

## CMPS 102 — Winter Quarter 2017 – Homework 2

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### Solution to Problem 1 - Box Picking

You are a box picker at SoftPillows and your job is to pick the boxes for the pillows your company is shipping out. The fit is rarely perfect, but if the box is too big you can stuff it with bubblewrap, and if the box is too small, you can compress the pillow. Specifically, if box  $b_j$  has length  $m_j$  and pillow  $a_i$  has length  $l_i$ . Given  $n$  boxes and  $n$  pillows each day, your job is to pick a box for each pillow, so that total cost is minimized.

One day your boss comes to you to let you know that they just bought the MTT BoxPicker3000, a machine that automatically matches  $n$  boxes to  $n$  pillows. It works by finding the pair  $(b_j, a_i)$  for which  $|m_j - l_i|$  is minimum, assigning  $a_i$  to  $b_j$ , and repeating, until all pillows have been assigned. Being a proud descendent of a long line of box pickers, it falls to you to try to do better. Is the algorithm for the BoxPicker3000 optimal, or can you find a counterexample? Either way, describe an efficient optimal greedy algorithm for the problem and prove its optimality.

**Algorithm.** *Greedy: Pick boxes and pillows with the smallest size first. The resulting pairings formed by  $(b_i, a_i)$  where  $b_i$  and  $a_i$  are the  $i^{\text{th}}$  box and pillow respectively.*

**Claim.** *Any pairing that claims to be strictly more optimal than our greedy pairing must have adjacent inversions in either the boxes or pillows.*

*Proof.* As argued in class, any sorted array is a long chain of consecutive in-order pairs. If there exists a different solution set of pairs that is not the output of our greedy algorithm, then there must be adjacent elements out of order in either array.  $\square$

Now that we know for sure there must be adjacent inversions of some kind in a different solution set, let's define an inversion for this problem.

**Definition.** *An inversion in schedule  $S$  is a pair of boxes (or pillows)  $i$  and  $j$  such that  $i < j$  but  $j$  is paired with the  $i^{\text{th}}$  pillow (or box).*

We will compare our solution set with the one that claims to be strictly more optimal than ours, i.e. has at least 1 less total cost. We scan both solution sets, and upon finding a disagreement in our pairings, we swap the inversion found at that pair.

**Claim.** *Swapping two adjacent, inverted jobs reduces the number of inversions by 1 and does not increase total cost.*

*Proof.* It is obvious that, given an adjacent inversion, swapping the two resolves the inversion, and thus reduces the total number of inversions by 1. The question is now: "How can we guarantee the resulting pairing does not increase total cost?"

First, recall our definition of an inversion in the schedule on the boxes array, for simplicities sake. We will define three "costs" to keep our eyes on. Let  $C_{i,j,k}$  to represent these costs.

- $C_k$  = Cost of all other pairs not in the inverted pair prior to swapping.
- $C'_k$  = Cost of all other pairs not in the inverted pair after swapping.

- $C_i$  = Cost of the left pair of boxes and pillows prior to swapping.
- $C'_i$  = Cost of the left pair of boxes and pillows after swapping.
- $C_j$  = Cost of the right pair of boxes and pillows prior to swapping.
- $C'_j$  = Cost of the right pair of boxes and pillows after swapping.

Now, let's analyze what happens when we swap the  $i^{th}$  and  $j^{th}$  entries.

$C'_k = C_k$  Since cost is calculated based on (box, pillow) pairs, if no changes are made to those pairs, then the total cost between those pairs stays the same. Thus,  $C_k = C'_k$ .

$C'_i \leq C_i$  For the left pair, observe that the cost can only get lower. This is because any swaps made only replaces a larger box with a smaller one, otherwise the pair of boxes  $(b_i, b_j)$  wouldn't have been scrutinized as being an inverted pair. Thus,  $[C'_i \leq C_i]$ .

$C'_j \leq C_i$  It is less obvious why this is true. By definition,  $C'_j = |m'_j - l_i|$ , where the pair now has a new box length  $m'_j$  to subtract with the old pillow length  $l_i$ .

**Claim 1.** *The new cost  $C'_j$  could be worse than  $C_j$ , but it will never be worse than the old cost of pairing the box with the previous pillow,  $C_i$ .*

*Proof.* Since  $C'_j$  is the difference between the larger box with a larger pillow, regardless of what the new cost of this pairing is, it cannot be greater than the old cost was, since the old cost was calculated with the same larger box with a smaller pillow.  $\square$

Thus, any solution set of pairs claiming to be more optimal than ours can, through surgery (correcting inversions) become our greedy solution set of pairs, with at least no increase in total cost. However, since their solution set is equal to ours, they couldn't possibly have a better total cost, so by proof of contradiction, our greedy algorithm must be the most optimal.  $\square$

### TT BoxPicker3000

While MTT BoxPicker3000 does minimize cost, its algorithm of calculating all possible costs for all combinations and then comparing all costs to pick the pair that yields the lowest one is a brute force solution, which runs in  $O(n^2)$  time. Our greedy algorithm utilizes sorting, which can be done in  $O(n \log n)$  time, so our algorithm is more optimal.

## Solution to Problem 2 - Parade Planning

You are scheduling a parade full for some **very** self-important groups, who have all submitted their float to the parade. Each group has a cutoff time,  $c_i$ , such that if their float comes on at time  $t \leq c_i$ , then all  $p_i$  members of the group will come to watch the parade. If the float comes on at time  $t > c_i$ , then all of them will stay home. All floats take exactly one minute to parade and can start as early as time 0: Your job is to maximize the number of people who show up. Give an efficient algorithm to schedule all floats given  $(c_i, p_i)$  for  $1 \leq i \leq n$ , where each  $c_i \in \mathbb{R}^+$  and each  $p_i \in \mathbb{N}$ .

**Algorithm.** *Sort all groups by cutoff time  $c_i$  in descending order. Then, starting with the latest  $c_i$ , begin with  $t = c_i$ , and if there is any overlap at  $t$ , pick the group with the largest  $p_i$ .  $t$  decreases in value, and the algorithm runs until  $t = 0$ .*

This means that for every time  $t$ , the most number of people who can be present at that time are present at that time. We claim this to be the most optimal algorithm.

**Claim.** *Our algorithm yields an optimal scheduling of groups in the parade such that number of attendees is maximized.*

*Proof.* Proof by Contradiction. If someone claims to have a strictly more optimal (at least 1 more person showing up) than the output of our greedy algorithm, this means that we differ at some point in time  $t$ , where they have chosen a group with a strictly lower  $p_i$  than us at that time, since we always choose the largest possible  $p_i$  at any given time  $t$ . So, for any given point in time where they don't choose the group with the highest  $p_i$ , we simply choose the greatest  $p_i$  for any  $t$ .  $\square$

Of course, this works great except with one glaring question: What if that group has been scheduled to go earlier?

**Claim.** *If a group has been scheduled to go earlier than it is scheduled to go in the greedy schedule, you can always schedule it to go later so to match the greedy algorithm's timing.*

*Proof.* If there is a conflict where a group was scheduled to go later in the greedy schedule, that means the group  $(c_i, p_i)$  can go at some later time  $t$ , at least when it goes in the greedy algorithm (it could still potentially go later, but the fact it doesn't means there's another group that will draw in a bigger crowd that has taken an even later spot). If it were to go at the time of the greedy algorithm, it would in fact at least keep the total number of people attending the event constant, or increase it. This is because at that time in the greedy algorithm, it is the group with the greatest  $p_i$ .  $\square$

What if a group simply isn't scheduled in the greedy schedule, but exists in the challenging schedule?

**Claim.** *If a group is scheduled in the challenging schedule and isn't in the greedy schedule while taking into account the previous conditions, then that group can simply be replaced by the group at that time slot in the greedy schedule.*

*Proof.* If this group is scheduled and the time is truly the latest the group can go, that means that at that point in time it is not the group with the largest  $p$ . This means there is some other group (namely, the one chosen in the greedy algorithm), with the largest  $p$  out of all groups for that time slot. By replacing this old group with the group for that time slot in the greedy algorithm. As the group in the greedy algorithm must have some  $p_j \geq$  the challenging group, so the total number of people can only stay constant or go up.  $\square$

This algorithm sorts the list of tuples  $(c_i, p_i)$  on  $c_i$  in descending order. This can be done in  $O(n \log n)$  time using MergeSort. Then, in a single pass through the amount of time allotted, if we assume the time to pick the group with the largest  $p_i$  is a constant time operation, then the algorithm runs with an upper bound of  $O(n \log n)$ , where the worst case run time is where the number of minutes (i.e.  $t_{max}$ ) is equal to the number of groups  $n$ , where one group is scheduled every minute.

### Solution to Problem 3 - Graph of Spanning Trees

Let  $G$  be a connected graph with two different spanning trees,  $T$  and  $T'$ . We say that  $T$  and  $T'$  are neighbors if  $T$  contains exactly one edge that is not in  $T'$  and  $T'$  contains exactly one edge that is not in  $T$ .

Now, from any graph  $G$ , we can build a graph  $H$  as follows: the vertices of  $H$  are the spanning trees of  $G$ , and there is an edge between two vertices of  $H$  if the corresponding spanning trees are neighbors. Is it true that if  $G$  is connected, then  $H$  is connected?

Given a connected graph  $G$ , we can say by definition that it contains at least one spanning tree, since otherwise the graph wouldn't be connected. With this graph  $H$ , our goal is to return a path between  $u, v$  in  $H$ . In order to prove connectivity of  $H$ , it is enough to prove that we can always return such a path connecting two vertices in  $H$ .

## Solution to Problem 4 - Mixed Trees

Let  $G = G(V, E)$  be an arbitrary, connected, undirected graph with vertex set  $V$  and edge set  $E$ . Assume that every edge in  $E$  represents either a road or a bridge. Give an efficient algorithm that takes an input  $G$  and decides whether there exists a spanning tree of  $G$  where the number of edges that represents roads is  $\lfloor |V|/\sqrt{2} \rfloor$ .

For this problem,  $\lfloor |V|/\sqrt{2} \rfloor$  is essentially a red herring, as its value does not actually bear any significance with regard to the graph (as stated in class). Instead, our goal should be to prove that a graph with these properties exists. To do this, we need to find a range corresponding to the minimum number of roads present in any vertex of  $H$ , and the maximum number of roads present in any vertex of  $H$ . We can apply Prim's algorithm to help us find these numbers.

**Algorithm.**    1. Assign weights of road = 1, and bridges = 0.

2. Run Prim's algorithm on  $G$  and return the number of bridges. Subtract this from  $E$  to get the maximum number of roads.

3. Assign weights of road = 0, and bridges = 1.

4. Run Prim's algorithm on  $G$  and this time, you'll return the minimum number of roads.