1 A Review of Functions

1.1 Functions

The **composition** of functions g and f is written as

$$f \circ g,$$

so

$$(f \circ g)(x) = f(g(x)).$$

An **inverse function** is a function, denoted as f^{-1} (do not confuse it with $\frac{1}{f}$), that reverses function $f(x)$.

Classes of functions:

- For constants $a_0, a_1, \ldots a_n, a_n \neq 0$, and non-negative integer n,

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

  is a **polynomial**.

- **Rational functions** are functions in the form $\frac{P(x)}{Q(x)}$, where P and Q are polynomials.

Proposition 1.1.1 (Pythagorean Trigonometric Identity)
$\sin^2 \theta + \cos^2 \theta = 1.$

Important trigonometric functions:

- $\tan \theta = \frac{\sin \theta}{\cos \theta}$
- $\cot \theta = \frac{\cos \theta}{\sin \theta}$
- $\sec \theta = \frac{1}{\cos \theta}$
- $\csc \theta = \frac{1}{\sin \theta}$

"Now all of these that involve ratios wind up having vertical asymptotes in their graphs. That is, places where the function is undefined and the denominator goes to zero."

Inverse of Trigonometric Functions:

- $\sin^{-1} = \arcsin$
- $\cos^{-1} = \arccos$
- $\tan^{-1} = \arctan$

The domain of the arcsin and arccos functions are restricted to the closed interval from -1 to 1, because sine and cosine can only take values in that interval. However, arctan has an infinite domain, yet it's range is limited to the closed interval from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

Exponential functions are functions of the form e^x.

Definition 1.1.2 (e)**.** The number e, known as Euler's number, is a mathematical constant approximately equal to 2.71828.

Algebraic Properties of the Exponential Function:

- $e^x e^y = e^{x+y}$
- $(e^x)^y = e^{xy}$

Differential/Integral Properties of the Exponential Function:

- $\frac{d}{dx} e^x = e^x$
- $\int e^x dx = e^x + c.$

Theorem 1.1.3 (Euler's Formula)
$e^{ix} = \cos x + i \sin x.$

1.2 Exponentials

Definition 1.2.1 (e^x)

$$e^x = \sum_{i=0}^{\infty} \frac{1}{ki} x^i,$$

where $i! = i(i-1)(i-2)(i-3)\cdots 3 \cdot 2 \cdot 1$.

"In general, the principle that you should follow in trying to understand statements such as the definition of e to the x is to pretend that this is a long polynomial; a polynomial of unbounded degree."

Recall when it comes to

- Differentiation: $\frac{d}{dx}x^k = kx^{k-1}$.
- Integration: $\int x^k dx = \frac{1}{k+1}x^{k+1} + c$, for c is an arbitrary constant.

Therefore,

$$\frac{d}{x}e^x = \frac{d}{dx}\sum_{i=0}^{\infty} \frac{1}{i!}x^i = \sum_{i=0}^{\infty} \frac{1}{i!} i x^{i-1} = e^x,$$

and

$$\int e^x dx = \int \left(\sum_{i=0}^{\infty} \frac{1}{i!} x^i \right) dx = \sum_{i=0}^{\infty} \frac{1}{i!} \frac{1}{i+1} x^{i+1} + c = e^x + c.$$

Consider Euler's Formula, $e^{ix} = \cos(x) + i\sin(x)$. We know

$$e^{ix} = \sum_{k=0}^{\infty} \frac{1}{k!} i^k x^k.$$

Since the powers of i repeat in a definite pattern $(i, -1, -i, 1)$, thus

$$e^{ix} = \left(\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \right) + i \left(\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \right).$$

Equating the coefficients,

$$\cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \text{ and } \sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}.$$

Therefore,

$$\frac{d}{dx}\sin(x) = \frac{d}{dx}\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = \cos(x).$$

Remark 1.2.2. 1) Finite polynomial approximations work best near zero.

2) Sometimes you will see the term MacLaurin series used to apply to infinite series of this form:

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}.$$