Assigment - 6
(Parameter Estimation)

Q1 Let $(x_1, x_2, \ldots)$ be a random sample of size $n$ taken from Normal Population with params mean $= \theta_1$, and Var $= \theta_2$. find Maximum Likelihood Estimates of these 2 parameters.

Sol $\theta_1 = \mu \quad \theta_2 = \sigma^2$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$$

$$f(x, \theta_1, \theta_2) = \frac{1}{\sqrt{2\pi\theta_2}} e^{-\frac{1}{2}\left(\frac{x-\theta_1}{\theta_2}\right)}$$

Now $\theta_1 \in (-\infty, \infty)$

$\theta_2 \in [0, \infty)$

$$L(\theta_1, \theta_2) = \prod_{i=1}^{n} f(x_i, \theta_1, \theta_2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta_2}} e^{-\frac{1}{2}\left(\frac{x-\theta_1}{\theta_2}\right)}$$

$$= \theta_2^{-n/2} \cdot (2\pi)^{-n/2} \cdot e^{-\frac{1}{2\theta_2}\sum_{i=1}^{n}(x_i-\theta_1)^2}$$

$$\log L(\theta_1, \theta_2) = -\frac{n}{2}\log\theta_2 - \frac{n}{2}\log(2\pi) - \frac{\sum(x_i-\theta_i)^2}{2\theta_2}$$

Taking Partial derivative (wrt $\theta_1$)

$$\frac{d \log L(\theta_1, \theta_2)}{d\theta_1} = \frac{-2\sum(x_i-\theta_1)(-1)}{2\theta_2} = 0$$

$$\theta_1 = \mu = \frac{\sum x_i}{n} = \bar{x}$$

w.r.t $\theta_2$)

$$\frac{d\log L(\theta_1,\theta_2)}{d\theta_2} = \frac{-n}{2\theta_2} + \frac{\Sigma(x-\theta_1)^2}{2\theta_2^2} = 0$$

$$-n\theta_2 + \Sigma(x-\theta_1)^2 = 0$$

$$\hat{\theta}_2 = \hat{\sigma}_2 = \frac{\Sigma(x_i - \bar{x})^2}{n}$$

$$\therefore \hat{\mu} = \frac{\Sigma x_i}{n} \quad \& \quad \hat{\sigma}^2 = \frac{\Sigma(x_i - \bar{x})^2}{n}$$

Q2 Let $x_1, x_2, \ldots, x_n$ be random sample from $B(m,\theta)$ distribution where $\theta \in \Theta = (0,1)$ is unknown and 'm' is known +ve integer. Compute value of $\theta$ using MLE

$${}^mC_{x_i}\, \theta^{x_i} (1-\theta)^{m-x_i} = B(m,\theta)$$

$$L(\theta | x_1, x_2 \ldots x_n) = \prod_{i=1}^{n} \binom{m}{x_i} \theta^{x_i} (1-\theta)^{m-x_i}$$

To compute log likelihood

$$\log L(\theta | x_1 \ldots x_n) = \sum_{i=1}^{n} \log\binom{m}{x_i} + x_i \log\theta + (m-x_i)\log(1-\theta)$$

$$\frac{dL}{d\theta} = \Sigma \frac{x_i}{\theta} + \frac{m-x_i}{1-\theta} = 0$$

$$\sum_{i=1}^{n} \frac{x_i - \theta(m)}{\theta(1-\theta)} = 0$$

$$\sum_{i=1}^{n} \frac{x_i}{\theta(1-\theta)} - \frac{\Sigma\theta(m)}{\theta(1-\theta)} = 0$$

$$\sum_{i=1}^{n} \frac{x_i}{\theta(1-\theta)} = \frac{m\cdot n}{\theta(1-\theta)} \Rightarrow \boxed{\theta = \frac{\sum_{i=1}^{n} x_i}{m\cdot n}}$$

$$\boxed{\therefore \text{MLE } B(m,\theta) = \frac{\sum_{i=1}^{n} x_i}{m\cdot n}}$$