# Fuzzy Logic

Neural Networks & Fuzzy Logic
(BITS F312)

# Fuzzy Set Theory

## What is Fuzzy Logic?

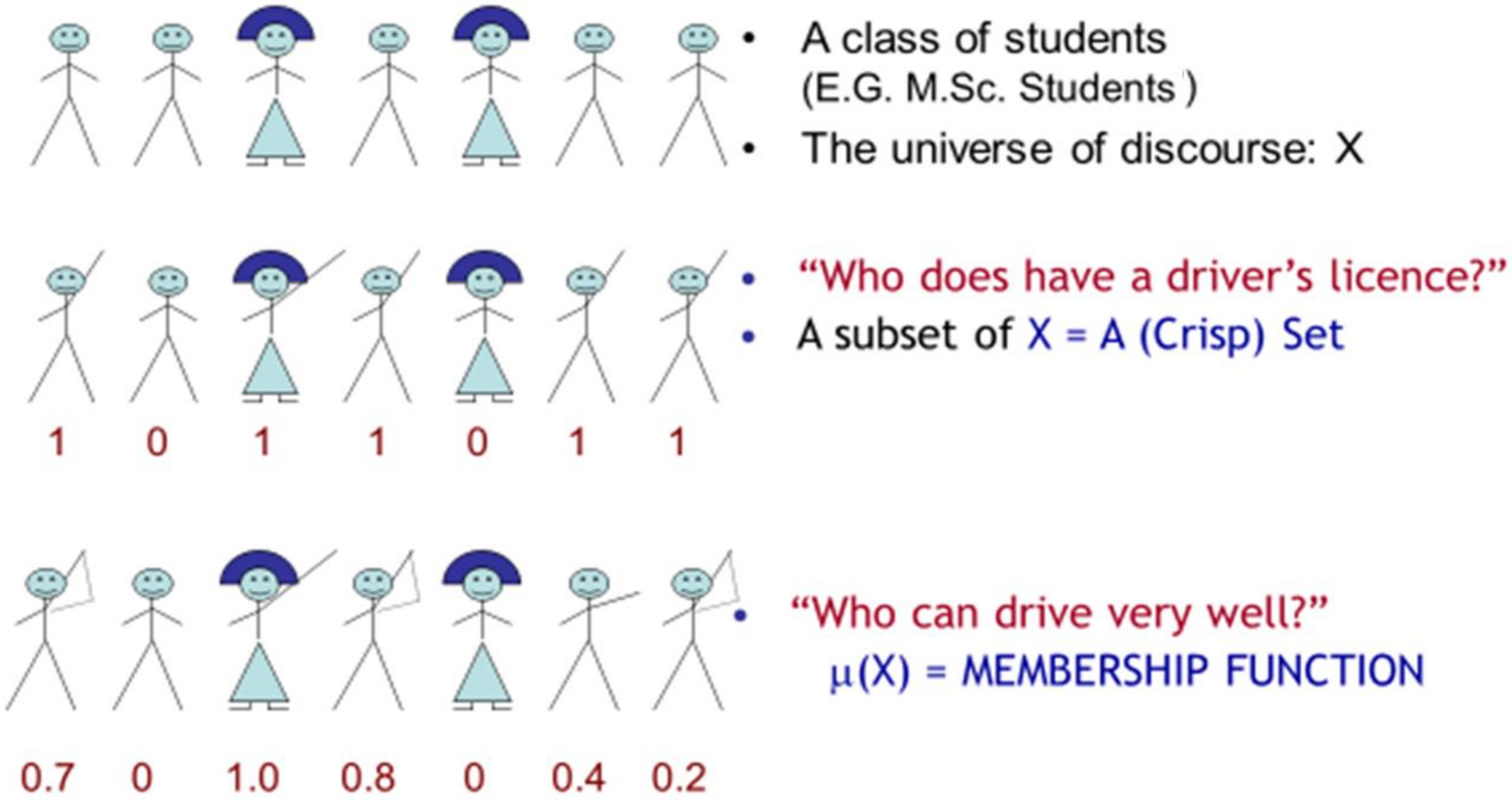
- FUZZY : not clear or precise; blurred; vague
- FUZZY LOGIC: A form of knowledge representation in vague linguistic terms

  (for phenomena which are difficult to describe mathematically)
- A generalization of conventional (Boolean) logic to handle the concept of partial truth i.e. truth values between “completely true” (membership value 1.0) and “completely false” (membership value 0.0)
- Age less than 40 years → YOUNG

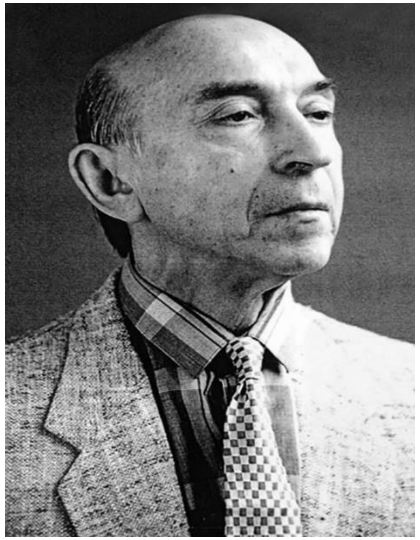
  Age more than 40 years → OLD
- All boundaries are Blurred!

No ambiguity

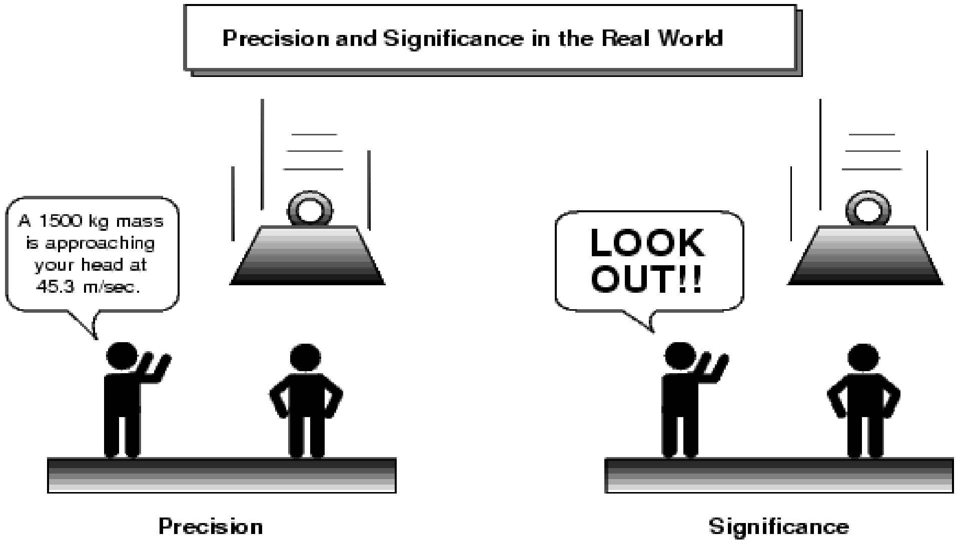
Ambiguous



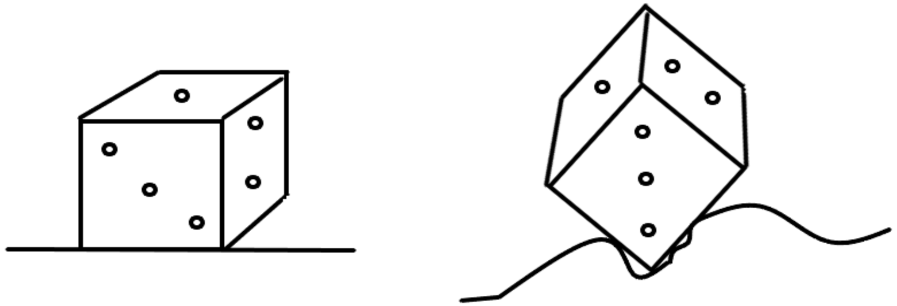
- Prof. Lotfi A. Zadeh (1921- 2017), UC Berkeley
  Proposed fuzzy set theory in 1965



- "As complexity rises precise statements lose meaning and meaningful statements lose precision"




- Fuzzy vs. Probability
  how much vs. whether



- ADVANTAGES:

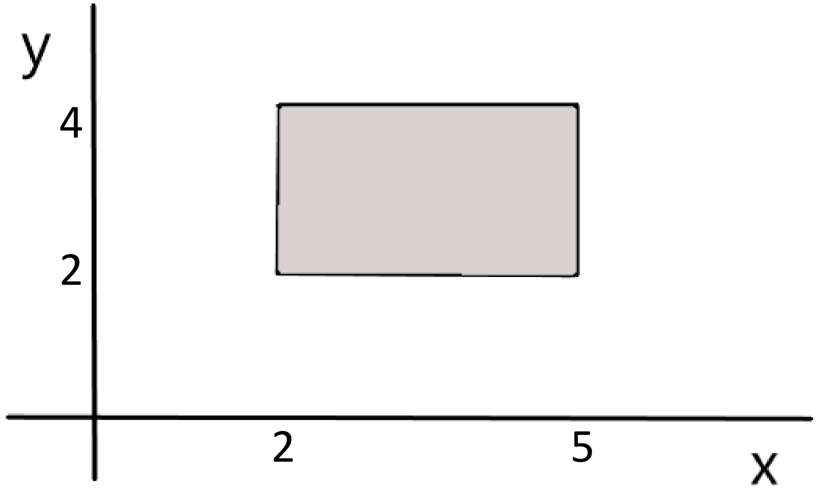
  - Mimics human reasoning and decision making process
  - Based on qualitative understanding of the phenomena under consideration (less dependant on input-output data)
  - Easy to conceptualize
  - Tolerant to imprecise data

- DISADVANTAGES:
  - Usually lacks accuracy
  - Computationally expensive
  - Describing some phenomena even linguistically requires a lot of domain expertise
  - Very often depends on some optimizer (such as GA) to be more efficient
  - Number of rules increases exponentially as number of variables increases

## A Brief Review of Crisp Sets

- A collection of objects

- A crisp set has sharp boundaries




E.g. A = $\{(x,y) \mid 2 \leq x \leq 5, 2 \leq y \leq 4\}$

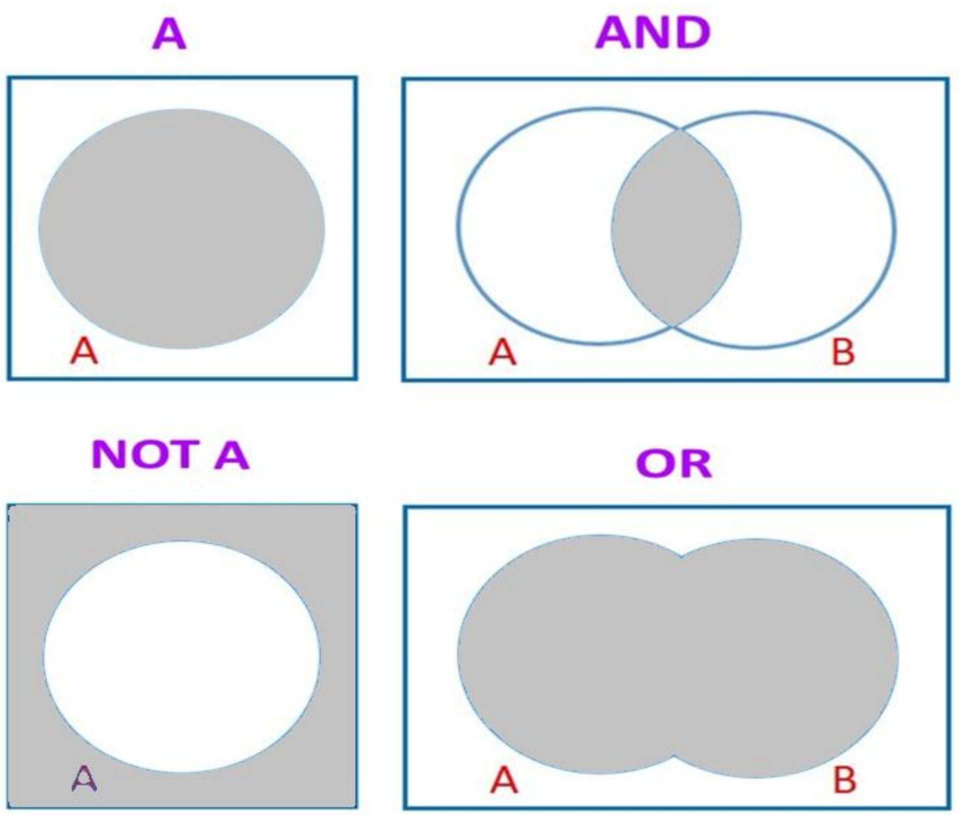
- Cardinal No.: The number of elements of a set

- Null Set: A set with no member (Ø)

- Universal Set: A set with all possible members in a given context (X)

## Three Fundamental Operations

- $\mathrm{A} \cap \mathrm{B} = \{\mathrm{x} \mid x \in A \ \ and \ x \in B\}$
- $\mathrm{A} \cup \mathrm{B} = \{\mathrm{x} \mid x \in A \ \ or \ x \in B\}$
- $\overline{\mathrm{A}} = \{\mathrm{x} \mid x \notin A\}$




## Properties of Crisp Sets

i) Commutativity: A· B = B· A ; A+B = B+A

ii) Distributivity: A+(B·C) = (A+B) · (A+C) ; A · (B+C) = (A·B) + (A·C)

iii) Associativity: $A \cdot (B \cdot C) = (A \cdot B) \cdot C$ ; $A+(B+C) = (A+B)+C$

iv) Involution: $\overline{\overline{A}} = A$

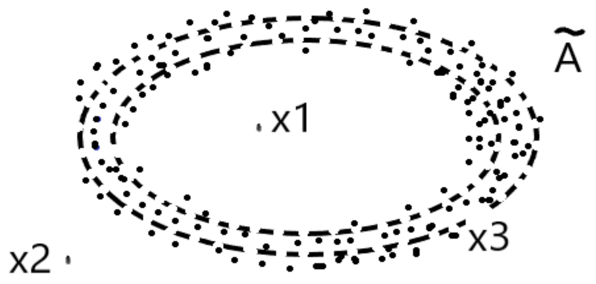
v) DeMorgan's laws: $\overline{A \cdot B} = \overline{A} + \overline{B}$ ; $\overline{A + B} = \overline{A} \cdot \overline{B}$

Law of Contradiction and Law of Excluded Middle:

vi) $A \cdot \overline{A} = \emptyset$

vii) $A + \overline{A} = X$

# Fuzzy Sets




- For discrete domains a fuzzy set is described by assigning a membership value (μ between 0 and 1) to each element

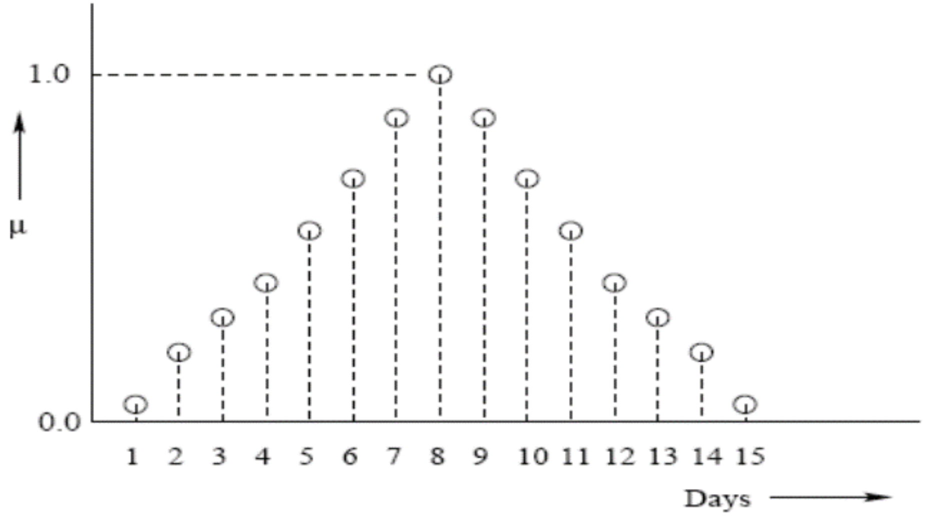
$$\widetilde{\mathrm{A}} = \left\{\frac{\mu(x_1)}{x_1} + \frac{\mu(x_2)}{x_2} + \cdots\right\}$$

E.g.   X = {1,2,3,4,5,6} be the set of houses with number of rooms.

‘comfortable house for a four member family’ ,

$$\widetilde{\mathrm{A}} = \left\{\frac{0.1}{1} + \frac{0.5}{2} + \frac{0.8}{3} + \frac{1.0}{4} + \frac{0.9}{5} + \frac{0.4}{6}\right\}$$

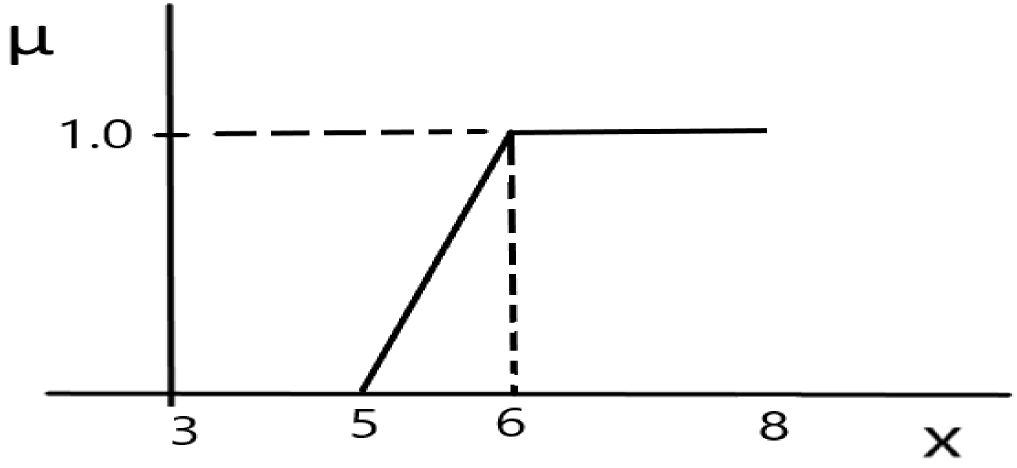
or ‘an enjoyable vacation’



- For continuous domains, a fuzzy set is described by the membership function

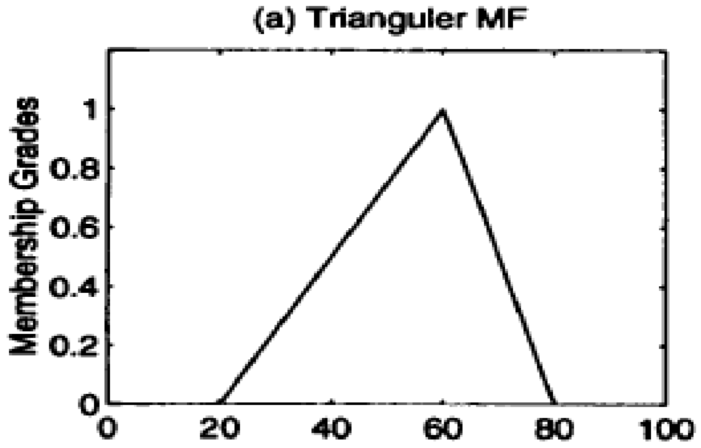
E.g. X = [3, 8] is the height of people in foot

'a tall man', $\widetilde{\mathrm{A}} = \left\{ \int \frac{\mu(x)}{x} \right\}$ where $\mu(x) = \begin{cases} 0.0\,, x \leq 5.0 \\ x - 5\,, \quad 5.0 \leq\ x \leq 6.0 \\ 1.0\,, x \geq 6.0 \end{cases}$

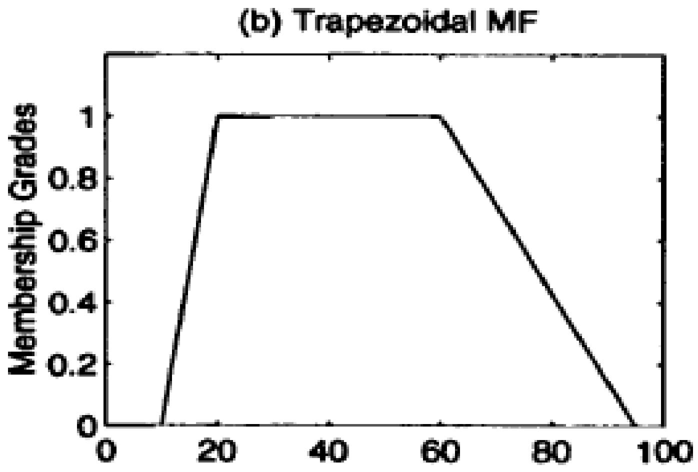


# Common Shapes of Membership Functions

a) $$\text{triangle}(x;a,b,c) = \begin{cases} 0, & x \le a. \\ \frac{x-a}{b-a}, & a \le x \le b. \\ \frac{c-x}{c-b}, & b \le x \le c. \\ 0, & c \le x. \end{cases}$$

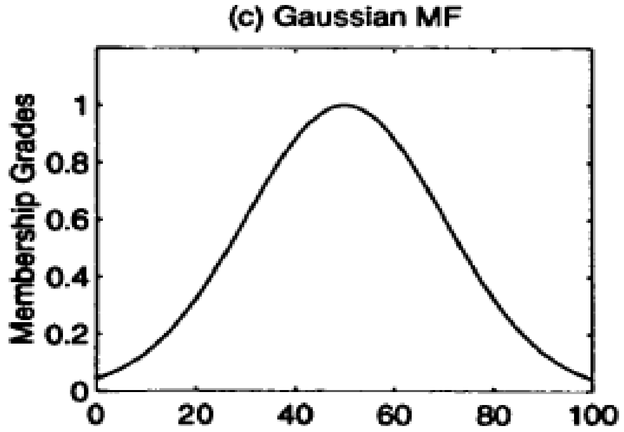



b) $$\text{trapezoid}(x;a,b,c,d) = \begin{cases} 0, & x \le a. \\ \frac{x-a}{b-a}, & a \le x \le b. \\ 1, & b \le x \le c. \\ \frac{d-x}{d-c}, & c \le x \le d. \\ 0, & d \le x. \end{cases}$$



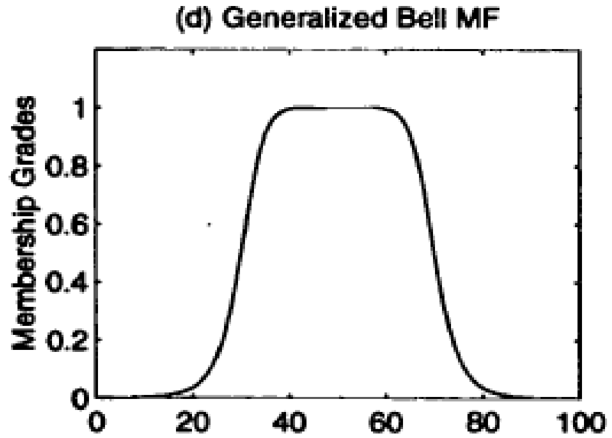
c)

$$\text{gaussian}(x; c, \sigma) = e^{-\frac{1}{2}\left(\frac{x-c}{\sigma}\right)^2}.$$



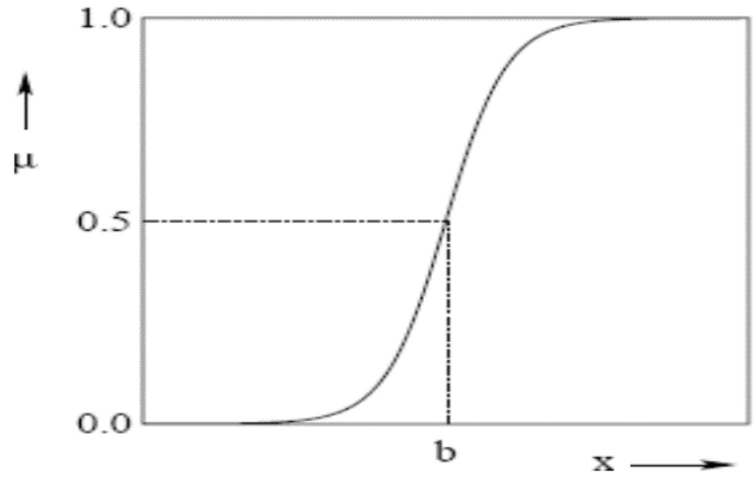

d)

$$\text{bell}(x; a, b, c) = \frac{1}{1 + \left|\frac{x-c}{a}\right|^{2b}}$$



**e)** **sigmoidal**

$$\mu_{Sigmoid} = \frac{1}{1+e^{-a(x-b)}}$$




**f)** **singleton**

# A Few More Terms:

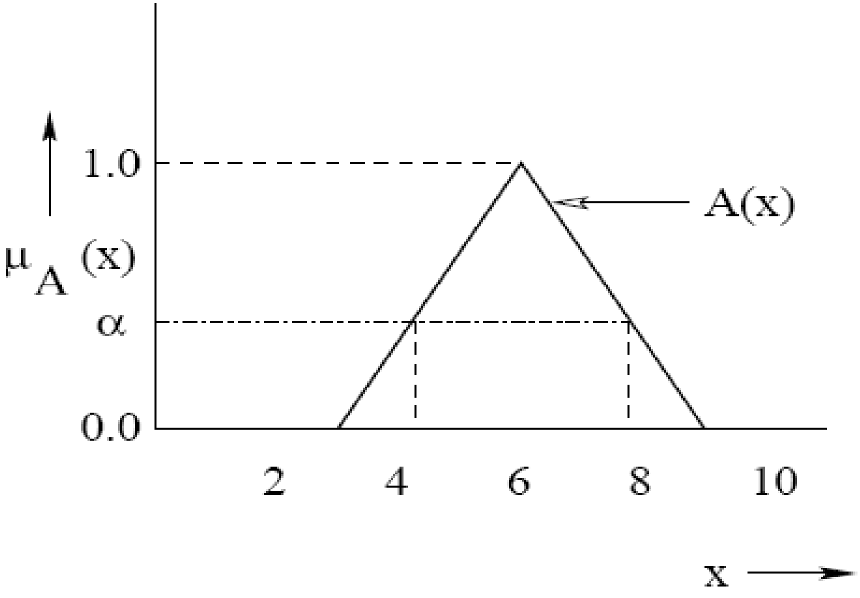
- **α- Cut of a fuzzy set** $\alpha_{\mu_A}(x)$

  A set consisting of elements *x* of the Universal set X, whose membership values are either greater than or equal to the value of $\alpha$

$$\alpha_{\mu_A}(x) = \{ \mid \mu_A(x) \geq \alpha \}$$

- **Strong α-Cut of a fuzzy set**

$$\alpha^{+}_{\mu_A}(x) = \{ \mu_A(x) > \alpha \}$$

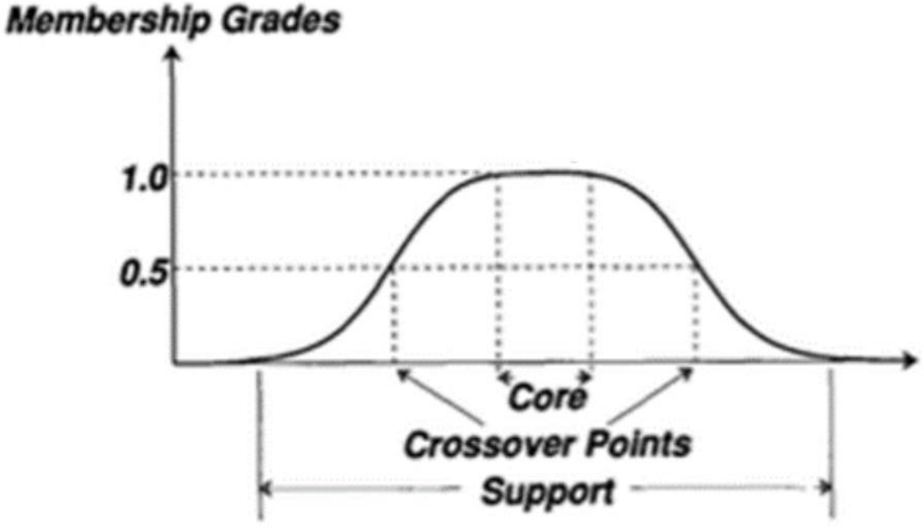


- **Support of a fuzzy set A(*x*)**

It is defined as the set of all $x \in X$, such that $\mu_A(x) > 0$

supp(A) $= \{x \in X \mid \mu_A(x) > 0\}$

It is nothing but its Strong 0- cut




- **Core of a fuzzy set A(*x*)**

core $(\tilde{A}) = \{ x \mid \mu_{\tilde{A}}(x) = 1.0 \}$

It is nothing but its 1-cut

- **Crossover points**

crossover $(\tilde{A}) = \{ x \mid \mu_{\tilde{A}}(x) = 0.5 \}$

- **Height of a fuzzy set A(*x*)**

  It is the largest membership values of the elements contained in that set

- **Normal fuzzy set**

  For a normal fuzzy set, h(A) = 1.0

- **Sub-Normal fuzzy set**

  For a sub normal fuzzy set, h(A) < 1.0

- **Concentration:** $\mu_{very\,\tilde{A}}(x) = (\mu_{\tilde{A}}(x))^{2}$

- **Dilation:** $\mu_{slightly\,\tilde{A}}(x) = (\mu_{\tilde{A}}(x))^{0.5}$

**Numerical Example:**

- Assume X={ a, b, c, d, e }

  A={ 1/a ,0.3/b , 0.2/c , 0.8/d, 0/e } ,

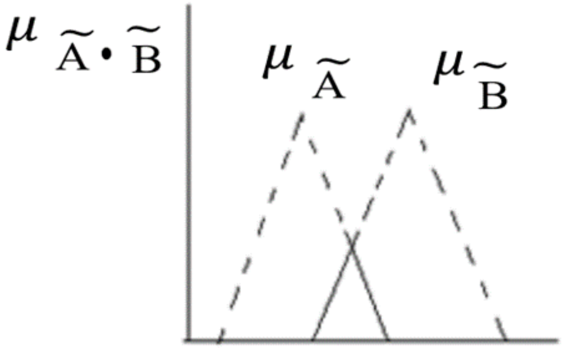
  B= { 0.6/a, 0.9/b , 0.1/c , 0.3/d, 0.2/e } .

- Find out whether A and B are normal sets or non-normal sets

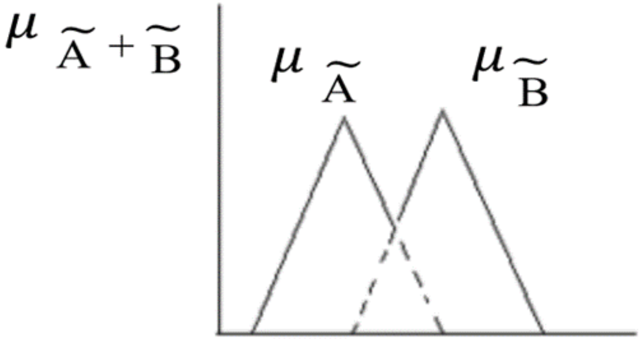
- Find out the height, support and core of A and B.

# Three Fundamental Operations on Fuzzy Sets

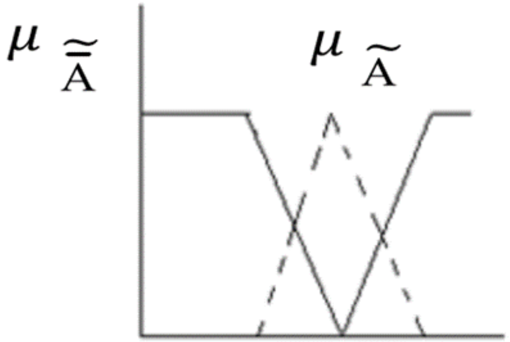
$$\mu_{\tilde{\mathrm{A}}\,.\,\tilde{B}}(x) = \min\left(\mu_{\tilde{\mathrm{A}}}(x), \mu_{\tilde{\mathrm{B}}}(x)\right)$$



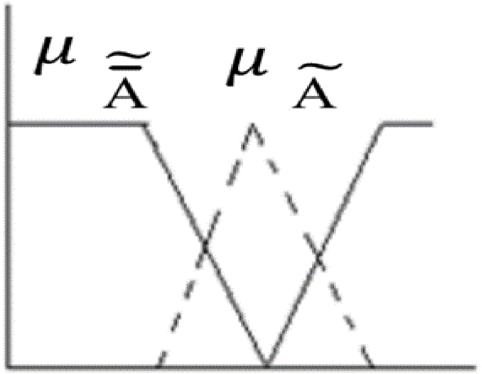

$$\mu_{\tilde{\mathrm{A}}\,.\,\tilde{B}}(x) = \min\left(\mu_{\tilde{\mathrm{A}}}(x), \mu_{\tilde{\mathrm{B}}}(x)\right)$$




$$\mu_{\tilde{\tilde{\mathrm{A}}}}(x) = 1 - \mu_{\tilde{\mathrm{A}}}(x)$$



**Properties of fuzzy sets**




Fuzzy sets follow all the properties of crisp sets except for the following two:

1. Law of excluded middle

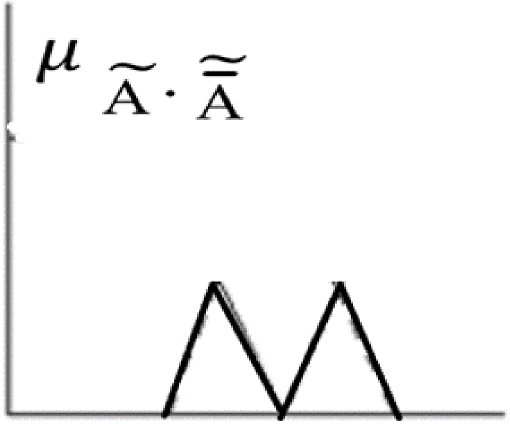
In crisp set, $A \cup \overline{A} = X$

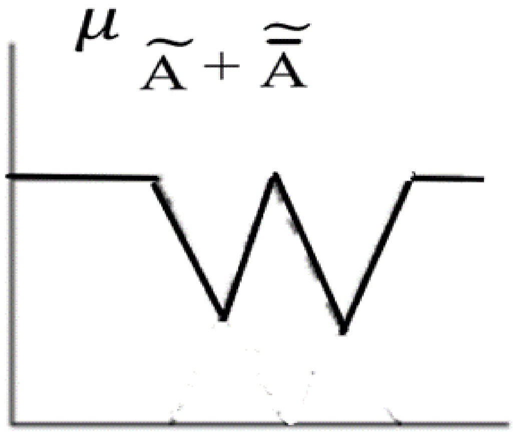
In fuzzy set, $A \cup \overline{A} \neq X$

2. Law of contradiction

In crisp set, $A \cap \overline{A} = O$

In fuzzy set, $A \cap \overline{A} \neq O$



**Numerical Example:**

Given the two fuzzy sets

$$\widetilde{\mathrm{A}} = \left\{\frac{1}{1} + \frac{0.6}{2} + \frac{0.3}{3} + \frac{0.2}{4}\right\} \quad \text{and} \;\; \widetilde{\mathrm{B}} = \left\{\frac{0.5}{2} + \frac{0.7}{3} + \frac{0.2}{4} + \frac{0.4}{5}\right\}$$

i) Verify DeMorgan's laws

ii) Compute $\widetilde{\mathrm{A}} \cdot \overline{\widetilde{\mathrm{A}}}$ and $\widetilde{\mathrm{B}} + \overline{\widetilde{\mathrm{B}}}$

**Various T-norms (AND Operators):**

T(0,0) = 0 ; T(a,1) = T(1,a) = a : boundary

T(a,b) $\leq$ T(c,d) if a $\leq$ c and b $\leq$ d : monotonicity

T(a,b) = T(b,a) : commutativity

T(a, T(b,c)) = T(T(a,b),c) : associativity

Minimum: T(a,b) = min(a,b)

Algebraic Product: T(a,b) = a b

Bounded Product: T(a,b) = max(0, a+b-1)

Drastic Product: T(a,b) = $\begin{cases} a\,, if\ b = 1 \\ b\,, if\ a = 1 \\ 0\,, if\ a, b < 1 \end{cases}$

**Various S-norms (OR Operators):**

S(1,1) = 1 ;  S(a,0) = S(0,a) = a : boundary

S(a,b) $\leq$ S(c,d)  if a $\leq$ c  and b $\leq$ d : monotonicity

S(a,b) = S(b,a) : commutativity

S(a, S(b,c)) = S(S(a,b),c) : associativity

Maximum: S(a,b) = max(a,b)

Algebraic Sum: S(a,b) = a+b-ab

Bounded Sum: S(a,b) = min(1, a+b)

Drastic Sum: $S(a,b) = \begin{cases} a\,, if\ b = 0 \\ b\,, if\ a = 0 \\ 1\,, if\ a, b > 0 \end{cases}$

**Sugeno's Complement Operator:**

N(0) = 1 ;  N(1) = 0 : boundary

N(a) $\geq$ N(b)  if a $\leq$ b : monotonicity

Sugeno: N(a) = $\frac{1-a}{1+ s\cdot a}$ where s > -1

**Numerical Example:**

Verify DeMorgan's laws for the two fuzzy sets

$\widetilde{\mathrm{A}} = \left\{\frac{1}{1} + \frac{0.6}{2} + \frac{0.3}{3} + \frac{0.2}{4}\right\} \quad and \;\; \widetilde{\mathrm{B}} = \left\{\frac{0.5}{2} + \frac{0.7}{3} + \frac{0.2}{4} + \frac{0.4}{5}\right\}$ using

i) Algebraic product T-norm and Maximum S-norm
ii) Minimum T-norm and Algebraic Sum S-norm

**Crisp Relation**

- Sometimes elements of a set may be related to each other

  E.g. A = {Delhi, Rupee, Hindi}

- Two sets defined on two different universes but their elements are related to each other

  E.g. A = {China, India, Germany, Japan}

  B = {Berlin, Beijing, Tokyo, Delhi}

- A binary relation (relation between two sets) is a structure that represents the presence or absence of interaction between the elements of the two sets

- Cartesian Product:

Let A = {a1, a2, a3} defined on the universe X

and B = {b1, b2, b3, b4} defined on the universe Y

then their Cartesian product defined on the cross space $X \times Y$ is given by

$$A \times B = \begin{array}{c c} & \begin{array}{cccc} b1 & b2 & b3 & b4 \end{array} \\ \begin{array}{c} a1 \\ a2 \\ a3 \end{array} & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

- Relation R is a subset of $A \times B$ with the elements being 0 or 1 depending on absence or presence of a relation

Let A = {India, Japan, Canada, Kenya}

B = {Tokyo, Cairo, Delhi}

then the relation 'capital of the country' is given by

$$R = \begin{array}{c c} & \begin{array}{ccc} T & C & D \end{array} \\ \begin{array}{c} I \\ J \\ C \\ K \end{array} & \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

- Composition: If R1 is the relation between X and Y and R2 is the relation between Y and Z then the relation R between X and Z is computed from the rule of composition as

$$R = R1 \circ R2 \quad \text{where } r(x,z) = \max\{ \min (r1(x,y), r2(y,z))\}$$

Let A = {India, Japan, Canada, Kenya}

B = {Tokyo, Cairo, Delhi}

C = {English, Hindi}

$$R1 = \begin{array}{c c} & \begin{array}{ccc} T & C & D \end{array} \\ \begin{array}{c} I \\ J \\ C \\ K \end{array} & \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array} \qquad R2 = \begin{array}{c c} & \begin{array}{cc} H & B \end{array} \\ \begin{array}{c} T \\ C \\ D \end{array} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \end{array}$$

$$R = \begin{array}{c c} & \begin{array}{cc} H & B \end{array} \\ \begin{array}{c} I \\ J \\ C \\ K \end{array} & \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array} \qquad R = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \circ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

**Fuzzy Relation**

- Crisp relation only reveals if the elements of a set or of different sets are related or not
- Fuzzy relation reveals if the elements are weakly or strongly related i.e. the strength of the relations
- The strength is represented by a membership value in the interval [0,1]

Let X = {Delhi, Bombay, Calcutta, Madras}

and Y = {Vizag, Pune, Kochi, Guwahati}

the relation 'far' may be given by

$$\tilde{R} = \begin{array}{c} \\ D \\ B \\ C \\ M \end{array} \begin{array}{c} \begin{array}{cccc} V & P & K & G \end{array} \\ \begin{bmatrix} 0.7 & 0.7 & 0.8 & 0.8 \\ 0.5 & 0.1 & 0.5 & 0.9 \\ 0.4 & 0.6 & 0.7 & 0.2 \\ 0.2 & 0.4 & 0.2 & 0.8 \end{bmatrix} \end{array}$$

Alternate representation,

$\tilde{R}$ =

$$\left\{\frac{0.7}{(D,V)},\frac{0.7}{(D,P)},\frac{0.8}{(D,K)},\frac{0.8}{(D,G)},\frac{0.5}{(B,V)},\frac{0.1}{(B,P)},\frac{0.5}{(B,K)},\frac{0.9}{(B,G)},\frac{0.4}{(C,V)},\frac{0.6}{(C,P)},\frac{0.7}{(C,K)},\frac{0.2}{(C,G)},\frac{0.2}{(M,V)},\frac{0.4}{(M,P)},\frac{0.2}{(M,K)},\frac{0.8}{(M,G)}\right\}$$

Cartesian Product of two fuzz y sets:

$$\mu_{\tilde{A}\times\tilde{B}}(x,y) = \min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y))$$

Let $\widetilde{\mathrm{A}} = \left\{\frac{0.2}{x_1} + \frac{0.5}{x_2} + \frac{1.0}{x_3}\right\}$ $and$ $\widetilde{\mathrm{B}} = \left\{\frac{0.3}{y_1} + \frac{0.9}{y_2}\right\}$

$$\tilde{A} \times \tilde{B} = \begin{array}{c c} & \begin{array}{cc} \text{y1} & \text{y2} \end{array} \\ \begin{array}{c} \text{x1} \\ \text{x2} \\ \text{x3} \end{array} & \begin{bmatrix} 0.2 & 0.2 \\ 0.3 & 0.5 \\ 0.3 & 0.9 \end{bmatrix} \end{array}$$

## Fuzzy Composition

- Fuzzy composition is defined in the same line as crisp composition
- Let $\widetilde{\mathrm{R}}$ be a fuzzy relation in the cross space $\mathrm{X} \times \mathrm{Y}$

  and $\tilde{\mathrm{S}}$ be a fuzzy relation in the cross space $\mathrm{Y} \times \mathrm{Z}$

  then the fuzzy relation $\widetilde{\mathrm{T}}$ in the cross space $\mathrm{X} \times \mathrm{Z}$ is given by

$$\widetilde{\mathrm{T}} = \widetilde{\mathrm{R}} \circ \tilde{\mathrm{S}}$$

where

$$\mu_{\tilde{T}}(x, z) = \max_{y} \ [\min\{\mu_{\tilde{R}}(x, y), \mu_{\tilde{S}}(y, z)\}]$$ : max-min composition rule

$$\mu_{\tilde{T}}(x, z) = \max_{y} \ [\mu_{\tilde{R}}(x, y) \ \mu_{\tilde{S}}(y, z)]$$ : max-product composition rule

**Numerical Example:**

Let X={x1, x2}, Y={y1, y2} and Z={z1, z2, z3} be three different universes of discourse.

Fuzzy relation matrices on the cross spaces $X \times Y$ and $Y \times Z$ are given by

$$\tilde{R} = \begin{bmatrix} 0.7 & 0.5 \\ 0.8 & 0.4 \end{bmatrix} \text{ and } \tilde{S} = \begin{bmatrix} 0.9 & 0.6 & 0.2 \\ 0.1 & 0.7 & 0.5 \end{bmatrix}$$

Obtain the relation matrix relating X and Z by

i) max-min rule of composition

ii) max-product rule of composition

$$\widetilde{T} = \widetilde{R} \circ \tilde{S}$$

$$= \begin{bmatrix} 0.7 & 0.6 & 0.5 \\ 0.8 & 0.6 & 0.4 \end{bmatrix} \qquad \text{(i)}$$

$$= \begin{bmatrix} 0.63 & 0.42 & 0.25 \\ 0.72 & 0.48 & 0.20 \end{bmatrix} \qquad \text{(ii)}$$

# Properties of Fuzzy Relations

**Fuzzy Extension Principle**

- A function (y=f(x)) maps one universe to another (in crisp sense)

- Same concept can be extended to fuzzy sets

- Let $\widetilde{\mathrm{R}}$ be the fuzzy relation between two universes X and Y.

  If $\widetilde{\mathrm{A}}_1$ is a fuzzy set from X then its mapping onto universe Y is given by

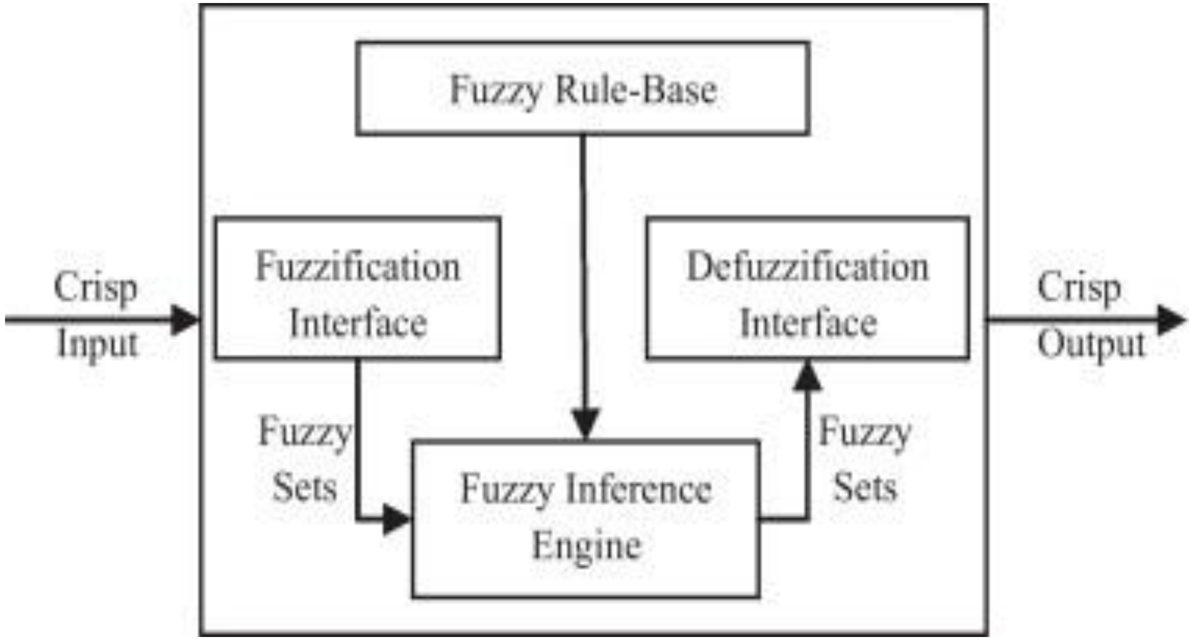
  $\widetilde{\mathrm{B}}_1 = \widetilde{\mathrm{A}}_1 \circ \widetilde{\mathrm{R}}$

## A Numerical Example:

# Fuzzy Reasoning Process

# Four Parts

- Fuzzification
- Fuzzy Rule Base
- Fuzzy Inference
- Defuzzification



**Fuzzification** (Crisp to fuzzy conversion)

- Deciding on the spaces for each i/p and o/p variable

  (may be normalized universes coupled with input scaling factors)

- Deciding on the number of fuzzy sets for each variable

N P

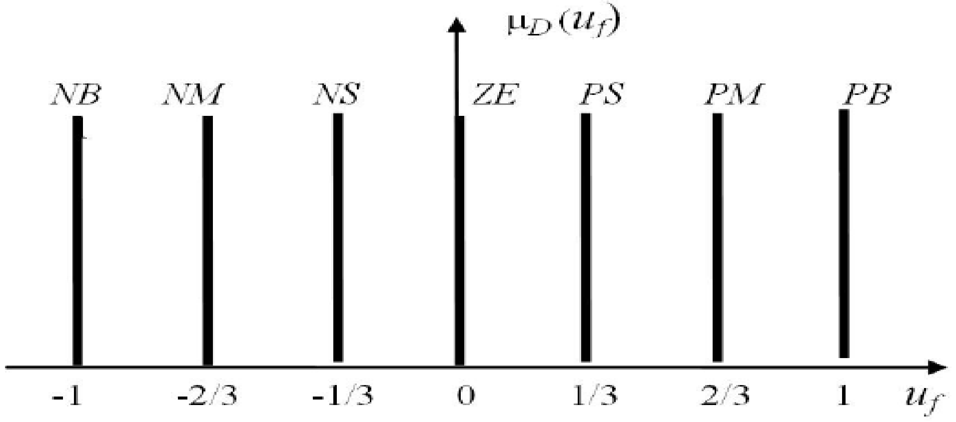
N Z P

NB NS Z PS PB

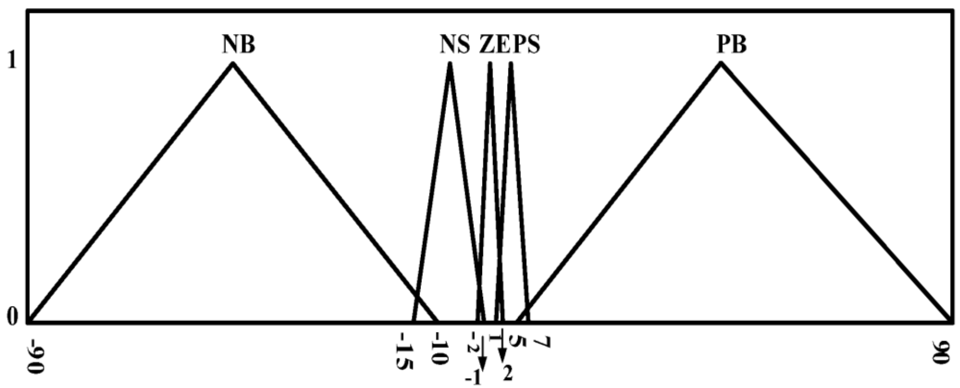
NB NM NS Z PS PM PB

- Deciding on the shapes of the fuzzy sets

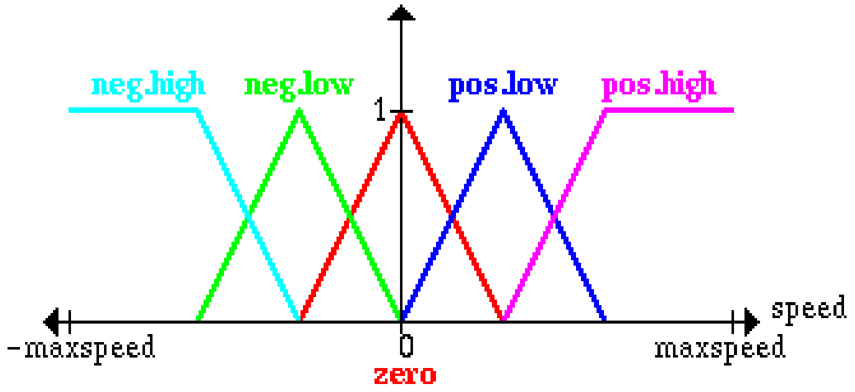
  (or the nature of the membership functions)




Singleton




Asymmetric




Trapezoidal at the ends

**Fuzzy Rule Base** (look up table for reasoning)

- Rule base for Mamdani inference (1976):

  If Temp. is High And Humidity is High Then Speed is High

  If Temp. is High And Humidity is Low Then Speed is Low

  …

| de/e | NB | NM | NS | ZE | PS | PM | PB |
|---|---|---|---|---|---|---|---|
| NB | NB | NB | NB | NM | NM | NS | ZE |
| NM | NB | NB | NM | NS | NS | ZE | PS |
| NS | NB | NM | NS | NS | ZE | PS | PM |
| ZE | NM | NS | NS | ZE | PS | PS | PM |
| PS | NM | NS | ZE | PS | PS | PM | PB |
| PM | NS | ZE | PS | PS | PM | PB | PB |
| PB | ZE | PS | PM | PM | PB | PB | PB |

- Linear Fuzzy Rules

  If $X_1$ is $A_i$ And $X_2$ is $B_j$ Then Y is $C_{i+j}$

  $-J \leq i, j \leq +J$

  (2J+1 numbers of fuzzy sets for each i/p variable and

  4J+1 numbers of fuzzy sets for o/p variable)

- Rule base for Sugeno (or TS) Inference (1985)

  If Temp. is High And Humidity is High Then Speed is f1(T, H)

  If Temp. is High And Humidity is Low Then Speed is f2(T, H)

  …

Zero-th order Sugeno model and Mamdani with singleton o/p fuzzification are same

**Fuzzy Inference**

- Mamdani Model

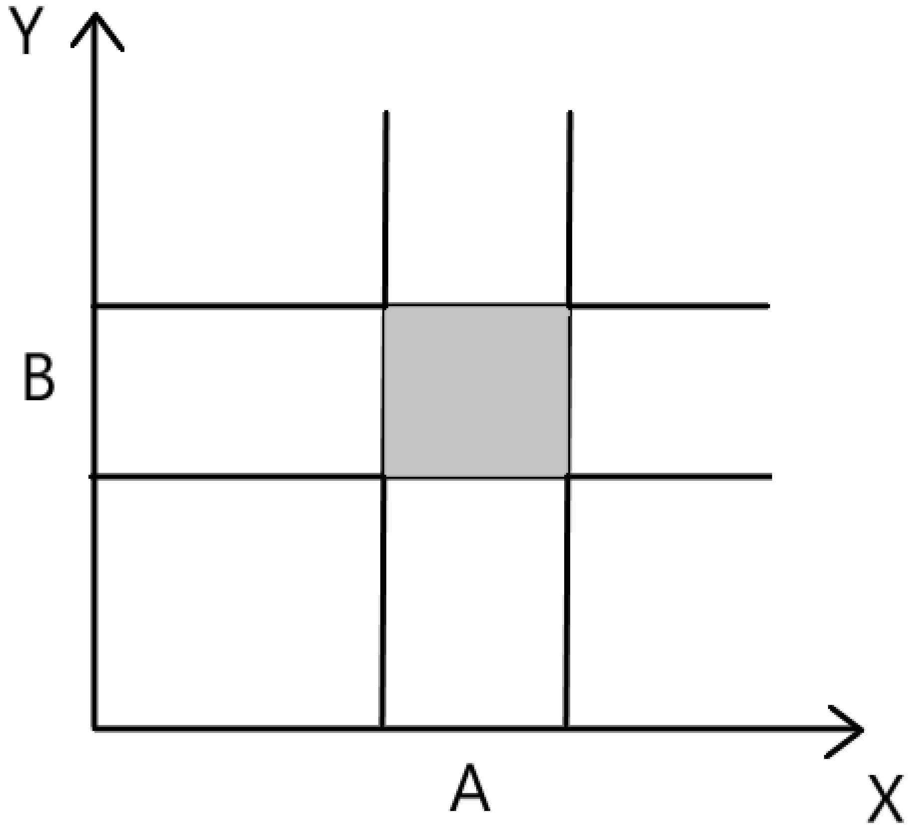
$$\underbrace{\text{IF } x \text{ is } A}_{\text{antecedent}} \quad \underbrace{\text{THEN } y \text{ is } B}_{\text{consequence}}$$

$A \rightarrow B = A \cdot B$ : A is coupled with B

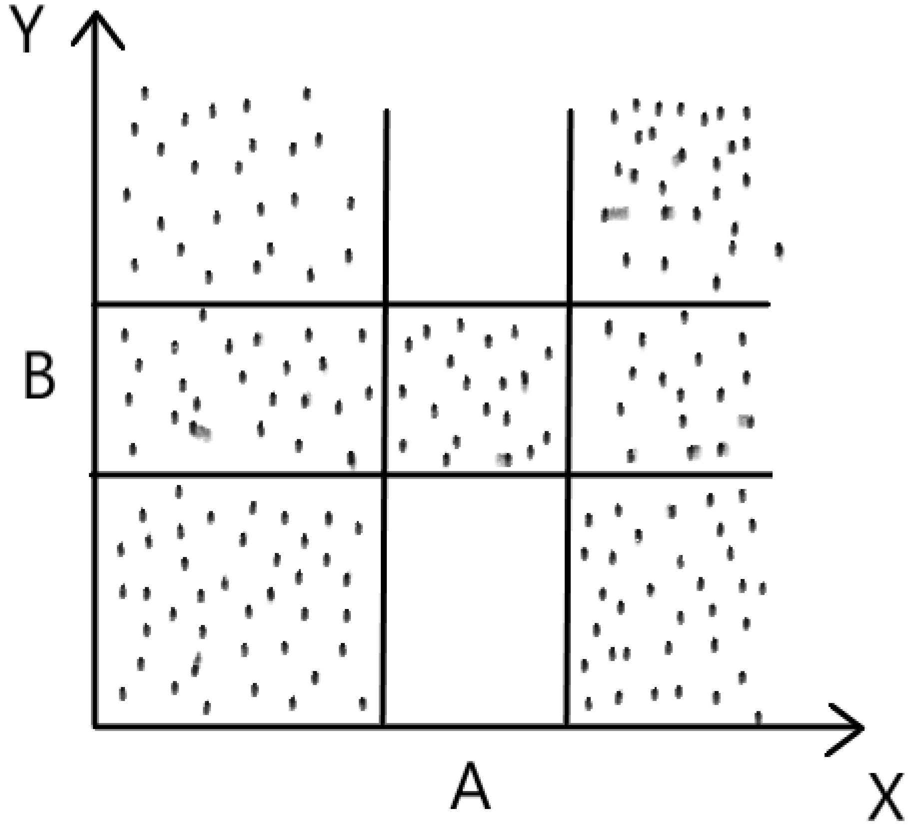
Another interpretation:

$A \rightarrow B = \bar{A} + B$ : A entails B

(i.e. If x is A Then y is not $\bar{B}$)




$A \to B = A \cdot B$




$A \to B = \overline{A} + B$

$A \rightarrow B = A \cdot B$

- Mamdani minimum inference
- Larsen product inference
- Bounded product inference
- Drastic product inference

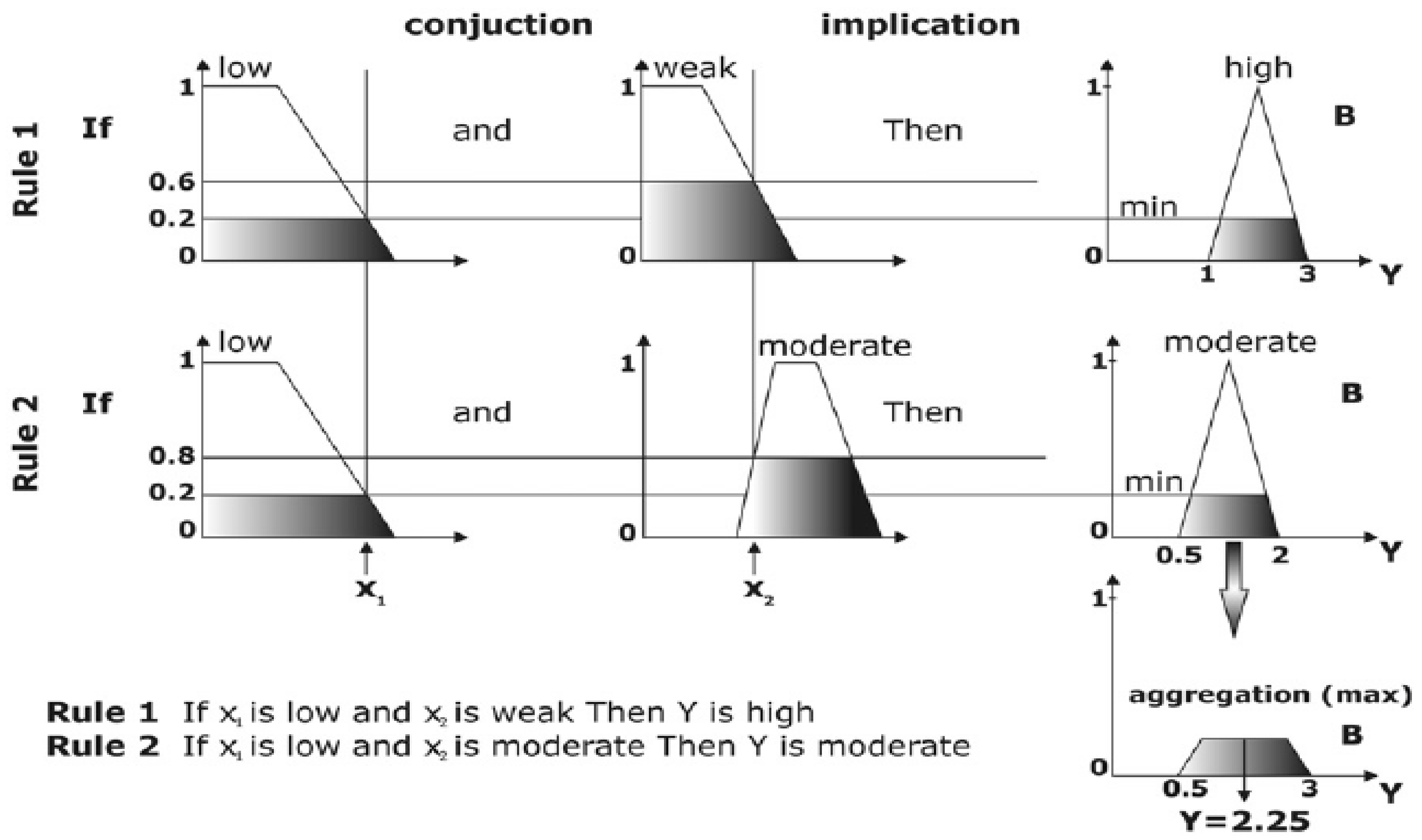
Logical OR of the outputs of the individual rules

$A \rightarrow B = \overline{A} + B$

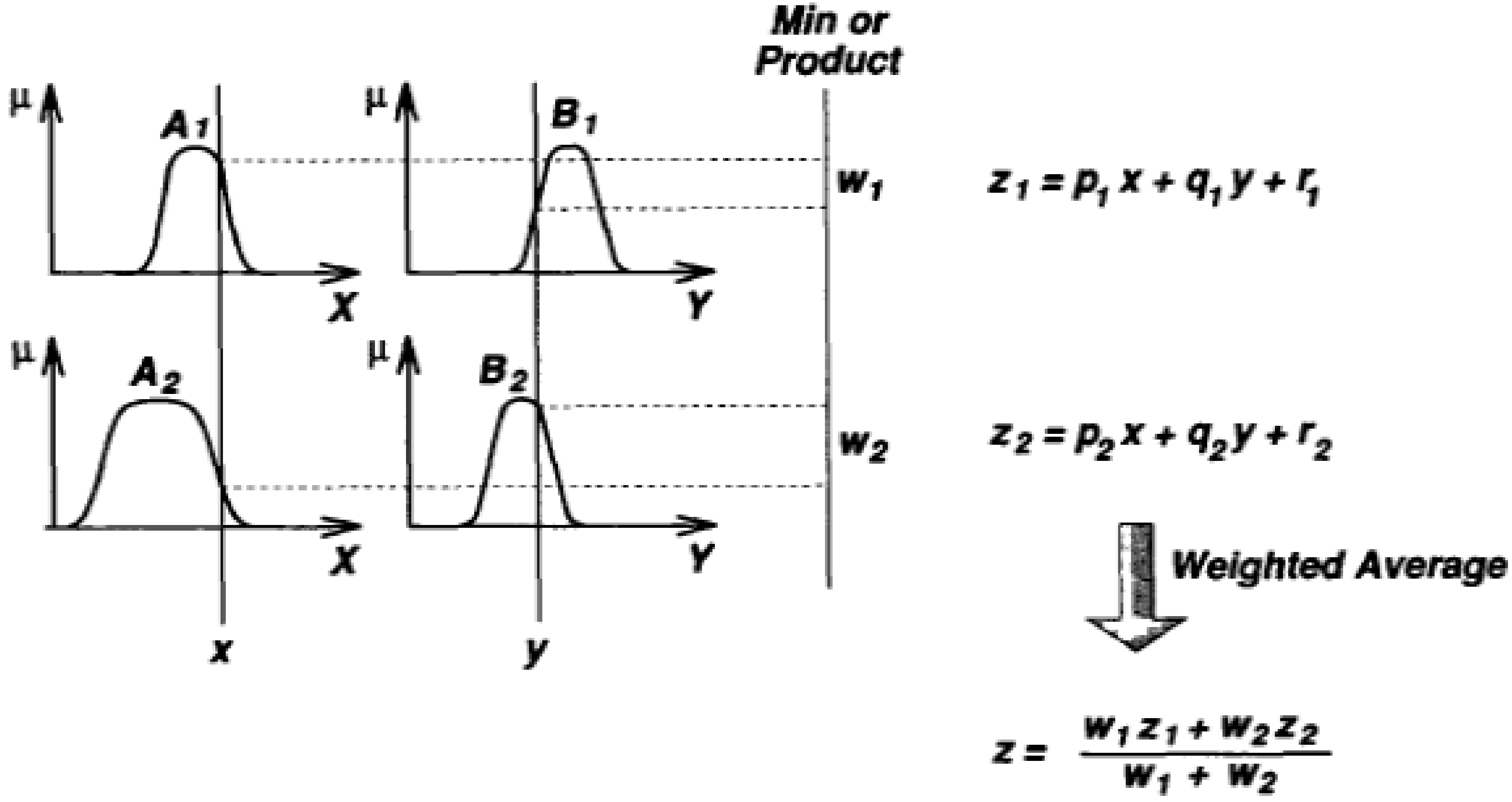
- Zadeh implication: $\overline{A} + (A \cdot B)$ ; max and min operators

- Lukasiewicz implication: Bounded sum operator for OR

- **Mamdani Inference**

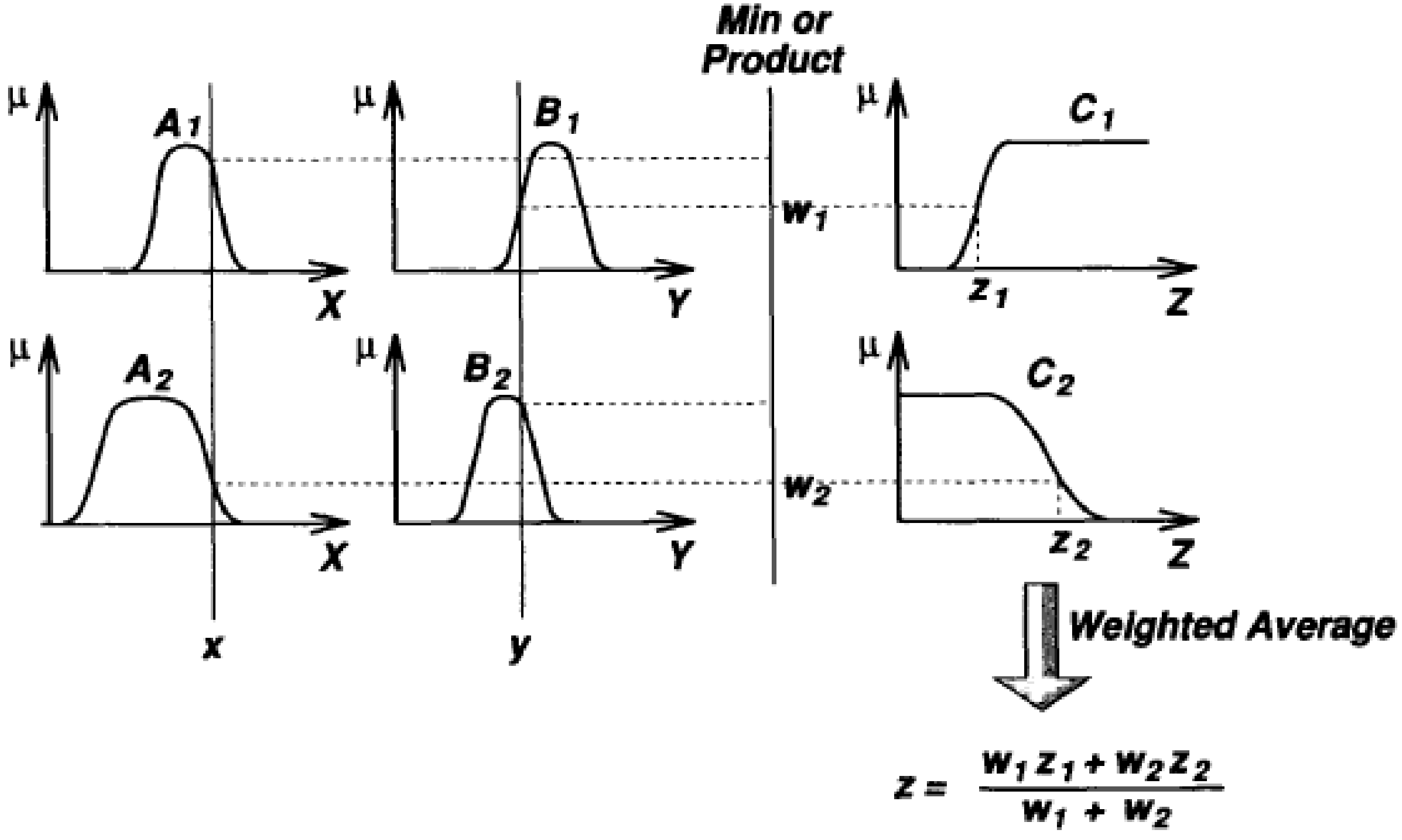


- **Sugeno Model (or Sugeno or TS Inference)**



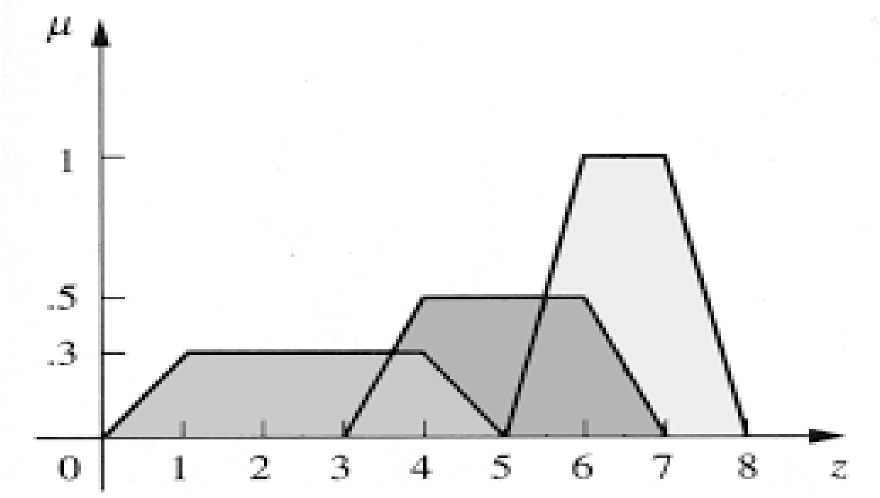
- **Tsukamoto Fuzzy Model**

  - output sets have monotonic membership functions



# **Defuzzification** (fuzzy to crisp conversion)

- Mamdani Model:




  - ✓ COG (Centre of Gravity) or COA
  - ✓ COS (Centre of Sums)
  - ✓ MOM (Mean of Maximum)

- Sugeno Model:

$$z^* = \frac{w_1 f_1 + w_2 f_2 + \cdots}{w_1 + w_2}$$

- An Application of TS Inference

  - Linearized models for dynamic systems at various operating points

  - A seamless transition from one model to another for control design

- Mamdani Infrence vs. Sugeno Inference

  - Mamdani is completely intuitive

  - Sugeno requires some mathematical formulation or input-output data to aid

- A Numerical Example on Mamdani Inference

- A Numerical Example on TS Inference