_0) = y_0}$$

Description

Make successive approximations as follows:

$x_{n+1} = x_n + \Delta x$	$y_{n+1} = y_n + f(x_n, y_n)\Delta x$

Superposition, Existence, Uniqueness

Superposition

Suppose y_1 and y_2 are two solutions of the homogeneous equation

$$y'' + p(x)y' + q(x)y = 0$$

Then $y(x) = C_1y_1(x) + C_2y_2(x)$ also is a solution

If p and q are continuous functions and y_1 and y_2 are linearly independent, then we can also say that it is the general (unique) solution.

Existence and Uniqueness

Suppose p, q, f are continuous functions on some interval I, a is within I, and a, b_0, b_1 are constants:

$$y'' + p(x)y' + q(x)y = f(x),$$

has exactly one unique solution $y(x)$ defined on the same interval I with initial conditions:

$$y(a) = b_0 \qquad y'(a) = b_1$$

Second Order Linear Homogeneous

Constant Coefficient

$$\text{For: } ay'' + by' + cy = 0$$

Try the substitution $y = e^{rx}$, then

$$e^{rx}(ar^2 + br + c) = 0$$

where $ar^2 + br + c = 0$ is the characteristic equation.
Solve for r using quadratic formula.

r **values**	**General Solution**
$r_1 \neq r_2$:	$y = C_1e^{r_1x} + C_2e^{r_2x}$
$r_1 = r_2$:	$y = (C_1 + C_2x)\, e^{r_1x}$
$r \notin \mathbb{R}, r = \alpha + \beta i$:	$y = C_1e^{\alpha x}\cos(\beta x) + C_2e^{\alpha x}\sin(\beta x)$

2nd Order Applications

Mechanical Vibrations

$$mx'' + cx' + kx = F(t)$$

Forced if $F \not\equiv 0$	**Damped** if $c > 0$
Unforced/Free if $F \equiv 0$	**Undamped** if $c = 0$

Undamped

$$mx'' + kx = 0 \implies x'' + \omega_0^2 x = 0,\ \omega_0 = \sqrt{\frac{k}{m}}$$

$$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t) = C\cos(\omega_0 t - \gamma)$$

$$\boxed{x(t) = C\cos(\omega_0 t - \gamma), \quad C = \sqrt{A^2 + B^2}, \quad \tan(\gamma) = \frac{B}{A}}$$

Damped:

$$x_c = \begin{cases} C_1e^{r_1t} + C_2e^{r_2t} & \text{if } c^2 > 4km, \\ C_1e^{-pt} + C_2te^{-pt} & \text{if } c^2 = 4km, \\ e^{-pt}\big(C_1\cos(\omega_1 t) + C_2\sin(\omega_1 t)\big) & \text{if } c^2 < 4km, \end{cases}$$

Overdamping:

$$\boxed{x(t) = C_1e^{r_1t} + C_2e^{r_2t}}$$

Underdamping:

$$r = p \pm \sqrt{p^2 - \omega_0^2} = \quad p \pm \omega_1 i$$

$$x(t) = e^{-pt}\big(A\cos(\omega_1 t) + B\sin(\omega_1 t)\big)$$

$$\boxed{x(t) = Ce^{-pt}\cos(\omega_1 t - \gamma)}$$

RLC Circuit

$$\boxed{LI''(t) + RI'(t) + \frac{1}{C}I(t) = E'(t)}$$

Pendulum Behavior

$$\boxed{\theta'' + \frac{g}{L}\sin\theta = 0}$$

Second Order Linear Nonhomogenous

Constant Coefficient

$$\text{For: } ay'' + by' + cy = g(x) \tag{1}$$

If y_p is a particular solution to the equation (1), and y_c is the complimentary solution to the associated homogeneous equation, then the general solution to (1) is:

$$y = y_p + y_c$$

Method of Undetermined Coefficients

If $g(x)$ above has finitely many linearly independent derivatives, then we can "guess" the proper solution:

$$y_p = A_1P_1(x) + A_2P_2(x) + ...A_nP_n(x)$$

Variation of Parameters

For $y_c = y_1 + y_2$ and $y'' + f(x)y' + h(x)y = g(x)$ **note that the factor of y'' is 1**, to find the particular solution y_p

$$\boxed{y_p = u_1y_1 + u_2y_2}$$

$$\mathcal{W}(y_1, y_2) = \det\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'$$

$$\boxed{u_1 = -\int \frac{y_2g(x)}{\mathcal{W}(y_1, y_2)} \qquad u_2 = \int \frac{y_1g(x)}{\mathcal{W}(y_1, y_2)}}$$

Undamped Forced Oscillations

$$\boxed{mx'' + kx = F_0\cos(\omega t)}$$

Solving for the case where $\omega \neq \omega_0$:

$$\boxed{x = C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t) + \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos(\omega t)}$$

Solving for the case where $\omega = \omega_0$:

$$\boxed{x = C_1\cos(\omega t) + C_2\sin(\omega t) + \frac{F_0}{2m\omega}t\sin(\omega t)}$$

Damped Forced Oscillations

$$\boxed{mx'' + cx' + kx = F_0\cos(\omega t)}$$

$$x_c = \begin{cases} C_1e^{r_1t} + C_2e^{r_2t} & \text{if } c^2 > 4km, \\ C_1e^{-pt} + C_2te^{-pt} & \text{if } c^2 = 4km, \\ e^{-pt}\big(C_1\cos(\omega_1 t) + C_2\sin(\omega_1 t)\big) & \text{if } c^2 < 4km, \end{cases}$$

Particular Solution

$$x_p = \frac{(\omega_0^2 - \omega^2)F_0}{m(2\omega p)^2 + m(\omega_0^2-\omega^2)^2}\cos(\omega t) + \frac{2\omega pF_0}{m(2\omega p)^2 + m(\omega_0^2-\omega^2)^2}\sin(\omega t)$$

$$\boxed{x_p = \frac{F_0}{m\sqrt{(2\omega p)^2 + (\omega_0^2-\omega^2)^2}}\cos(\omega t - \gamma)}$$

For all x_c, the value tends to zero as $t \to \infty$ because of damping. The particular solution becomes dominant over time, with maximum amplitude for the particular solution being reached at:

$$\boxed{\omega = \sqrt{\omega_0^2 - 2p^2}}$$

If $\omega_0^2 - 2p^2 < 0$ then there is no maximum resonance

Systems Of ODEs

Converting 2nd Order to 1st Order System

$$y'' + f(x)y' + g(x)y = h(x)$$

$$x_1 = y, \quad x_2 = y'$$

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -g(x) & -f(x) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ h(x) \end{bmatrix}$$

More generally, $\vec{x}'(t) = A(t)\vec{x}(t) + \vec{k}(t)$

Eigenvalue Analysis

Eigenvalues and vectors

Find the eigenvalues first then solve for vectors:

$$\det(A - \lambda I) = 0$$

$$(A - \lambda I)\vec{v} = \vec{0},$$

$$\boxed{\vec{x} = c_1\vec{v}_1e^{\lambda_1 t} + c_2\vec{v}_2e^{\lambda_2 t} + \cdots + c_n\vec{v}_ne^{\lambda_n t}}$$

$$\vec{x}(t) = \underbrace{\begin{bmatrix} \vec{v}_1e^{\lambda_1 t} & \vec{v}_2e^{\lambda_2 t} & \cdots & \vec{v}_ne^{\lambda_n t}\end{bmatrix}}_{\text{Fundamental Matrix}} \times \underbrace{\vec{c}}_{c_1, c_2 .., c_n}$$

Summary of Two Dimensional Systems	
Eigenvalues	Behaviour
Real and positive	Source/Unstable Node
Real and negative	Sink/Stable Node
Real and opposite sign	Saddle
Purely imaginary	Center Point/ Ellipses
Complex and (+) real	Spiral Source
Complex and (-) real	Spiral Sink

Higher Multiplicity Eigenvalues

Find two linearly independent eigenvectors for the same eigenvalue if the matrix happens to be a scalar matrix. Otherwise, apply the following technique:

$$\vec{x_1} = e^{\lambda_1 t}\vec{v_1}$$

$$\vec{x_2} = e^{\lambda_1 t}(t\vec{v_1} + \vec{v_2}) \text{ where } (A - \lambda_1 I)\vec{v_2} = \vec{v_1}$$

Non-Homogeneous Systems

Undetermined Coefficients

$$\vec{x'} = A\vec{x} + \vec{f}(t)$$

First find the complementary solution x_c for the associated homogeneous system. The particular solution x_p can be solved for many different components of $\vec{f}$ to simplify:

$$\vec{f}(t) = \vec{f_1} + \vec{f_2} ... + \vec{f_n}$$

$$\vec{x}(t) = \vec{x_c} + \vec{x_{p1}} + \vec{x_{p2}} ... + \vec{x_{pn}}$$

Solving for: $\vec{x_{pn}}'(t) = A\vec{x_{pn}} + \vec{f_n}(t)$

2nd Order Systems

Homogeneous

Manipulate the system into the form $\vec{x}'' = A\vec{x}$ and guess that $\vec{x} = \vec{v}e^{\alpha t}$.
Find the eigenvalues of A and then $\alpha^2 = \lambda$

λ values	Solution
$\lambda_i = 0$:	$\vec{x_i}(t) = \vec{v_i}(a_i + b_i t)$
$\lambda_i < 0$:	$\vec{x_i} = \vec{v_i}\left(a_i\cos\sqrt{\lvert\lambda_i\rvert}t + b_i\sin\sqrt{\lvert\lambda_i\rvert}t\right)$
General:	$\vec{x}(t) = \sum_{i=1}^{n}\vec{v_i}\big(a_i\cos(\omega_i t) + b_i\sin(\omega_i t)\big)$

Forced System

$$\vec{x}'' = A\vec{x} + \vec{F}\cos(\omega t)$$

Find the general solution to homogeneous system $\vec{x_c}$, then solve for $\vec{x_p}$ by finding $\vec{c}$

$$\vec{x}_p'' = -\omega^2\vec{c}\cos(\omega t).$$

$$\overbrace{-\omega^2\vec{c}\cos(\omega t)}^{\vec{x}_p''} = \overbrace{A\vec{c}\cos(\omega t)}^{A\vec{x}_p} + \vec{F}\cos(\omega t).$$

If ω matches a natural resonance than we guess:

$$\vec{x}_p = \vec{c}\,t\sin(\omega t) + \vec{d}\cos(\omega t).$$

Non-Homogeneous Systems

Variation of Parameters

$$\vec{x}' = A(t)\,\vec{x} + \vec{f}(t)$$

Find the complementary solution x_c and fundamental matrix for the associated homogeneous system.

$$\vec{x_c} = X(t)\vec{c}$$

$$\boxed{\vec{x}_p = X(t)\int [X(t)]^{-1}\vec{f}(t)\,dt.}$$

$$\begin{Bmatrix} a & b \\ c & d \end{Bmatrix}^{-1} = \frac{1}{\det(X)}\begin{Bmatrix} d & -b \\ -c & a \end{Bmatrix}$$

Non-Linear Systems

Linearization Autonomous System

$$\underbrace{\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}}_{\text{Autonomous System}} \quad \underbrace{\begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix} = \vec{0}}_{\text{Critical Points}}$$

Critical points correspond to equilibrium solutions, inspect the linearized system at these critical points (x_0, y_0):

$$\begin{bmatrix} u \\ v \end{bmatrix}' = \begin{bmatrix} \frac{\partial f}{\partial x}(x_0,y_0) & \frac{\partial f}{\partial y}(x_0,y_0) \\ \frac{\partial g}{\partial x}(x_0,y_0) & \frac{\partial g}{\partial y}(x_0,y_0) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

$$u = x - x_0, \qquad v = y - y_0.$$

Terminology

Critical Points Points where $x' = 0 \ \wedge \ y' = 0$

Isolated Only critical point in some small "neighborhood"

Almost Linear Critical point is isolated and the Jacobian matrix at the point is invertible

Laplace Transforms

Definitions

$$\boxed{\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt}$$

If $\exists M, c, t_0 \in \mathcal{R}, \forall t \in \mathcal{R}(t \geq t_o) \implies |f(t)| \leq Me^{ct}$
Then $f(t)$ is of exponential order
If $f(t)$ is a piecewise continuous function of exponential order for some c, then $\mathscr{L}\{f(t)\}$ is defined for all $s > c$

Application to ODEs

Apply the transform to entire ODE equation, solve for $Y(s)$, apply the reverse transform to $Y(s)$ to get the solution of $y(t)$

Convolution and Products

$$\boxed{(f * g)(t) \stackrel{\text{def}}{=} \int_0^t f(\tau)g(t-\tau)\ d\tau}$$

$$\boxed{\mathcal{L}\{(f*g)(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}.}$$

Table of Laplace Transforms

Laplace Transforms	
$f(t)$	$\mathscr{L}\{f(t)\} = F(s)$
1	$\frac{1}{s}$
$e^{at}f(t)$	$F(s-a)$
$\mathcal{U}(t-a)$	$\frac{e^{-as}}{s}$
$f(t-a)\mathcal{U}(t-a)$	$e^{-as}F(s)$
$\delta(t)$	1
$\delta(t-t_0)$	e^{-st_0}
$t^n f(t)$	$(-1)^n \frac{d^n F(s)}{ds^n}$
$f'(t)$	$sF(s) - f(0)$
$f^n(t)$	$s^n F(s) - s^{(n-1)} f(0) - \cdots - f^{(n-1)}(0)$
$\int_0^t f(x)g(t-x)dx$	$F(s)G(s)$
$t^n \ (n = 0,1,2,\dots)$	$\frac{n!}{s^{n+1}}$
$t^x \ (x \geq -1 \in \mathbb{R})$	$\frac{\Gamma(x+1)}{s^{x+1}}$
$\sin kt$	$\frac{k}{s^2+k^2}$
$\cos kt$	$\frac{s}{s^2+k^2}$
e^{at}	$\frac{1}{s-a}$
$\sinh kt$	$\frac{k}{s^2-k^2}$
$\cosh kt$	$\frac{s}{s^2-k^2}$
$\frac{e^{at}-e^{bt}}{a-b}$	$\frac{1}{(s-a)(s-b)}$
$\frac{ae^{at}-be^{bt}}{a-b}$	$\frac{s}{(s-a)(s-b)}$
te^{at}	$\frac{1}{(s-a)^2}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$e^{at}\sin kt$	$\frac{k}{(s-a)^2+k^2}$
$e^{at}\cos kt$	$\frac{s-a}{(s-a)^2+k^2}$
$e^{at}\sinh kt$	$\frac{k}{(s-a)^2-k^2}$
$e^{at}\cosh kt$	$\frac{s-a}{(s-a)^2-k^2}$
$t\sin kt$	$\frac{2ks}{(s^2+k^2)^2}$
$t\cos kt$	$\frac{s^2-k^2}{(s^2+k^2)^2}$
$t\sinh kt$	$\frac{2ks}{(s^2-k^2)^2}$
$t\cosh kt$	$\frac{s^2+k^2}{(s^2-k^2)^2}$