UNIVERSITÀ DEGLI STUDI DI TORINO

DIPARTIMENTO DI Scienze Economico-Sociali e Matematico-Statistiche

Corso di Laurea Triennale in

Economia

RELAZIONE DI LAUREA

TITOLO

Financial Data Analysis of Italian Stocks

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Anno Accademico 2021/2022

To my family and friends,
for giving me the most valuable gift,
genuine moral values.

Financial Data Analysis of Italian Stocks

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July 2022

Abstract

This dissertation focuses on the implementation of financial models using Python. Starting from the methodology, it gradually covers topics such as the statistical analysis of financial data, the Capital Asset Pricing Model, the Portfolio Selection Model and option pricing via Black-Scholes or Monte Carlo simulations. Despite the aforementioned concepts have already been widely discussed, the goal is to assess what can be learned by applying these models to different contexts, in this case, the Italian stock market.

Contents

1 Introduction

1.1 Preamble

This work is the result of a fervent interest in the world of quantitative finance. The themes and approach have been significantly inspired by a series of lectures held by Mauricio Labadie Martinez, former quant trader at Credit Suisse, at the National Autonomous University of Mexico.

For each topic I start with an explanation of the underlying theory and then I procede by commenting on the main results and intuitions that can be gained by analyzing those results. As John Maynard Keynes said: "the master-economist must possess a rare combination of gifts. He must reach a high standard in several different directions and must combine talents not often found together. He must be mathematician, historian, statesman, philosopher, in some degree." Initially, my intention was to provide a more thorough treatment on the actual implementation of the code since there are many interesting aspects behind it that shed additional light on the subjects of interest, but given the lenght constraint and the fact that it would have turned more into a discussion concerning computer science, I decided to omit that part. For further deepening, the source code and data set are available at: https://github.com/chiqui-quant/QuantPadawan.

Despite the rise of sophisticated mathematical approaches, in finance, differently from mathematics, not always a solution exists, and if it does, it is definitely not unique. The challenge, thus, lies in combining a vast set of methods and tools to a given application in order to solve specific problems. Just as economics is the study of the allocation of scarce resources which have alternative uses, and of the decisions that lead to such allocations, so do the fields of statistics and finance present us with tradeoffs: in the first case between model accuracy and explanability, in the second case between risk and return.

1.2 Data

Fortunately, in finance, the data wrangling process is less challenging compared with other applications in statistics. The main reasons are inherent to the price system, due to the evolution of technology and the concerns about informational asymmetries.

The dataset that has been used consists in 5 year historical daily Italian stock prices (from June 2nd 2017 to June 2nd 2022) downloaded from yahoo finance with the addition of an ETF of the FTSE MIB Index. The purpose is to compare the fexibility and assess the validity of the models being used by applying them to a different context from the usual American market.

As of June 2022, there are 40 components in the FTSE MIB Index[1]. However, for the seek of this dissertation, I focused mainly to a few randomly selected stocks to carry on my analysis. Finally, and more importantly, apart from specific metrics, what I try to convey is an explanation such that the same observations and rationales can be applied to other cases.

[1] https://www.borsaitaliana.it/borsa/azioni/ftse-mib/lista.html

2 Are stock returns normally distributed?

2.1 Jarque-Bera normality test

There have been a lot of concerns and polemics about the normality of stock returns.[2] Rather than relying on intuition alone, how can we assess wether a distribution is normal or not? One possible approach is to use a Jarque-Bera normality test. The JB test statistic is defined as:

$$JB = \frac{n}{6}\left(S^2 + \frac{1}{4}(K-3)^2\right)$$

where n is the number of observations, S is the sample skewness and K is the sample kurtosis. Intuitively, when skewness is zero and kurtosis is 3, as in the case of a standard normal distribution, then the JB statistic will be close to zero, otherwise deviations from those values will lead to an increase in the JB statistic. If data comes from a normal distribution, the JB statistic is asymptotically distributed as a chi-squared distribution with two degrees of freedom. This can be reduced at testing the joint null hypothesis that skewness and excess kurtosis are both zero. As a rule of thumb, any time a new algorithm or method is implemented it is useful to see how it works in a simpler and more controlled setting, where mistakes or unexpected results are easier to detect. Therefore I start by performing the test in a setting of simulated random variables.

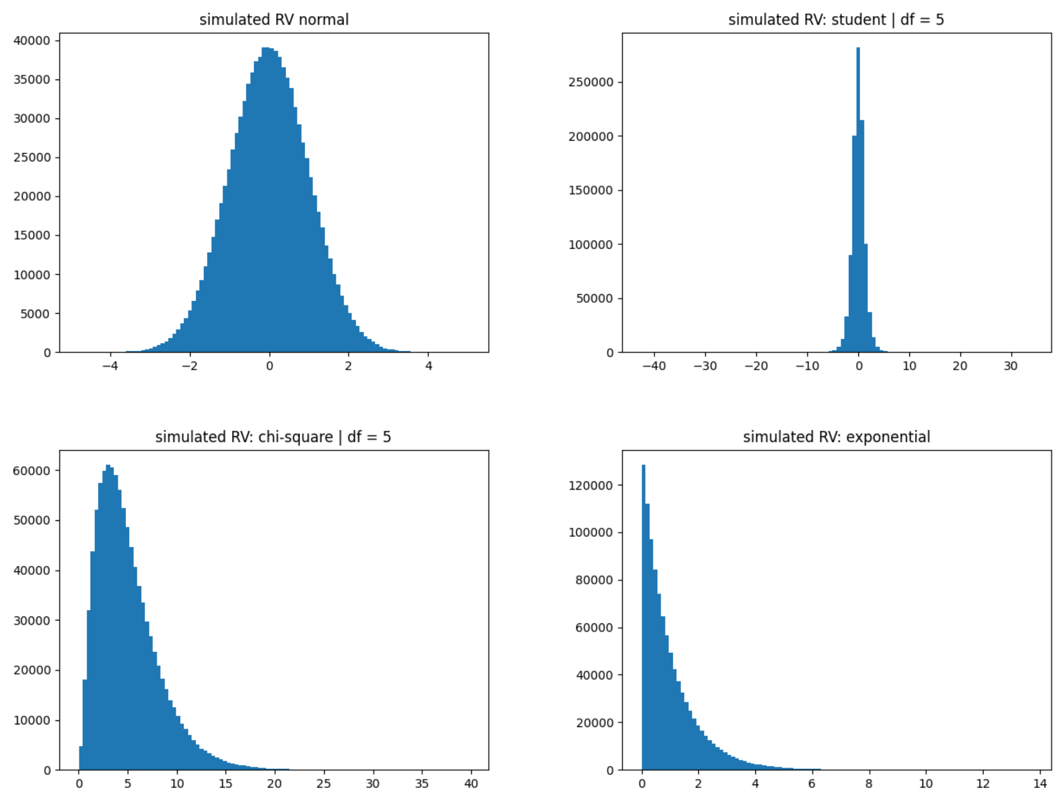

Figure 1: Different simulated distributions with 10^6 observations each.

One might reasonably ask, why are the degrees of freedom equal to 5 if the sample size is 1 million? The answer is that they have been chosen arbitrarily and the reason is that the simulation in

[2] This is due to the fact that the most commonly used models relied on that assumption, despite returns being actually not normally distributed.

Python allows to draw observations as if directly drawn from a given 'population distribution' with pre-specified degrees of freedom. As we shall see, this will turn to be particularily useful for the question of interest of this section. One additional and relevant observation is that the samples are actually pseudo-randomly generated. In the case of common computers, numbers are determined by an initial value called 'seed' and therefore not truly random.

Once the random simulations have been generated, it is possible to compute a series of useful statistical metrics and use the Jarque-Bera formula to test for normality. The critical value for a chi-squared distribution with 2 degrees of freedom at the 5% significance level is 5.99, therefore we reject the null hypothesis that the distribution is normal when the JB statistic is larger than this value. The associated p-value is the area on the right side of the critical value and we can calculate it as 1 minus the cumulative density function of the chi-squared distribution, with two degrees of freedom, up to the value we computed.

The output from the console (in the case of the normal distribution) is the following:

```
simulated RV: normal
mean is -0.0006352440319614184
standard deviation is 1.0001160793114432
skewness is -0.002692937851235039
excess kurtosis is -0.003512381715584656
JB statistic is 1.7226867666012902
p-value is 0.4225939951077672
is normal True
```

As expected, the computed statistical metrics are close to the ones of a normally distributed random variable (mean 0, standard deviation 1, excess kurtosis 0, skweness 0) leading to a JB statistic lower than the critical value of 5.99. The p-value is 0.42 which means there is not enough empirical evidence for rejecting the null hypothesis that the distribution is normal.

Having defined a 5% significance level, that means that 5% of the time the test will reject the null when it is actually true. For instance, by repeatedly simulating a normal random variable, one of the outputs was the following:

```
simulated RV: normal
mean is -0.0016239200778600778
standard deviation is 0.999860759582705
skewness is -0.007109991897447154
excess kurtosis is 0.002045368334968156
JB statistic is 8.599644614697796
p-value is 0.013570970259647597
is normal False
```

Here, despite having generated a normal distribution, the computed JB statistic is larger than the critical value of 5.99 and the corresponding p-value is less than 5% leading to a rejection of the null hypothesis of the distribution being normal (even though it was actually drawn form a normally distributed population). This serves to highlight a relevant question concerning any application, that is, that knowledge about the assumptions and implications, along with judgement are required when conducting a statistical analysis.

2.2 Real market data

By applying a few adjustments, it is possible to check wether real market data are normally distributed or not.

Let's consider for example the stock A2A.MI[3]:

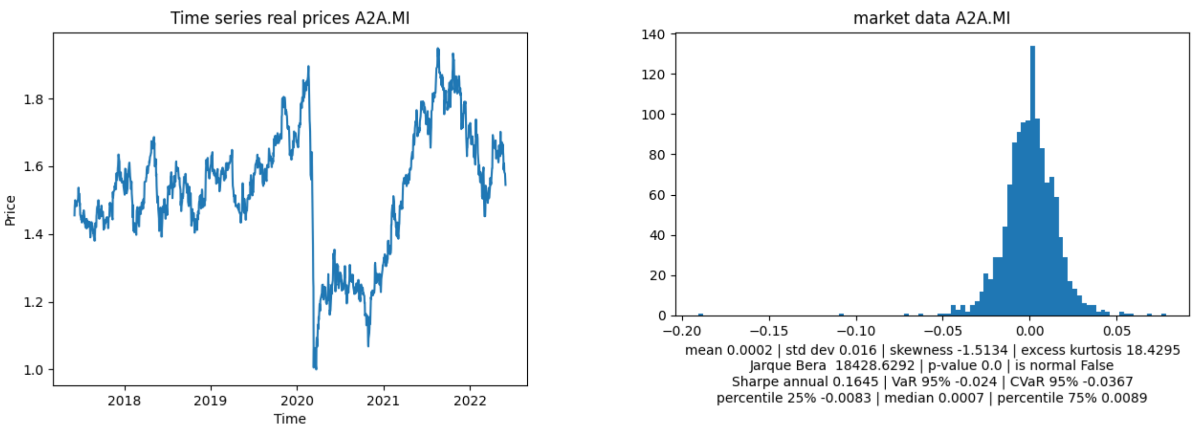

Figure 2: Timeseries and distribution of returns of A2A

```
---Real market data---
Ric is A2A.MI
mean is 0.00016557498162846543
standard deviation is 0.015978161935158598
skewness is -1.513417176329895
excess kurtosis is 18.42946380505845
JB statistic is 18428.629226984514
p-value is 0.0
is normal False
```

Here the story is quite different: at first glace, mean and standard deviation may seem close to those of a normal distribution, however the negative skewness and remarkable kurtosis lead to a large JB statistic and a small p-value. Therefore the null hypothesis of A2A.MI being normally distributed is rejected.

Performing the test multiple times for different stocks yields similar results, therefore empirical evidence suggests that the returns of Italian stocks are not normally distributed. For a more general view, it is possible to test wether the returns of the entire market are normally distributed. The mean and standard deviation of the market returns are similar of those of A2A.MI, but in this case the distribution is more negatively skewed and has larger kurtosis [4]. Moreover, the Jarque Bera statistic is even larger that that of A2A.MI.

One might argue that it was 'clearly visible' that the distributions treated in this section were not normal: "Effectively, intuition is a fundamental guide for the search of new results, but is sometimes

[3]Since the topics of this dissertation focus exclusively on quantitative considerations I refer to securities simply with their tickers.

[4]More informally one might say that it has 'fatter tails'.

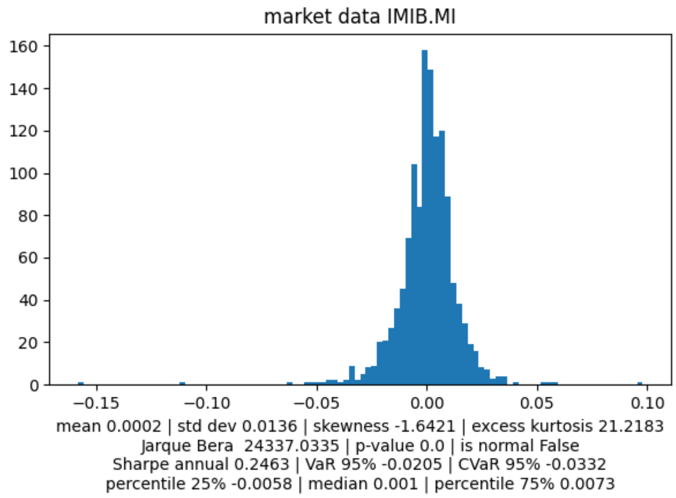

Figure 3: Distribution of returns of IMIB.MI

deceptive. Not infrequently, apparently intuitively obvious properties have even turned out to be false."[3]

3 Implementing the CAPM

The Capital Asset Pricing Model is used to define the expected rate of return of an asset by considering its sensitivity to non-diversifiable risk, that is systemic or market risk, and is formally defined as:

$$E(R_i) = R_f + \beta_i(E(R_m) - R_f)$$

where $E(R_i)$ is the expected return of the individual asset, R_f is the risk free rate of interest, $E(R_m)$ is the expected return of the market, $E(R_m) - R_f$ is known as market premium and $E(R_i) - R_f$ is known as risk premium.

From the equation above it is possible to derive the beta coefficient, the volatility-adjusted correlation, that corresponds to the sensitivity of the asset with respect to the market:

$$\beta_i = \frac{\mathrm{Cov}(R_i, R_m)}{\mathrm{Var}(R_m)} = \rho_{i,m}\frac{\sigma_i}{\sigma_m}$$

The computed results are the following:

```
Linear regression | security A2A.MI | benchmark IMIB.MI
alpha (intercept) 0.0 | beta (slope) 0.7871
p_value 0.0 | null hypothesis False
correl (r-value) 0.6698 | r-squared 0.4487
```

The estimated beta coefficient for A2A.MI is 0.7871, which means that the asset is less volatile with respect to the market. The p-value computed above is used to test the hypothesis that the slope of the regression line is zero, since the null was rejected it means that the regression has explanatory power.

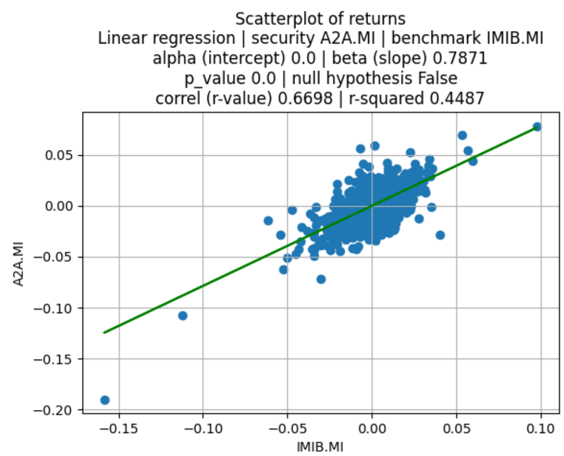

Figure 4: normalized time series and linear regression of A2A.MI with respect to the market IMIB.MI

The values of correlation and r-squared provide additional insightful information. In particular, the r-squared coefficient of 0.45 suggests that only 45% of the variance of A2A.MI is explained by the market. Therefore, the error term, that is the sum of other variables that may influence the value of interest, is significant in this case.

The table in the next page reports all the computed betas and respective r-squared values. In general, there is no 'right or wrong' value for the r-squared, but it is intuitively easy to see that in this case the model has poor explanatory power.

Now, we know the volatility of a specific stock with respect to the market. But how can we use this knowledge?

3.1 Delta and beta hedging

One aspect that has not been directly mentioned in the previous section is the fact that the CAPM assumes that the rate of return of a security beyond risk (or absolute return) commonly referred to as α, that is, the fraction of the total return of the asset that cannot be explained by the market, is equal to zero.

Security	Beta	R-Squared
A2A.MI	0.7871	0.4487
AMP.MI	0.7315	0.2139
ATL.MI	0.9617	0.2891
AZM.MI	1.1075	0.5252
BAMI.MI	1.3548	0.4855
BGN.MI	1.0151	0.5505
BMED.MI	1.0168	0.5239
BPE.MI	1.2601	0.3545
CNHI.MI	1.1786	0.4628
CPR.MI	0.6094	0.2502
DIA.MI	0.3222	0.0421
ENEL.MI	0.8772	0.5745
ENI.MI	1.0123	0.5606
EXO.MI	1.2326	0.6279
FBK.MI	1.0393	0.4851
G.MI	0.7957	0.6369
HER.MI	0.7213	0.3668
IG.MI	0.6523	0.3591
INW.MI	0.4556	0.1312
IP.MI	0.8171	0.3152
ISP.MI	1.1946	0.7008
IVG.MI	1.3135	0.4711
LDO.MI	1.1132	0.3668
MB.MI	1.1247	0.6074
MONC.MI	0.9378	0.356
NEXI.MI	0.9394	0.3415
PIRC.MI	1.0508	0.4124
PRY.MI	0.9548	0.3934
PST.MI	1.0361	0.5878
RACE.MI	0.8143	0.3763
REC.MI	0.6421	0.2422
SPM.MI	1.0846	0.2726
SRG.MI	0.721	0.4199
STLA.MI	1.3273	0.5214
STM.MI	1.1896	0.3935
TEN.MI	1.0282	0.3403
TIT.MI	0.9954	0.3034
TRN.MI	0.6243	0.3442
UCG.MI	1.4352	0.6041
UNI.MI	1.0478	0.4946

Figure 5: Table of computed betas and r-squared values after performing a linear regression of individual stock returns with respect to the FTSE MIB index as the reference market.

However, as we have seen, the CAPM did not yield satisfactory results in terms of explanability. Performing a simplified investment strategy as the one that will be proposed here, especially when the underlying model is not accurate, would be like building castles in the air. Therefore, since this section relies on the results previously obtained, what should be valued more is the underlying methodology rather than its conclusions.[5]

By taking into account only α and β it is possible to define 4 different kinds of investment strategies:

- Index tracker: whose goal is to replicate the performance of a benchmark ($\beta = 1, \alpha = 0$).
- Long-only asset management: whose goal is to outperform the market and achieve extra, uncorrelated return ($\beta = 1, \alpha > 0$).
- Smart beta: which aims to outperform the market by dynammically adjusting the weights of a portfolio ($\beta > 1$ when the market is up, $\beta < 1$ when the market is down, $\alpha = 0$).
- Hedge fund: that focuses on delivering absolute returns uncorrelated with the market ($\beta = 0, \alpha > 0$).

Given this non-exhaustive list of approaches, I procede by discussing about the last one. As a first example, which will be extended and generalized in the next section, suppose that the current holdings in a portfolio consist only in one stock. Is it possible to hedge the exposure of such an asset by buying or selling another security, or even better, a number of different securities?

[5]The reason is that once one builds a more accurate and reliable model, such as a multifactor model taking into account more relevant variables, conceptually one can apply a similar approach.

Although the concept is most commonly associated with options, on a general level, delta neutrality refers to a portfolio whose value remains unchanged, whereas beta neutrality defines a portfolio whose return is uncorrelated with the market. But why would an individual (wether a retail investor or institution) employ such a strategy in the first place? More specifically, why would an investor choose to hold a portfolio whose value stays constant over time instead of increasing? These are perfectly reasonable questions that highlight an important aspect about financial markets, that is, the fact that there is a variety of players with different goals.

For instance, a player that might be interested in the beta-delta neutrality is a market maker, whose function is to enhance market liquidity and efficiency by matching orders. To perform such a function, a market maker holds an inventory of securities and profits from the difference between bid and ask prices.

Mathematically, a hedging strategy with one security can be defined as follows. Let S_0 be the value of the initial holding in a portfolio (in euros):

- Delta neutral: find S_1 such that $S_0 + S_1 = 0$ (the value of the portfolio does not change).
- Beta neutral: find S_1 such that $\beta_0 S_0 + \beta_1 S_1 = 0$ (the sensitivity of the portfolio to the market is null).

More in general, a hedging strategy with N securities is:

- Delta neutral: find $S_1, ..., S_n$ such that $S_0 + \sum_{i=1}^{N} S_i = 0$.
- Beta neutral: find $S_1, ..., S_n$ such that $\beta_0 S_0 + \sum_{i=1}^{N} \beta_i S_i = 0$.

For example, if $N = 2$ the problem consists in finding $x = \begin{bmatrix} S_1 & S_2 \end{bmatrix}$, that is, the allocation of the notional of the 2 hedging stocks, such that:

$$\begin{bmatrix} 1 & 1 \\ \beta_1 & \beta_2 \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} = - \begin{bmatrix} S_0 \\ \beta_0 S_0 \end{bmatrix}$$

If $\beta_1 \neq \beta_2$ then the solution exits, is unique and is given by:

$$\begin{bmatrix} S_1 \\ S_2 \end{bmatrix} = - \begin{bmatrix} 1 & 1 \\ \beta_1 & \beta_2 \end{bmatrix}^{-1} \begin{bmatrix} S_0 \\ \beta_0 S_0 \end{bmatrix}$$

As a practical example, suppose the current portfolio holds 10 million euro worth of ISP.MI[6], whose sector is that of financial services, and suppose we want to hedge it with other two stocks of unrelated sectors: EXO.MI and ENEL.MI (respectively consumer cyclical and utilities).[7]

```
------
Input portfolio:
Delta mlnEUR for ISP.MI is 10
Beta for ISP.MI vs IMIB.MI is 1.1946
Beta mlnEUR for ISP.MI vs IMIB.MI is 11.946000000000002
------
```

[6] Which is the stock whith the highest r-squared (0.70). If we have to bet on a horse, let's at least pick the last among the worst.

[7] The computed r-squared of EXO.MI was 0.62 and that of ENEL.MI 0.57. There were a number of stocks whose r-squared values were higher than ENEL.MI, the choise is due to the fact that they all operated in fields related to the financial sector. A more accurate approach based on correlations will be discussed later.

```
Input hedges:
Beta for hedge[0] = EXO.MI vs IMIB.MI is 1.2326
Beta for hedge[1] = ENEL.MI vs IMIB.MI is 0.8772
------
Optimization result - exact solution
------
Delta portfolio: 10
Beta portfolio EUR: 11.946000000000002
------
Delta hedge: -10.0
Beta hedge EUR: -11.946000000000002
------
Optimal hedge:
[[-8.93078222] [-1.06921778]]
------
```

As shown above, the inputs display the delta to hedge, that is the 10 million euros of the starting porfolio, the beta of ISP.MI, computed from the market index, and the beta to hedge in terms of euros, which is a fictitious but practical measure, since by simply multiplying the notional for the beta of the initial asset we can get the total amount of beta to hedge.

The second step entails the computation of the betas for the two stocks we want to use for the hedging strategy. Finally, by applying the previously discussed theoretical approach, the optimal hedge results in selling around 8.93 million euros of EXO.MI and selling 1.07 million of ENEL.MI.

3.2 Introducing numerical solutions

There is one limitation about the analytical approach. If we want to hedge the initial holded security by using more than 2 assets, that is $N > 2$, the system is undefined and there are infinitely many solutions and 'Python' is not able to find an exact solution using linear algebra, yielding an error message. In such a case, it would be ideal to choose the one with smallest weights. This leads to finding the weights with the smallest norm $||x||$ (either l_1 or l_2). Why? Recall that the unitary l_1 and l_2 norms are given by:

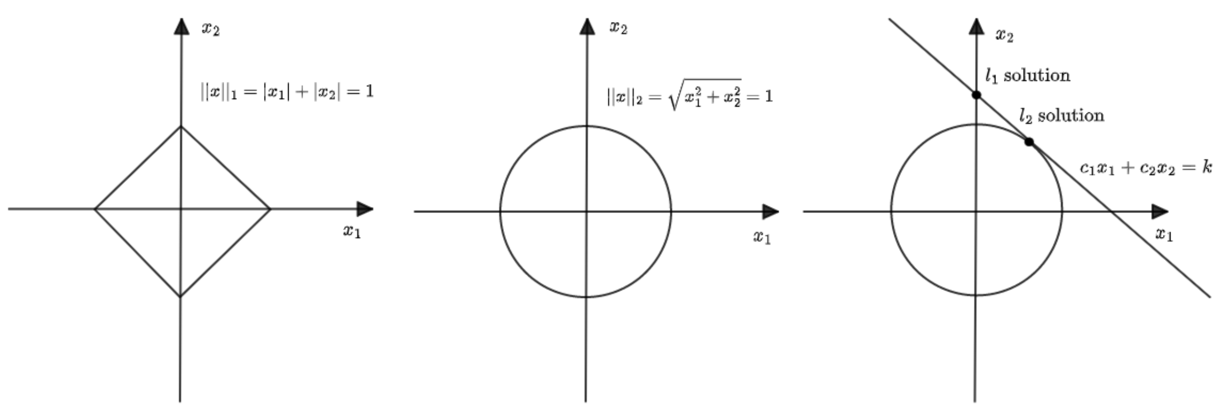

Figure 6: example of l_1 and l_2 norms in R^2, the line in the right panel in the set of infinite solutions, given that set we would like to find the solutions that minimize either the l_1 or l_2 norm.

Therefore one could try to minimize the norm under the constraints of beta and delta neutrality. However this would still present a problem which has not been hitherto mentioned. For practical clarity, suppose that instead of ISP.MI we had to hedge INW.MI, the result would be:

```
------
Input portfolio:
Delta mlnEUR for INW.MI is 10
Beta for INW.MI vs IMIB.MI is 0.4556
Beta mlnEUR for INW.MI vs IMIB.MI is 4.556
------
Input hedges:
Beta for hedge[0] = EXO.MI vs IMIB.MI is 1.2326
Beta for hedge[1] = ENEL.MI vs IMIB.MI is 0.8772
------
Optimization result - exact solution
------
Delta portfolio: 10
Beta portfolio EUR: 4.556
------
Delta hedge: -10.0
Beta hedge EUR: -4.556000000000001
------
Optimal hedge:
[[ 11.86268993] [-21.86268993]]
------
Optimization result - numerical solution
------
Delta portfolio: 10
Beta portfolio EUR: 4.556
------
Delta hedge: -10.000001429148787
Beta hedge EUR: -4.555998170096741
------
Optimal hedge:
[ 11.8626986  -21.86270003]
```

It is easy to see that the optimal hedge (both in the exact and numerical case) suggests to buy 11.86 million euros of EXO.MI and sell 21.86 million of ENEL.MI. Differently from the previous example, the strategy would require to employ more than 30 million euros, which may be unfeasible in practice. Moreover, a 'one-sided' hedge would, in general, be preferred.[8]

In order to solve this it is possible to add a new parameter that penalizes for having a large norm. Mathematically, let $x = \begin{bmatrix} S_1 & \cdots & S_n \end{bmatrix}$, $\beta = \begin{bmatrix} \beta_1 & \cdots & \beta_n \end{bmatrix}$ and $e = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}$, the function to minimize is:

$$f(x, \varepsilon) = f_{\text{delta}}(x) + f_{\text{beta}}(x) + f_{\text{penalty}}(x, \varepsilon)$$

where:

$$f_{\text{delta}}(x) = (e^{\mathrm{T}} x + S_0)^2$$
$$f_{\text{beta}}(x) = (\beta^{\mathrm{T}} x + \beta_0 S_0)^2$$
$$f_{\text{penalty}}(x, \varepsilon) = \varepsilon ||x||^2$$

The parameter $\varepsilon \geq 0$ makes the penalty larger as the norm of x grows. Although the solution will not be perfectly beta and delta neutral as before, the solution is easier to implement in a real world setting.

[8] One simple practical insight is to look at the betas, note that in this case the one of the stock to hedge is 0.46 and the ones of the stocks used for the strategy are respectively 1.23 and 0.88, therefore there is no convex linear combination of them that allows to get 0.46.

To end the conceptual example, by adding a regularization of 0.1 the new solution becomes:

```
------
Optimization result - numerical solution
------
Delta portfolio: 10
Beta portfolio EUR: 4.556
------
Delta hedge: -7.474187566427238
Beta hedge EUR: -6.5961013011958745
------
Optimal hedge:
[-0.11182883 -7.36235873]
```

which, as commented before, does not perfectly hedge but may be more convenient.

4 Portfolio selection in practice

4.1 Computing the efficient frontier

The portfolio optimization problem consists in computing the efficient frontier, that is, computing the set of portfolios with the maximum rate of return for every given level of risk (or alternatively the minimum risk for every level of return). In the absence of a risk-free security the problem can be mathematically stated as:

$$\text{min. } x^{\mathrm{T}} V x$$
$$\text{sub. } x^{\mathrm{T}} \mu = m, \; x^{\mathrm{T}} e = 1$$

where: e is a vector of ones (to make notation consistent), m is a fixed value of mean return, x is the vector of weights for a unitary portfolio such that $\sum_{i=1}^{n} x_i = 1$, whose mean return and variance are respectively:

$$\mathrm{E(P)} = m = x^{\mathrm{T}}\mu = \mu^{\mathrm{T}}x \qquad \mathrm{Var(P)} = \sigma_P^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \sigma_{i,j} = x^{\mathrm{T}} V x$$

The variance-covariance matrix V has some important properties: it is symmetric and semidefinite positive. Hence, the first order conditions are sufficient for finding the minimum. Finally, the Lagrangian function that has been implemented in the code is:

$$L(x, \lambda_1, \lambda_2) = x^{\mathrm{T}} V x + \lambda_1(m - x^{\mathrm{T}}\mu) + \lambda_2(1 - x^{\mathrm{T}} e)$$

The actual implementation using Python is straightfoward since there are methods which allow for optimization by simply declaring the objective function and the underlying contraints. Back in 1952, when the portfolio selection model of Markowitz was first published, the efficient frontier was expensive in terms of computation, with the advance of technology that is not an issue anymore. Since reporting the computed results of the entire frontier of the top 40 Italian stocks might be a tedius, I will focus on a simplified case whose principles can easily be extended.

As it can be seen in the figure below, each portfolio in the efficient frontier is associated to a given Sharpe ratio. The Sharpe ratio defines how many units of risk premium are present in the portfolio for every unit of standard deviation (or risk):

$$S = \frac{R_p - R_f}{\sigma_p}$$

where R_p and R_f are respectively the returns of the portfolio and of the risk free security and σ_p is the standard deviation of the portfolio.

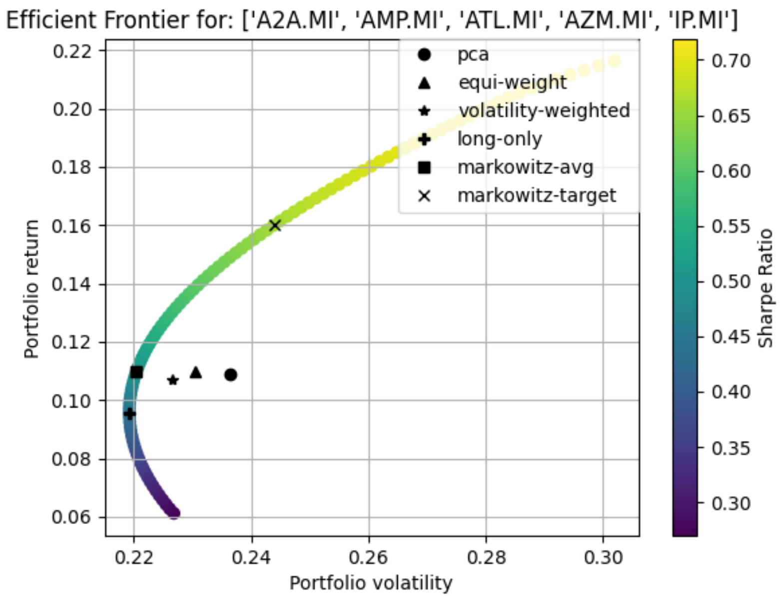

Figure 7: Markowitz's efficient frontier of 5 Italian stocks

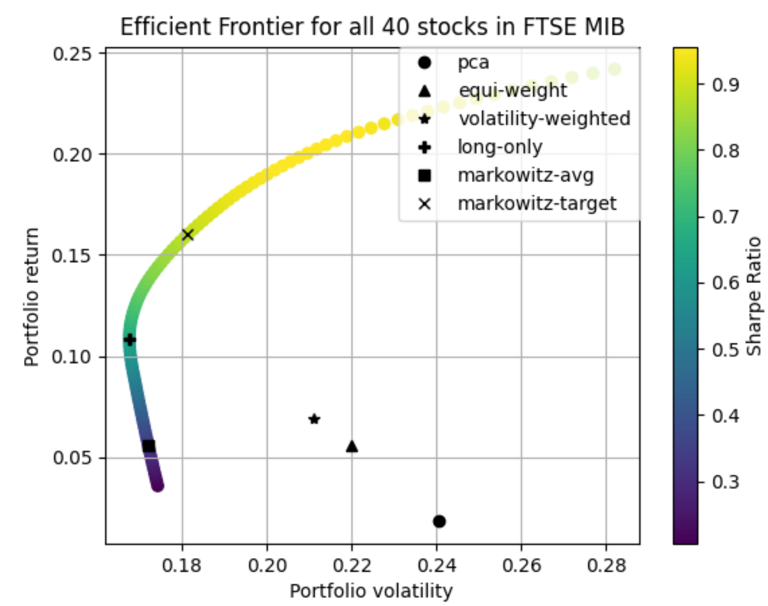

Figure 8: Markowitz's efficient frontier of the main 40 Italian stocks

Although I omitted the results about the frontier of all the stocks in the FTSE MIB Index, there is an important observation that is worth mentioning. As the number of stocks taken into account grows, the efficient shifts to the left, yielding portfolios with higher returns for a given level of risk.

This can be proved by simply looking at the Markovitz target return portfolio (in which the target return has been arbitrarily defined as 0.16 in both figures), in the first case the volatility of the portfolio is is more than 0.24, whereas in the second case it is 0.18. Another important observation, related with the previous one, is that in the second frontier it is possible to obtain portfolios that yield a higher Sharpe ratio, hence a better mix of risk and return.

Finally, the preference about which kind of portfolio to choose according to the mean-variance principle may change as the universe of securities expands. By looking at the figures above it is easy to notice that the long-only and markowitz average portfolios 'switched places' and the 4 portfolios that initially lied on the same level of return, are now well ordered in terms of strong mean-variance dominance.

4.2 Computing different portfolios

In the previous section I mentioned the 'Markowitz target portfolio', here I explain the rationale of the different portfolios that I reported in the figures of the efficient frontier. As it has already previously highlighted, different individuals might have different investment prefereces. In order to satisfy such preferences it is possible to construct different kinds of portfolios:

- Min-variance using eigenvectors: computed by diagonalizing the variance-covariance matrix and picking the eigenvector associated to the smallest eigenvalue (Jordan decomposition [5]).
- PCA or max-variance using eigenvectors: which instead picks the eigenvector associated with the largest eigenvalue. Why would one construct a portfolio with the highest variance? The main reason is for replication purposes.
- Equi-weighted: in which the notional is equally splitted between securities.
- Volatility-weighted: which weights more the securities with lower volatility.
- Long-only: the minimum variance portfolio that only admits long positions.
- Markowitz average: the classical Markowitz portfolio whose return is the average return of the stocks.
- Markowitz target: minimum variance portfolio given a target return.

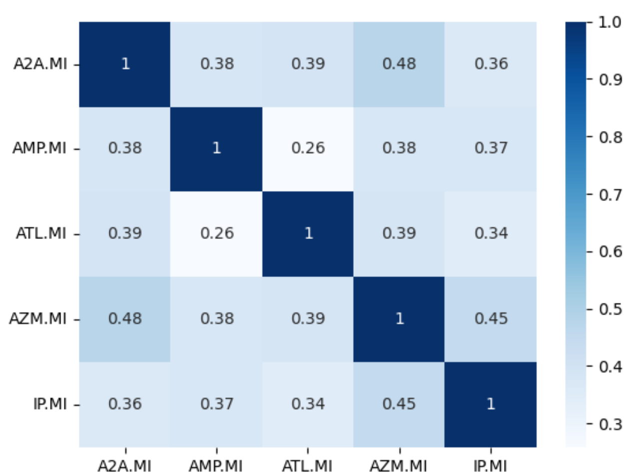

Figure 9: Matrix of correlations between stock returns. Surprisingly, the returns are quite uncorrelated.

The computed portfolios taking into account the 5 stocks considered for the first efficient frontier are the following:

```
------
Portfolio type: min-variance
Rics:
['A2A.MI', 'AMP.MI', 'ATL.MI', 'AZM.MI', 'BAMI.MI']
Weights:
[ 5.97808895 -1.03199789 -0.74840773 -1.98060661 -0.26089882]
Notional (mln EUR): 10.000000000000002
Delta (mln EUR): 1.9561779000472335
Variance explained: 0.06299448174082709
Profit and loss annual (mln EUR): -0.22415920417722016
Volatility annual (mln EUR): 1.2661412982994997
Return annual: -0.022415920417722016
Volatility annual: 0.12661412982994993
Sharpe ratio annual: -0.1770412231859737
```

The first one is the optimal portfolio considering the lowest overall volatility (using eigenvectors), however the historical return of this portfolio has been negative and since common sense suggests that any investor prefers a positive return, it is not worth spending further words.

```
------
Portfolio type: min-variance-l2
Rics:
['A2A.MI', 'AMP.MI', 'ATL.MI', 'AZM.MI', 'BAMI.MI']
Weights:
[ 5.97771194 -1.03267415 -0.74823322 -1.98225261 -0.25912808]
Notional (mln EUR): 10.0
Delta (mln EUR): 1.9554238830079715
Profit and loss annual (mln EUR): -0.22426706668044422
Volatility annual (mln EUR): 1.2661755283537062
Return annual: -0.022426706668044423
Volatility annual: 0.1266175528353706
Sharpe ratio annual: -0.1771216246550259
```

The portfolio minimizing the norm of the weights, which as previously seen is only a numerical (not exact) solution yields very similar results as before and therefore presents the same issue of negative returns.

```
------
Portfolio type: min-variance-l1
Rics:
['A2A.MI', 'AMP.MI', 'ATL.MI', 'AZM.MI', 'BAMI.MI']
Weights:
[5.22882917 2.01821135 1.08884199 1.25504747 0.40907001]
Notional (mln EUR): 10.0
Delta (mln EUR): 10.0
Profit and loss annual (mln EUR): 0.8955487473140936
Volatility annual (mln EUR): 2.265556811289524
Return annual: 0.08955487473140937
Volatility annual: 0.22655568112895247
Sharpe ratio annual: 0.39528858550422286
```

On the other hand, minimizing using the l_1 norm, which is another numerical solution, yields remarkably different results. This time the portfolio is profitable, yielding a positive annualized return of almost 9%.

```
------
Portfolio type: long-only
Rics:
```

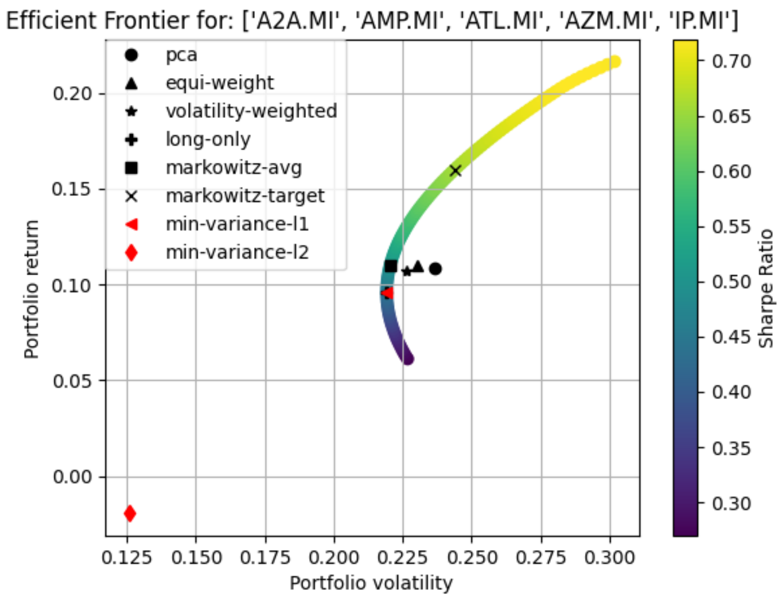

Figure 10: Efficient frontier with the addition of min-variance-l1 and min-variance-l2 portfolios. Note that the min-variance l1 almost coincides with the long-only portfolio.

```
['A2A.MI', 'AMP.MI', 'ATL.MI', 'AZM.MI', 'BAMI.MI']
Weights:
[5.22882915 2.01821137 1.08884201 1.25504748 0.40907    ]
Notional (mln EUR): 9.999999999999996
Delta (mln EUR): 9.999999999999996
Profit and loss annual (mln EUR): 0.8955487516919051
Volatility annual (mln EUR): 2.2655568112877
Return annual: 0.08955487516919051
Volatility annual: 0.22655568112877003
Sharpe ratio annual: 0.39528858743687467
```

The long-only portfolio lies in the same point of the efficient frontier than the min-variance-l1, indeed the portfolio weights, returns and volatilities are almost identical.

```
------
Portfolio type: pca
Rics:
['A2A.MI', 'AMP.MI', 'ATL.MI', 'AZM.MI', 'BAMI.MI']
Weights:
[1.36473264 1.53229535 2.17626302 2.1300496  2.79665939]
Notional (mln EUR): 10.0
Delta (mln EUR): 10.0
Variance explained: 0.5085770084985362
Profit and loss annual (mln EUR): 1.010186705743057
Volatility annual (mln EUR): 2.582219964385345
Return annual: 0.10101867057430569
Volatility annual: 0.2582219964385345
Sharpe ratio annual: 0.39120861881474733
```

This portfolio replicates 50% of the total variance of the variance-covariance matrix. The idea behind the computation is to pick the largest eigenvalue of the variance-covariance matrix and divide it by the absolute value of the sum of all the eigenvalues. The corresponding eigenvector represents the weights of the portfolio.

```
------
Portfolio type: equi-weight
Rics:
['A2A.MI', 'AMP.MI', 'ATL.MI', 'AZM.MI', 'BAMI.MI']
Weights:
[2. 2. 2. 2. 2.]
Notional (mln EUR): 10.0
Delta (mln EUR): 10.0
Profit and loss annual (mln EUR): 1.03894
Volatility annual (mln EUR): 2.445286077333284
Return annual: 0.10389400000000001
Volatility annual: 0.24452860773332843
Sharpe ratio annual: 0.4248746229042534
```

Paradoxically, despite its simplicity, the equi-weight approach yields the highest Sharpe ratio between all the portfolios hitherto considered.

```
------
Portfolio type: volatility-weighted
Rics:
['A2A.MI', 'AMP.MI', 'ATL.MI', 'AZM.MI', 'BAMI.MI']
Weights:
[2.65034562 1.96814335 1.74165354 2.03792703 1.60193046]
Notional (mln EUR): 10.0
Delta (mln EUR): 10.0
Profit and loss annual (mln EUR): 1.005354995199421
Volatility annual (mln EUR): 2.3801366899371157
Return annual: 0.10053549951994209
Volatility annual: 0.23801366899371154
Sharpe ratio annual: 0.42239380597338
```

This portfolio weights more the securities with lower volatility, however in this case, the portfolio weights are close to the equi-weight approach, hence yielding similar return and volatility.

```
------
Portfolio type: markowitz
Rics:
['A2A.MI', 'AMP.MI', 'ATL.MI', 'AZM.MI', 'IP.MI']
Weights:
[1.97290381 4.45811289 0.12118309 0.41499271 3.03280749]
Notional (mln EUR): 10.0
Delta (mln EUR): 10.0
Profit and loss annual (mln EUR): 1.5999999999862542
Volatility annual (mln EUR): 2.43880216079541
Target return: 0.16
Return annual: 0.1599999999986254
Volatility annual: 0.24388021607954097
Sharpe ratio annual: 0.6560597762732906
```

Finally, this is the markovitz target return portfolio, which allows to get any point in the 'smooth curve' of the efficient frontier, that is, any dominant portfolio according to the mean-variance principle. In this case the rate of return was arbitrarily fixed at 16%, however a risk-prone investor may prefer to move to the right side of the curve so as to get a portfolio with an even larger Sharpe ratio, represented by the yellow dots in the plot.

5 An informal introduction to option pricing

5.1 Black-Scholes

We have seen the workhorse models for analyzing stock returns and building investment portfolios. Now, I would like to touch (on the surface) a more sophisticated topic: derivatives. To give an idea of their respective relevance, as of 2020 the estimated world market capitalization of listed domestic companies was \$93.69 trillion[9], whereas the notional value of outstanding derivatives contracts at the end of 2021 was \$600 trillion and the gross market value, that is the "sum of the absolute values of all outstanding derivatives contracts with either positive or negative replacement values evaluated at market prices prevailing on the reporting date" stood at \$12.4 trillion.[10]

Derivatives are broadly divided into futures, forwards, swaps and options, the latter being the subject of interest. The standard definition says that an option is "a contract which conveys to its owner the right, but not the obligation, to buy or sell an underlying asset or instrument at a specified strike price on or before a speficied date". There is a wide variety of option contract types, here we will only discuss European call and put options (for further deepening see [6] [8] [7]).

Why is it useful determine the theoretical value of an option? Just as the map is not the territory, a model is not reality, yet it provides a useful guide to better understand it. In our case, an option pricing model provides a starting point for determining an appropriate strategy and more importantly for quantifying the risks that the strategy entails.

The Black-Scholes formula for determining the value of an european call option for a non-paying stock is[11]:

$$C(S_t, t) = N(d_1)S_t - N(d_2)Ke^{-r(T-t)}$$

and the price of a corresponding put option based on the put-call parity:

$$P(S_t, t) = N(-d_2)Ke^{-r(T-t)} - N(-d_1)S_t$$

where $d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S_t}{K}\right) + (r + \frac{\sigma^2}{2})(T-t)\right]$ and $d_2 = d_1 - \sigma\sqrt{T-t}$

Therefore there are five characteristics that are needed for the computation, namely: strike price, time to maturity (or expiration), price of the underlying, interest rate and volatility. In practice, the parameters, and accordingly the portfolio positions, are perpetually recalibrated as market conditions change (this is known as dynamic hedging). Here I will simply present a static and extremely simplified approach.

Let's consider a practical example. Suppose we are interested in determining the price of: ENI Call 13, Dec 2022 (whose market price at the time of writing is settled at €0.205). After implementing the Black-Scholes model in Python, the output that we get is the following:

```
------
Option Inputs
Price underlying: 10.97
```

[9] https://data.worldbank.org/indicator/CM.MKT.LCAP.CD?locations=US-1W-CN&view=chart

[10] https://www.bis.org/publ/otc_hy2205.pdf

[11] For a concise and practical look at the core assumptions and derivation see [9], for a more theoretical treatment there are many other valid books in the references section.

```
Starting date (in years): 0.0
Interest rate: 0.005
Maturity date (in years): 0.4166666666666667
Strike price: 13
Option type: call
------
Price using Black-Scholes formula | call
0.2291095420725866
------
```

Where the price of the underlying is the current price of ENI.MI's stock, the starting date means literally 'now' (which in the formula was defined by t), the interest rate is assumed to be 0.5%[12] and the maturity date is $5/12$ years from now $(T - t)$.

The result is a price of around €0.23. Therefore, trusting the model and its various assumptions, since the market price for the option was lower, one may consider the option as undervalued and will procede by buying it. At this point, by shorting the underlying it would be theoretically possible to make a riskless profit, given by the difference between the price we computed and the market price of the option.

It would be great if things were this simple. A peculiar fact is that one may lose money despite having all the right and perfectly calibrated parameters, or may also gain with completely wrong ones. But as mentioned throughout this work, one can limit his margin of error by combining different methods and this is what we shall attempt in the next section.

5.2 Pricing options through Monte Carlo simulation

In essence, a Monte Carlo Simulation (or Monte Carlo method) is a way of estimating the possible outcome of an uncertain event by using random numbers. One simple example that helps to clarify how this approach works is the following: assume we want to determine the value of π, to achieve that it is possible to start with a unit circle inscribed in a square and generate random points in the space. Since we know the ratio of the area of the circle with respect to the square we can estimate the value of π by counting how many points fall inside the circle compared to the total number of points. The answer will approximate the true value and will converge to it as the number of generated points increases.

In our context, the goal is to estimate the value of the theoretical price of a given option by simulating realizations of a random variable (the price of the option based on the random behavior of the underlying). Assuming that the behavior of stock prices follows a Geometric Brownian Motion, that is:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

where W_t is a standard Brownian motion (or Wiener process), μ the drift (return) and σ the volatility, then we can determine the price of a stock at time t (that is, S_t) given an arbitrary initial value of the stock S_0 at time $t = 0$. The solution is given by:

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right)$$

[12] The ECB recently hiked interest rates by 0.50 basis point each, the one implemented in the model is the main refinancing rate: https://www.ecb.europa.eu/stats/policy_and_exchange_rates/key_ecb_interest_rates/html/index.en.html

This way we can simulate a given number of future stock prices and then use them to find the discounted expected option payoffs.

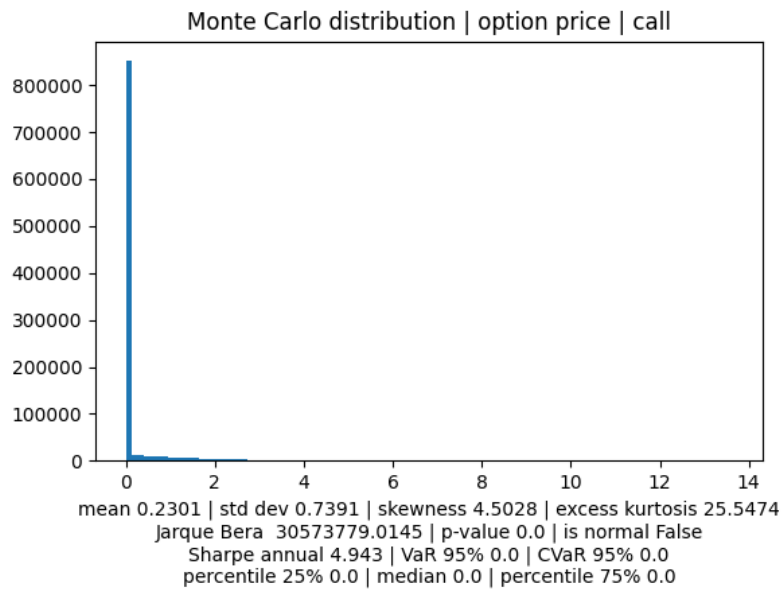

Figure 11: distribution of option prices given by the Monte Carlo simulation. The large frequency of value 0 is due to the fact that the price of the option is often, at least in this case, worthless at expiration (out of the money).

```
------
Price using Black-Scholes formula | call
0.2291095420725866
------
Monte Carlo simulation for option pricing | call
number of simulations 1000000
confidence radius 0.0014486371172774007
confidence interval [0.22869461 0.23159189]
price 0.23014325080062217
probability of exercise 0.162402
probability of profit 0.140603
```

The average price of the option using the Monte Carlo simulation is very close to the one given by the Black Scholes formula. Since the computed value is a mean, by the law of large numbers, as the sample size increases, the sample mean converges to the real mean, moreover, by the central limit theorem, the mean converges in distribution to $N(\mu, \frac{\sigma^2}{n})$. Therefore (as reported in the output above) it is possible to construct a 95% confidence interval for the average option price as:

$$S_n \pm 1.96SE = S_n \pm 1.96 \times \frac{\sigma}{\sqrt{n}}$$

Along with this information it is possible to estimate: (1) the probability of exercising the option, that is the proportion of the expected value of simulated prices larger/smaller than the strike price of the call/put and (2) the probability of profit as the frequency of the expected value of simulated payoffs greater/lower than the computed mean price for the call/put.

To sum up, although the model is still dependent on the validity of the values assumed for the parameters[13], the Monte Carlo simulation allowed to gain relevant additional information about the option price.

5.3 An intuitive approach to greeks

To conclude the discussion, I will comment on some insights that can be gained by simply looking at a figure on the price of an option as the price of the underlying changes. In finance, the Greeks represent the sensitivity of the price of a derivative with respect to several parameters and are used as measures of risk.

The basic interpretation of the most common Greeks is the following:

- Delta: sensitivity of the option price to a change in the price of the underlying ($\frac{\partial V}{\partial S}$).
- Gamma: sensitivity of the option delta to a change in the price of the underlying ($\frac{\partial \Delta}{\partial S} = \frac{\partial^2 V}{\partial S^2}$).
- Vega: sensitivity of the option price to a change in the volatility of the underlying ($\frac{\partial V}{\partial \sigma}$).
- Theta: expected change in the option price as time passes ($\frac{\partial V}{\partial \tau}$).
- Rho: sensitivity of the option price to interest rates or dividend payout ($\frac{\partial V}{\partial \rho}$).

As an example let's consider a put option this time:

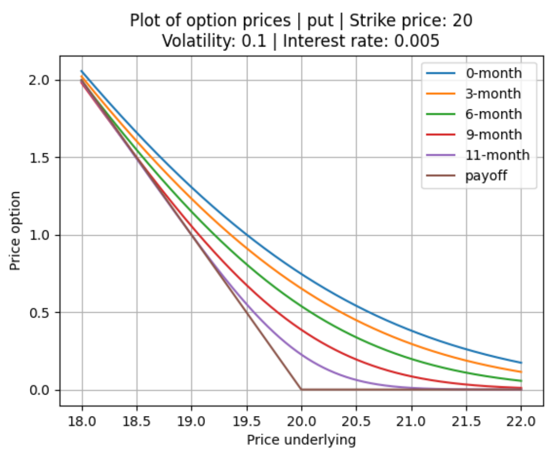

Figure 12: plot of put option prices with different time to maturity and initial arbitrary values.

What is the delta? As previously said, the delta is simply the derivative of the option price to the price of the underlying. In the plot above the payoff function is not differentiable, but the others are. The slope of the function is initially -1 and as it gets closer to the strike price it rapidly goes to 0. How rapid is determined by the gamma (or curvature) which in the figure above is larger as the time to expiration gets closer to the payoff date. The interpretation of this behavior is as follows: the payoff function determines that the price of an option increases linearly as the price of the underlying goes down (for a put) and as it becomes larger than the strike price the option

[13] As stated by Natenberg [6]: "A wide variery of forces can affect an option's value. If a trader uses a theoretical pricing model to evaluate options, any of the inputs into the model can represent a risk because there is always a chance that the inputs have been estimated incorrectly." Coincidentally, the same day of writing this, I later noticed a 'simple and innocent' typo in the inputs, that is 0.5 (50%) interest rate instead of 0.005 (0.5%), yielding to an option price of 1 euro both using the Black-Scholes model and the Monte Carlo method. As a consequence the associated probabilities of exercise and profit were much larger (around 54% and 35% respectively). I say this to stress out, at least to myself, the importance of always checking if numbers make sense.

is worthless. Now, instead of the payoff function consider one of the others in the plot. At the extremes the holder of the option is quite confident that the option price will be in the money or out of the money, because it is unlikely that the price of the underlying will change by a lot.[14] However, when an option is at the money there is high uncertainty and it not possible to tell a priori if the option will end up being in or out of the money.

What is the vega? That is, how sensitive is the option price to a change in volatility? A lager

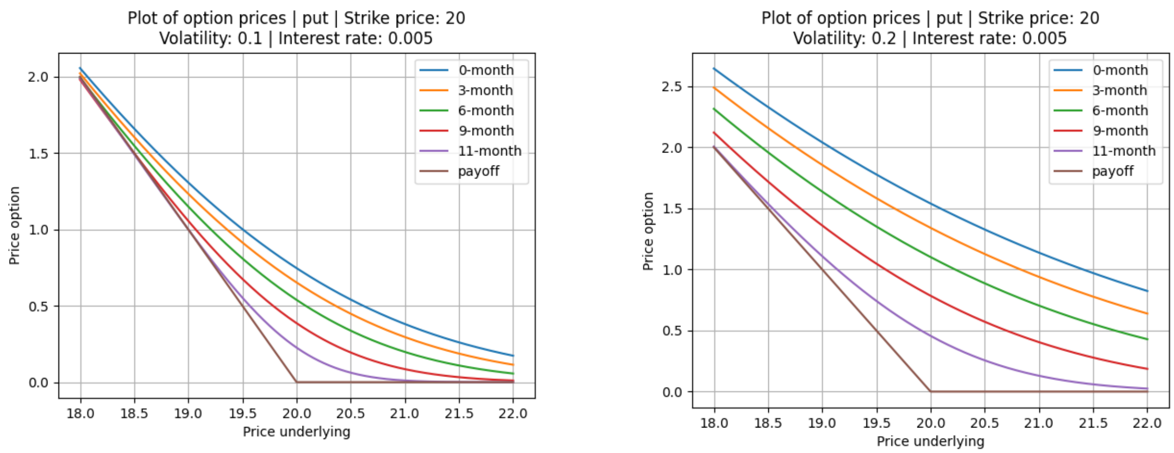

Figure 13: plots of the initial put option and of the same put with double volatility of the underlying.

volatility implies more uncertainty, therefore it will take more to the curves to assume the asymptotic behavior described above. Moreover, holding constant other factors, the option price will be larger.

What is the theta? How does the price of an option change as time passes? The value of an option is made up of intrinsic value and time value. As the time to expiration shortens, the time value of the option will be lower, moreover, since the Black-Scholes formula assumes Brownian motion, the trajectories of the underlying will be less spread out leading to potentially lower option prices. This behavior is also known as 'time decay'.

What is the rho? How sensitive is the option price to changes in interest rates? In general, the way interest rates affect the price of a derivative depends on the type of the underlying, moreover it is usually the less impactful input in determining the theoretical value of an option.

A peculiar behavior that can be noticed from the figure in the next page is that the option prices on the left side are below the intrinsic option value, meaning that the option has negative time value. As time passes, the option value rises to its intrinsic value. This also implies a positive theta, that is, the value of the option will be worth more tomorrow than today.

[14] An option is said to be 'in the money' when its value is positive, 'out of the money' when its value is zero or 'at the money' when is is exacly at the edge, that is when the strike price is close to the current market price.

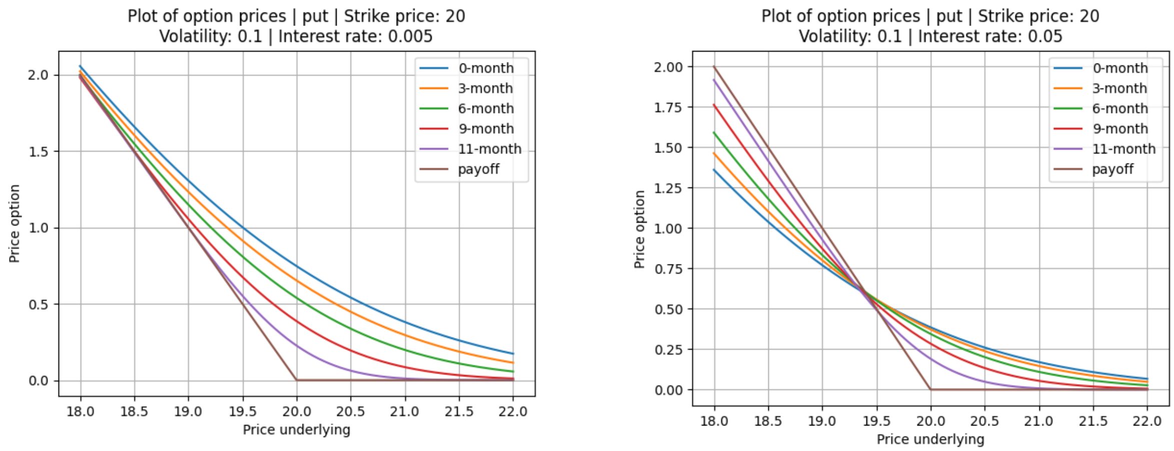

Figure 14: plots of the initial put option and of the same put with tenfold interest rate (to make the differences more visible).

6 Conclusion

There is a saying stating that: "In theory there is no difference between theory and practice, but in practice there is." What I tried to do with this dissertation is to bridge what may sometimes seem to be two separate worlds, rather than two sides of the same coin. In order to achieve such a task, along with the theoretical explanations, I provided motivating examples that, hopefully, helped to clarify the topics of interest and the practical aspects behind them. But, apart from epistemological matters, the main results can be summarized as follows.

First, thanks to the initial environment of randomly generated distributions and the computation of statistical metrics, it was possible to perform a Jarque-Bera test at the 5% significance level and it has been shown that Italian stock returns were not normally distributed. In particular, the distributions of market data presented, generally, negative skewness and large kurtosis.

Second, the beta coefficients estimated via the CAPM, despite the poor explanatory accuracy due to the low r-squared values, allowed to define a simple delta and beta hedging strategy. In addition, the limitation of an exact solution, that is the impossibility of hedging the initial holding with more than two stocks, has been overcome via numerical methods, thus expanding the available solutions and achieving more diversification.

Third, the computation of the efficient frontier yielded results consistent with the underlying theory: a larger number of stocks implied a leftwards shift of the efficient frontier and therefore better possible mixes of risk and return (in terms of Sharpe ratios). Moreover, a wide selection of portfolios (using a restricted number of stocks) has been constructed in order to match the preferences of a given investor.

Finally, the treatment of option pricing, despite being probably less rigorous and accurate, served as an entry point for further analysis of such a wide subject called finance. In this instance, the Black-Scholes model and the Monte Carlo method provided two complementary ways for determining what the fair price of an European option should be. But, especially as experienced in recent years, economic and market conditions may change quickly and drastically. As for options, there are measures of risk known as Greeks, whose interpretations have been informally examined and explained in the last section of this work.

The most relevant limitation of the approach I presented here is its static setting. Although in reality any decision is actually made in a given snapshot of time and based on the available information, that is to say statically, what I really mean by ‘limitation’ is the lack of methods and adjustments for dealing with possible different scenarios that may happen. The fact that Italian stock returns were not normally distributed does not prevent them from being so in the future (even if it seems very unlikely). The estimated beta coefficient of a given stock may change for a variety of reasons (higher or lower earnings, increased or decreased market share, positive or negative management decisions etc.). Choosing the most efficient portfolio does not guarantee that it will achieve the same performance in the future, in fact it has been shown that past returns can often be a dangerous bias [14].

How should one react to such possible changes then? This problem is what gave rise to the increased popularity of Bayesian techniques and probability based approaches. One of the many examples is the Black-Litterman model, which combines the subjective views of an investor with the observed expected returns given by the market equilibrium to form a new mixed estimate and rebalance the asset allocation accordingly [15]. In this regard, the amount of literature on the operational value of a probabilistic framework is so vast that it would be impossible to effectively summarize in a single paragraph. I shall thus close simply by quoting Bruno de Finetti[15] who, devoting much of his life to the study and development of probability theory, socratically declared: “Probability does not exist except for me and according to the degree of ignorance in which I find myself momentarily. [...] The probability of an event is therefore relative to our degree of ignorance.”

[15] Known for his constributions to the ‘operational subjective’ conception of probability, he was publicly credited in 2007 by Harry Markowitz himself for anticipating the ideas underlying the Portfolio Selection Model (see [16] [17] [18]).

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