FALL 2016

SPLASH! AT CORNELL

MUSICAL GROUPS

Chirag Bharadwaj

"Mathematics is a foreign language."

–Chirag Bharadwaj, *et. al*.

秋2016

コーネル大学でスプラッシュ

おんがく　　りろん

音楽の理論のグループ

坂本ひかる

BRIDGING THE GAP

- How can we overcome an overwhelming sea of symbols and extract meaning from first principles?
- How can we use these abstract notions to model real-life phenomena and make meaningful connections?
- How can we learn to think beyond our boundaries and create original work?

COLOR SCHEME

- In this presentation…
  - *Green text* refers to things I think you should know already given your past experiences
  - *Blue text* refers to things you may know or be able to reason about given enough time
  - *Red text* refers to things that you most likely do not know yet and will hopefully learn sometime soon!
  - *Magenta text* mostly refers to definitions/emphasis

ABOUT ME

- From Flushing, NY
- Went to high school in Princeton, NJ
- Senior at Cornell University
  - B.Sc., *Computer Science*
- Currently applying to graduate schools for CS
- Mathematics and music are my side interests!
- Other than that, just like you (except maybe a little older):
  - 19 years old
  - Interested in self-learning and teaching others

ABOUT ME

- From Flushing, NY
- Went to high school in Princeton, NJ
- Relevant math classes I took in high school:
  - AP Calculus-AB, BC
  - Multivariable calculus, linear algebra
  - Differential equations, complex analysis
  - Real analysis (two semesters)
- I’ve been playing the piano since April 2005 (~11-12 years)
- What we will cover today is related to basic algebra

ABOUT ME

- From Flushing, NY
- Went to high school in Princeton, NJ
- Things I've taught at Splash before:
  - *Spring 2015:* Modern Complex Analysis
  - *Fall 2015:* Introduction to Japanese Linguistics
  - *Spring 2016:* Special Polynomials in Differential Equations
  - *Fall 2016: Mathematical Groups in Music*
- Pattern in my teaching?
  - *Spring* = more technical material; *fall* = more accessible material

OVERVIEW

- **120** minutes to get a quick introduction to some interesting applications of modern mathematics
- Focus: Applications of some intermediate-level math to music and music theory
- Pace: **reasonably fast**
    - Then again, 120 minutes is a *lot* of time…
- Holism vs reductionism: age-old question/answer

BACKGROUND

- I will assume *complete* familiarity with a few things:
  - Algebra at the level of a second high-school class
  - How to read music written on 2 clefs (e.g. for the piano)
  - How to think using your brain (it must be yours!)
- Things I do *not* expect you to have seen before:
  - Group theory and sets
  - Music theory and musical counterpoint
  - Paying attention to one guy for two straight hours

BACKGROUND

- Don't worry if you don't understand *everything*!
  - The idea is to gain exposure to unfamiliar concepts
- Not everything you see here will be immediately useful in your high school mathematics
  - But it will teach you a little bit about how to think for yourself and teach yourself new things from old
- This is a **challenge**—get ready to be splashed!

STRUCTURE

- First third: Mathematical toolbox
  - “The boring stuff”
  - Interesting new things you can do with what you already know from high school!
- Second third: Musical toolbox
  - The other side of boring
  - Some of you probably know this already…
- Last third: Applications of what we just learned
  - Fun ways to apply newfound knowledge

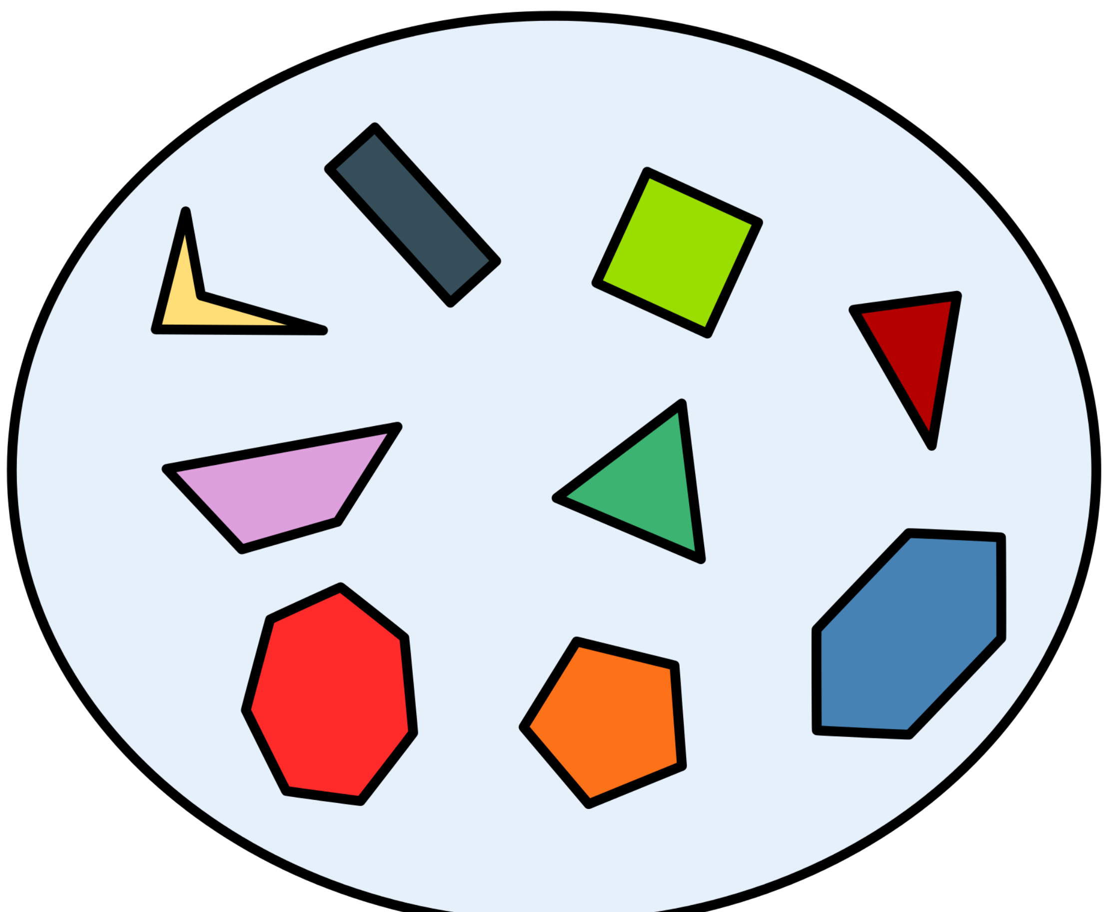
STRUCTURE

- We need to develop the right kind of *framework* to study music in proper theoretical detail
  - *I suspect many of you are not used to going into this much depth with any topic… **that's okay!***
- We will start with basic ideas from math and build up the notion of groups
  - Slowly the connections to other fields will become apparent over time!

PRELUDE

SETS

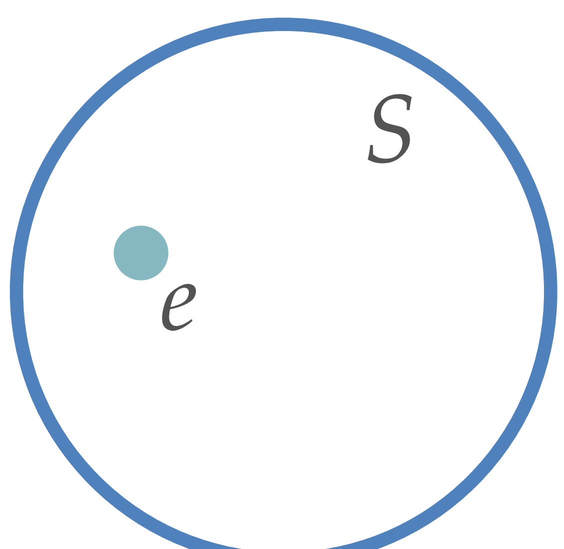
- What is a *set* in mathematics?
  - Colloquial definitions? "Primary school" definition?
  - How I always thought about it:



heterogeneous collection of certain kinds of objects

SETS

- Let us now be a bit more formal…
  - A *set* is an unordered collection of objects
  - These objects are called *members* (or *elements*)
  - What does it mean for it to be *unordered*?



e is a member of S

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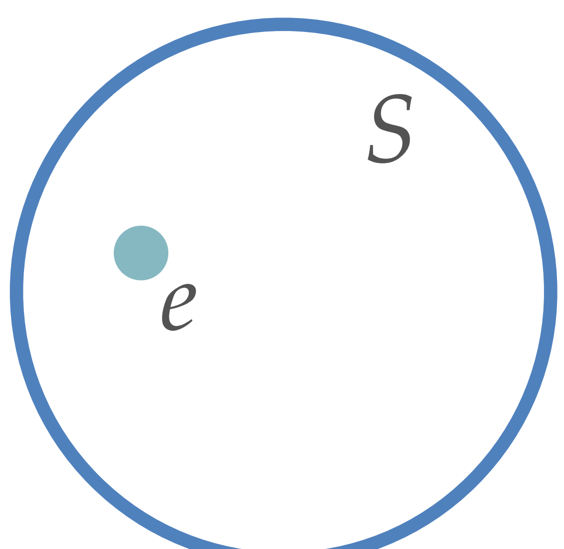

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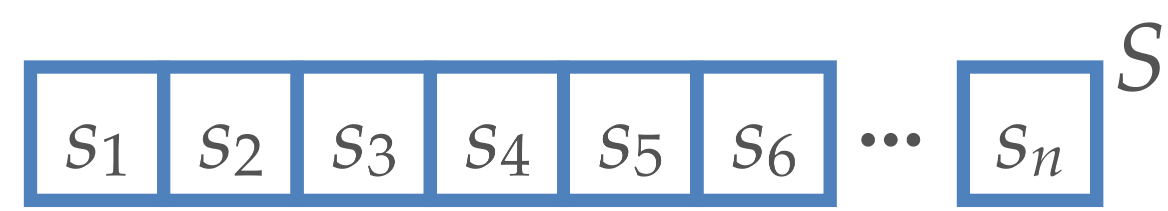


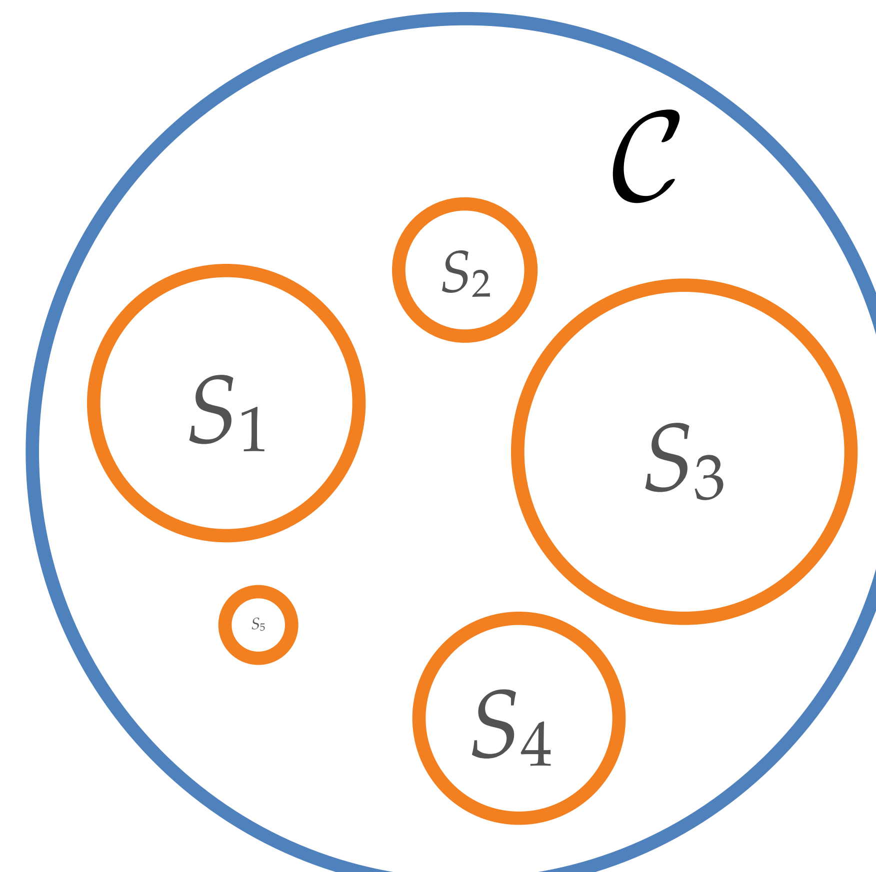
it means that this is not our model of organization!

SET CONSTRAINTS

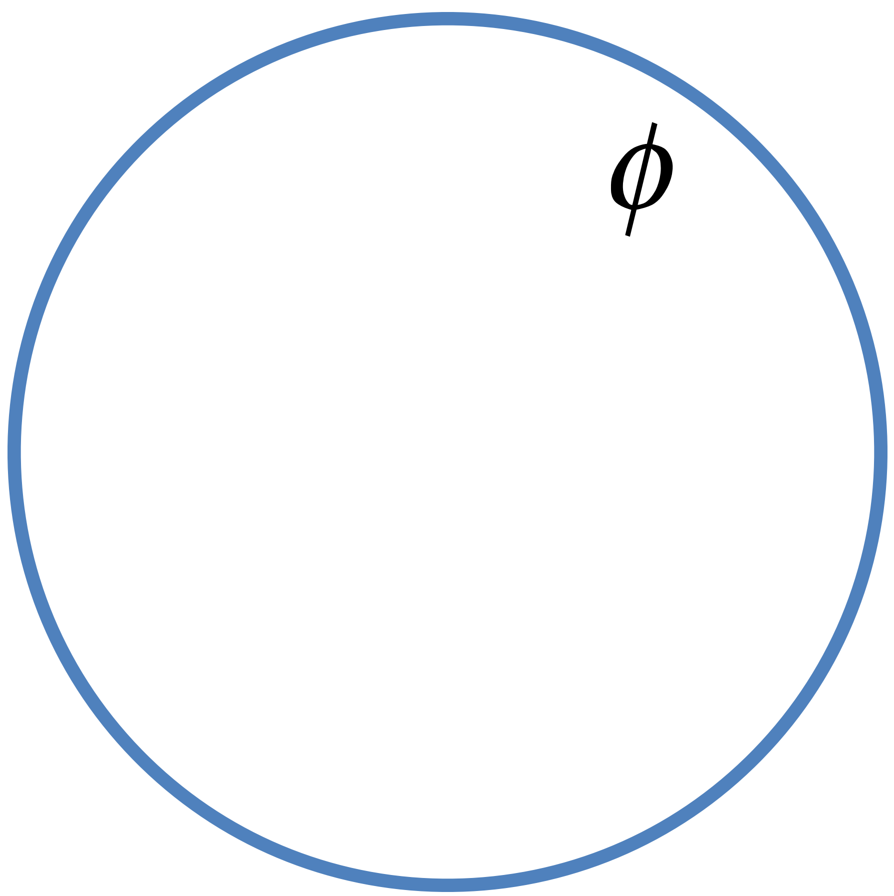
- What kinds of objects are *valid* members of sets?
  - Do they all have to be unique?
  - Do they all have to be the same "kind" of object?
- Let's think about it…
  - Real life: *No two objects are physically equal*
  - Real applications use *homogeneous collections*
- Leads to two important rules (*axioms*):
  - (*Uniqueness.*) **All elements are necessarily unique.**
  - (*Homotypicity.*) **All elements have the same *type*.**

SET CONSTRAINTS

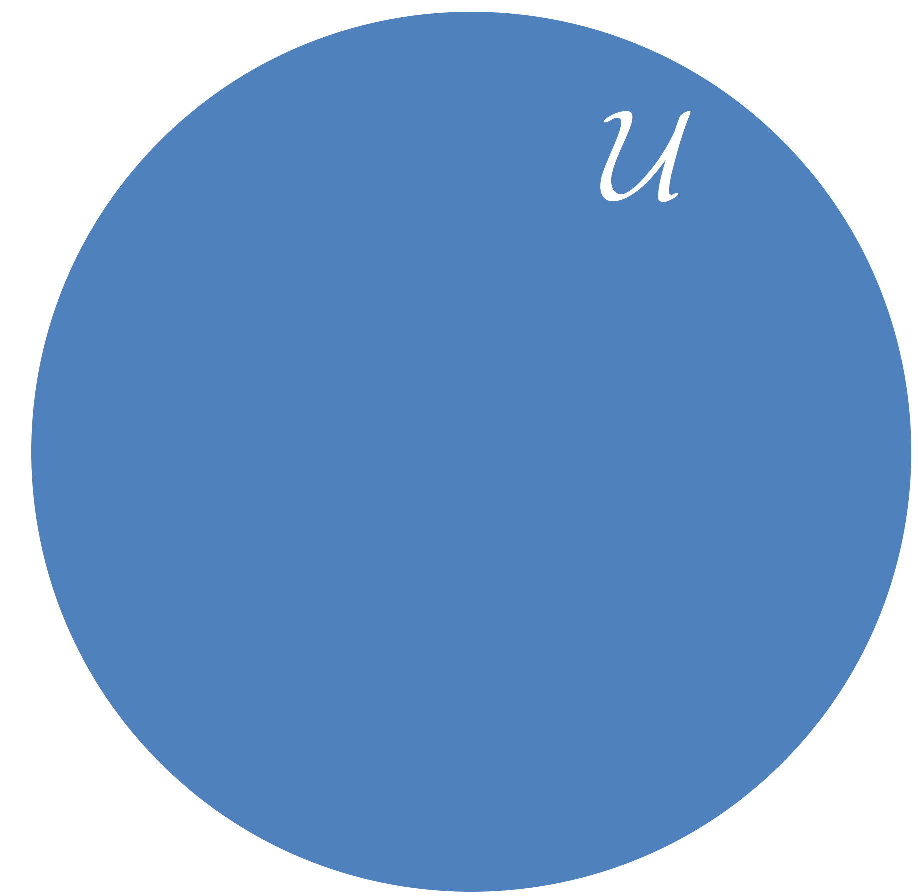
- What kinds of objects are *valid* members of sets?
  - Since sets are themselves objects, we can entertain the idea of a set of sets (i.e. *higher-order sets*)
  - A *class* is a set of sets that all share a property
    - Classes are also *not ordered*!
  - What does this even look like?
    - high school *classes*
    - *set* of *units* of study
    - *unit* of study = *set* of topics



SET CONSTRAINTS



The empty set $\varnothing$



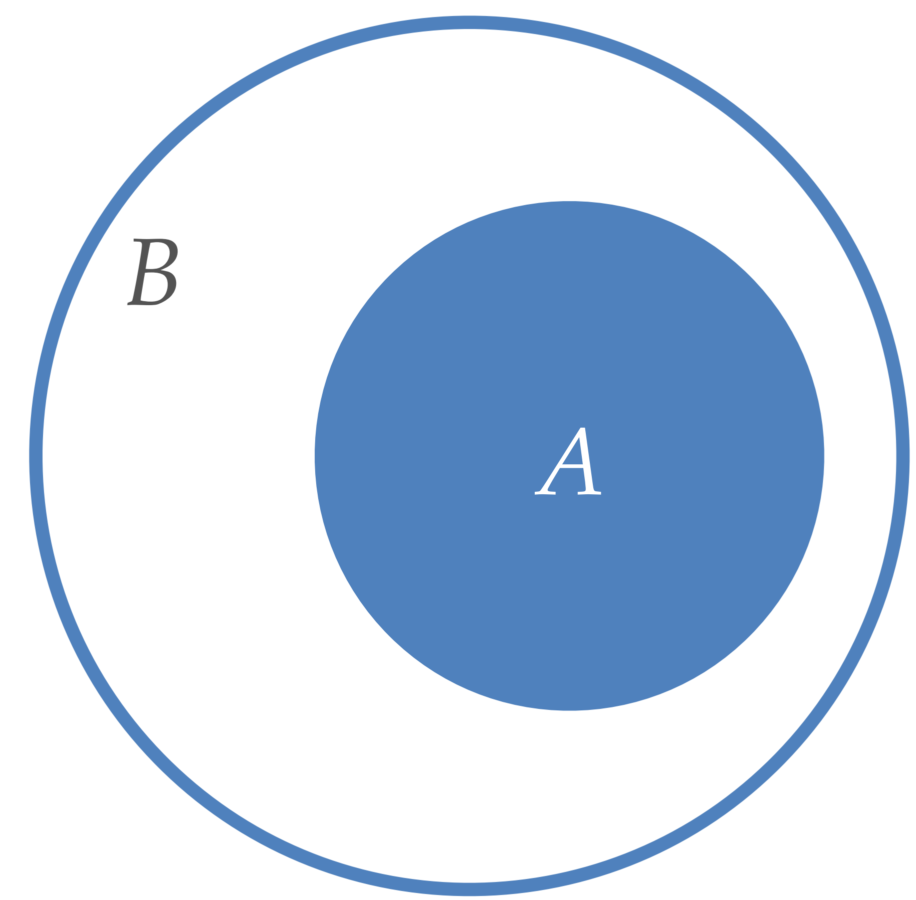
The universal set $\mathfrak{U}$

SET COMPARISONS

- Two sets A and B are *equal* if and only if they contain the exact same elements
  - That is, A contains every element of B **and** B contains every element of A
- This is sometimes known as *structural equality*
  - The two sets are still not physically equal!
  - Recall: No two objects are equal

SET COMPARISONS

- A set A is a *subset* of a set B if B contains all of the elements of A
  - Note that this definition is non-restrictive!
    - In particular, it could be the case that A and B are equal sets
    - If A is a subset of B but A and B are not equal, then we say that A is a *proper subset* of B

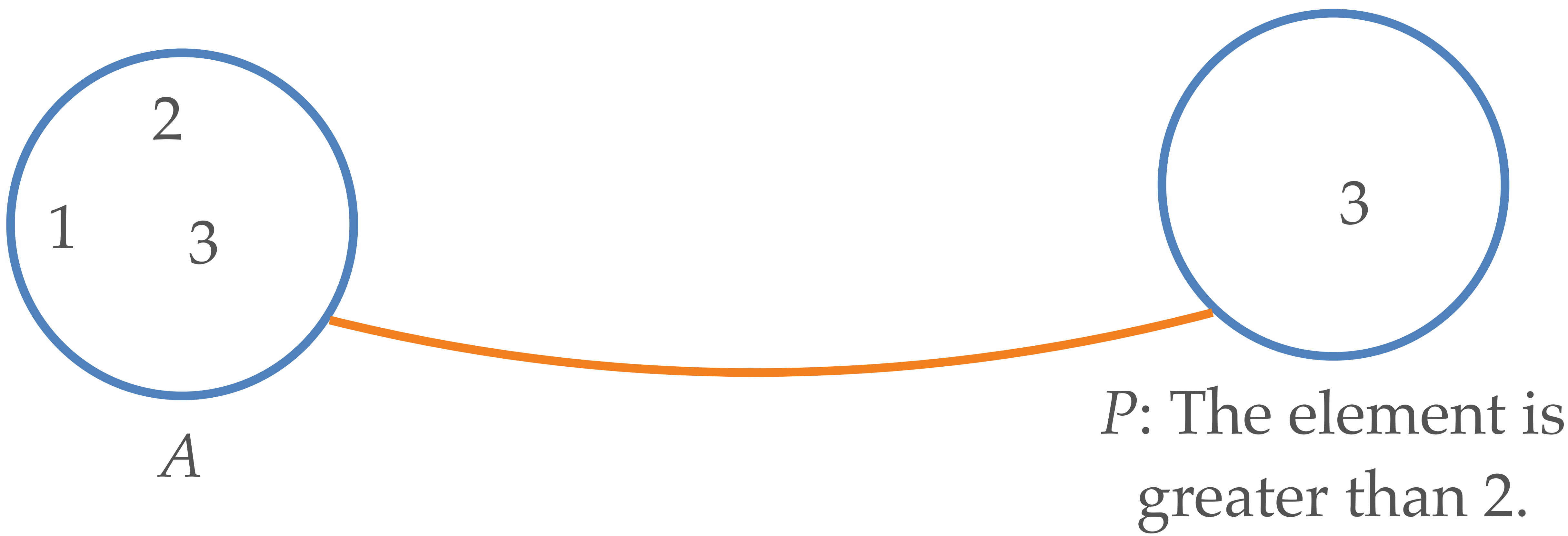


SET COMPARISONS

- A set *A* is a *subset* of a set *B* if *B* contains all of the elements of *A*
  - Our intuition tells us that this is a weaker condition than that of set equality
- Formally, what does it mean for one condition to be *stronger* or *weaker* than another?
  - We need the notion of a *property*
    - This, in turn, helps us to know other things…

PROPERTIES

- A *property* P of a set A is a subset of A wherein all elements of the subset satisfy some condition c



PROPERTY STRENGTH

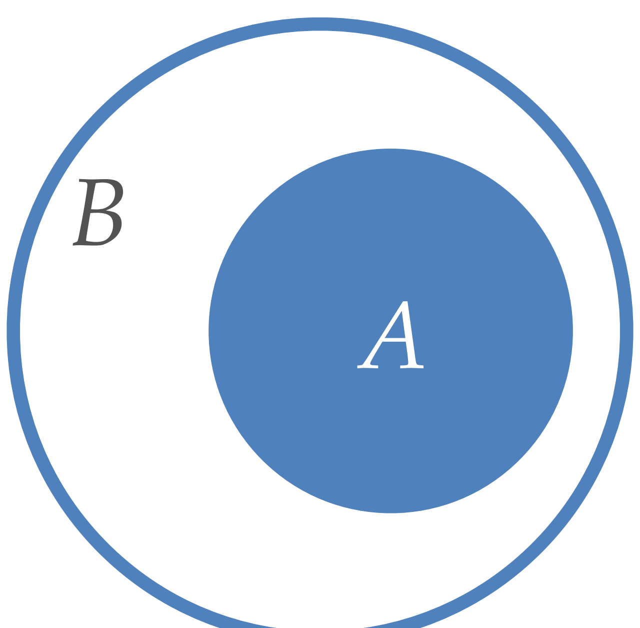
- Consider a set X and two properties of X: P_1 and P_2
- We measure the strength of a property by how tight of a *filter* it is on a set
  - P_1 is *strictly weaker* than P_2 if P_2 is proper subset of P_1
  - P_1 is *no stronger* than P_2 if P_2 is a subset of P_1
  - P_1 and P_2 are *equipotent* if P_1 and P_2 are equal
  - P_1 is *no weaker* than P_2 if P_1 is a subset of P_2
  - P_1 is *strictly stronger* than P_2 if P_1 is proper subset of P_2

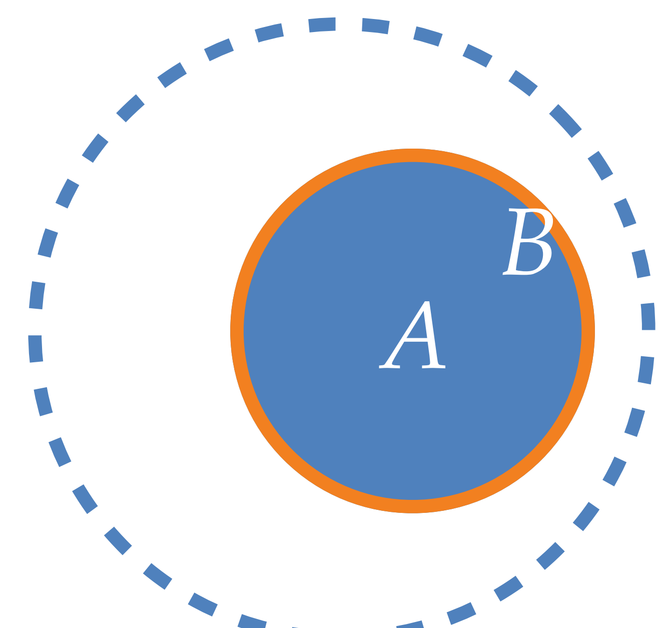
SET COMPARISONS

- Consider a set X and two properties of X: P_1 and P_2
- We measure the strength of a property by how tight of a *filter* it is on a set
  - The *weakest property* of a set X is "the element is in X" — $\bot$ *bottom*
  - The *strongest property* of a set X is "the element is not in X" — $\top$ *top*

SET COMPARISONS

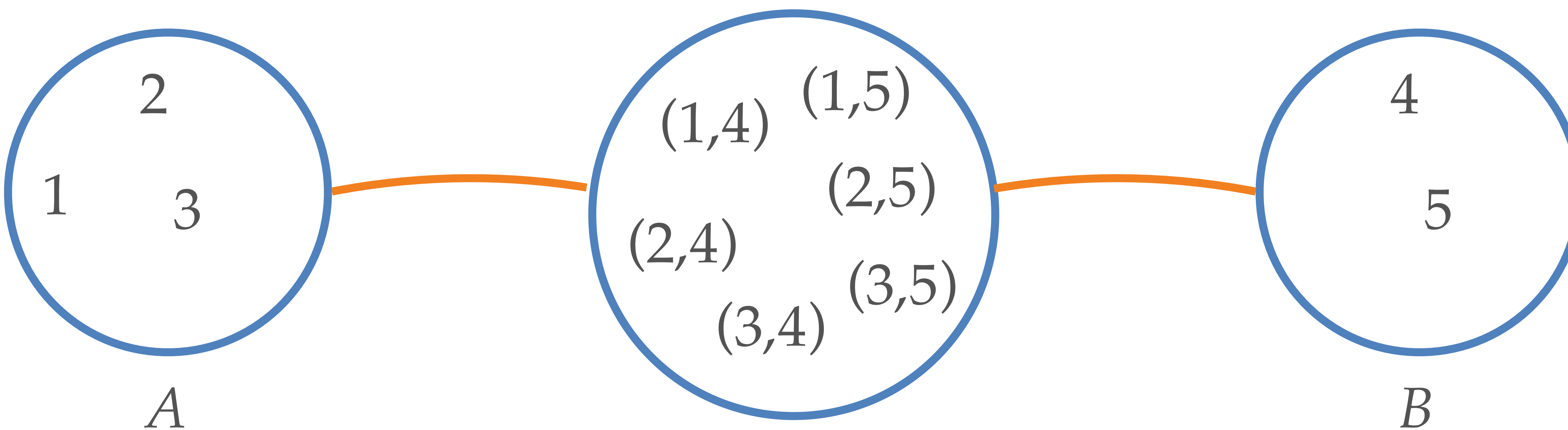
- Back to subset and equality of sets:
  - Recall: Two sets A and B are equal if and only if A is a subset of B and B is a subset of A
  - Thus, equality is at least as tight of a filter as subset
  - This tells us that *subset is no stronger than set equality*





PRODUCTS

- The *Cartesian product* of two sets is a set containing all possible ordered pairs of elements from both sets
  - The first element in the pair is from the first set
  - The second one in the pair is from the second set



Cartesian Product of A and B

PRODUCTS

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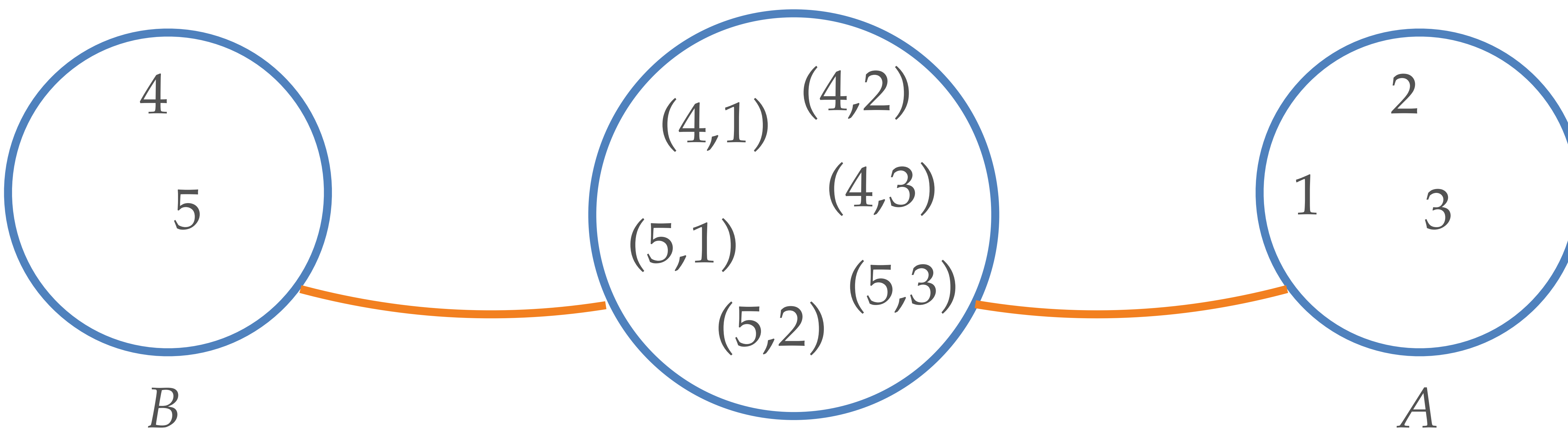


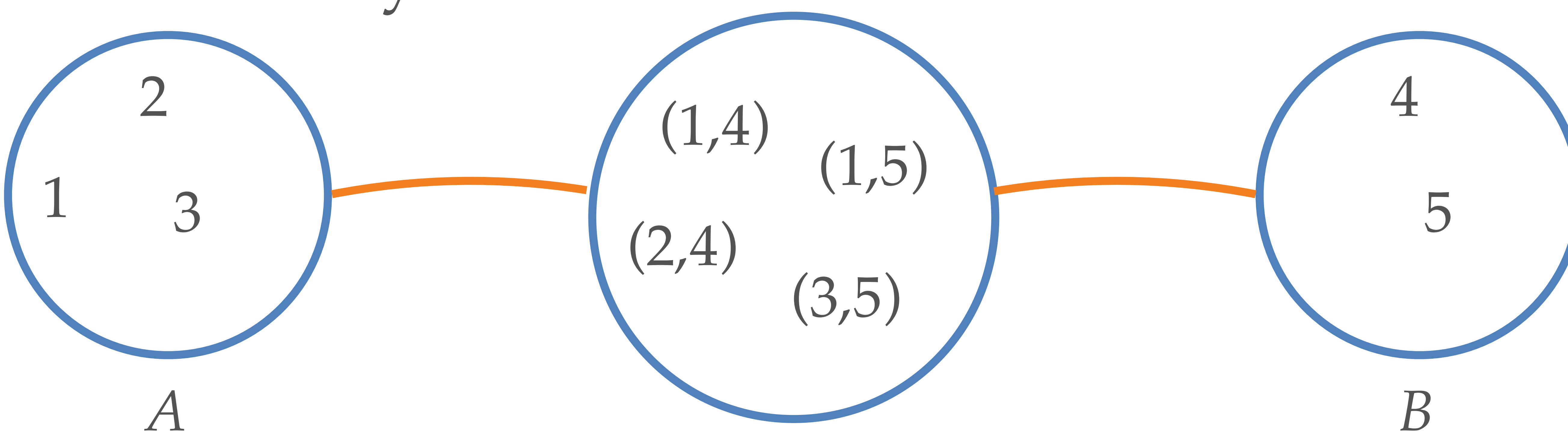
Cartesian Product of B and A

PRODUCTS

- The *Cartesian product* of two sets is a set containing all possible ordered pairs of elements from both sets
  - The first element in the pair is from the first set
  - The second one in the pair is from the second set
- These pairs *are* ordered:
  - The Cartesian product is not commutative
  - If we swap the order of the first and second set, then the ordered pairs change (as we saw!)
    - But both Cartesian products have the same *size*

RELATIONS

- A (binary) *relation* R from a set A to a set B is a subset of the Cartesian product of A and B
  - Elements are either in the relation or not
- If a pair (a, b) is in the relation R, then we say that a is related to b by R



Relation between A and B

RELATIONS

- A (binary) *relation* R from a set A to a set B is a subset of the Cartesian product of A and B
- Consider a set X and a relation R from X to X
  - If for all elements a in X, it holds that (a, a) is in R, then we say that R is *reflexive*
  - If for all a and b in X, it holds that (a, b) is in R and (b, a) is in R, then we say that R is *symmetric*
  - If for all a and b in X, it holds that the joint membership of (a, b) and (b, a) in R implies that $a = b$, then we say that R is *antisymmetric*
  - If for all elements a, b, and c in X, it holds that the joint membership of (a, b) and (b, c) in R implies the membership of (a, c) in R, then we say that R is *transitive*

NOTATION

- Time for shorthand!
- Too cumbersome to write everything out in English… we need a standard way of denoting these concepts
- We introduce the *canonical set notation*
- This notation is standard in the literature

NOTATION

$$S$$

S is a *set*

NOTATION

$$S = \{1, 2, 3\}$$

S *contains* 1, 2, and 3

NOTATION

$$s \in S$$

s is a *member* of S

NOTATION

$$A \subseteq B$$

A is a *subset* of *B*

NOTATION

$$A \subset B$$

A is a *proper subset* of B

NOTATION

$$A = B$$

A and B are *equal*

NOTATION

$$P = c(x)$$

Property P: element x satisfies condition c

NOTATION

$$P(x)$$

Property P: shorthand notation

NOTATION

$$S = \left\{ x \,\middle|\, P(x) \right\}$$

Set builder: S is the set of all x that satisfy property P

NOTATION

$$A \times B$$

The *Cartesian product* of A and B

NOTATION

$$\left\{(a,b) \middle| a \in A, b \in B\right\}$$

The *Cartesian product* of A and B

NOTATION

$$R$$

R is a relation

NOTATION

$$a \, R \, b$$

a is *related* to *b* *by* *R*

NOTATION

$$(a, b) \in R$$

(a,b) is *in the relation* R

NOTATION

$$\forall x; P(x)$$

for all x, x satisfies property P

NOTATION

$$\exists x; P(x)$$

there exists an x that satisfies property P

NOTATION

- P_1 is *strictly weaker* than P_2: $P_1 \prec P_2$
- P_1 is *no stronger* than P_2: $P_1 \preceq P_2$
- P_1 and P_2 are *equipotent*: $P_1 = P_2$
- P_1 is *no weaker* than P_2: $P_1 \succeq P_2$
- P_1 is *strictly stronger* than P_2: $P_1 \succ P_2$

$\forall P; \bot \preceq P$ $\quad \forall P; P \preceq \top$ $\quad \nexists P; \top \prec P$ $\quad \nexists P; P \prec \bot$

Boundedness of top/bottom — *Another way to say the same thing*

SUPPLEMENTS

- Some additional definitions:
  - A *partial order* on a set X is a relation R from X to X that is *reflexive*, *antisymmetric*, and *transitive*
  - A *total order* is a partial order R that is also *total*, i.e. for all $x \in X$ and $y \in X$, either $x\ R\ y$ or $y\ R\ x$
  - An *equivalence relation* on a set X is a relation R from X to X that is *reflexive*, *symmetric*, and *transitive*
  - Typical symbols:
    - partial order: $\leq$, total order: $\leq_+$, eq. relation: $=$

CANTUS FIRMUS

Introduction to the Melody

META-STRUCTURES

- A (finitary) *operation* is an operation that takes in some number of inputs and produces an output
  - A binary operation takes in two inputs and produces some output
  - You have certainly seen binary operations before…

META-STRUCTURES

- A (finitary) *operation* is an operation that takes in some number of inputs and produces an output
  - For example: $+$ is an operation and so is $\cdot$
  - What are the *types* of those operations above?
    - $(+)$ takes in two reals and returns a real
    - $(\cdot)$ takes in two reals and returns a real

META-STRUCTURES

- A (finitary) *operation* is an operation that takes in some number of inputs and produces an output
  - For example: $+$ is an operation and so is $\cdot$
  - What are the *types* of those operations above?
    - $(+)$ takes in *a pair of* reals and returns a real
    - $(\cdot)$ takes in *a pair of* reals and returns a real
    - Mathematically: $op : t$ means "op has type t"
      - Types: $A \to B$ means "takes in something of type A and returns something of type B"

$$(+) : \mathfrak{R} \times \mathfrak{R} \to \mathfrak{R} \qquad (\cdot) : \mathfrak{R} \times \mathfrak{R} \to \mathfrak{R}$$

META-STRUCTURES

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    - $(+)$ takes in *a pair of* reals and returns a real
    - $(\cdot)$ takes in *a pair of* reals and returns a real
    - Mathematically: $op : t$ means "*op* has type *t*"
      - Types: $A \to B$ means "takes in an element of set A and returns an element of set B"

$$(+) : \mathfrak{R} \times \mathfrak{R} \to \mathfrak{R} \qquad\qquad (\cdot) : \mathfrak{R} \times \mathfrak{R} \to \mathfrak{R}$$

META-STRUCTURES

- A (finitary) *operation* is an operation that takes in some number of inputs and produces an output
  - For example: the negation operator – (e.g. *unary op.*)
  - What is the *type* of this operator?
    - – takes in a real number and returns its opposite
      - *Note: 0 is the fixpoint of the negation operator*
      - *In other words, –0 is still 0*
    - The negation of a real number is also real

$$(-) : \mathfrak{R} \to \mathfrak{R}$$

META-STRUCTURES

- A (finitary) *operation* is an operation that takes in some number of inputs and produces an output
  - A binary operation takes in two inputs and produces some output
  - A binary operation op is *closed* on a set X if it has the type $op : X \times X \to X$
  - That is, it takes in two elements of X and the result it returns is also an element of X (so the operation is closed… it never produces anything outside of X)

META-STRUCTURES

- An *algebraic structure* is a set with one or more operations defined over it
  - Each operation must satisfy a list of *axioms*
- There are many, many algebraic structures
  - We will limit our study to *group-like structures*
    - *Groupoids*
    - *Semigroups*
    - *Monoids*
    - *Groups*

META-STRUCTURES

- An *algebraic structure* is a set with one or more operations defined over it
  - Each operation must satisfy some properties
- There are many, many algebraic structures
  - We will limit our study to *group-like structures*
    - i.e. fancy terms for structural building blocks

EARLY GROUPS

- A *groupoid* $M = (S, \bullet)$ is an algebraic structure
  - *Note:* $\bullet$ is a binary operator that is closed over set S
  - Mathematically speaking, the definition implies that

$$\forall a,b \in S;\ a \bullet b \in S$$

- In some sense, this is a very *weak* definition:
  - No properties are asserted about the nature of $\bullet$
    - Formally: $\mathbf{st}(\bullet) = \perp$, *where* **st** *is the relative strength*
  - In general: $a \bullet b \neq b \bullet a$ and $(a \bullet b) \bullet c \neq a \bullet (b \bullet c)$
- Sometimes a groupoid is also called a *magma*

SEMIGROUPS

- A *semigroup* SG is a groupoid $M = (S, \bullet)$ whose binary operator $\bullet$ is also associative
  - In this case, it *is* true that $(a \bullet b) \bullet c = a \bullet (b \bullet c)$
  - Note that we retain all the properties of a groupoid (all zero of them) and then add associativity also
- Unlike with groupoids, there are actually some interesting examples of semigroups that we can take a look at to get a better understanding of the structure

SEMIGROUPS

- Some examples of semigroups:
  - The empty semigroup
  - Semigroups with one element
  - Semigroups with two elements
  - $(N,+)$, where N is the set of natural numbers $\{1,2,\ldots\}$
    - I won’t show *why*, but in a few minutes you will already understand how to prove this one, too
- Let us now look at each of these in more detail!

SEMIGROUPS

- The *empty semigroup* $SG_\varnothing = (\varnothing, \textbf{empty})$
  - The *empty function* has type
  $$\textbf{empty} : \varnothing \to \varnothing \text{ or } \textbf{empty} : \varnothing \times \varnothing \to \varnothing$$
  and takes no inputs and produces no outputs.
  - Some theorists claim that this is an invalid semigroup
  - …but does **empty** satisfy the properties required of $\bullet$?
    - *Closed:* **empty** produces no output, so all outputs are in $\varnothing$
    - *Associative:* **empty** takes no input, so it behaves the same on all inputs: it does nothing (so it is associative, as any operation gives the same answer: nothing)
  - This semigroup is not very interesting…

SEMIGROUPS

- The *trivial semigroup* $SG_{id} = (\{e\}, \textbf{id})$
  - All other one-element semigroups are equivalent (*isomorphic*) to this one since e is just some element
  - The *identity function* has type

$$\textbf{id} : \{e\} \times \{e\} \to \{e\}$$

    and is defined as $\textbf{id}(e, e) = e$
  - Does **id** satisfy the properties required of $\bullet$?
  - *Closed*: Always takes in e (a pair of e's) and returns e
  - *Associative*: $\textbf{id}(\textbf{id}(e, e), e) = \textbf{id}(e, e) = \textbf{id}(e, \textbf{id}(e, e))$

SEMIGROUPS

- The *trivial semigroup* $SG_{id} = (\{e\}, \textbf{id})$
  - All other one-element semigroups are equivalent (*isomorphic*) to this one since e is just some element
  - The *identity function* has type
$$\textbf{id} : \{e\} \times \{e\} \to \{e\}$$
    and is defined as $\textbf{id}(e, e) = e$
  - Does **id** satisfy the properties required of $\bullet$?
  - This one is also not particularly interesting, since there is only one possible trivial (order-1) semigroup

SEMIGROUPS

- Semigroups of *two* elements:
  - An interesting case to consider, as there is a lot more variety here that we must account for!
  - It turns out that there are **5** distinct such semigroups:
    - The null semigroup
    - The left-zero and the right-zero semigroups
    - The classical two-element boolean algebra
    - A fancy special-case semigroup (more later)
  - We won't prove it, but these are the ONLY five
    - Other *order-two semigroups* are isomorphic to these

SEMIGROUPS

- Absorbing elements of an operation:
  - Consider some *absorber* **abs** and a binary operation •
  - Then, for all s it holds that **abs** • $s = s$ • **abs** = **abs**
  - Nomenclature: Absorbs the other value into itself
- A *left-absorber* has a weaker property:

  for all s, it holds that **abs** • s = **abs**
- A *right-absorber* has a similar weaker property:

  for all s, it holds that s • **abs** = **abs**
- Clearly, an element is an absorber if and only if it is both a left-absorber and a right-absorber

SEMIGROUPS

- *Null semigroup*
  - SG whose set contains an absorber
- *Left-zero semigroup*
  - SG whose set contains a left-absorber
- *Right-zero semigroup*
  - SG whose set contains a right-absorber
- Absorbers are generalizations of the notion of 0 for sets
- The absorbing element of a semigroup is unique! (*Why?*)
- We will now take a look at null, left-zero, and right-zero semigroups of order-2

SEMIGROUPS

- The *null semigroup* of order-2: $O_2 = (\{0,1\}, \textsf{zero})$
  - All other null semigroups of order-2 are isomorphic
  - Here, 0 is the absorber and **zero** has the type

$$\textsf{zero} : \{0,1\} \times \{0,1\} \to \{0,1\}$$

  and is defined as follows:

$$\textsf{zero}(\_, \_) = 0$$

  - Here, the underscore (_) notation means "any input"
  - So the full definition is technically the following:

$$\textsf{zero}(0,0) = \textsf{zero}(0,1) = \textsf{zero}(1,0) = \textsf{zero}(1,1) = 0$$

SEMIGROUPS

- The *null semigroup* of order-2: $O_2 = (\{0,1\}, \textbf{zero})$
  - Does **zero** satisfy the properties required of •?
    - *Closed*: **zero** always outputs 0, which is in $\{0,1\}$
    - *Associative*:

$$
\begin{aligned}
&\textbf{zero}(\textbf{zero}(a,b), c) \\
&= \textbf{zero}(0,c) \\
&= 0 \\
&= \textbf{zero}(a,0) \\
&= \textbf{zero}(a, \textbf{zero}(b,c))
\end{aligned}
$$

SEMIGROUPS

- The *left-zero semigroup* of order-2: $LO_2 = (\{0,1\}, \textbf{left})$
  - All other such semigroups of order-2 are isomorphic
  - Here, 0 is the left-absorber and **left** has the type

    $$\textbf{left} : \{0,1\} \times \{0,1\} \to \{0,1\}$$

    and is defined as follows:

    $$\textbf{left}(0, \_) = 0 \text{ and } \textbf{left}(1, \_) = 1$$
  - Recall the use of the underscore as a *wildcard*
  - **left** is quite the eponymous function:
    - Output is the same as the left argument

SEMIGROUPS

- The *left-zero semigroup* of order-2: $LO_2 = (\{0,1\}, \textbf{left})$
  - Does **left** satisfy the properties required of •?
    - *Closed*: **left** always outputs 0 or 1, which is in {0,1}
    - *Associative*:

$$\begin{aligned} &\textbf{left}(\textbf{left}(a,b), c) \\ &= \textbf{left}(0,c) \\ &= 0 \\ &= \textbf{left}(0, \textbf{left}(b,c)) \\ &= \textbf{left}(a, \textbf{left}(b,c)) \end{aligned}$$

Case 1: $a = 0$

$$\begin{aligned} &\textbf{left}(\textbf{left}(a,b), c) \\ &= \textbf{left}(1,c) \\ &= 1 \\ &= \textbf{left}(1, \textbf{left}(b,c)) \\ &= \textbf{left}(a, \textbf{left}(b,c)) \end{aligned}$$

Case 2: $a = 1$

SEMIGROUPS

- The *right-zero semigroup* of order-2: $RO_2 = (\{0,1\}, \textbf{right})$
  - All other such semigroups of order-2 are isomorphic
  - Here, 0 is the right-absorber and **right** has the type

$$\textbf{right} : \{0,1\} \times \{0,1\} \rightarrow \{0,1\}$$

  and is defined as follows:

$$\textbf{right}(\_, 0) = 0 \text{ and } \textbf{right}(\_, 1) = 1$$

  - Recall the use of the underscore as a *wildcard*
  - **right** is quite the eponymous function:
    - Output is the same as the right argument

SEMIGROUPS

- The *right-zero semigroup* of order-2: $RO_2 = (\{0,1\}, \textbf{right})$
  - Does **right** satisfy the properties required of •?
    - *Closed*: **right** always outputs 0 or 1, which is in {0,1}
    - *Associative*:

$$\begin{aligned} &\textbf{right}(\textbf{right}(a,b), c) \\ &= \textbf{right}(\textbf{right}(a,b),0) \\ &= 0 \\ &= \textbf{right}(a, 0) \\ &= \textbf{right}(a, \textbf{right}(b,c)) \end{aligned}$$

Case 1: $c = 0$

$$\begin{aligned} &\textbf{right}(\textbf{right}(a,b), c) \\ &= \textbf{right}(\textbf{right}(a,b),1) \\ &= 1 \\ &= \textbf{right}(a, 1) \\ &= \textbf{right}(a, \textbf{right}(b,c)) \end{aligned}$$

Case 2: $c = 1$

SEMIGROUPS

- The *boolean semigroup*: $B_2 = (\{\bot, \top\}, \wedge)$
  - All other such semigroups of order-2 are isomorphic
  - Here, $\wedge$ has the type
    $$\wedge : \{\bot, \top\} \times \{\bot, \top\} \to \{\bot, \top\}$$
    and is defined as follows:
    $$\bot \wedge \_ = \_ \wedge \bot = \bot \text{ and } \top \wedge \top = \top$$
  - Recall the use of the underscore as a *wildcard*
  - $\wedge$ is sometimes called **and**:
    - Output is $\top$ only if *both* inputs are $\top$

SEMIGROUPS

- The *boolean semigroup*: $B_2 = (\{\bot, \top\}, \wedge)$
  - Does $\wedge$ satisfy the properties required of $\bullet$?
    - *Closed*: $\wedge$ always outputs $\bot$ or $\top$, which is in $\{\bot, \top\}$
    - *Associative*: A more interesting proof…

$$
\begin{aligned}
&(a \wedge b) \wedge c \\
&= (\bot \wedge b) \wedge c \\
&= \bot \wedge c = \bot \\
&= \bot \wedge (b \wedge c) \\
&= a \wedge (b \wedge c)
\end{aligned}
$$

Case 1: $a = \bot$

$$
\begin{aligned}
&(a \wedge b) \wedge c \\
&= (\top \wedge b) \wedge c \\
&= b \wedge c \\
&= \top \wedge (b \wedge c) \\
&= a \wedge (b \wedge c)
\end{aligned}
$$

Case 2: $a = \top$

SEMIGROUPS

- The *unit semigroup*: $U_2 = (\{-1,1\}, \cdot)$
  - All other such semigroups of order-2 are isomorphic
  - Here, $\cdot$ has the type (subset of $\times$ over $\mathfrak{R}$)

    $$\cdot : \{-1,1\} \times \{-1,1\} \rightarrow \{-1,1\}$$

    and behaves the same as the standard integer multiplication operator (over real numbers)

SEMIGROUPS

- The *unit semigroup*: $B_2 = (\{-1,1\}, \cdot)$
  - Does $\cdot$ satisfy the properties required of $\bullet$?
    - *Closed*: $\cdot$ always outputs -1 or 1, which is in $\{-1,1\}$
    - *Associative*: Proof by *perfect induction*:

$$\boldsymbol{a = -1, b = -1, c = -1}\text{:}\ (a \cdot b) \cdot c = 1 \cdot -1 = -1 = -1 \cdot 1 = a \cdot (b \cdot c)$$

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MONOIDS

- A *monoid* N is a semigroup $SG = (S, \bullet)$ whose set S also contains an identity element **id**:

  $$\exists\ \mathbf{id} \in S \text{ such that } \forall s \in S,\ \mathbf{id} \bullet s = s \bullet \mathbf{id} = s$$

  - Note that we retain all the properties of a semigroup (i.e. associativity) and then add the identity element
- The identity element of a monoid MUST be unique
- There are literally TONS of uses for (free) monoids:
  - Finite-state machines
  - Process calculus/concurrent computing
  - Transition table for a linear system

MONOIDS

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  - Note that we retain all the properties of a semigroup (i.e. associativity) and then add the identity element
- Going through examples of monoids would take us WAY too long…
  - Exponential blow-up in properties and interesting applications when compared to just semigroups

MONOIDS

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  - Note that we retain all the properties of a semigroup (i.e. associativity) and then add the identity element
- Were some of the semigroups we saw *also* monoids?
  - Trivial semigroup (**id** = the only element e)
  - Boolean semigroup (**id** = $\top$)
  - Unit semigroup (**id** = 1)

GROUPS

- A *group* G is a monoid $N = (S, \bullet, \mathbf{id})$ whose set S is also complete under invertibility
  - That is, for all elements s in S, there exists an *inverse element* s^{-1} in S such that $s \bullet s^{-1} = s^{-1} \bullet s = \mathbf{id}$
  - Note that we retain all the properties of a monoid (i.e. identity and associativity) and then add invertibility on top
- Each element has a unique inverse in S
- If $\bullet$ is also commutative, then G is called an *abelian group*
- There are even more uses for groups than for monoids:
  - Music theory and musical counterpoint
  - Geometries in crystallization of solids in chemistry
  - Quantum mechanics and wave physics

GROUPS

- A *group* G is a monoid $N = (S, \bullet, \mathbf{id})$ whose set S is also complete under invertibility
  - That is, for all elements s in S, there exists an *inverse element* s^{-1} in S such that $s \bullet s^{-1} = s^{-1} \bullet s = \mathbf{id}$
  - Note that we retain all the properties of a monoid (i.e. identity and associativity) and then add invertibility on top
- Each element has a unique inverse in S
- If $\bullet$ is also commutative, then G is called an *abelian group*
- Were some of the semigroups we saw *also* groups?
  - Trivial semigroup ($e = e^{-1}$)
  - Unit semigroup (1′s inverse is 1 and –1′s inverse is –1)

GROUPS

- A *group* G is a monoid $N = (S, \bullet, \textbf{id})$ whose set S is also complete under invertibility
  - That is, for all elements s in S, there exists an *inverse element* s^{-1} in S such that $s \bullet s^{-1} = s^{-1} \bullet s = \textbf{id}$
  - Note that we retain all the properties of a monoid (i.e. identity and associativity) and then add invertibility on top
- Each element has a unique inverse in S
- If $\bullet$ is also commutative, then G is called an *abelian group*
- Were some of the semigroups we saw also *abelian* groups?
  - Trivial semigroup ($e \bullet e = e \bullet e = e$)
  - Unit semigroup ($\cdot$ is commutative by default)

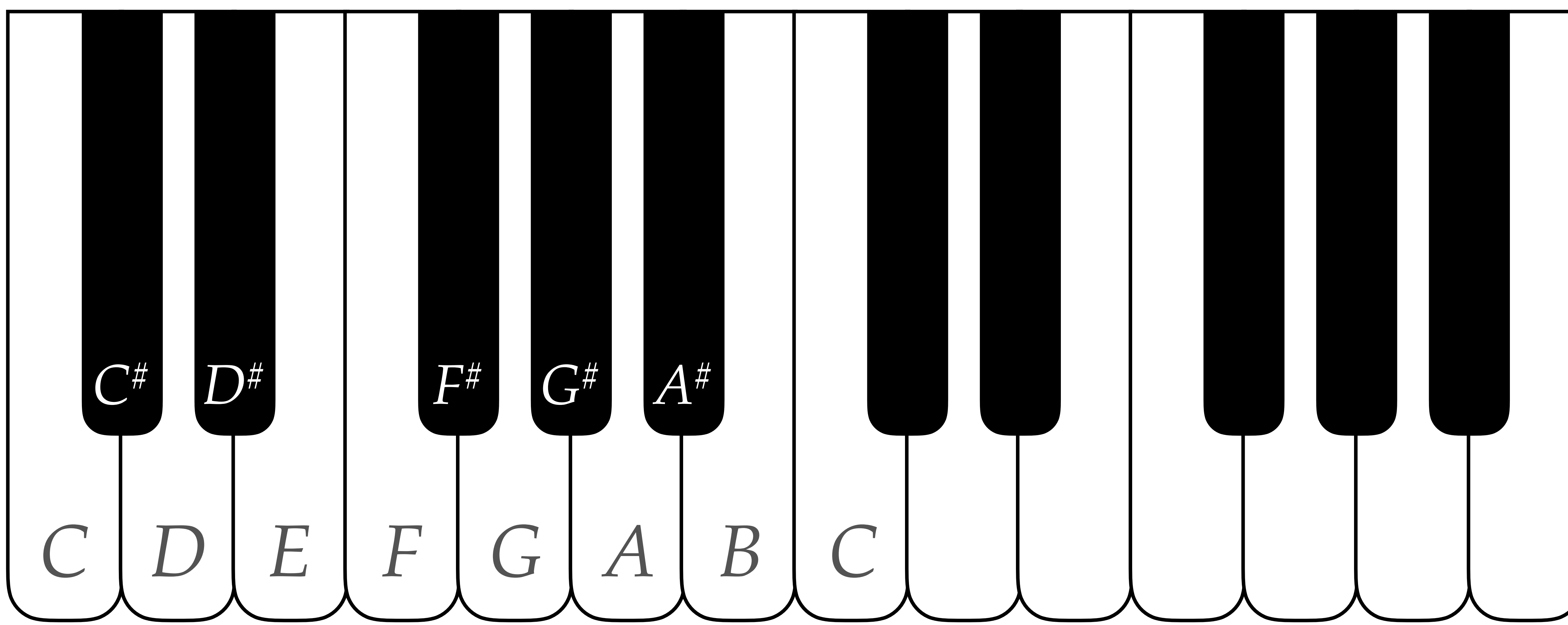
COUNTERPOINT

Presentation of the Main Melody

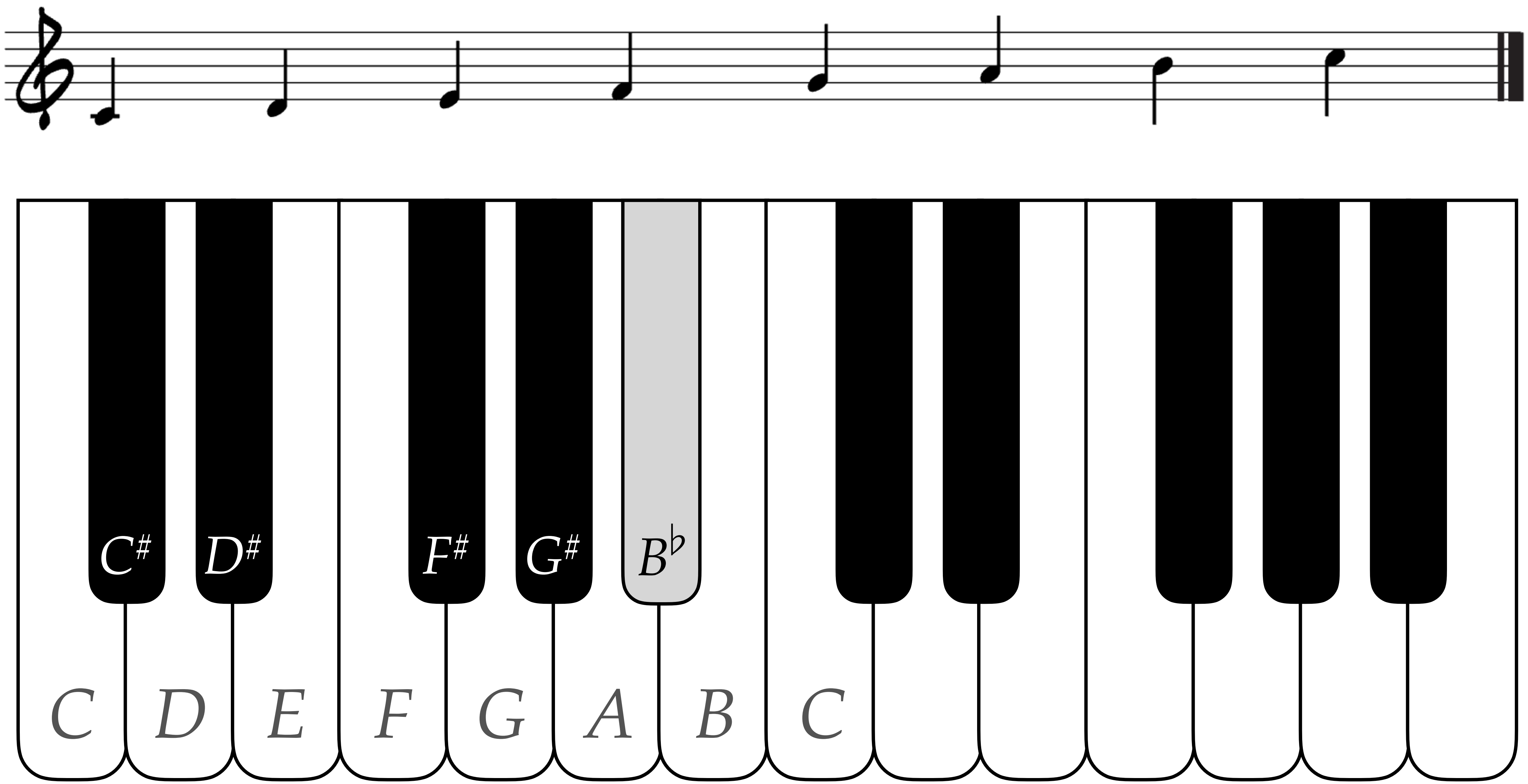
THEORY

- To begin understanding the applications, we need to prepare ourselves musically
  - Just as we prepared ourselves mathematically
- We will begin with a general discussion of the canonical music theory:
  - Interval spelling
  - Chord spelling
- Then we can delve into the details of cyclical properties and other aspects of musical literature

INTERVALS: STEPS

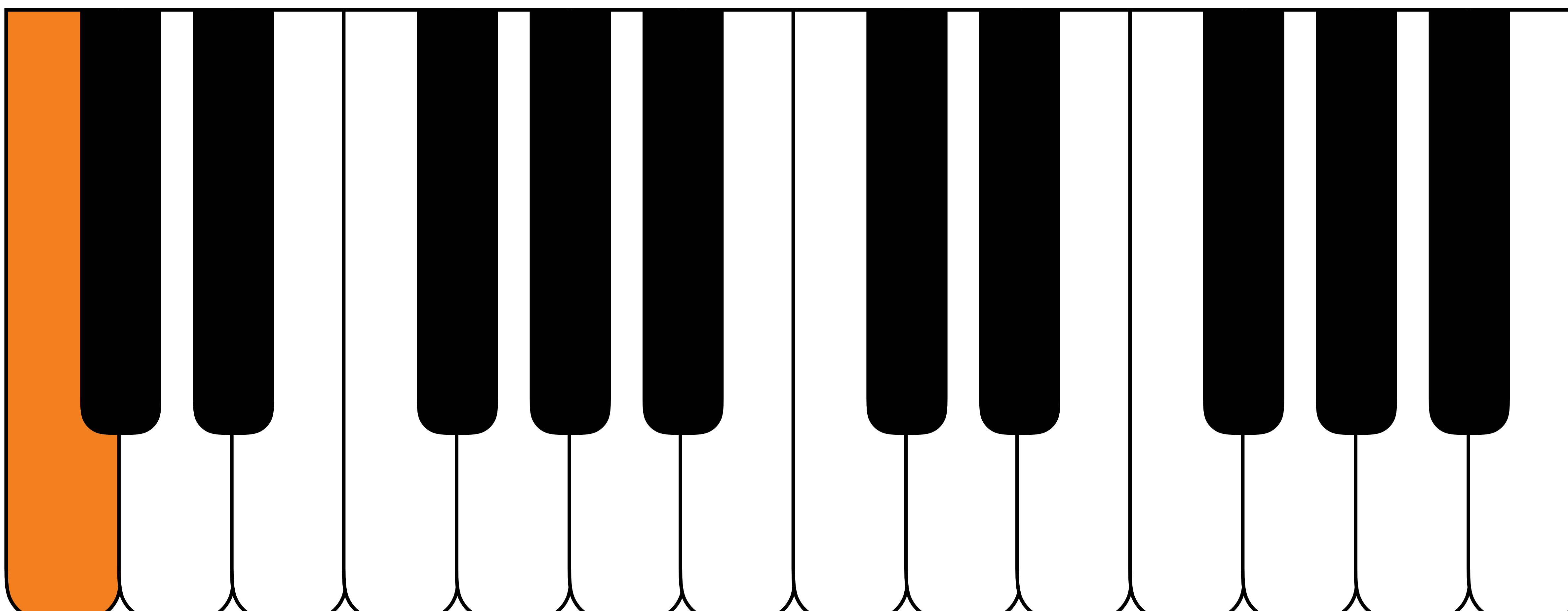


INTERVALS: STEPS



INTERVALS: STEPS

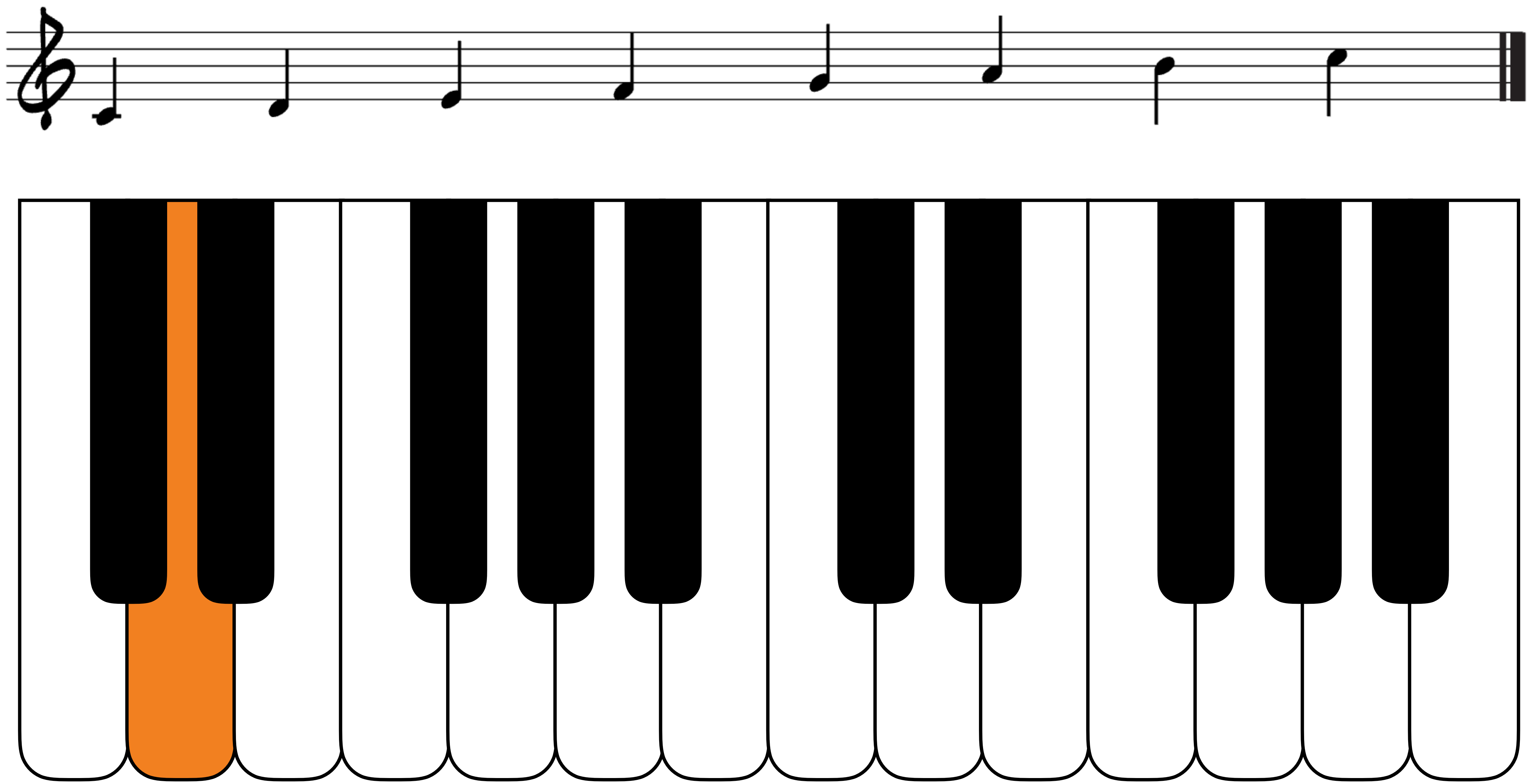
C



Basis note of scale

INTERVALS: STEPS

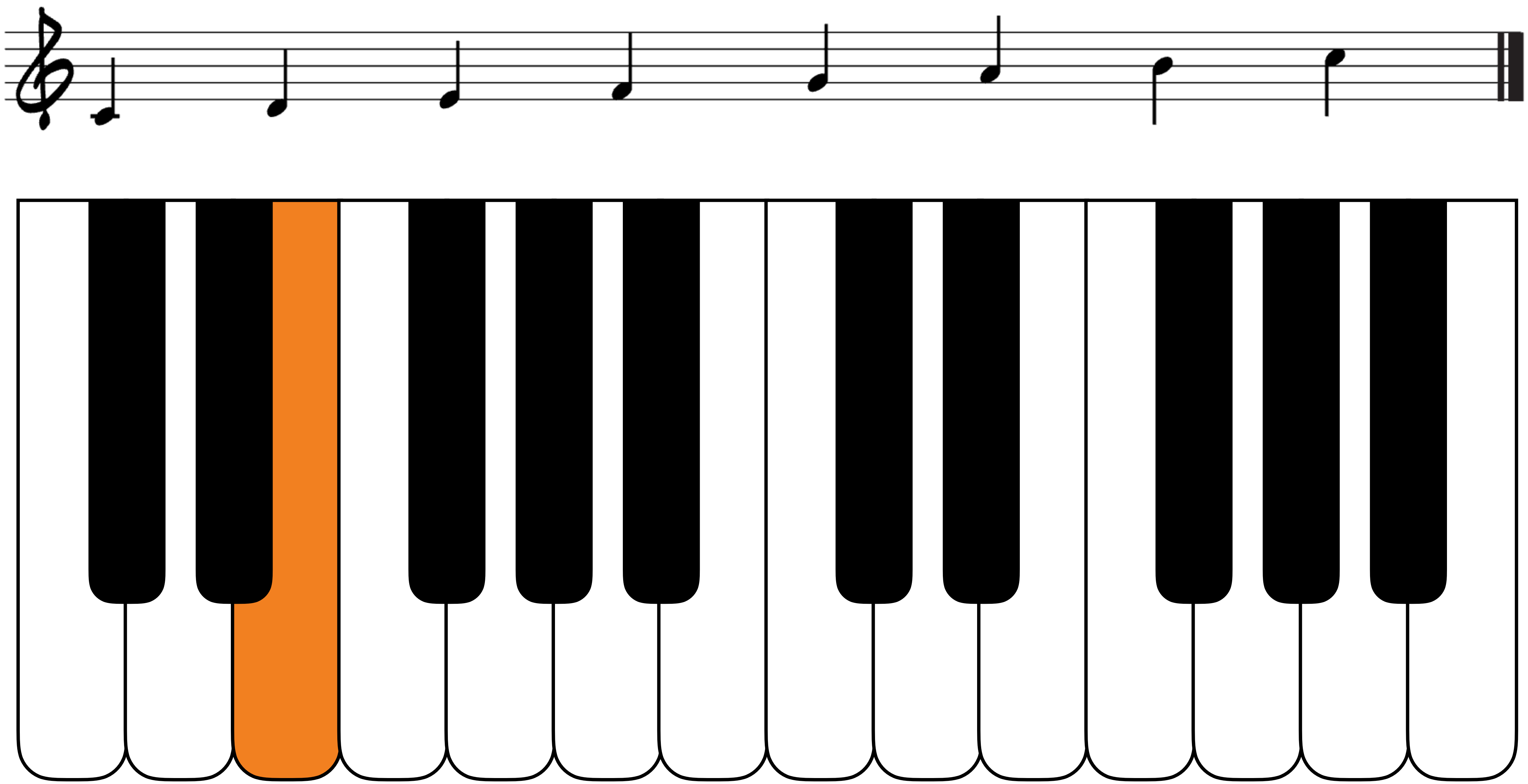
D



Whole step up from C

INTERVALS: STEPS

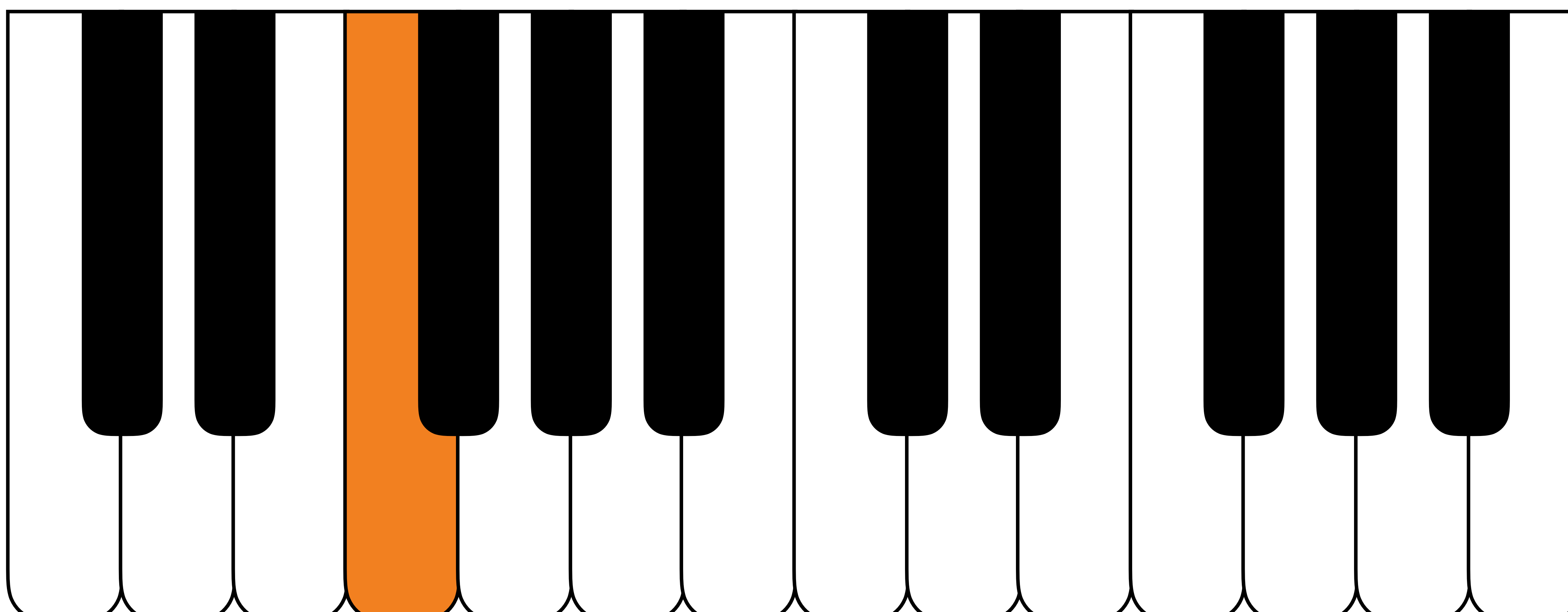
E



Whole step up from *D*

INTERVALS: STEPS

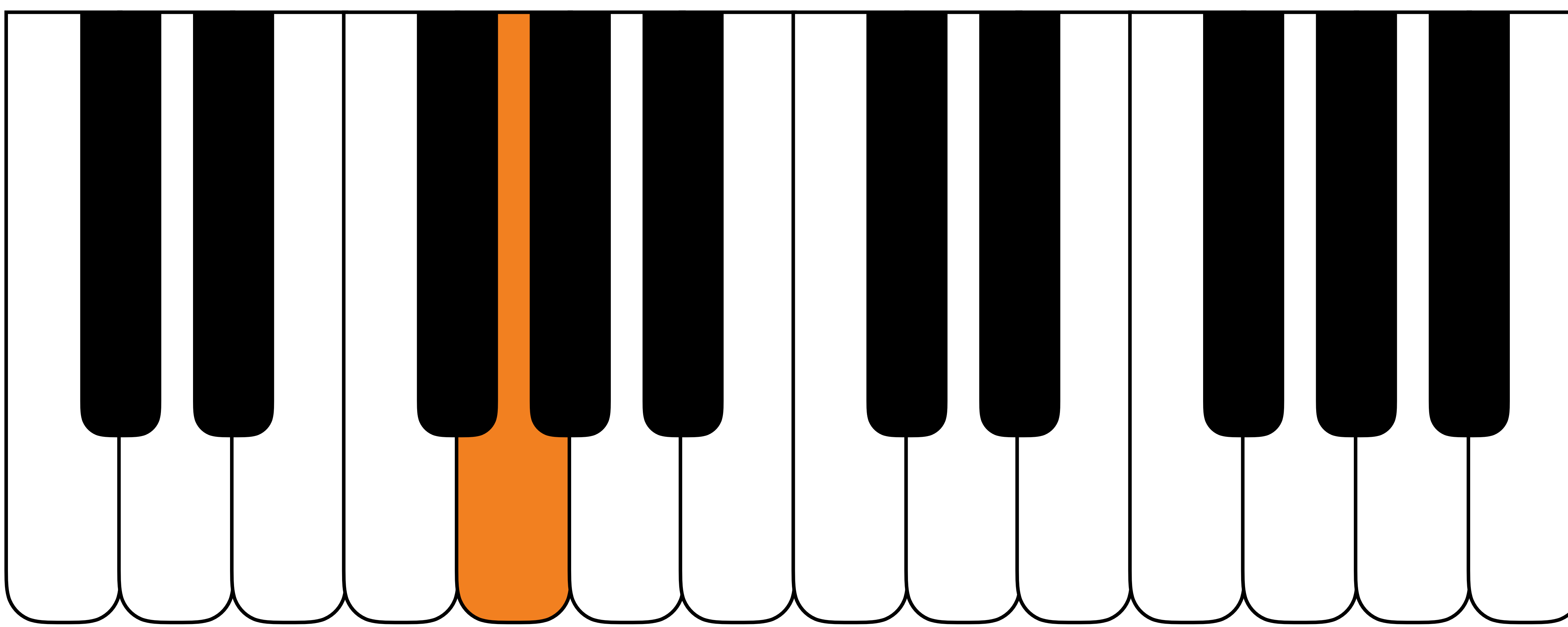
F



Half step up from *E*

INTERVALS: STEPS

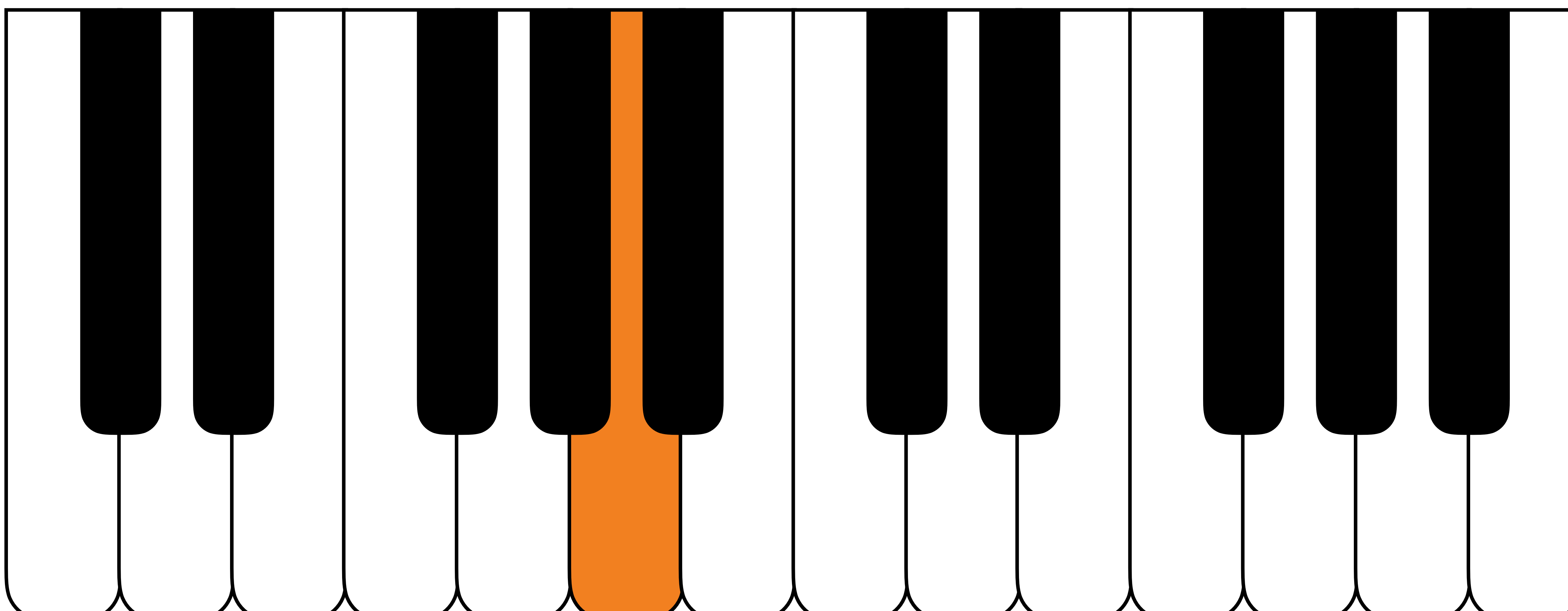
G



Whole step up from *F*

INTERVALS: STEPS

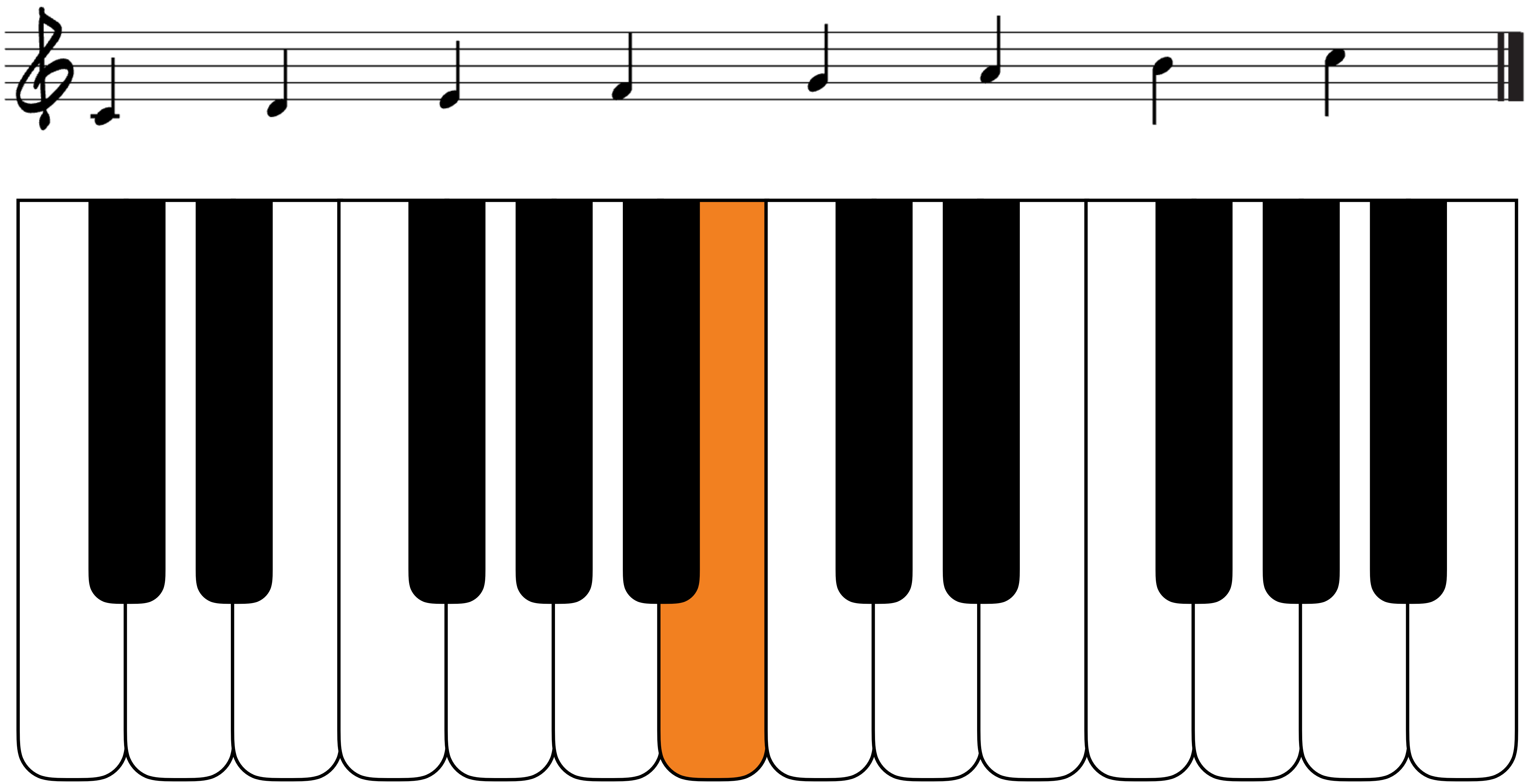
A



Whole step up from *G*

INTERVALS: STEPS

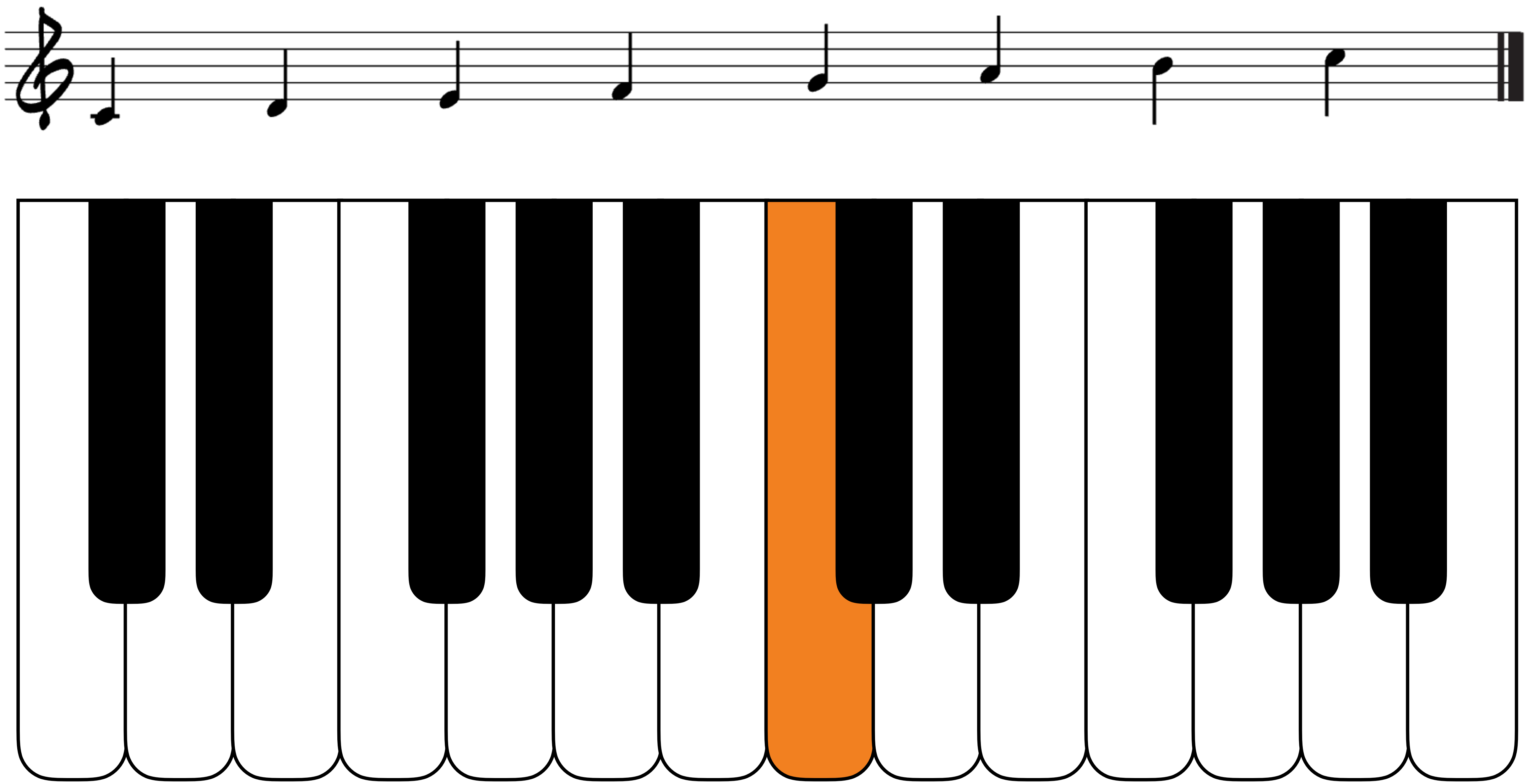
B



Whole step up from *A*

INTERVALS: STEPS

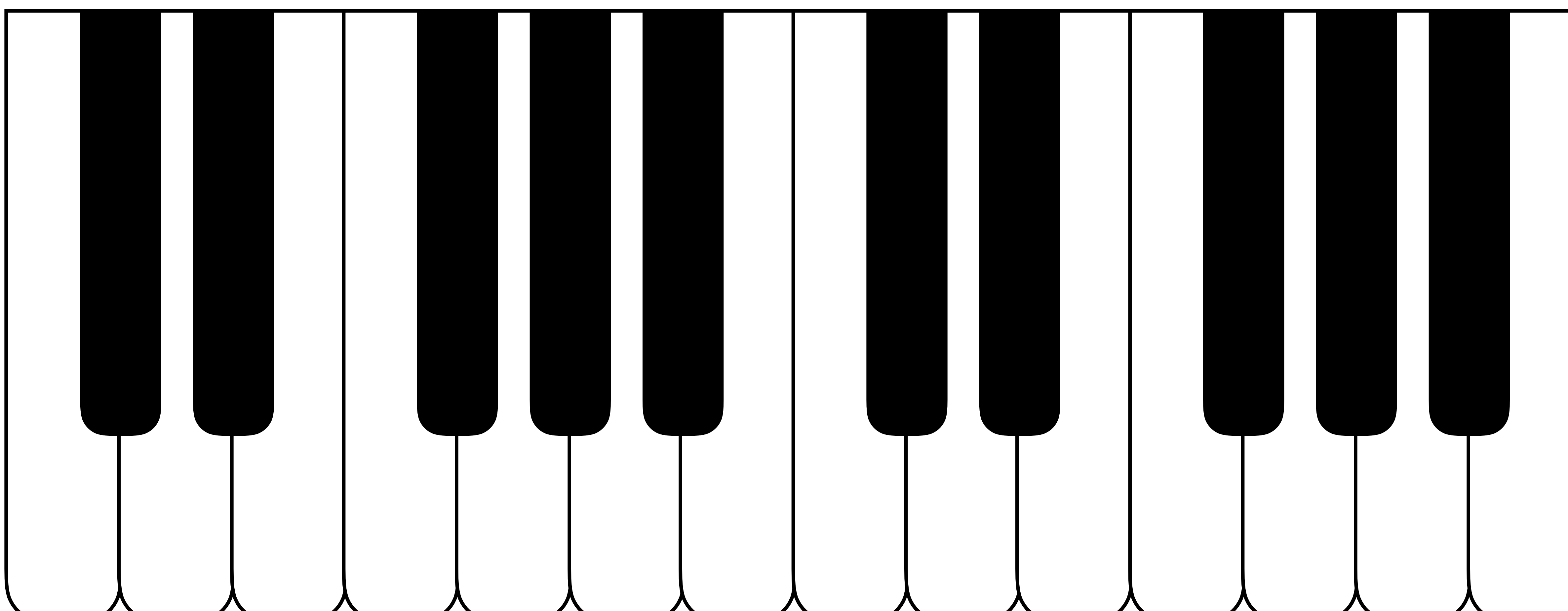
C^{8va}



Half step up from *B*

INTERVALS

- Standard interval terminology:
  - Two notes can be various distances apart
  - There are essentially only 13 possible distances:



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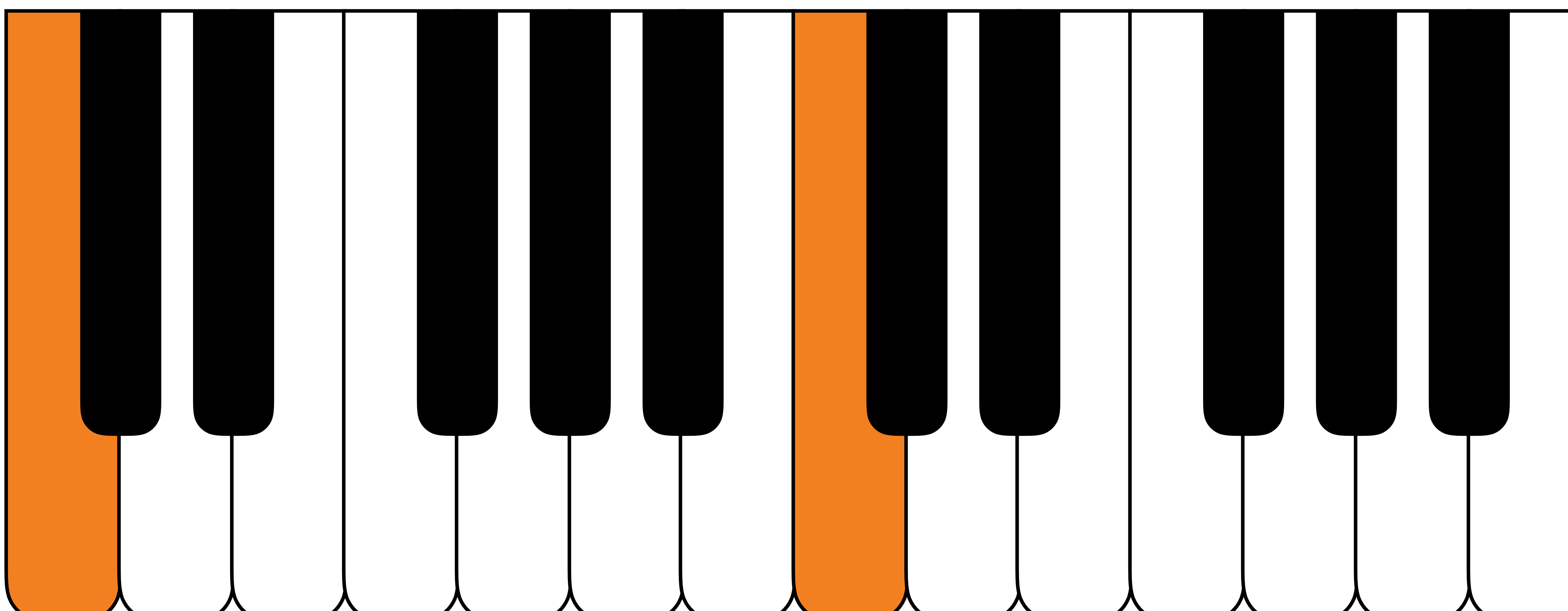

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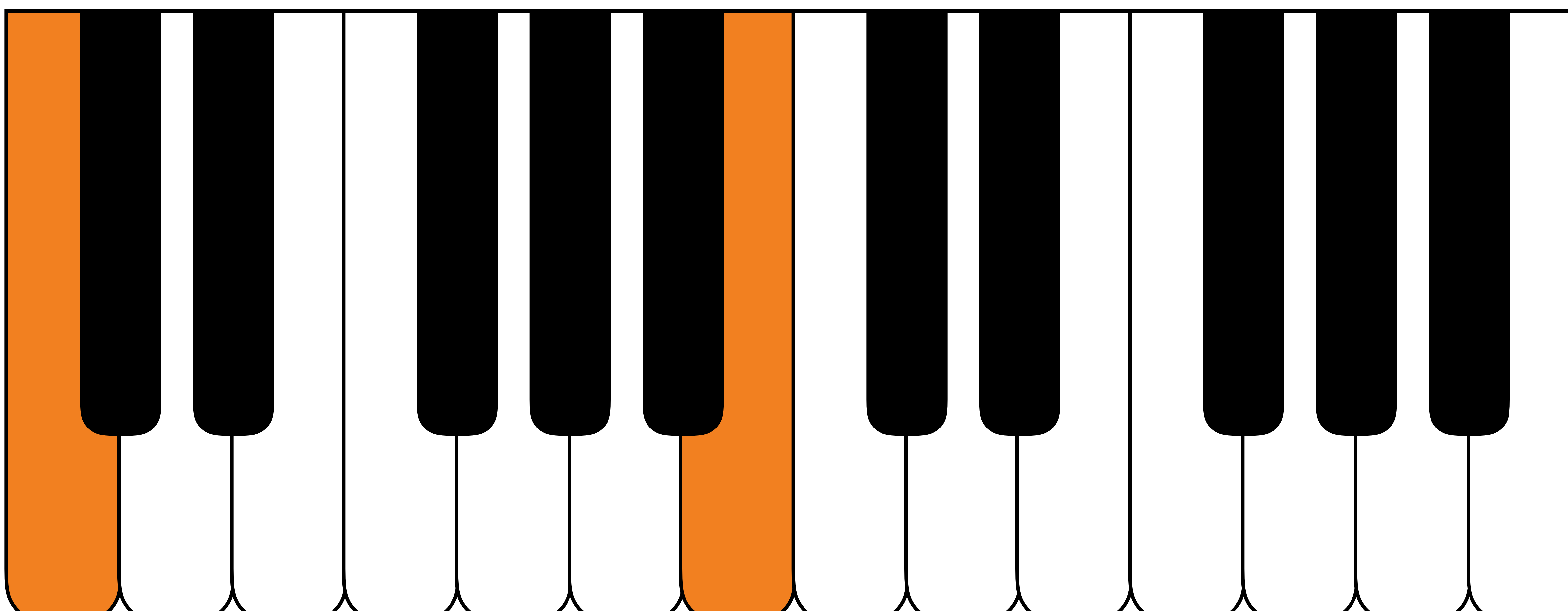

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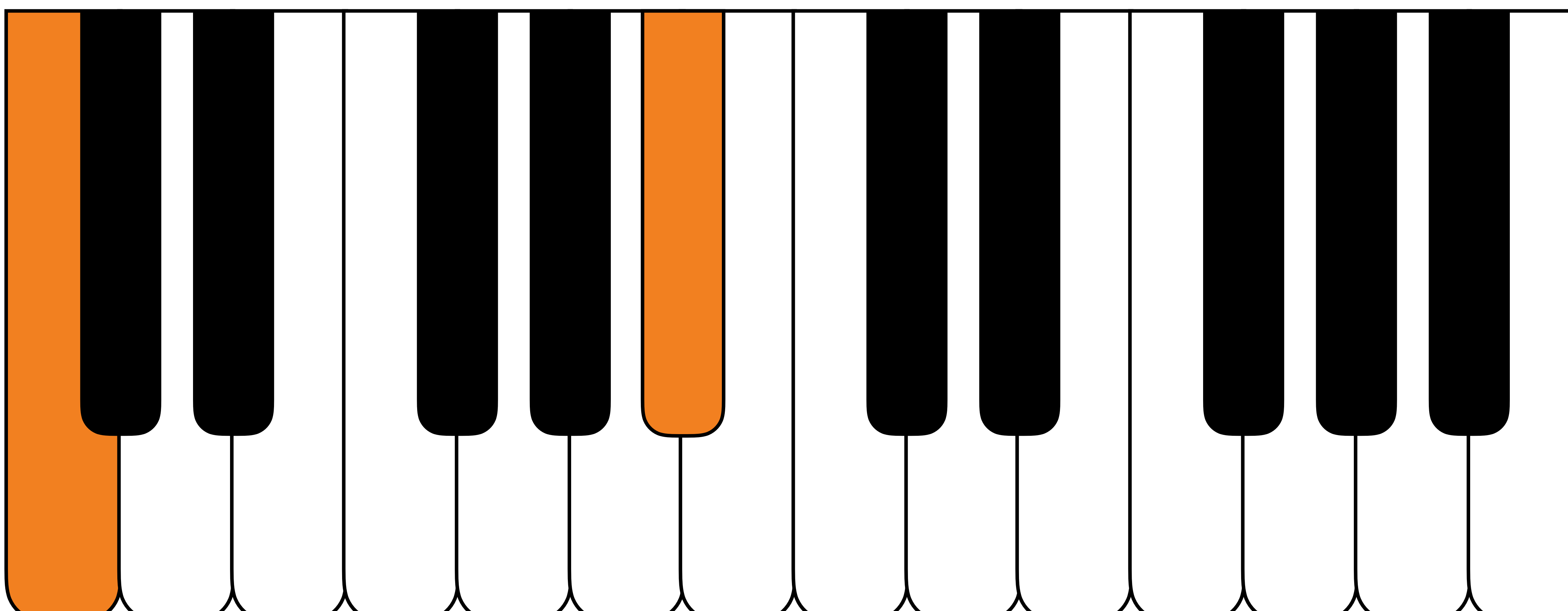

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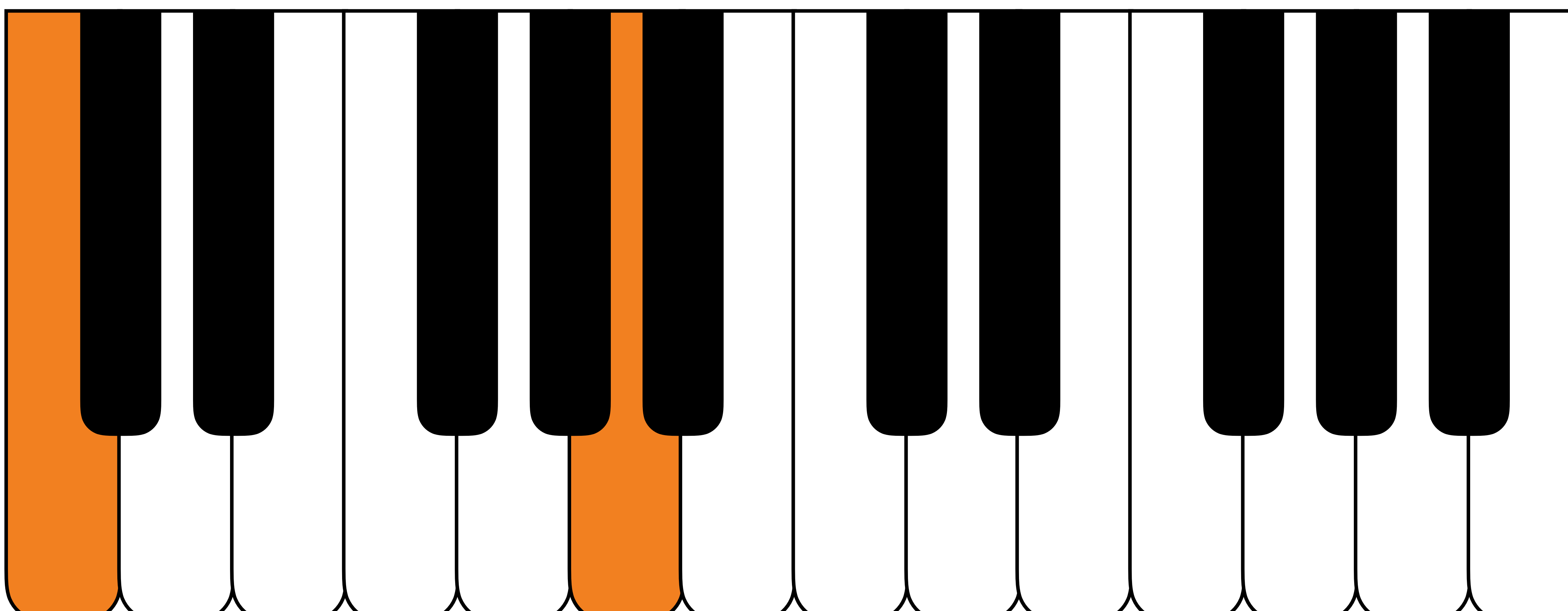

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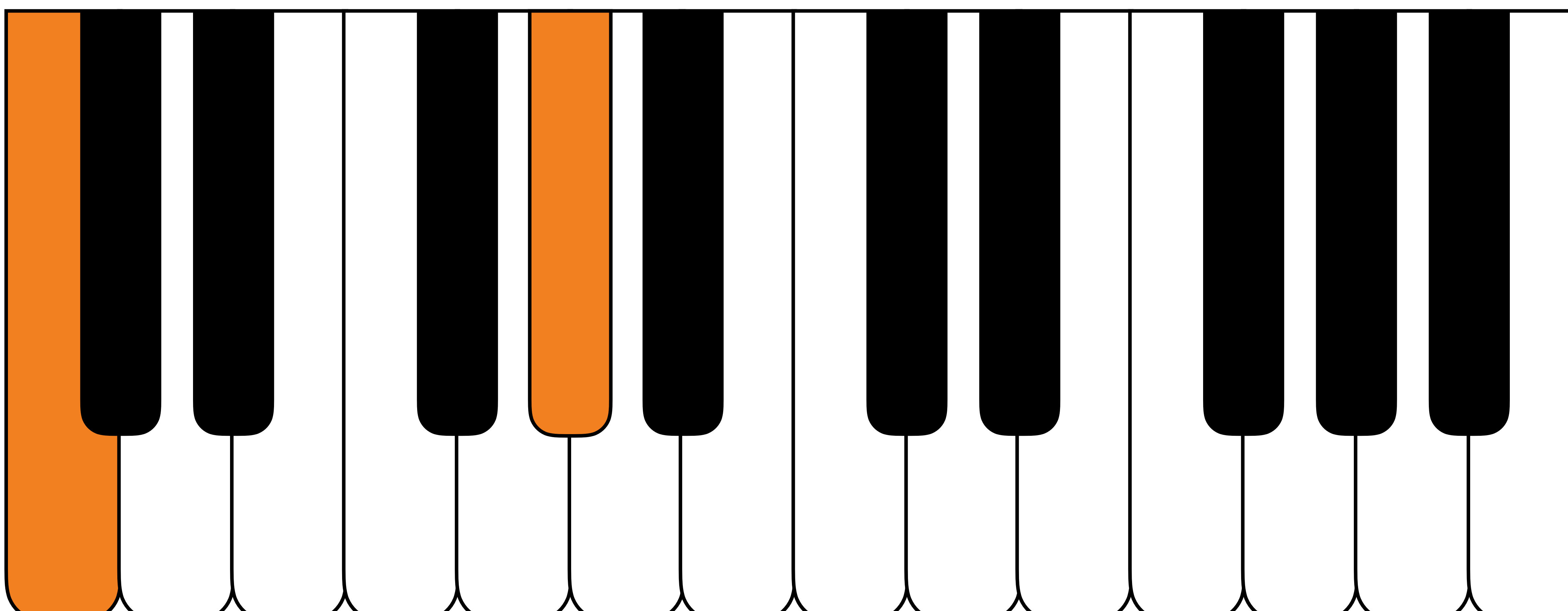

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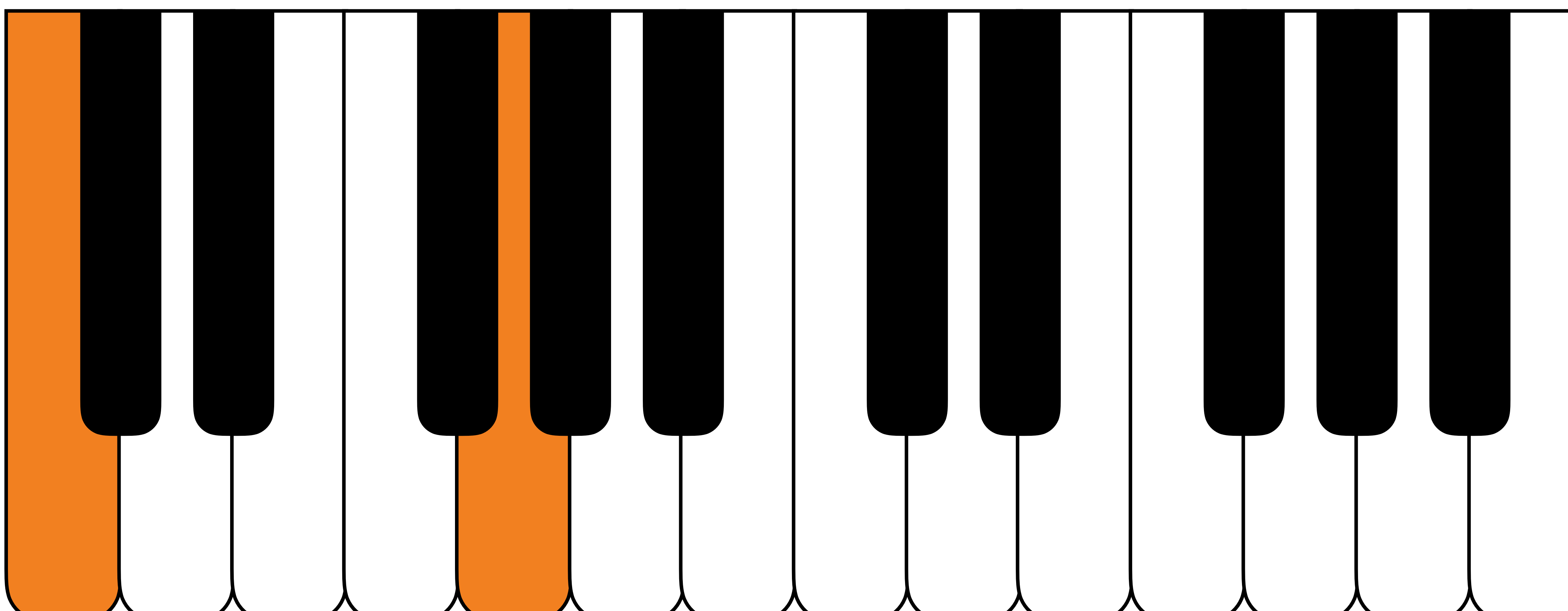

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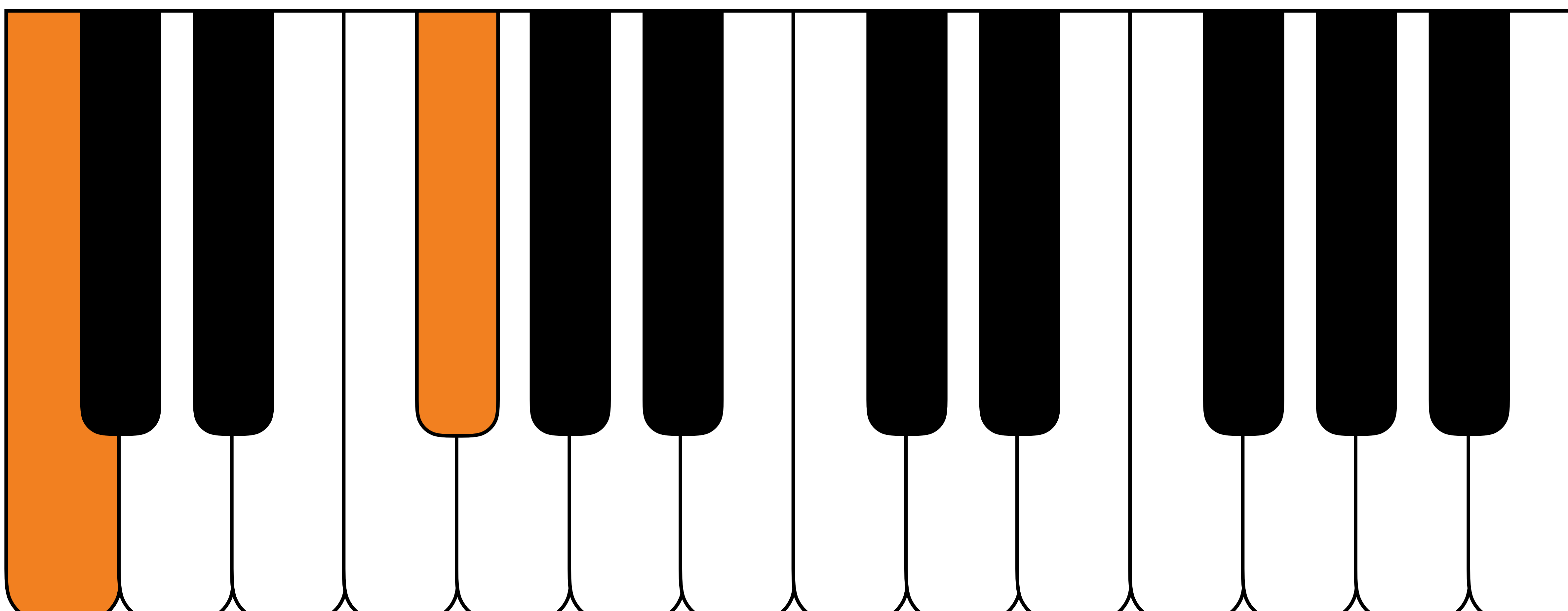

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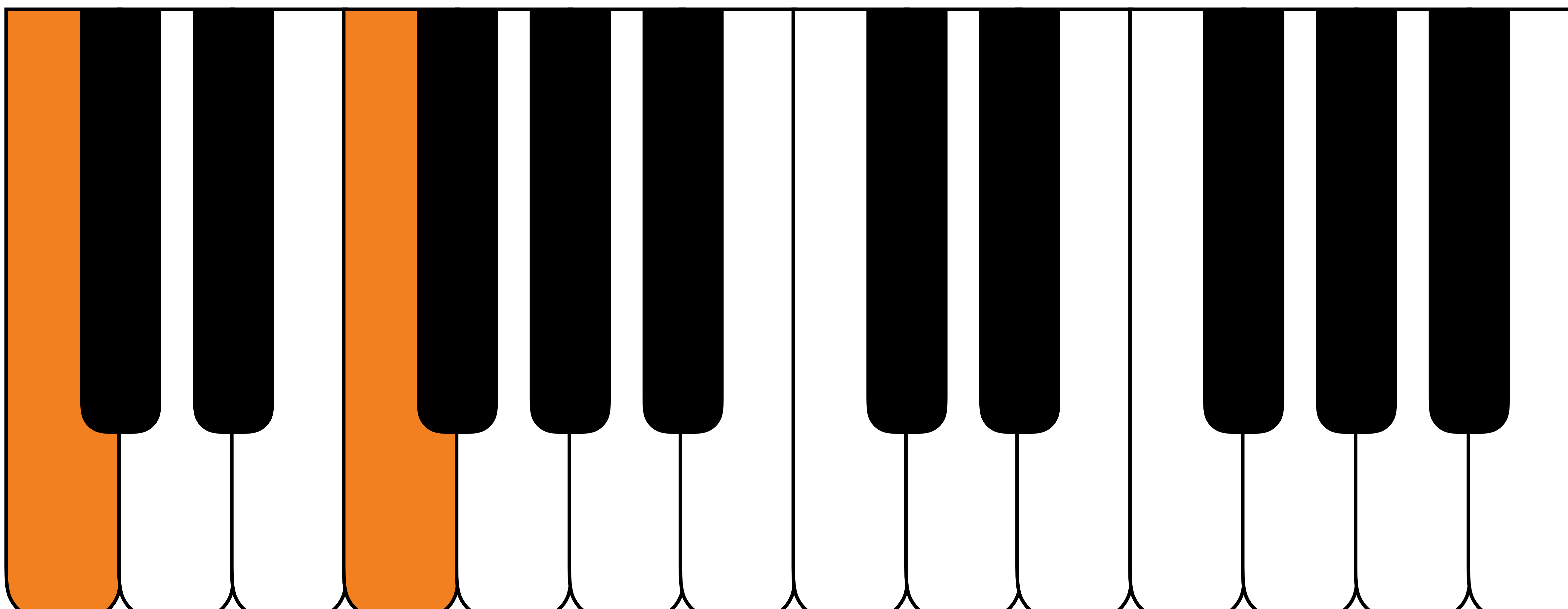

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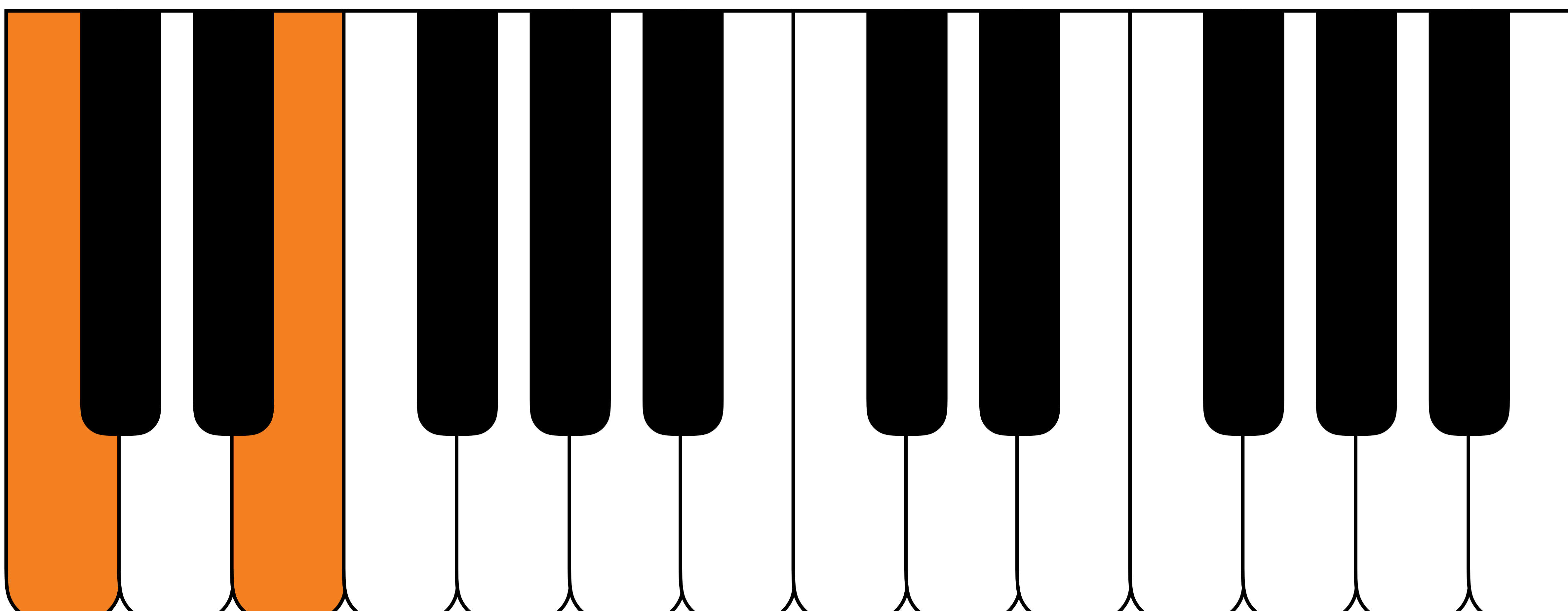

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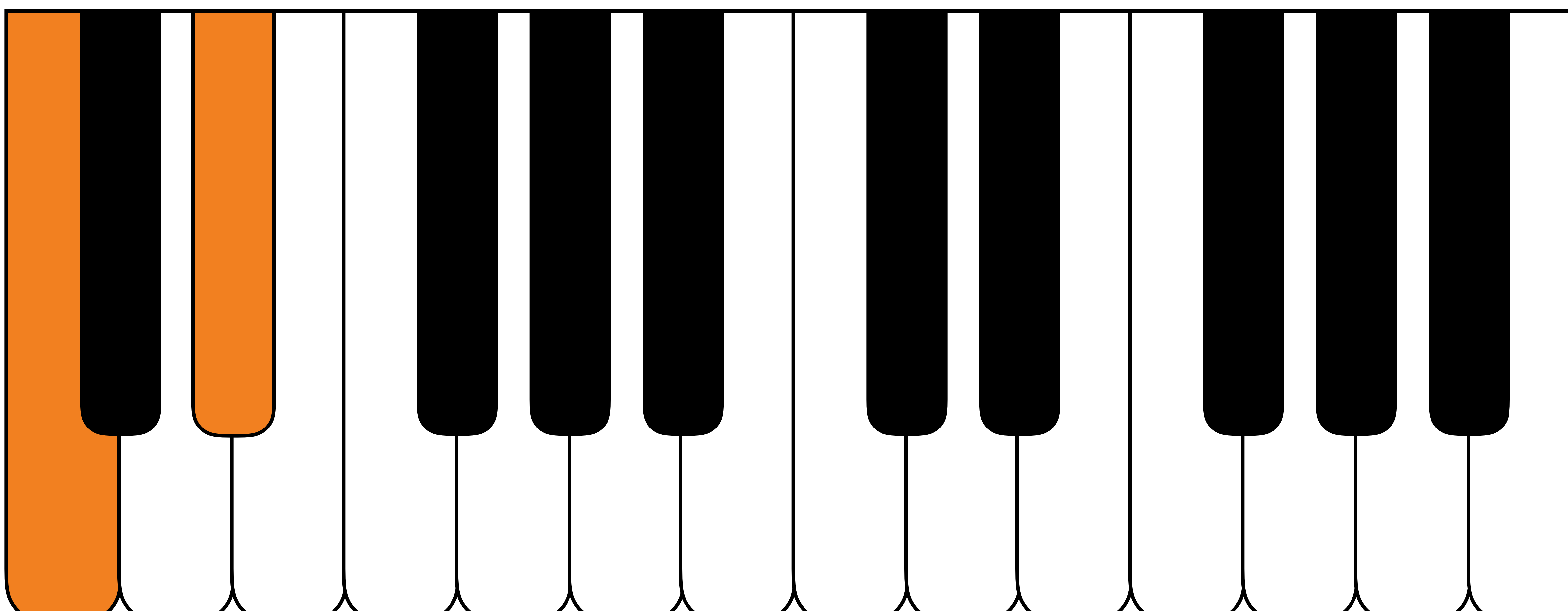

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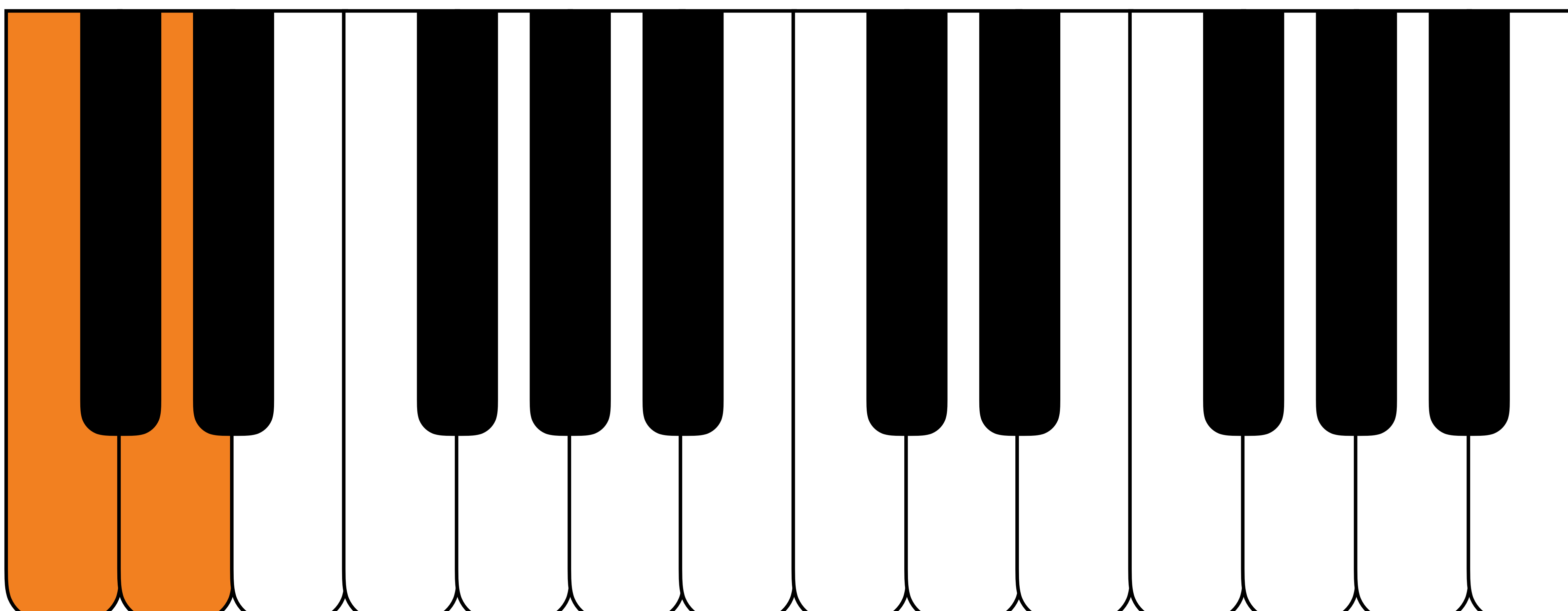

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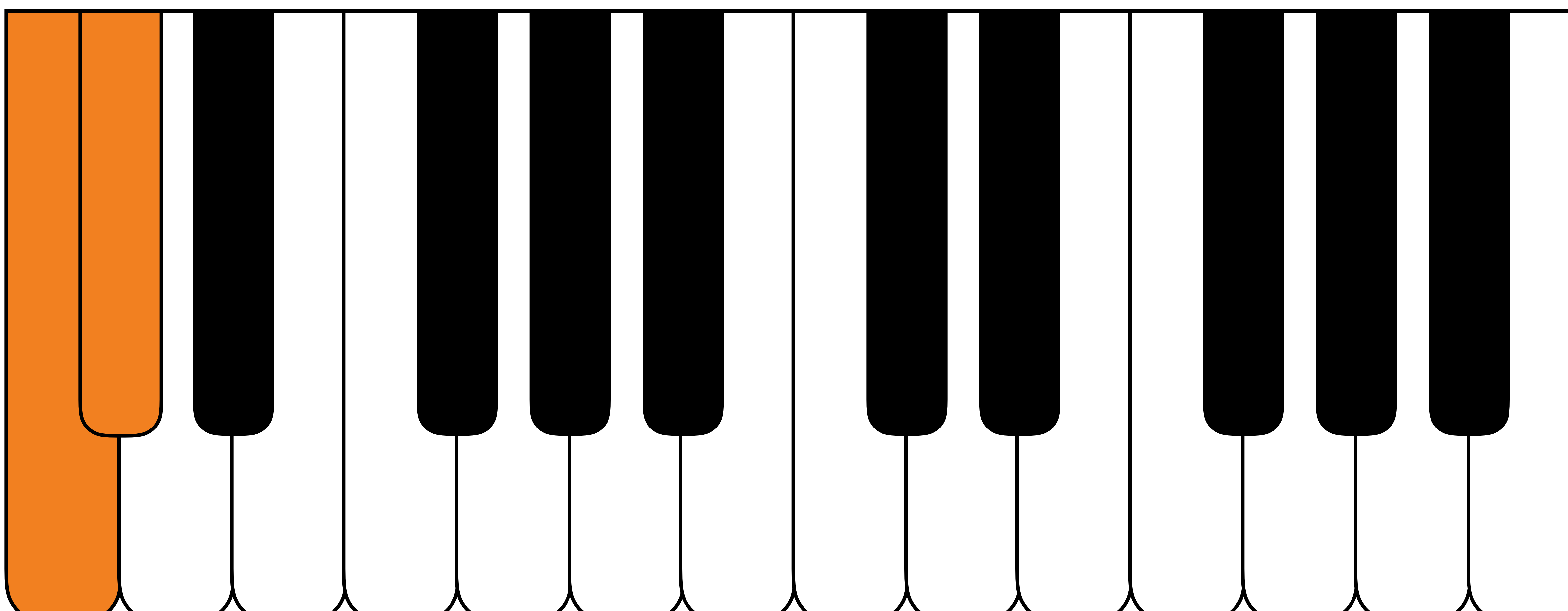

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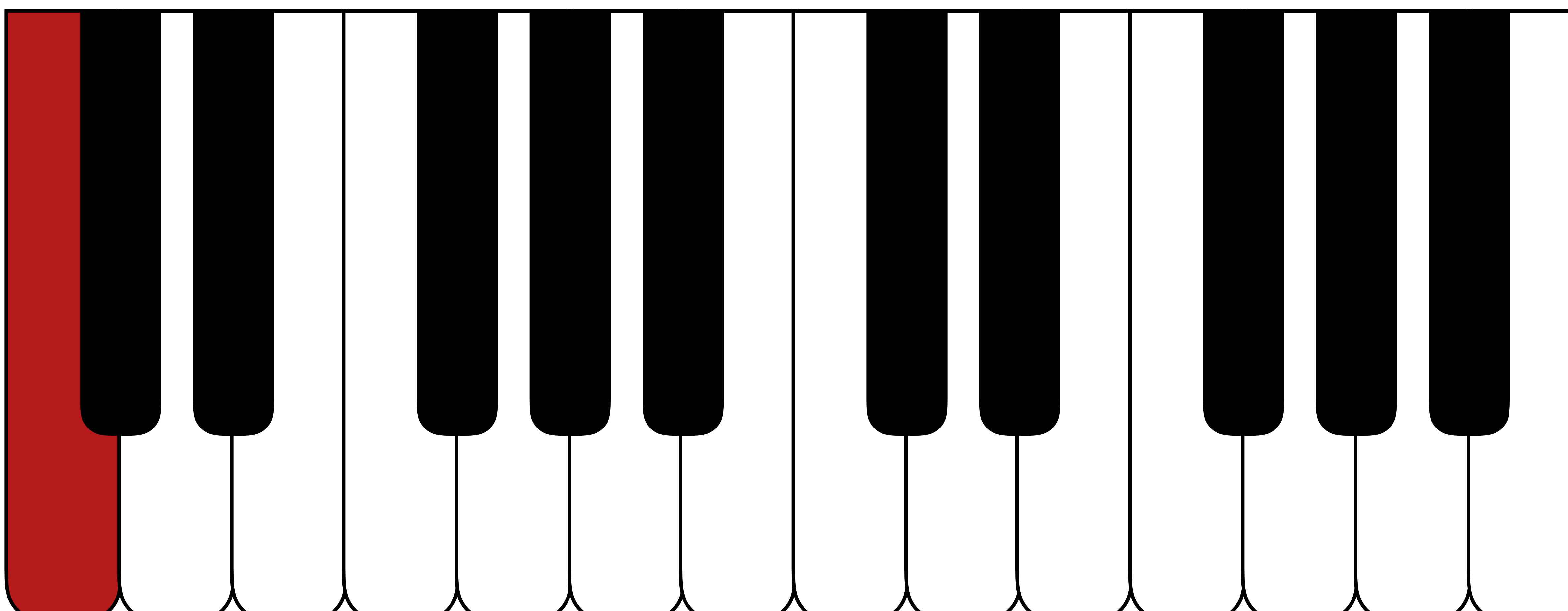
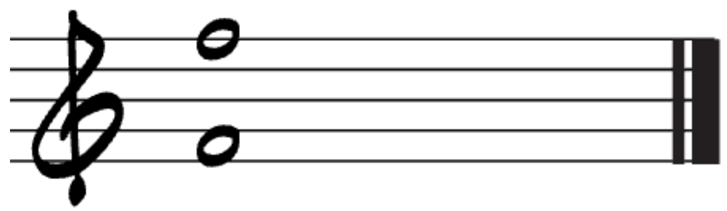


INTERVALS

- Standard interval terminology:
  - Two notes can be various distances apart
  - There are essentially only 13 possible distances
  - All other distances are *isomorphic by translation* or by *modulo 12*
  - What do we call these intervals?
    - Each one has a certain distance apart in *half*-steps
    - Does each have a unique name?
      - Yes!

PERFECT INTERVALS

- *Perfect interval*: More hollow / consonant sound
- At a distance of 12 half-steps apart:
  - Same note, one octave higher
    - e.g. *F* and *F*^{8va}
  - (Perfect) *octave*



- At a distance of 0 half-steps apart:
  - Same name
    - e.g. *G* and *G*
  - (Perfect) *unison*

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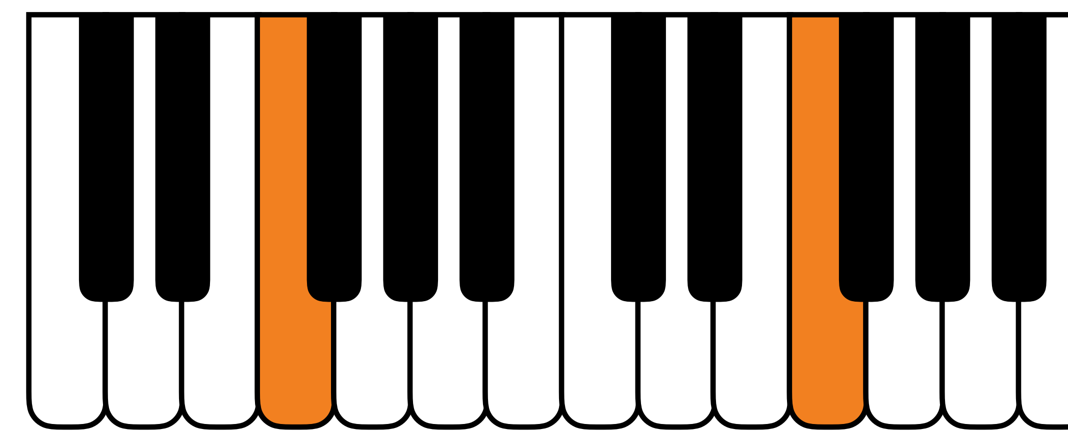

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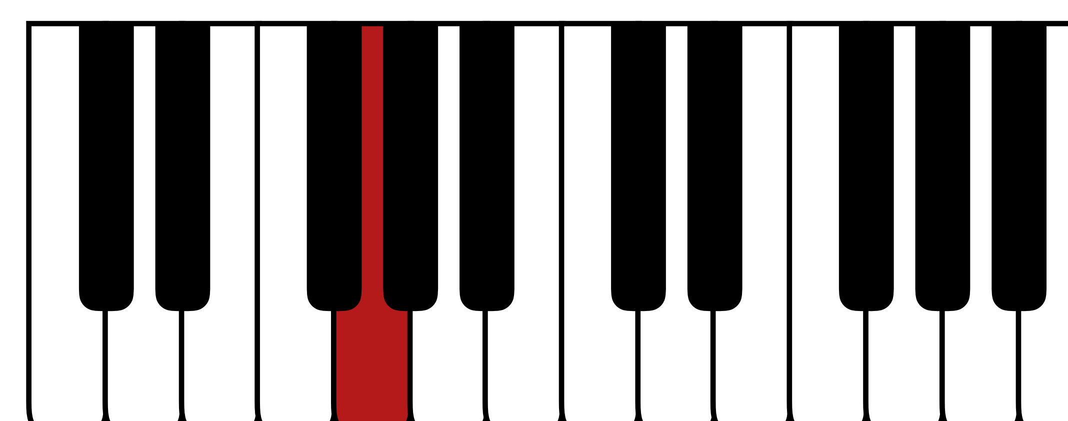
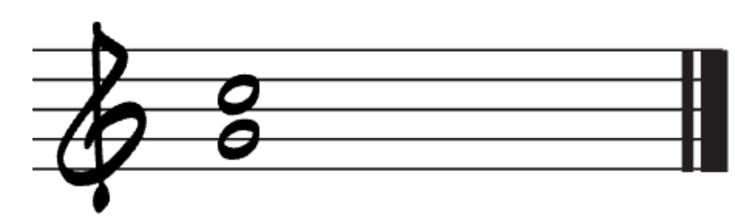
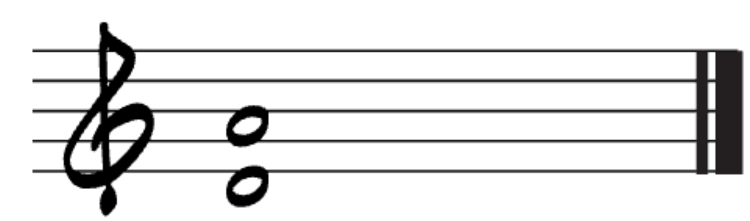


PERFECT INTERVALS

- Other perfect intervals:
- At a distance of 5 half-steps apart:
  - Four “notes” apart (*usually*)
    - e.g. *G* and *C*
  - *Perfect fourth*



- At a distance of 7 half-steps apart:
  - Five “notes” apart (*usually*)
    - e.g. *D* and *A*
  - *Perfect fifth*



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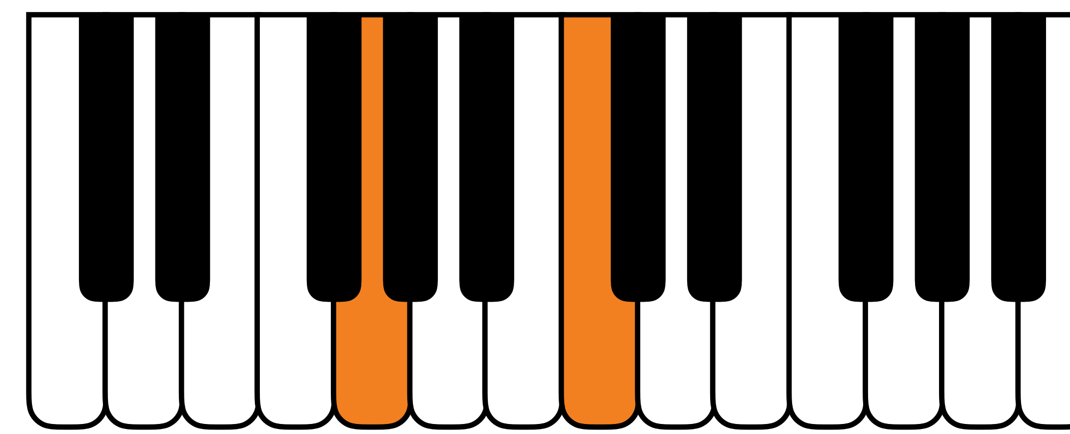

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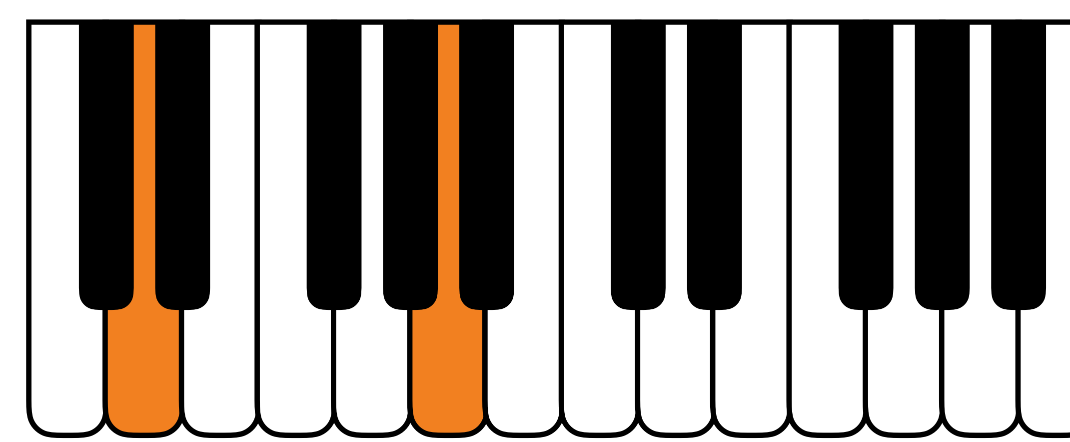
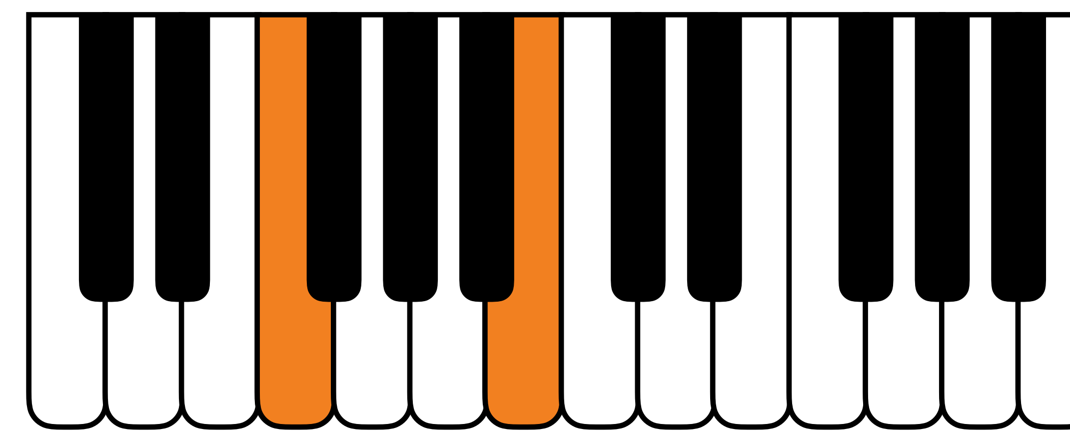


PERFECT INTERVALS

- Need to be careful! Even in *C* major, need accidentals:
- At a distance of 5 half-steps apart:
  - Four "notes" apart (***usually***)
    - e.g. *F* and *B*
  - NOT a perfect fourth!



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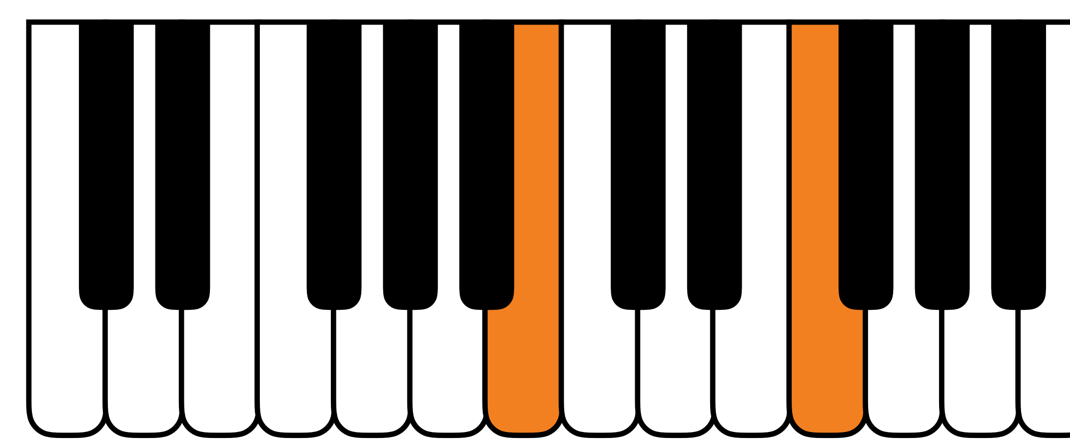

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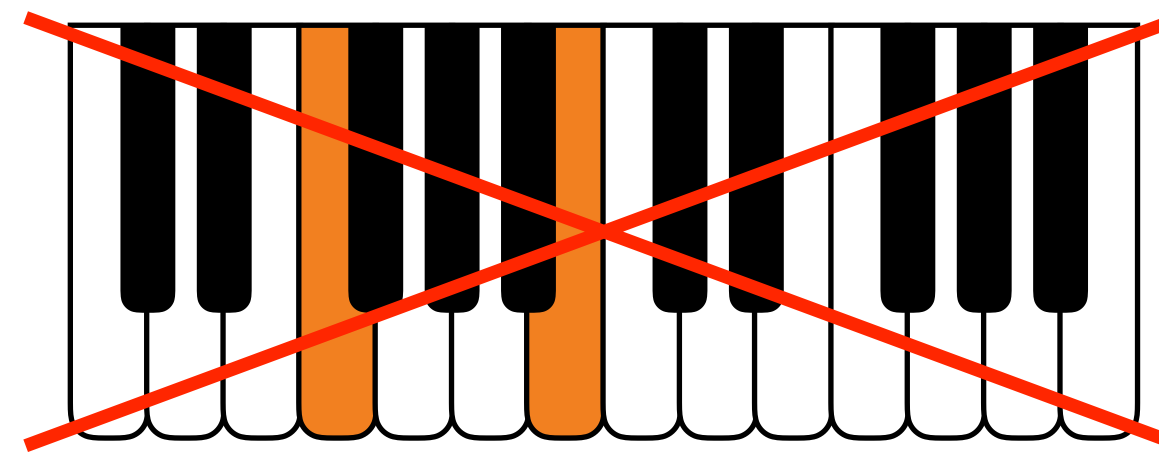
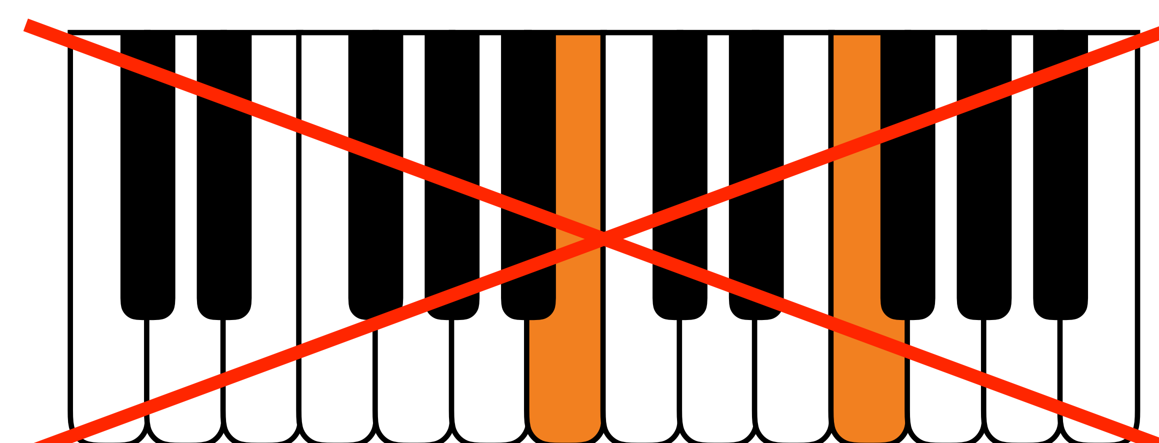
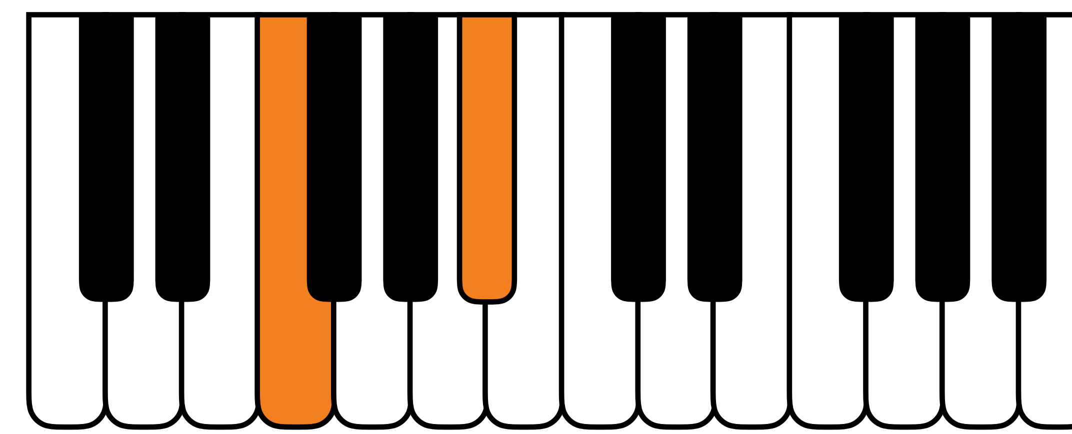


- At a distance of 7 half-steps apart:
  - Five "notes" apart (***usually***)
    - e.g. *B* and *F*
  - NOT a perfect fifth!



PERFECT INTERVALS

- Need to be careful! Even in *C* major, need accidentals:
- At a distance of 5 half-steps apart:
  - Four "notes" apart (***usually***)
    - e.g. *F* and $B^\flat$
  - IS a perfect fourth!



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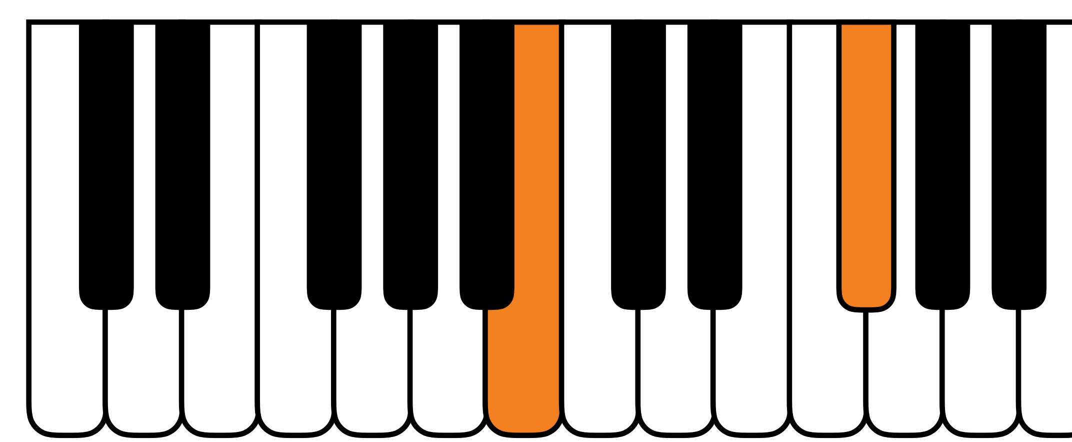
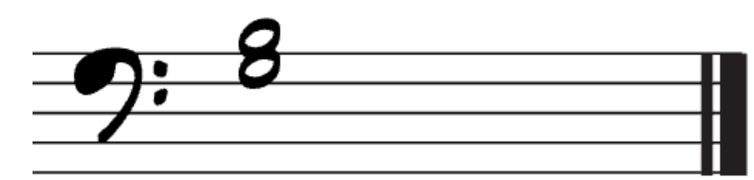
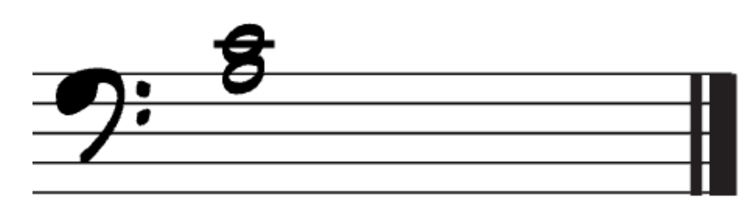


IMPERFECT INTERVALS

- *Imperfect interval*: Dissonant or somewhat-consonant
- At a distance of 4 half-steps apart:
  - Three notes apart
    - e.g. *G* and *B*
  - *Major third*



- At a distance of 3 half-steps apart:
  - Almost three notes apart
    - e.g. *A* and *C*
  - *Minor third*



IMPERFECT INTERVALS

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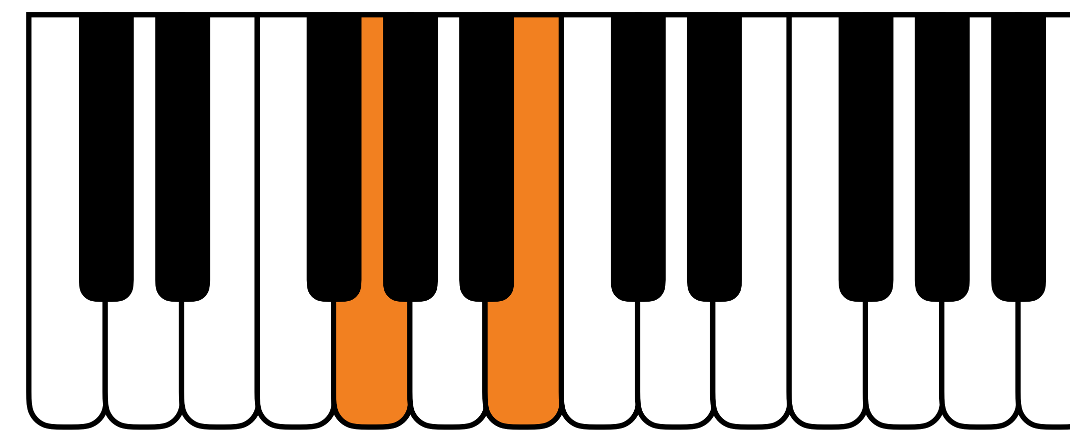
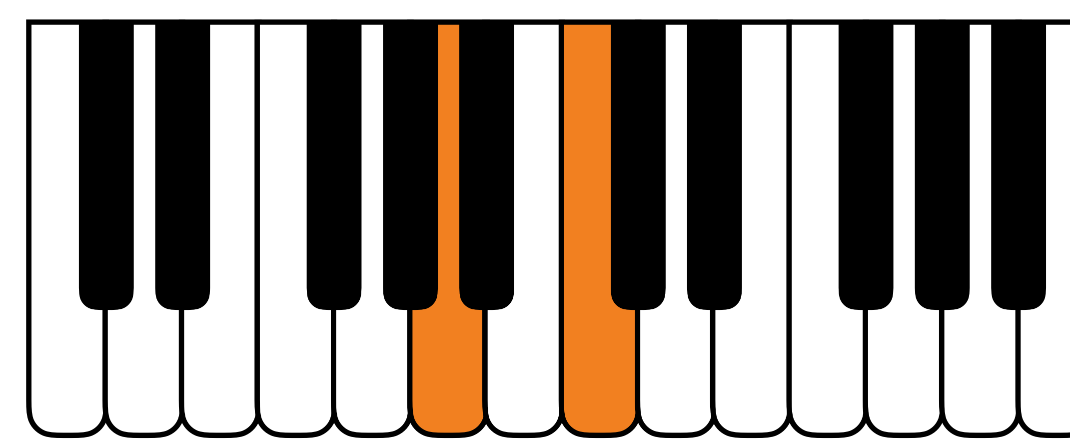


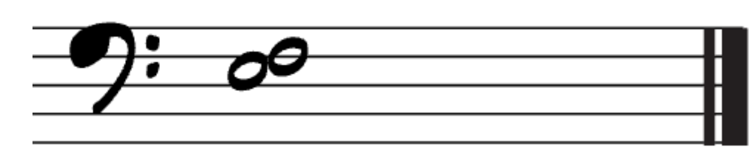
- At a distance of 3 half-steps apart:
  - Almost three notes apart
    - e.g. *A* and *C*
  - *Minor third*



IMPERFECT INTERVALS

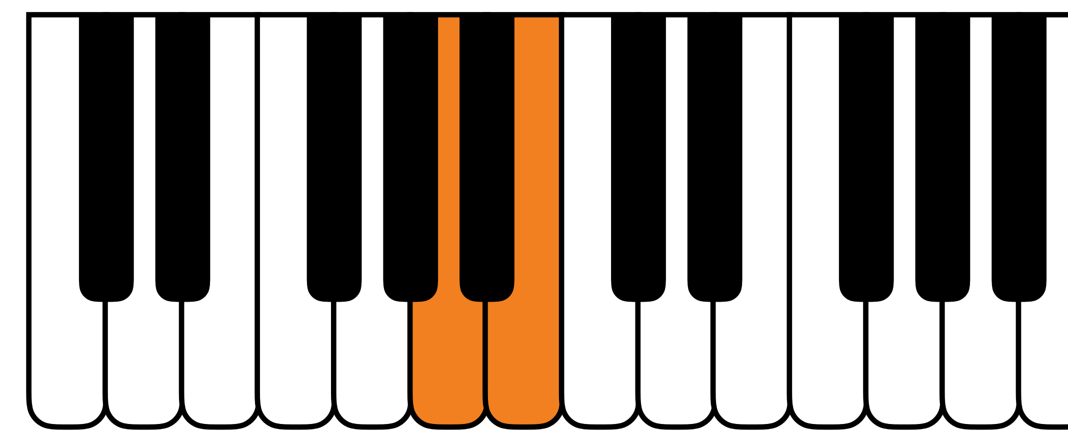
- *Imperfect interval*: Dissonant or somewhat-consonant
- At a distance of 2 half-steps apart:
  - Two notes apart
    - e.g. *A* and *B*
  - *Major second*

- At a distance of 1 half-step apart:
  - Almost two notes apart
    - e.g. *E* and *F*
  - *Minor second*

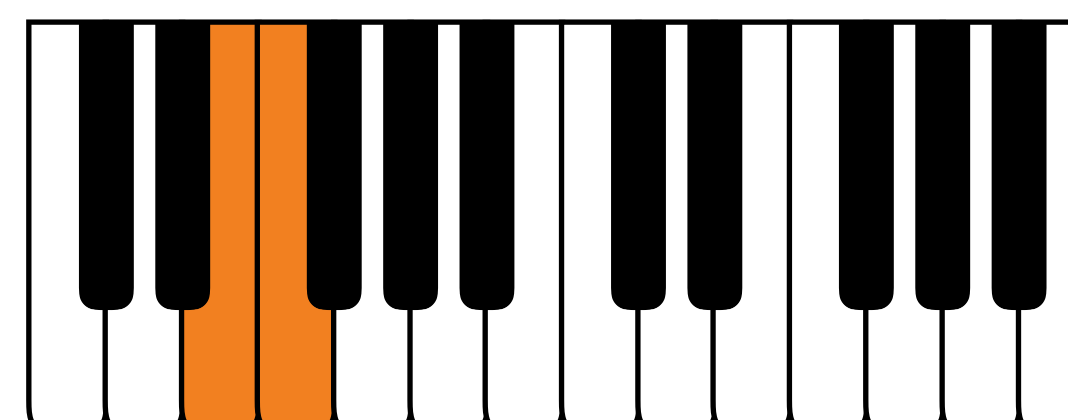


IMPERFECT INTERVALS

- *Imperfect interval*: Dissonant or somewhat-consonant
- At a distance of 2 half-steps apart:
  - Two notes apart
    - e.g. *A* and *B*
  - *Major second*

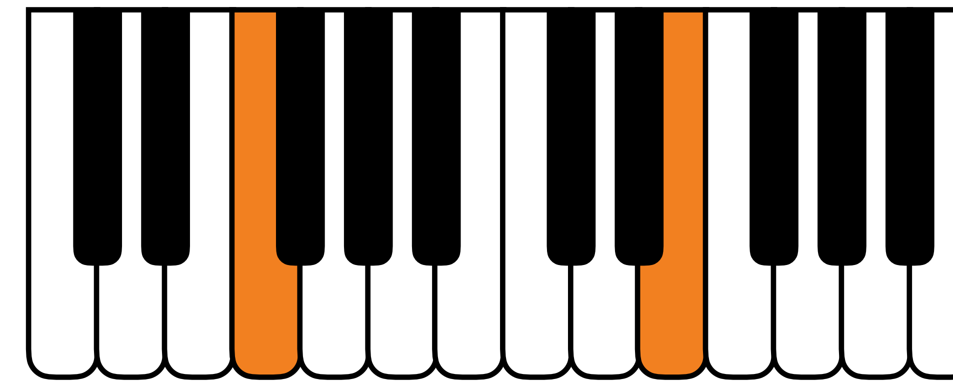


- At a distance of 1 half-step apart:
  - Almost two notes apart
    - e.g. *E* and *F*
  - *Minor second*

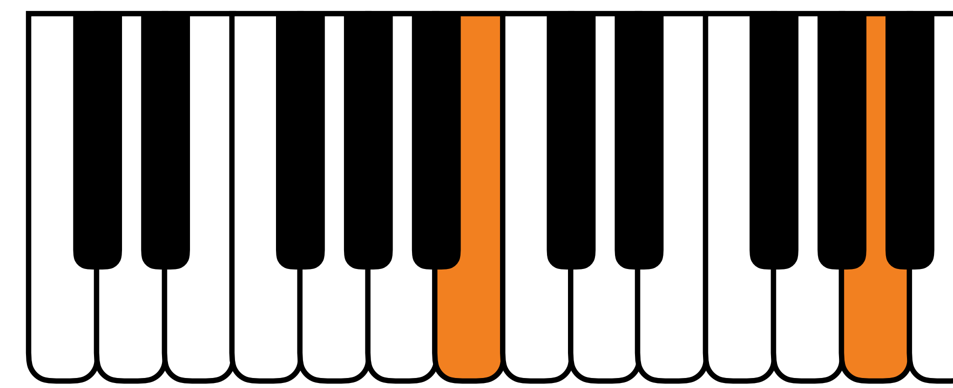


IMPERFECT INTERVALS

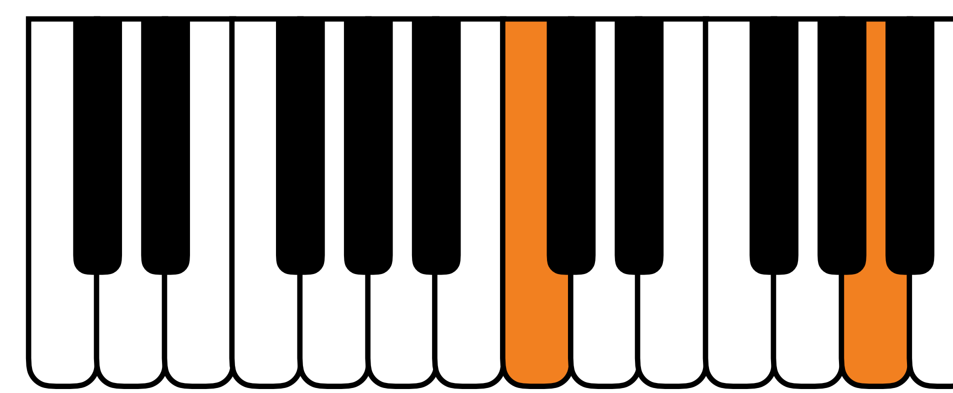
- At a distance of 11 half-steps apart:
  - Six notes apart
  - *Major seventh*



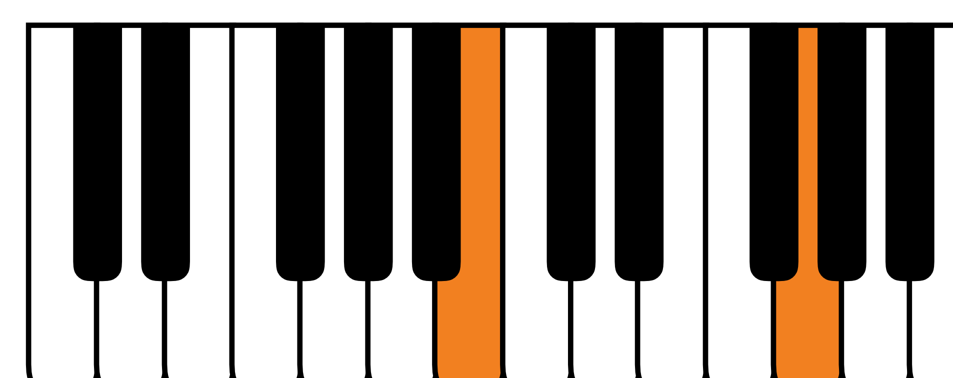
- At a distance of 10 half-steps apart:
  - Almost six notes apart
  - *Minor seventh*



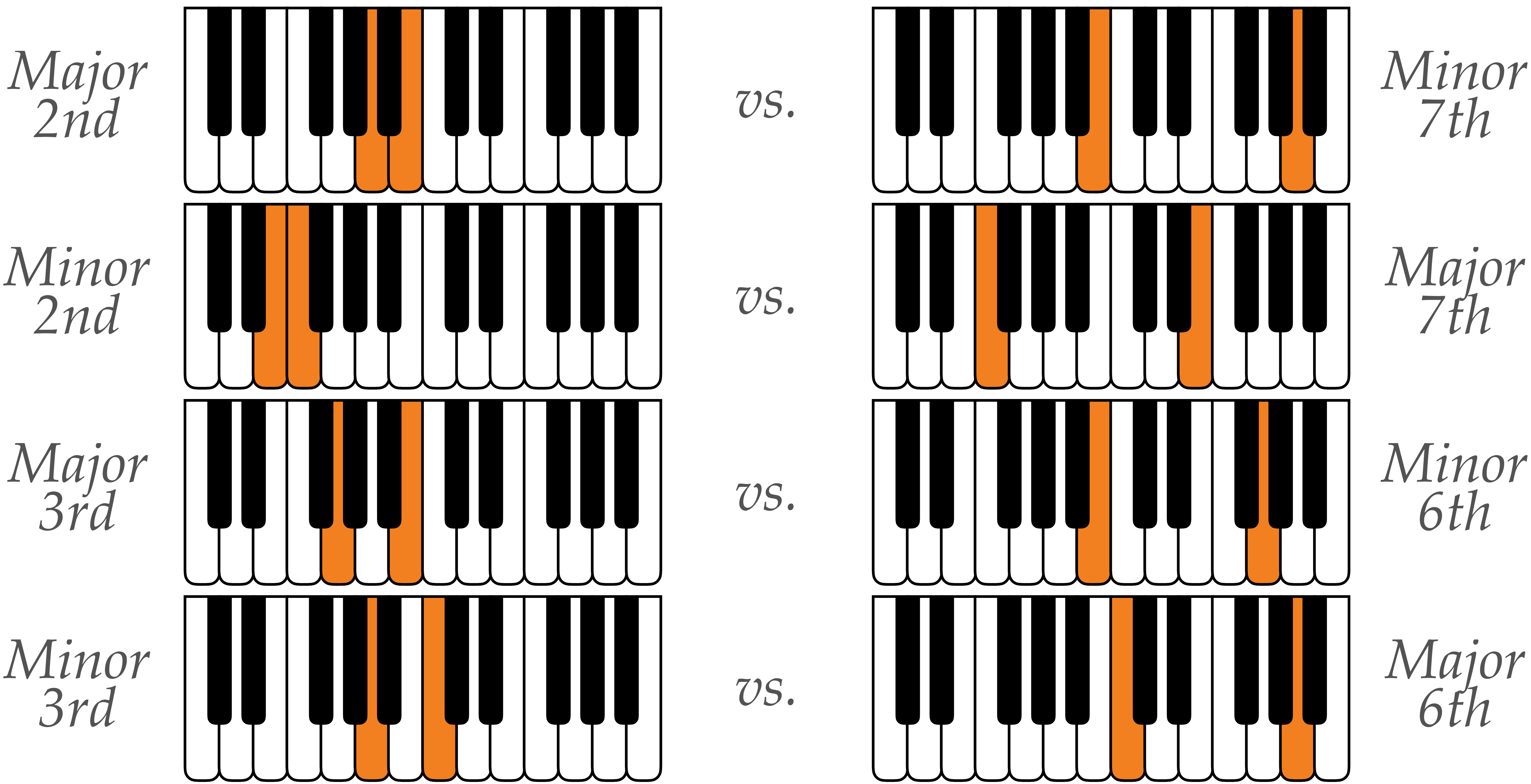
- At a distance of 9 half-steps apart:
  - Six notes apart
  - *Major sixth*



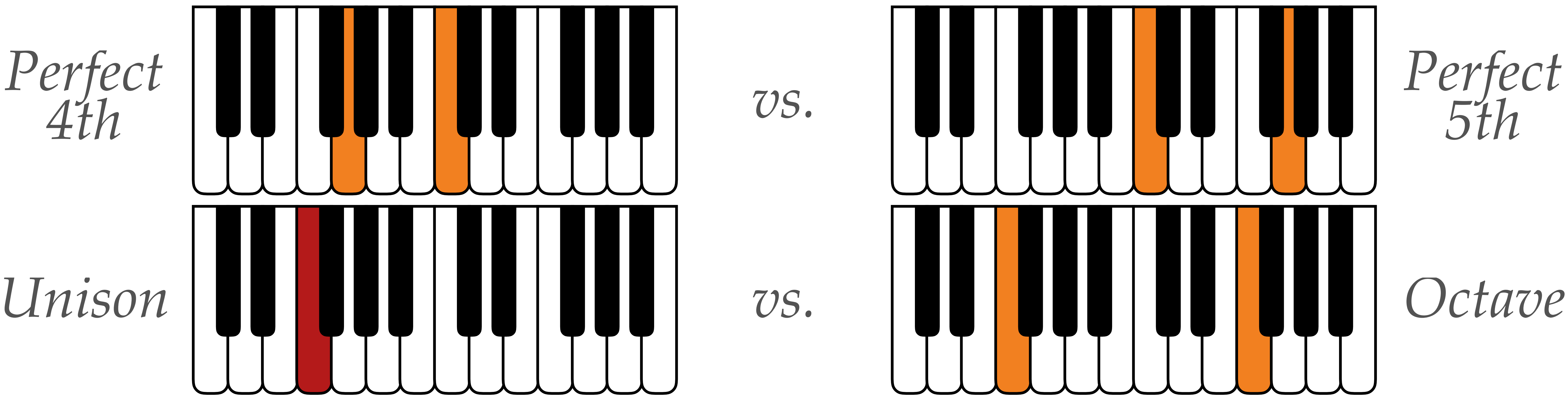
- At a distance of 8 half-steps apart:
  - Almost six notes apart
  - *Minor sixth*



PRINCIPLE OF INVERSION

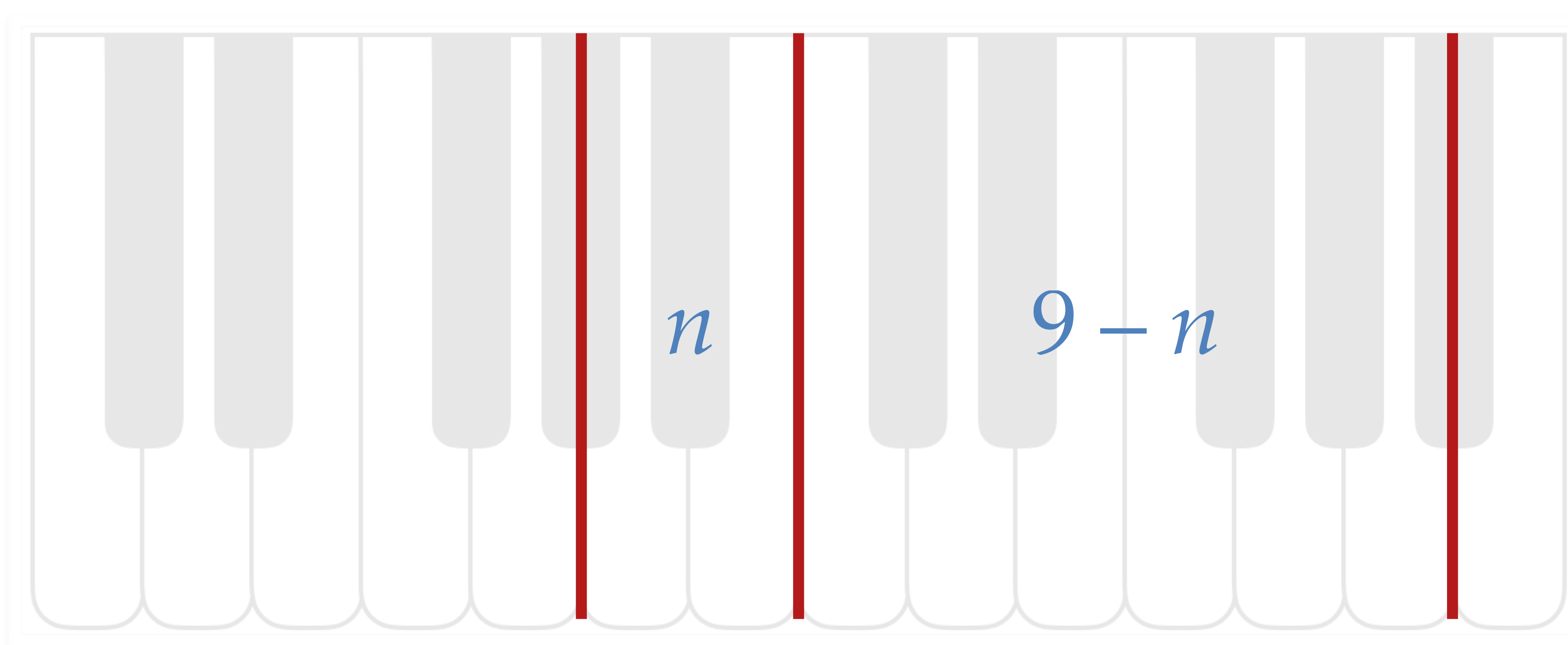


PRINCIPLE OF INVERSION



PRINCIPLE OF INVERSION

- The *principle of inversion* tells us a few things:
  - Intervals are read from the bottom-up
  - We only need to know the names of some of the intervals, as the rest follow logically:



1-indexed, modulo 8

SIX HALF-STEPS?

- What about the interval of six half-steps?
  - We hinted at this earlier…
  - This is a raised perfect fourth (by a half-step)
  - …or a lowered perfect fifth (by a half-step)
  - Thus, this is often called one of the following:
    - *Augmented fourth*
    - *Diminished fifth*
  - to **augment** = to lengthen, to **diminish** = to shorten
  - Another common name for this is a *tritone*
    - Three whole steps = three *tones* (cf. *semitones*)

SIX HALF-STEPS?

- The principle of inversion tells us intuitively that the inverse of a tritone is… a tritone!
- Mathematically:

  $n = 9 - n$ if $2n = 9$, i.e. $n = 4.5$

  where n measures whole steps (i.e. tones)
- So $n = 4.5$ is the fixpoint of the principle of inversion
- But 4.5 is in between 4 and 5
  - So the tone in between a 4th and a 5th
  - i.e. augmented 4th or diminished 5th
- Tritone also has a *diminished sound*, so often called d5

CONNOTATIONS

- *Minor chords:* Sad, empty, hanging sounds
- *Major chords:* Full, harmonious, definite sounds
- Are there other types of sounds?
  - Those that don't conform strictly to what we've mentioned so far
  - Require at least another note so we can form a three-note basis
  - Triangle of sound changes in these other types
    - Forms *triads* / chords for more interesting stuff

TRIADS

- Let's stack together the intervals we've seen earlier:
  - Major 3rd on top of a Minor 3rd
    - 4 + 3 = 7 half-steps from base to top
  - Minor 3rd on top of a Major 3rd
    - 3 + 4 = 7 half-steps from base to top
  - Minor 3rd on top of a Minor 3rd
    - 3 + 3 = 6 half-steps from base to top
  - Major 3rd on top of a Major 3rd
    - 4 + 4 = 8 half-steps from base to top

TRIADS

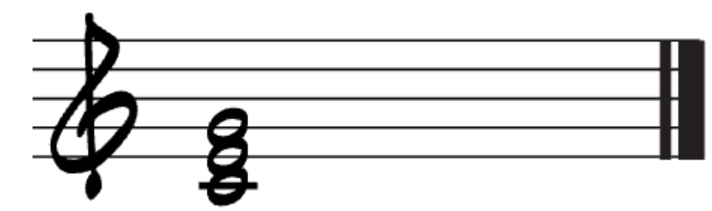
- Let's stack together the intervals we've seen earlier:
  - Major 3rd on top of a Minor 3rd
    - Perfect fifth with a major basis
  - Minor 3rd on top of a Major 3rd
    - Perfect fifth with a minor basis
  - Minor 3rd on top of a Minor 3rd
    - Diminished fifth with a minor basis
  - Major 3rd on top of a Major 3rd
    - Minor sixth with a major basis

TRIADS

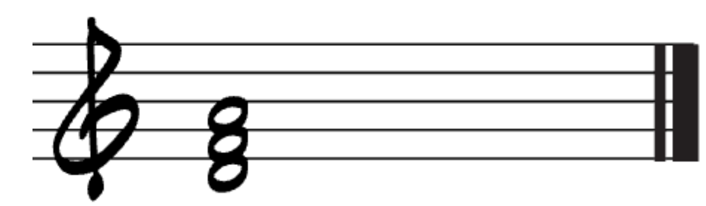
- Let's stack together the intervals we've seen earlier:
  - Major 3rd on top of a Minor 3rd
    - Perfect fifth with a major basis
  - Minor 3rd on top of a Major 3rd
    - Perfect fifth with a minor basis
  - Minor 3rd on top of a Minor 3rd
    - Diminished fifth with a minor basis
  - Major 3rd on top of a Major 3rd
    - *Augmented fifth* with a major basis

TRIADS

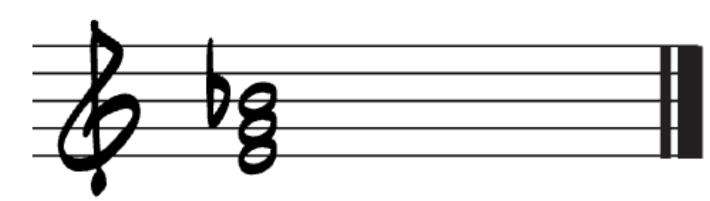
- Let's stack together the intervals we've seen earlier:
  - Major 3rd on top of a Minor 3rd
    - *Major triad*



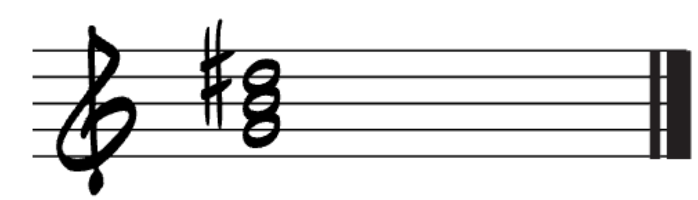
  - Minor 3rd on top of a Major 3rd
    - *Minor triad*



  - Minor 3rd on top of a Minor 3rd
    - *Diminished triad*



  - Major 3rd on top of a Major 3rd
    - *Augmented triad*



TRIADS

- Triads are known as *tertian chords*, as they use thirds to create sound instead of any other interval (n.b. below)
  - *1st inversion*: Move basis note above the top note from root position of chord
  - *2nd inversion*: Move basis note above the top note from 1st inversion of chord
  - Implication: tertian = 3rds or 6ths
- Non-tertian chords use 2nds/4ths as well
  - Not as widely used, but still exist (obviously)

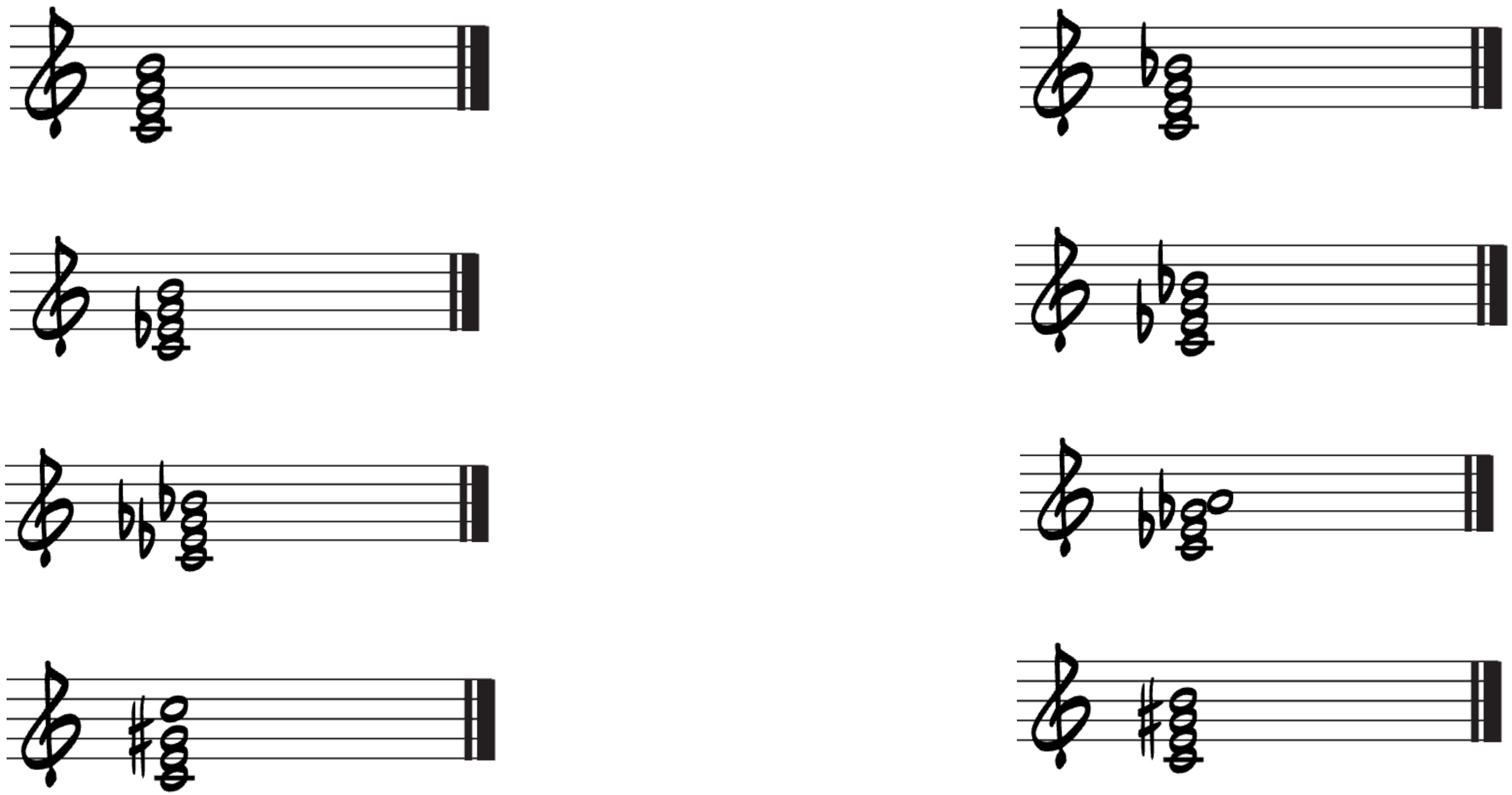
SEVENTH CHORDS

- (Tertian) *seventh chords* can be created by sticking major or minor thirds on top of (tertian) triads
- Since there are four possible tertian triads, there are $4 \times 2 = 8$ possible tertian seventh chords (since we could put either a major or a minor third above each)
- Let's take a closer look at all of the possible sevenths
  - Preview: There will actually only be 7 possible tertian seventh chords, very interestingly

SEVENTH CHORDS

- major triad + major 3rd = *major seventh*
- major triad + minor 3rd = *dominant seventh*
- minor triad + major 3rd = *minor-major seventh*
- minor triad + minor 3rd = *minor seventh*
- diminished triad + major 3rd = *half-diminished seventh*
- diminished triad + minor 3rd = *diminished seventh*
- augmented triad + major 3rd = *doubly-augmented seventh**
- augmented triad + minor 3rd = *augmented seventh*
- *the 7th above the root here is an octave, so it is effectively just an augmented triad… so it is not widely called this

SEVENTH CHORDS



FUGUE

CYCLICAL STRUCTURES

- Let us now consider the following set:

$$\mathbf{ch} = \{C, C^{\#}, D, D^{\#}, E, F, F^{\#}, G, G^{\#}, A, B^{\flat}, B\}$$

- As we know from our musical lessons, these are the twelve notes in any scale
  - Notice that this is a *set*, so we don't include the top C that completes the chromatic scale

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- As we know from our musical lessons, these are the twelve notes in any scale
  - Notice that this is a *set*, so we don't include the top C that completes the chromatic scale
- Another way to write this set might be:

$$\textbf{ch} = \{\mathrm{P1, m2, M2, m3, M3, P4, d5, P5, m6, M6, m7, M7}\}$$

where P = perfect, m = minor, M = major, d = diminished

CYCLICAL STRUCTURES

- Start on middle-C and ascend chromatically
  - Eventually, the notes generated will be part of the same sequence as one that occurred before
  - The notes on a piano are thus *cyclical*
  - They are a *modular system*, much like the decimal one
- Can we define an operation on this chromatic set so we can make it a group?
  - Does such an operation exist? If so, is it unique?
  - If yes to both, can we prove that the set and operation together actually *do* form a group?

MUSICAL GROUPS

- Let's consider the following binary operation on **ch**:

  $a \bullet b$ ::= the note that is an interval of b above a
- For example, let us suppose that a = P4 and b = m3
  - Then, $a \bullet b$ = P4 • m3 = m6
- Another way to think of this is via an *isomorphism* to the addition operation in Z_{12} (i.e. the integers from 0 to 11 only):

  ch = {P1, m2, M2, m3, M3, P4, d5, P5, m6, M6, m7, M7}

  ~ {0 , 1 , 2 , 3 , 4 , 5 , 6 , 7 , 8 , 9 , 10 , 11 }
- Then, our example above is quite nice, as • is simply + modulo 12: P4 • m3 ~ $5 +_{12} 3 = (5 + 3) \bmod 12 = 8 \bmod 12$ ~ m6
- Why does the isomorphism work? *Hint: half-step counts in* **ch**

MUSICAL GROUPS

- So what would the identity element be?
  - Addition, so a good idea seems to be 0 ~ P1 for **id**
- Is (**ch**, •, P1) a valid group definition? Alternatively, is (Z_{12}, $+_{12}$, 0) a valid group definition?
  - Yes!
  - Let's prove it…

MUSICAL GROUPS

- Have you learned? Help fill in the formal details:
- *Closure*:
  - **Intuition**: No matter what we do, we end up somewhere on the keyboard, which is still in **ch**
- *Associativity*:
  - **Intuition:** We can stack in any order to get to an interval
- *Identity*:
  - **Intuition:** Every note on the keyboard (**ch**) is its own P1
- *Inverse*:
  - **Intuition:** Maybe we can extend the principle of inv. …

MUSICAL GROUPS

- It turns out that • is also commutative over **ch**, so **ch** is actually an abelian group
  - Intuition: Both paths will still lead to the same sum
- Well, we have a musical group!
  - What now?
  - Some interesting properties:
    - $P1 = P1^{-1}$ and $d5 = d5^{-1}$
    - **Cyclical property**
    - **Permutation property**

CYCLICAL GROUPS

- A *cyclical group* is a group $G = (S, \bullet, \mathbf{id})$ that contains a *generating element* g for which if $\bullet$ is applied to it in succession, the resulting sequence enumerates all of the elements of the group at least once before cycling
  - What does this mean?
  - In other words, $(((g \bullet g) \bullet g) \cdots) \bullet g = s_g$, where for all e in S, e is enumerated by the sequence s_g
- If $G = (\mathrm{S}, \bullet, \mathbf{id})$ is cyclical with generating element g, then we write $G = \langle g \rangle$ over $\bullet$
- Applications: g is an *enumeration machine* of G under $\bullet$

CYCLICAL GROUPS

- For example, $G = (Z_3, +_3, 0)$, where $Z_3 = \{0, 1, 2\}$
  - There are actually two generating elements of G:
    - $G = \langle 1 \rangle$ over $+_3$:
      $1 +_3 1 = \mathbf{2} +_3 1 = \mathbf{0} +_3 1 = \mathbf{1} \ldots$
    - $G = \langle 2 \rangle$ over $+_3$:
      $2 +_3 2 = \mathbf{1} +_3 2 = \mathbf{0} +_3 2 = \mathbf{2} \ldots$
    - Notice that $G \neq \langle 0 \rangle$ over $+_3$:
      $0 +_3 0 = 0 +_3 0 = 0 +_3 0 = 0 \ldots$
- **Intuition:** In general, $g_G \neq \mathbf{id}$ unless $S = \{\mathbf{id}\}$ (*Why?*)

CYCLICAL GROUPS

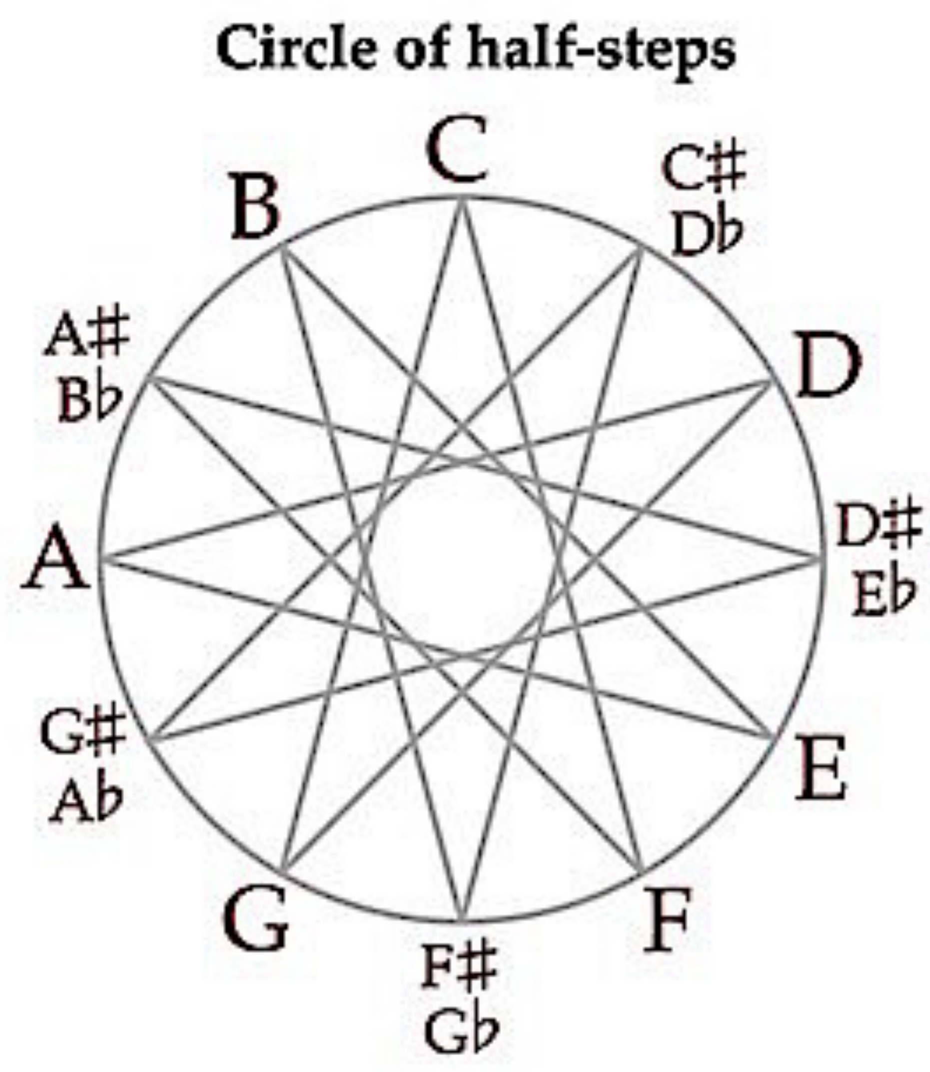
- As it turns out, **ch** is cyclic with these generators:
  - **ch** $= \langle 1 \rangle$ over $\bullet$:
    $s_1 = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 0, 1, \ldots$
  - **ch** $= \langle 5 \rangle$ over $\bullet$:
    $s_5 = 5, 10, 3, 8, 1, 6, 11, 4, 9, 2, 7, 0, 5, \ldots$
  - **ch** $= \langle 7 \rangle$ over $\bullet$:
    $s_7 = 7, 2, 9, 4, 11, 6, 1, 8, 3, 10, 5, 0, 7 \ldots$
  - **ch** $= \langle 11 \rangle$ over $\bullet$:
    $s_{11} = 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0, 11 \ldots$
- Interestingly, they all end on 0 (our **id**) before cycling

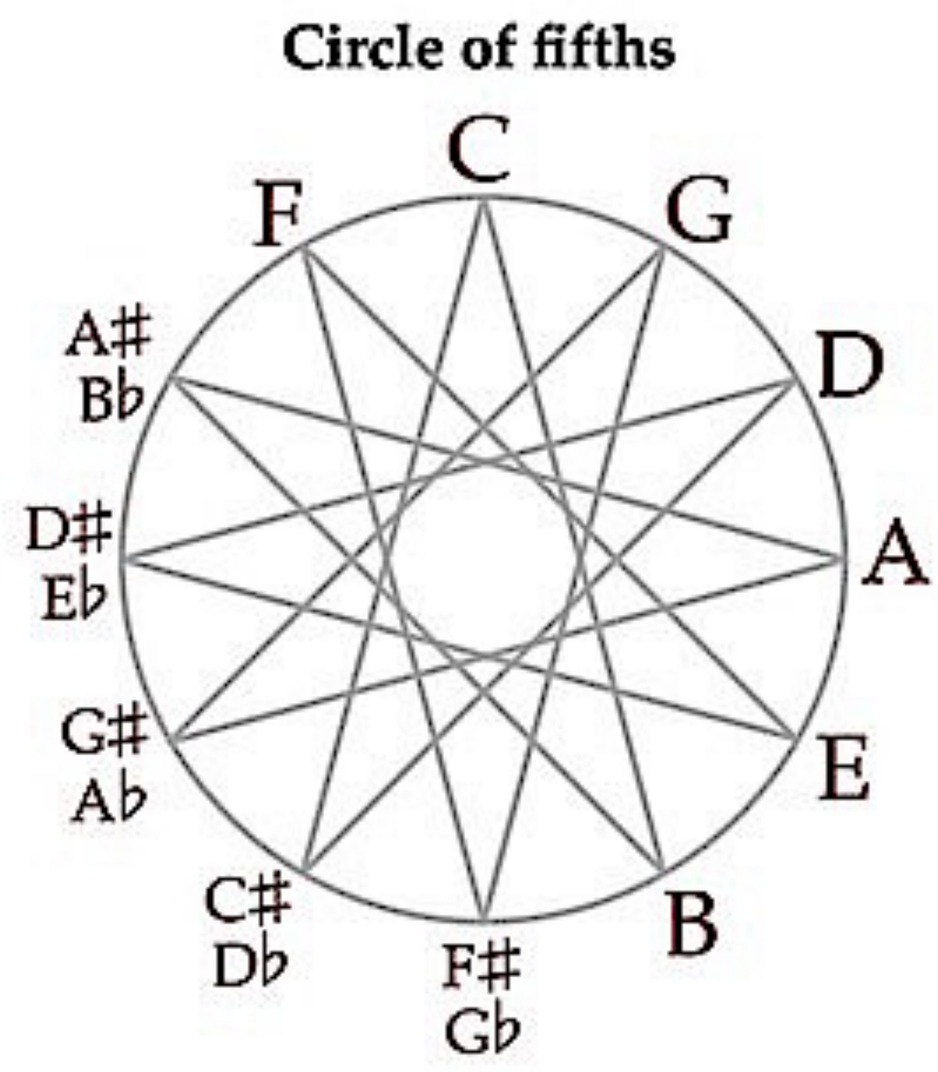
CYCLICAL GROUPS

- Removing the isomorphism abstraction, we see that
$$\mathbf{ch} = \langle \mathrm{m2} \rangle = \langle \mathrm{P4} \rangle = \langle \mathrm{P5} \rangle = \langle \mathrm{M7} \rangle$$
- Interesting properties:
  - The m2 generates an increasing chromatic scale
  - The M7 generates a decreasing chromatic scale
  - The P5 generates the circle of fifths
    - $C \to G \to D \to A \to E \to B \to F^{\#} \to C^{\#} \to G^{\#} \to D^{\#} \to A^{\#} \to F \to C$
  - The P4 generates the circle of fourths
    - $C \to F \to B^{\flat} \to E^{\flat} \to A^{\flat} \to D^{\flat} \to G^{\flat} \to B \to E \to A \to D \to G \to C$
- These generators directly hint at the invertibility of **ch**

CYCLICAL GROUPS

- The circles also both generate each other!
  - Reveals a deep connection in the underlying group





Pachelbel's [Canon in D]() is a popular work that employs what is essentially a circle of fifths to drive its foundation.

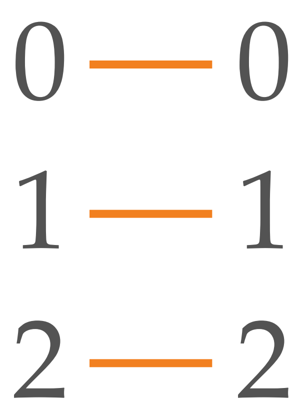
Bass clef:
$D \rightarrow A$
$B \rightarrow F^{\#}$
$G \rightarrow D$

Treble clef:
$F \rightarrow C$
$D \rightarrow A$
$B \rightarrow F^{\#}$

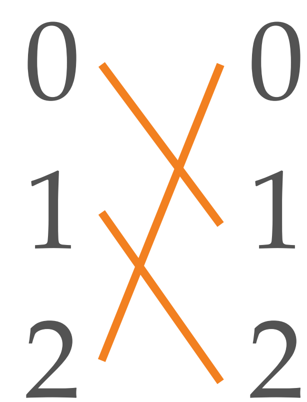
PERMUTATION GROUPS

- A *permutation* is a rearrangement of elements in a group
- For example, let's look at Z_3 again:

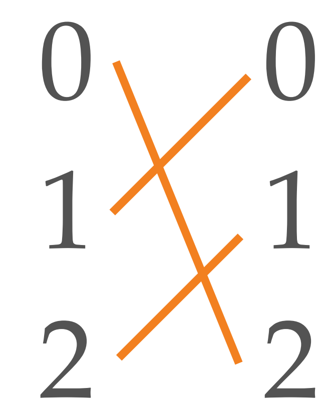
id



α



α^2



- Cauchy notation:

$$\mathbf{id} = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} \qquad \alpha = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix} \qquad \alpha^2 = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}$$

PERMUTATION GROUPS

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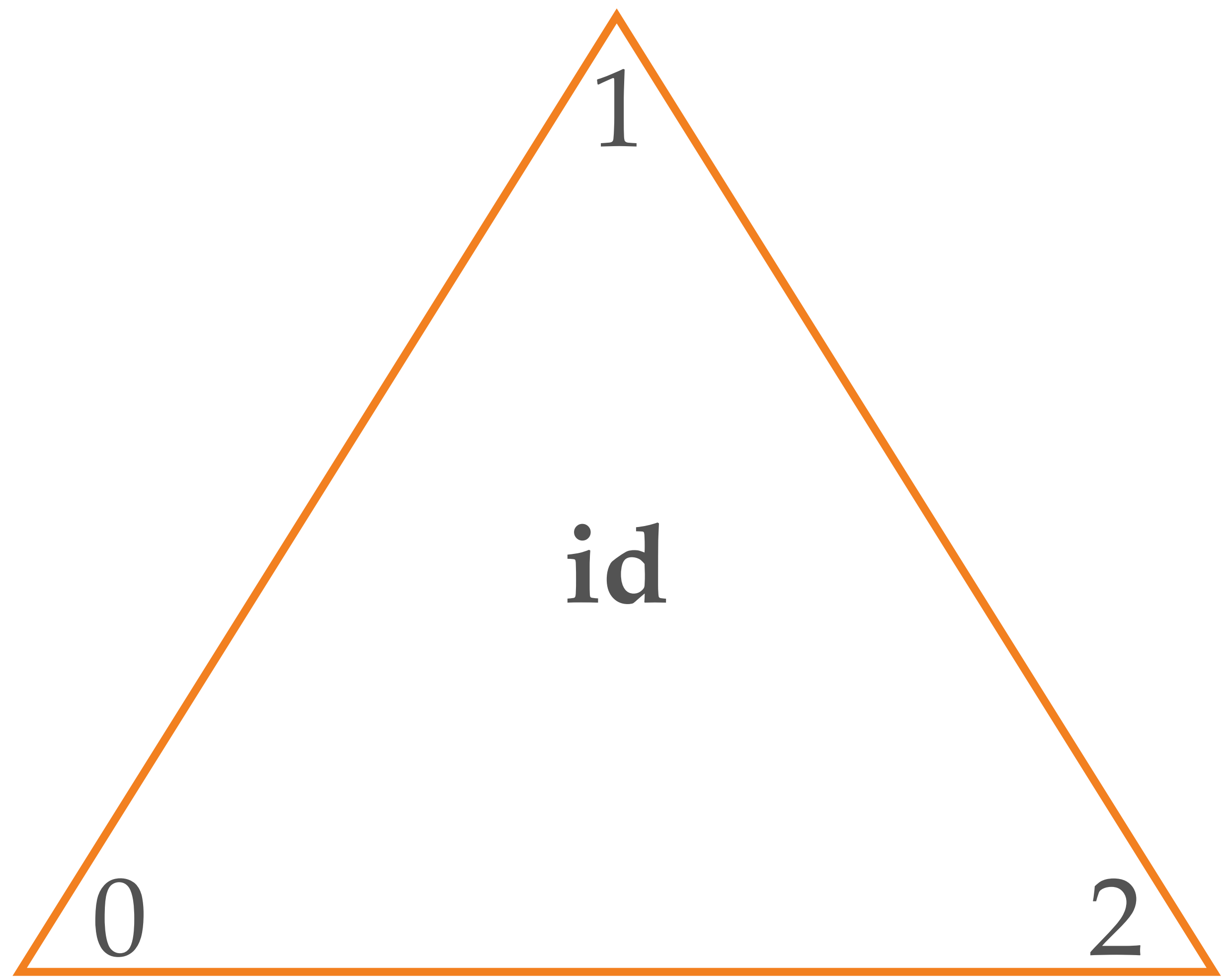

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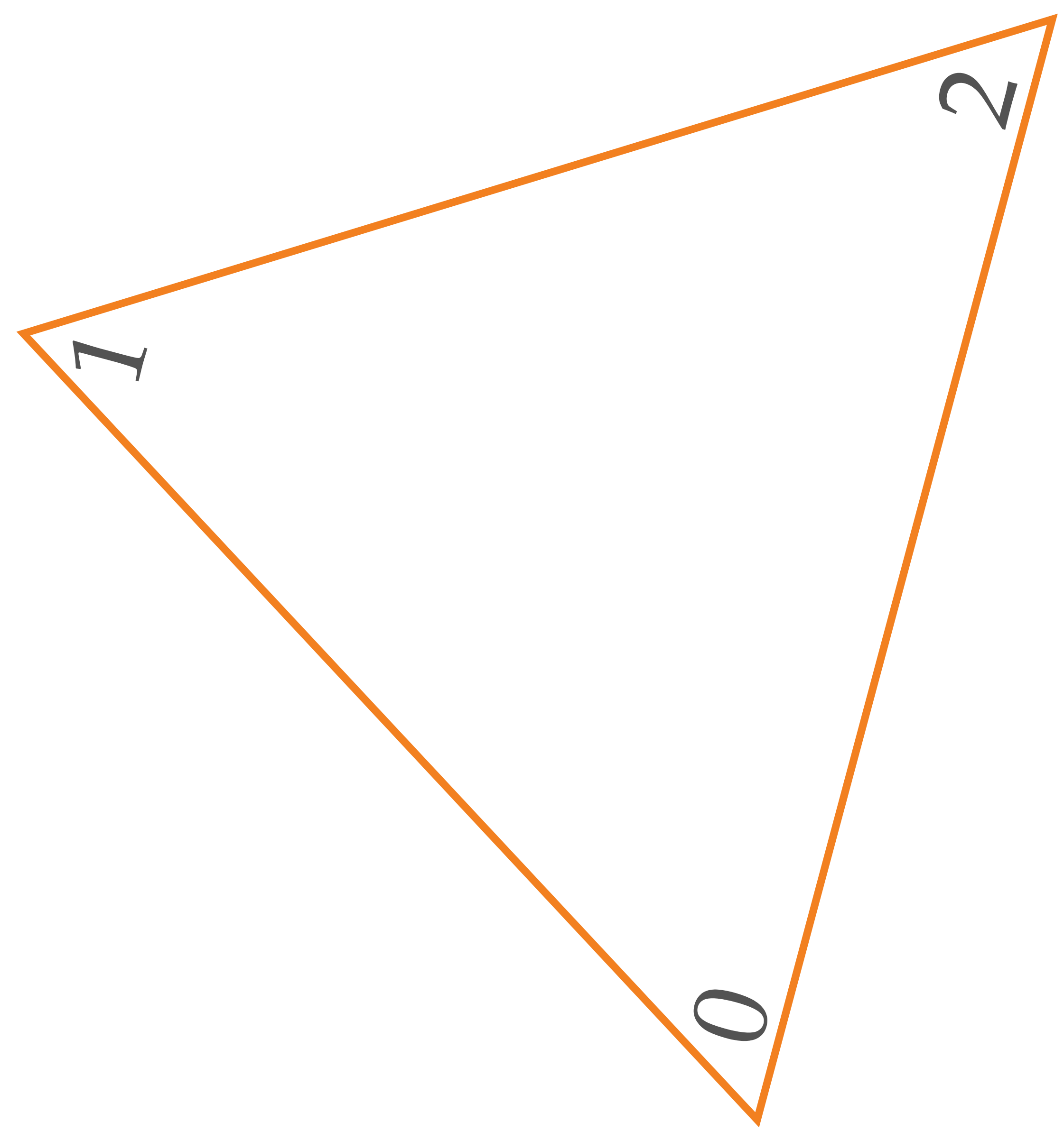

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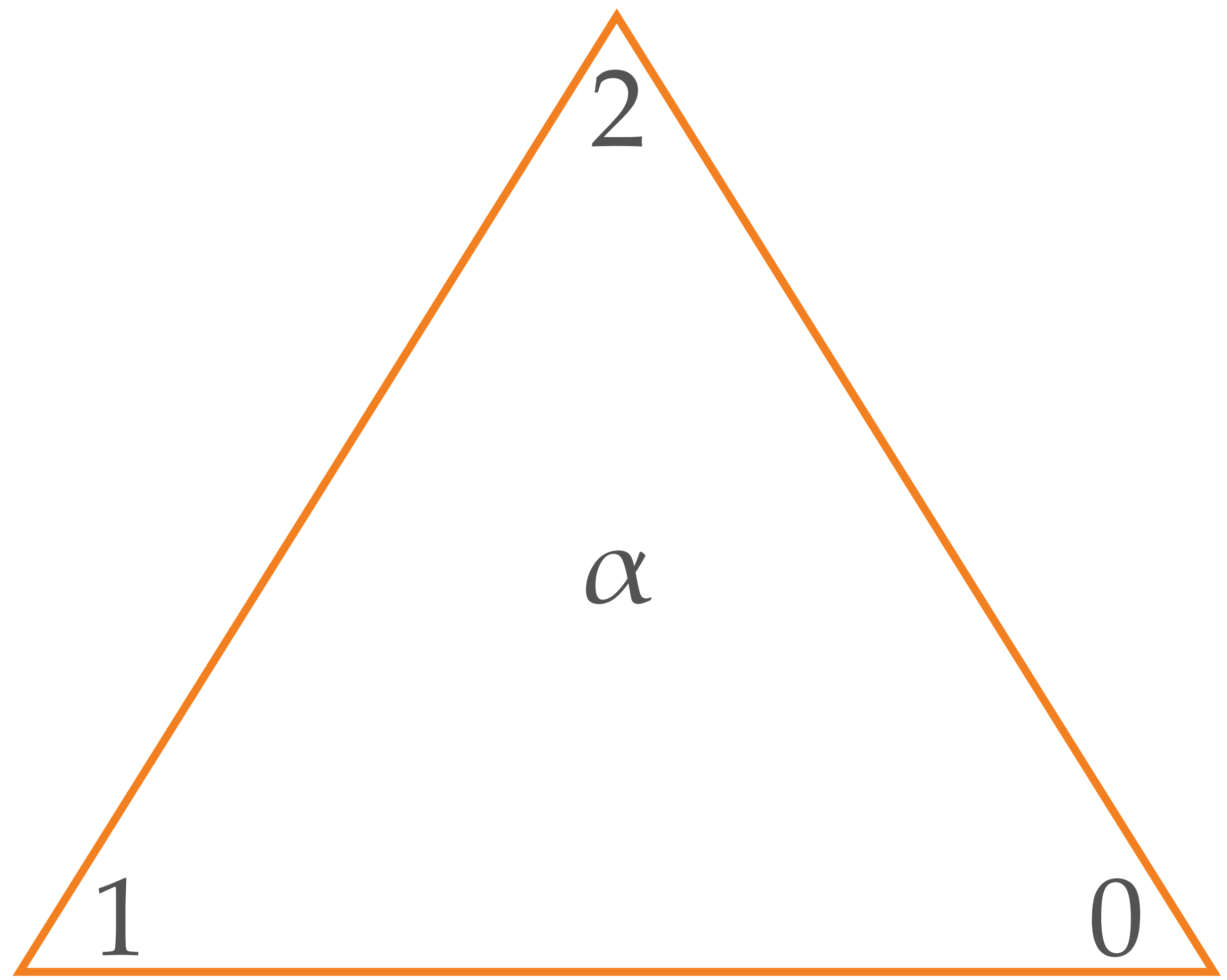

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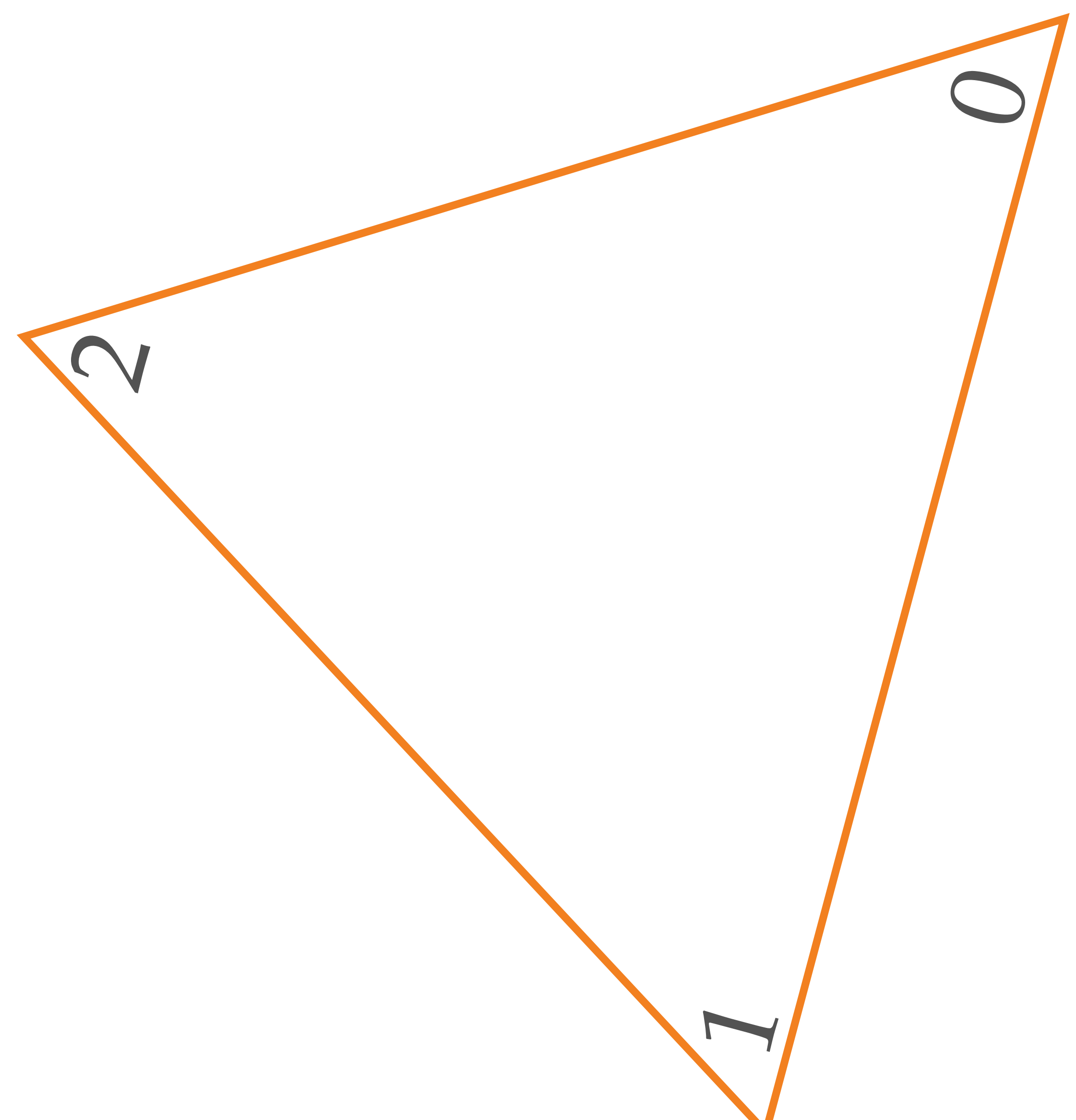

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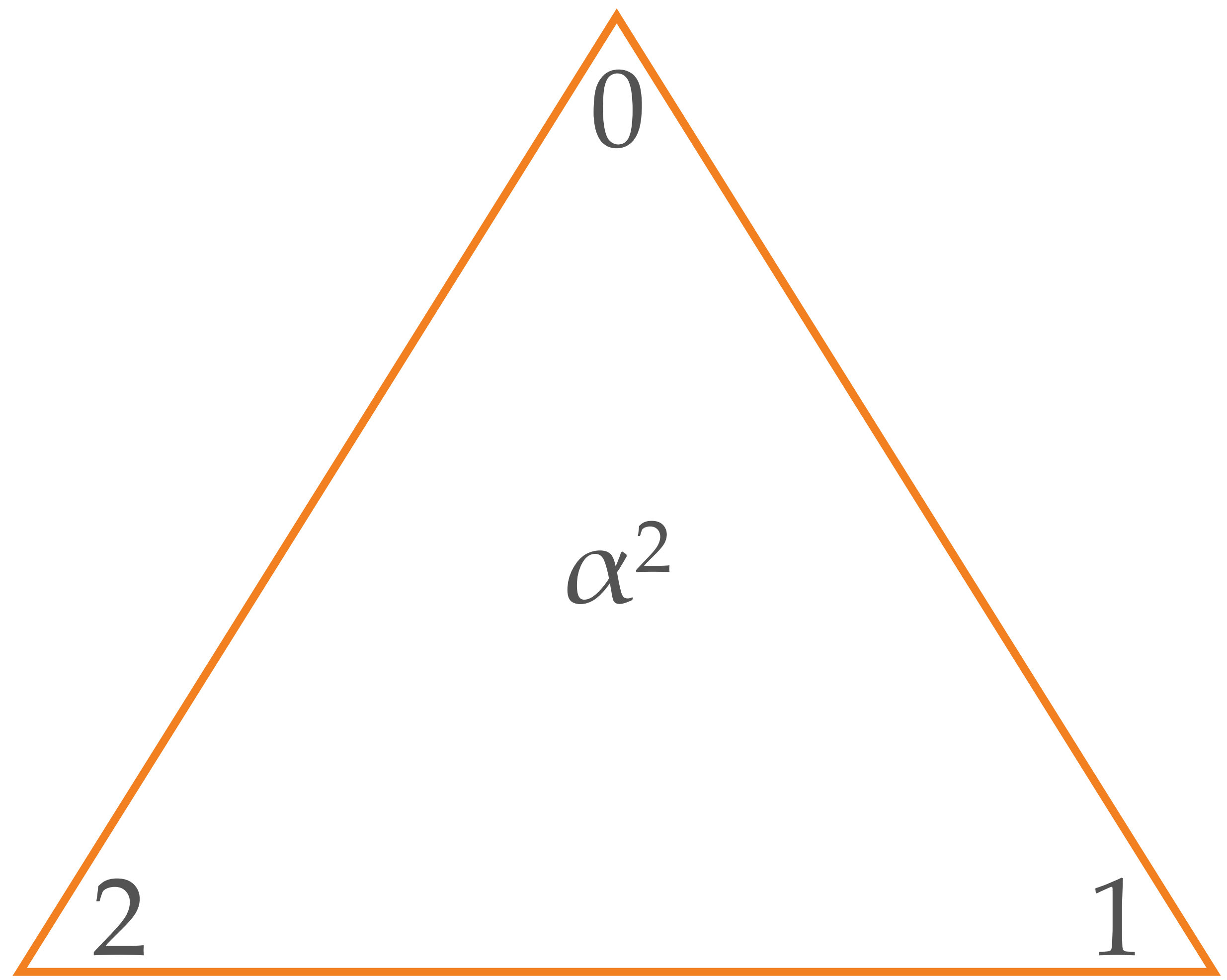

PERMUTATION GROUPS

- A *permutation* is a rearrangement of elements in a group
- Clearly, those were *rotation* permutations
- What about *reflection* permutations?
  - One number stays the same, the other swap places
  - That number is the *reflection axis*

PERMUTATION GROUPS

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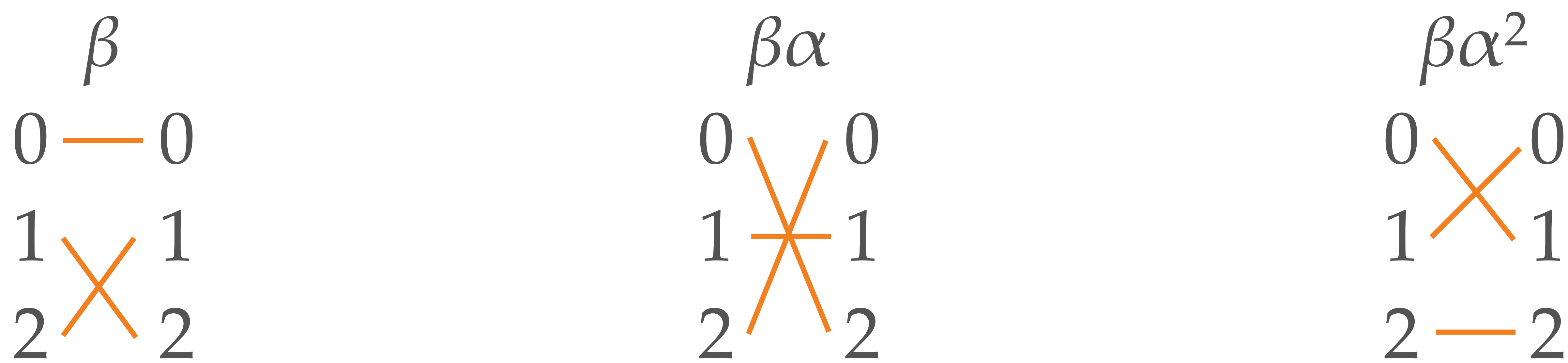

- Cauchy notation:

$$\beta = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix} \qquad \beta\alpha = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \end{pmatrix} \qquad \beta\alpha^2 = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix}$$

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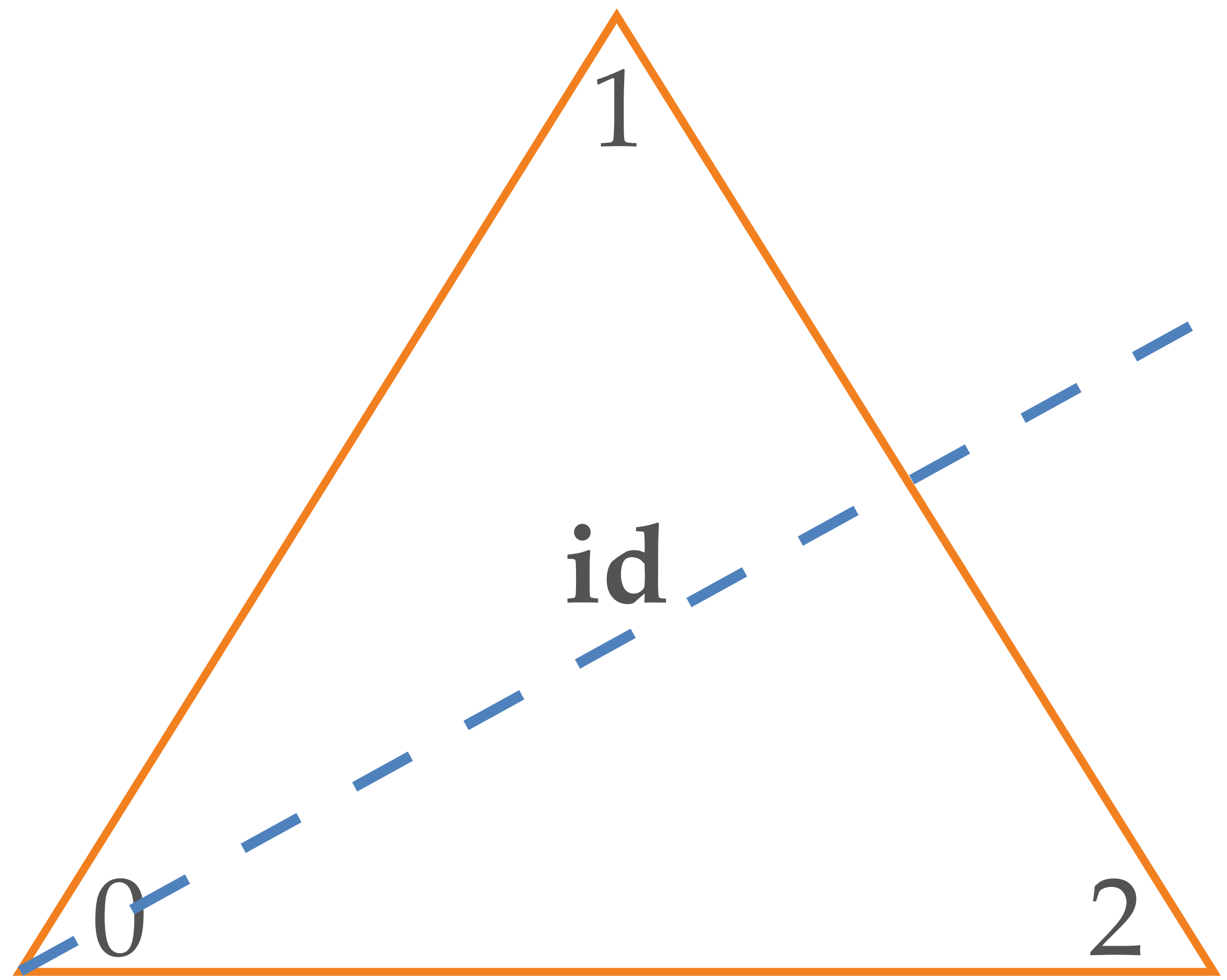

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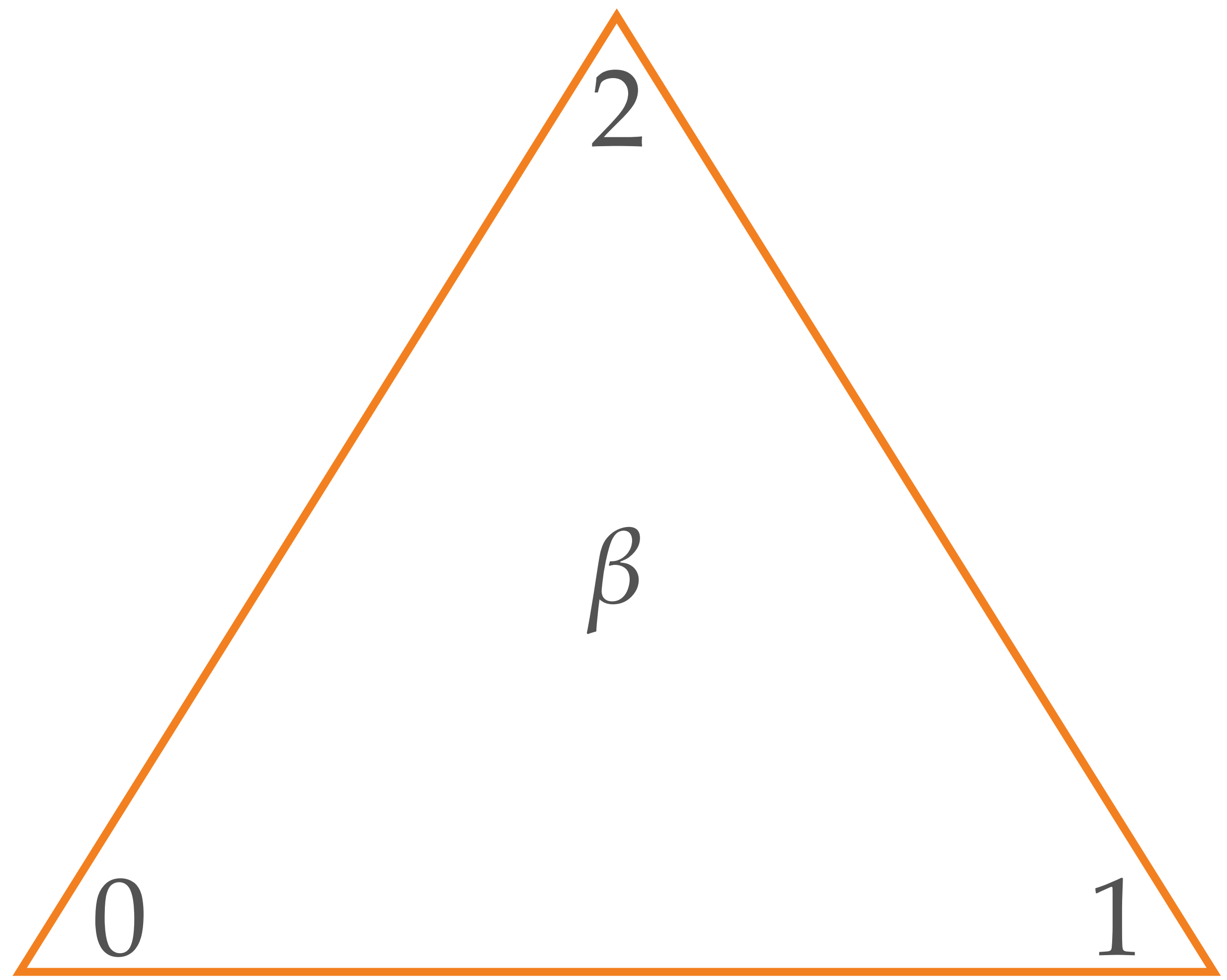

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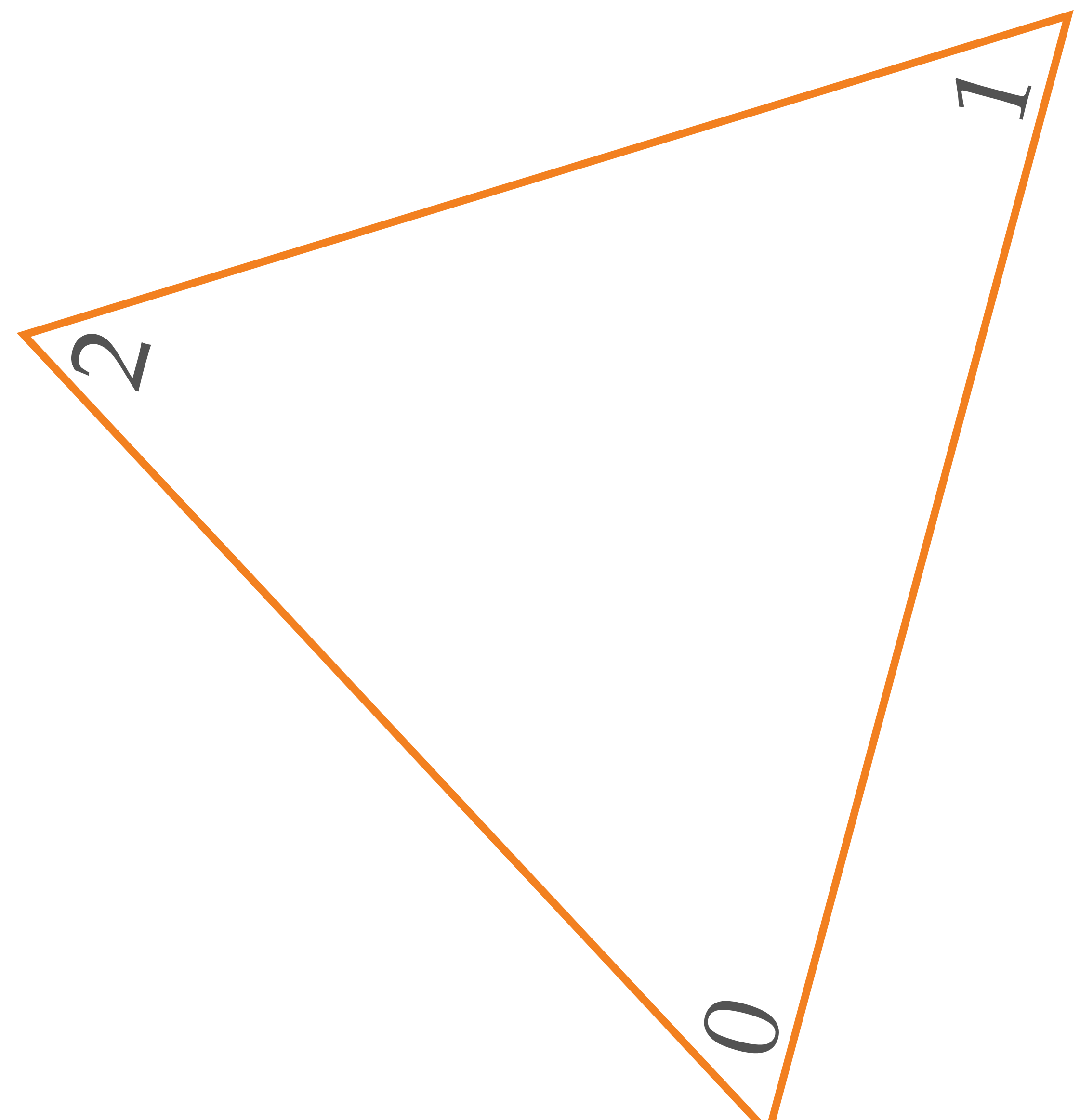

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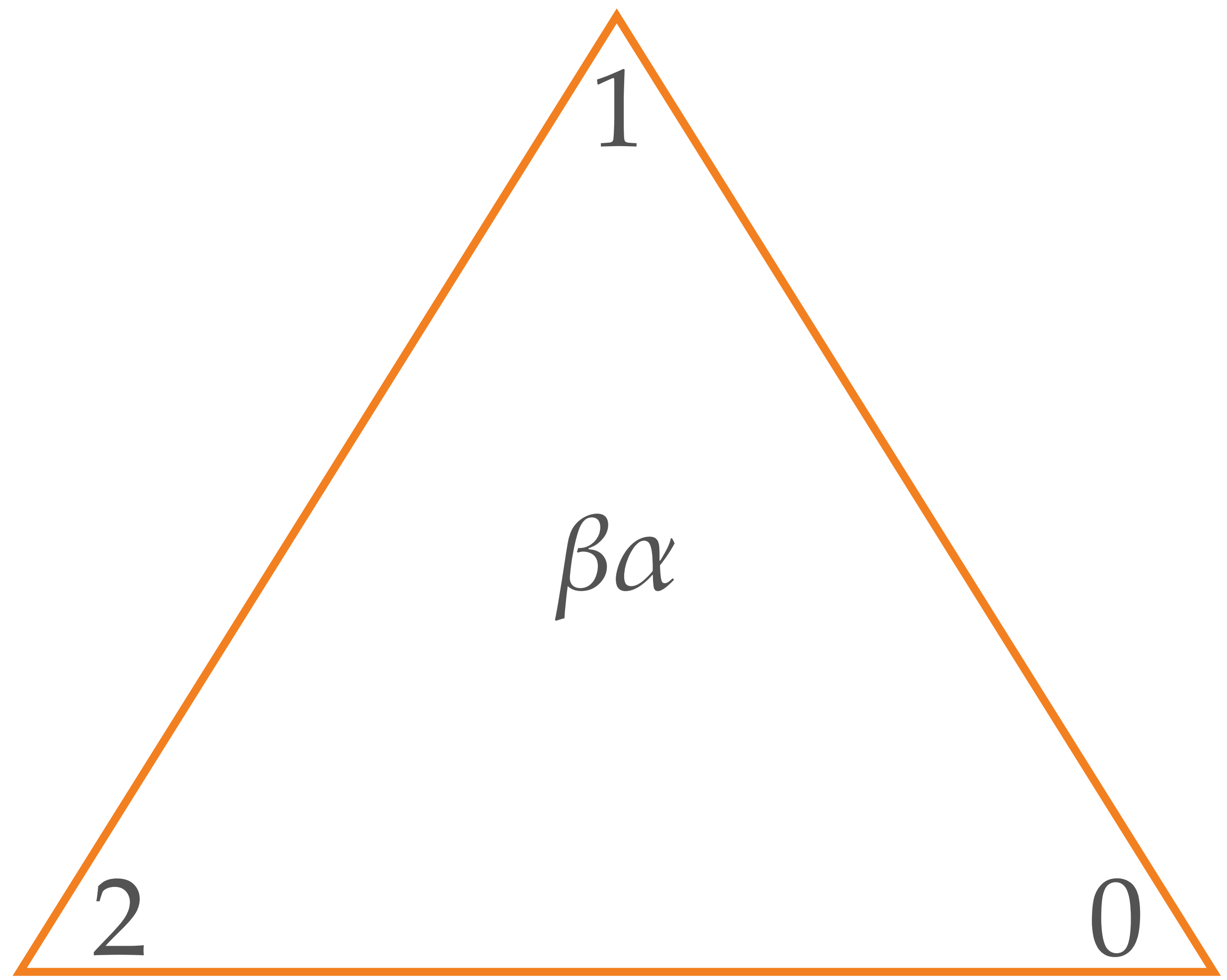

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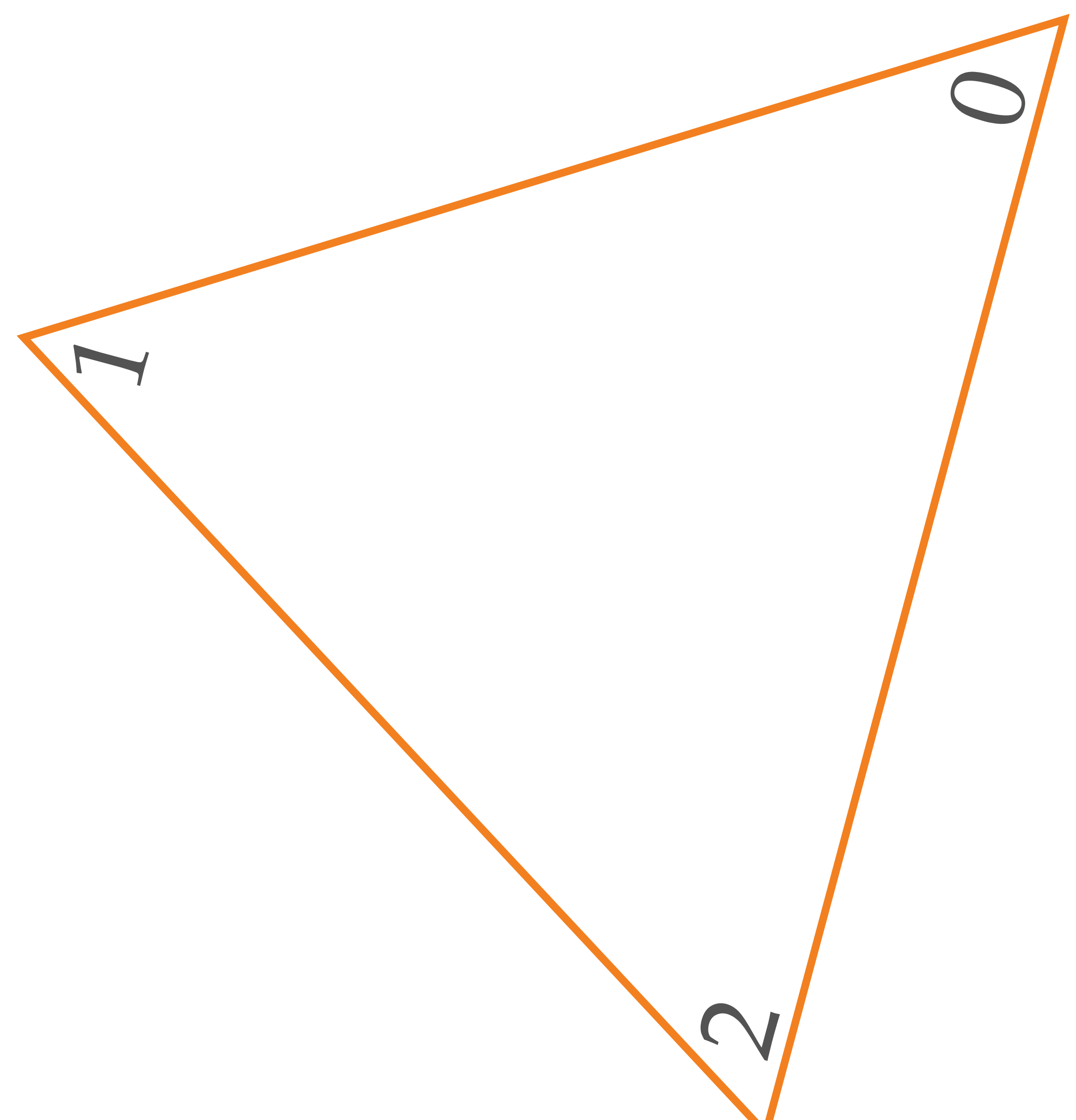

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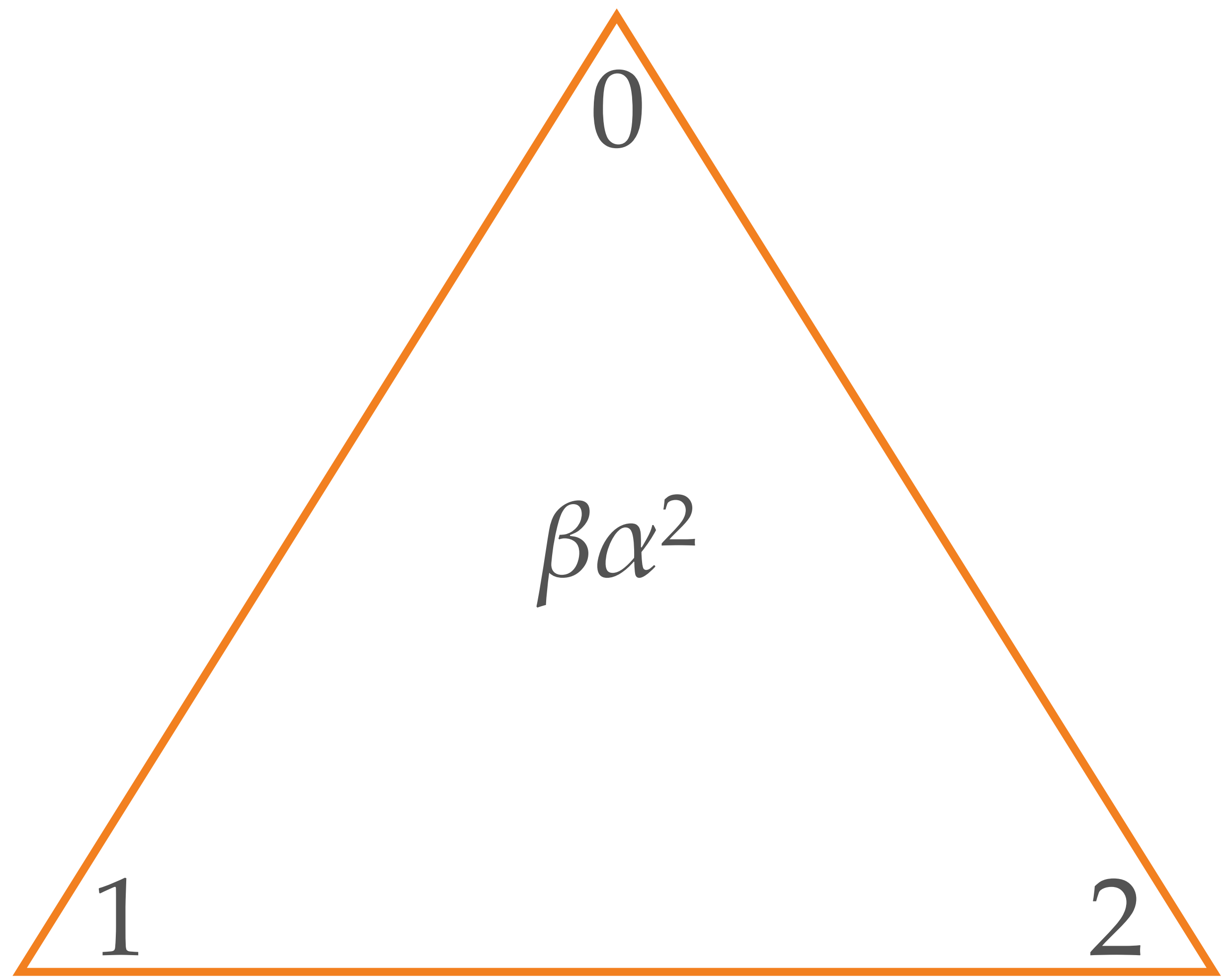

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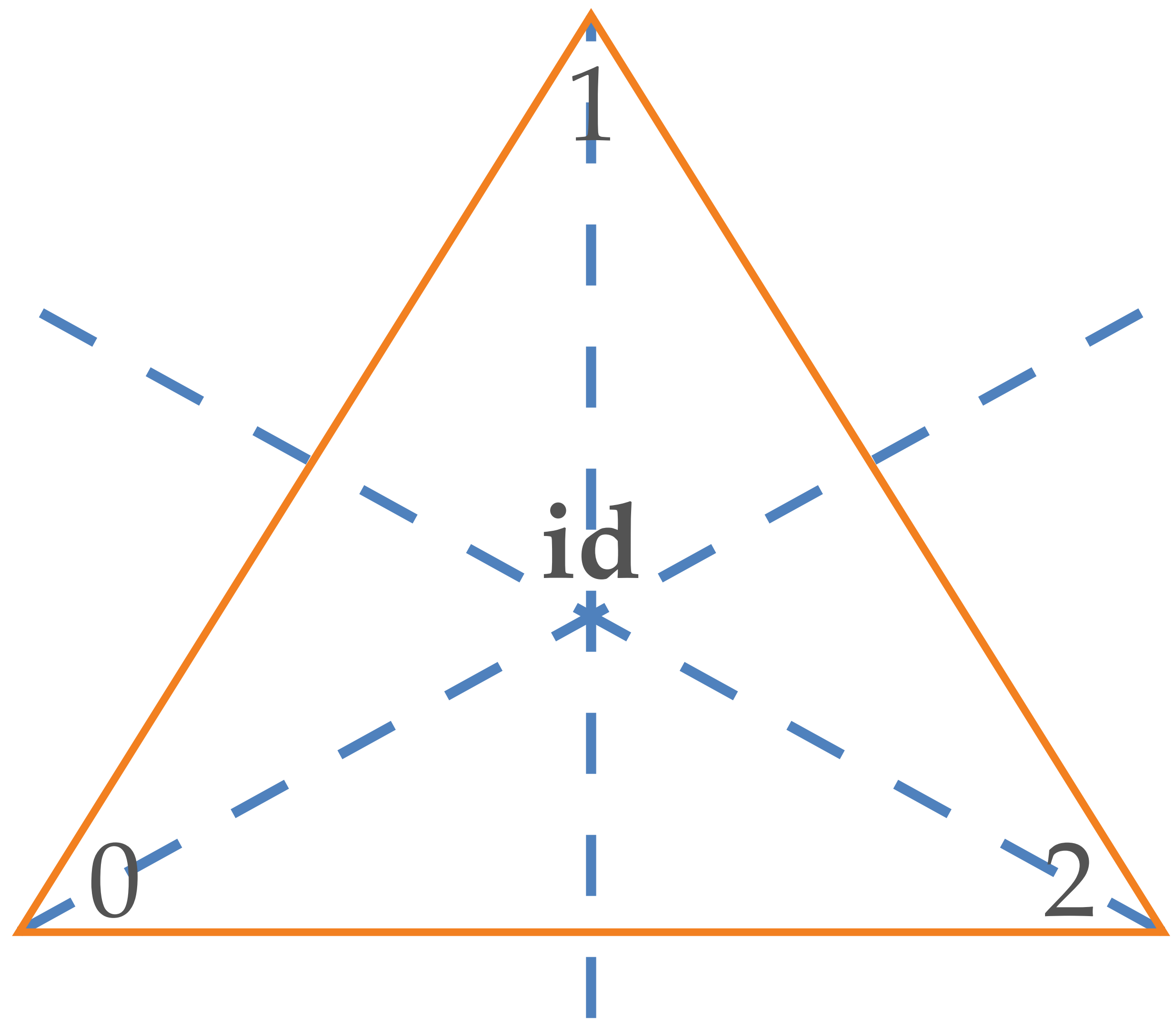
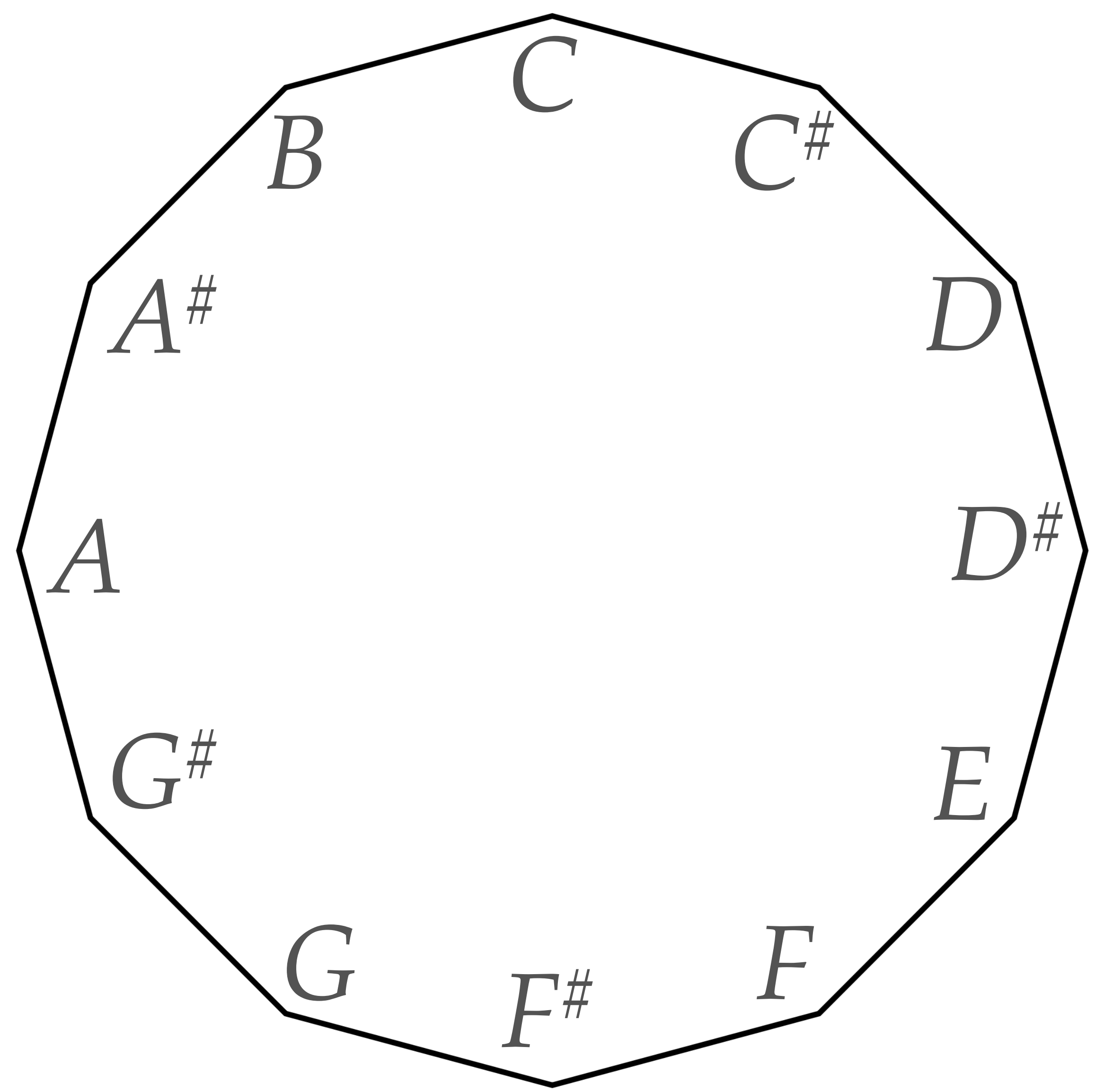


PERMUTATION GROUPS

- A *permutation* is a rearrangement of elements in a group
- Those were a combination of reflection and rotation
  - i.e. a permutation
  - No translations in our model (doesn't make sense)
  - Thus, permutation groups are a subset of *symmetry groups* (which offer translation + reflection + rotation)

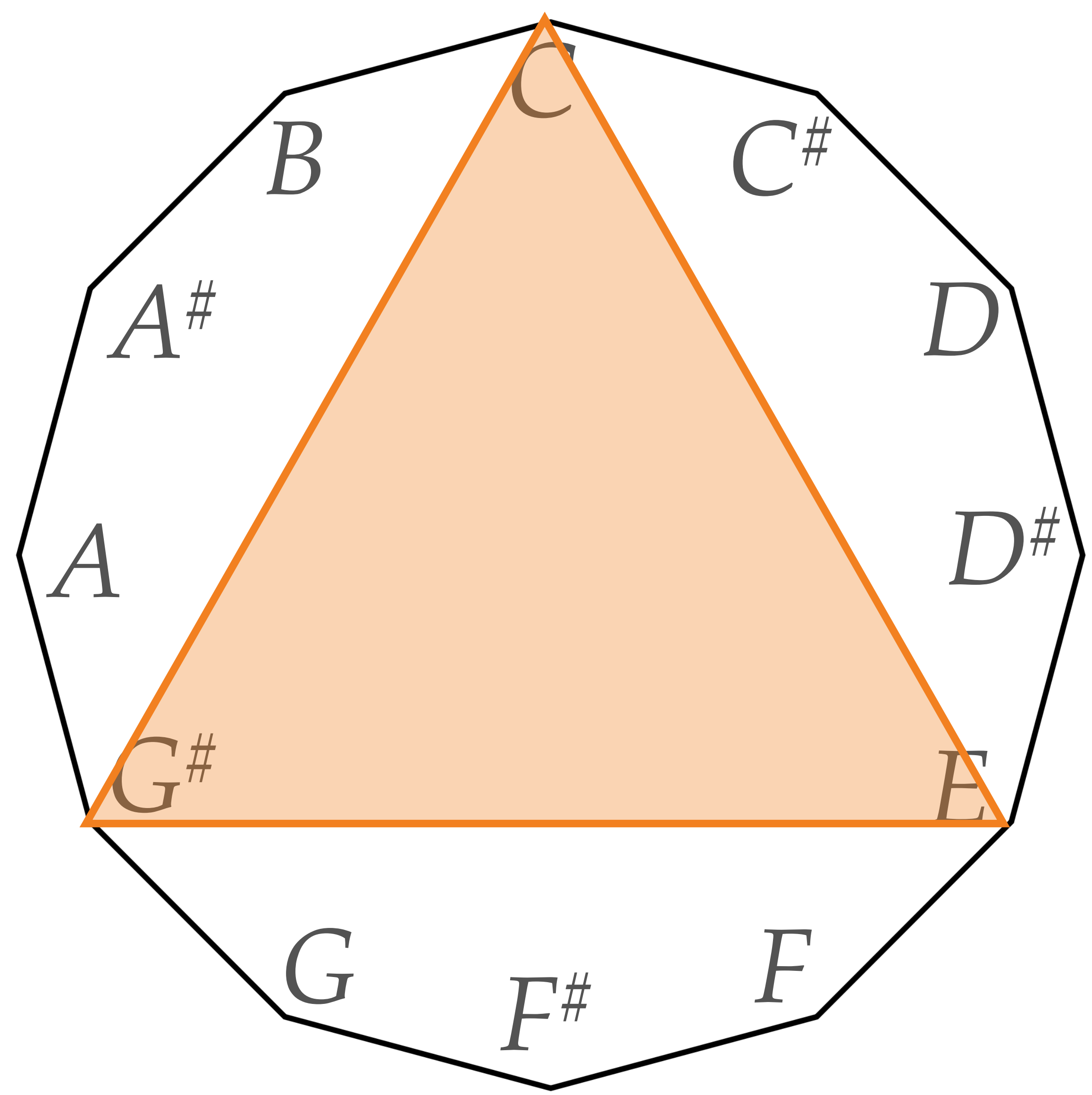
PERMUTATION GROUPS

- Using m2 as a generator, we can create a cyclic group that is representable as a regular dodecagon:



PERMUTATION GROUPS

- We can then overlay regular triangles to obtain chords in identity form:



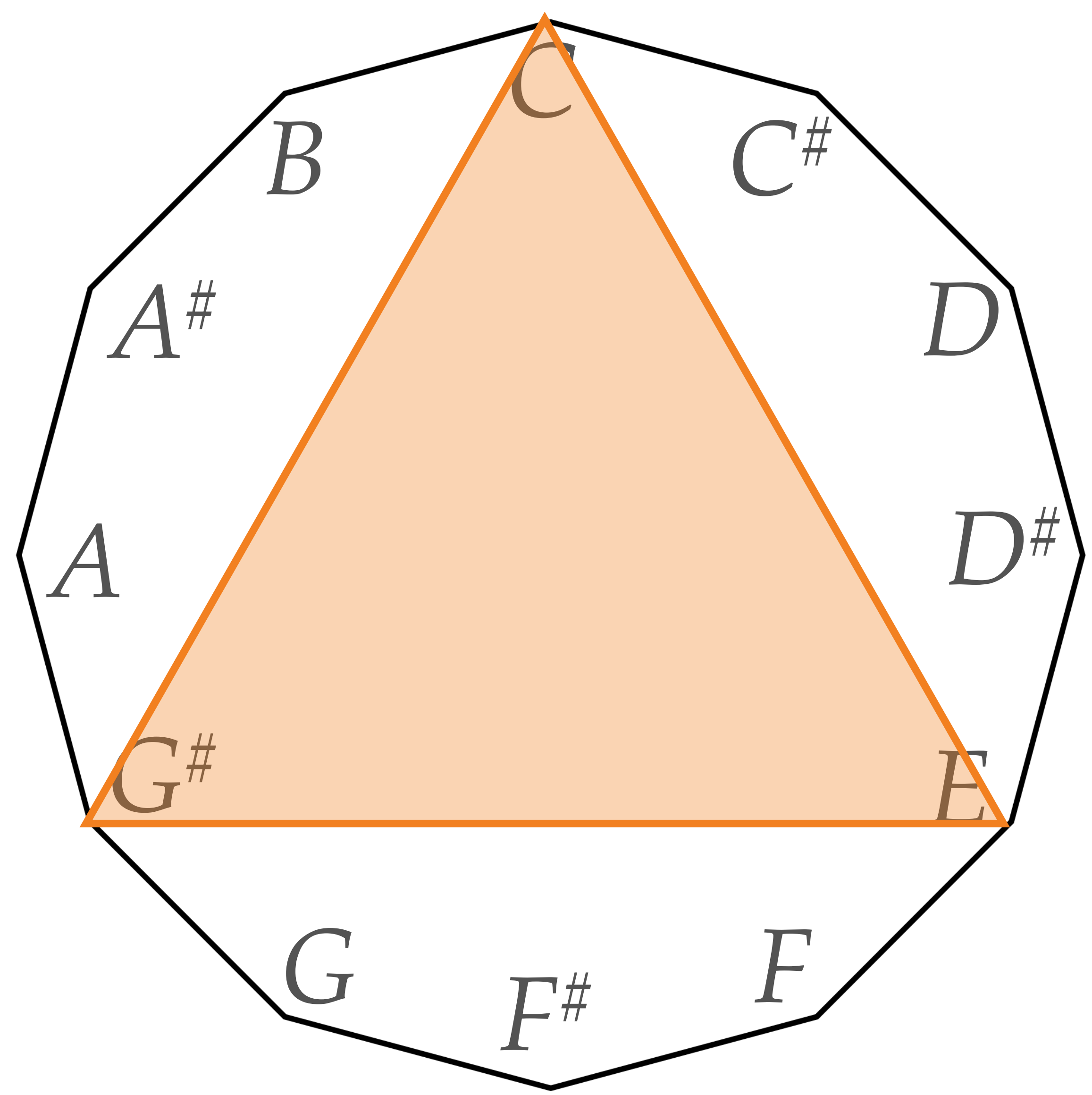
PERMUTATION GROUPS

- Permutations of this triangle via the process described earlier lead to very interesting results:

id = augmented triad in root position

α = 1st inversion

α^2 = 2nd inversion



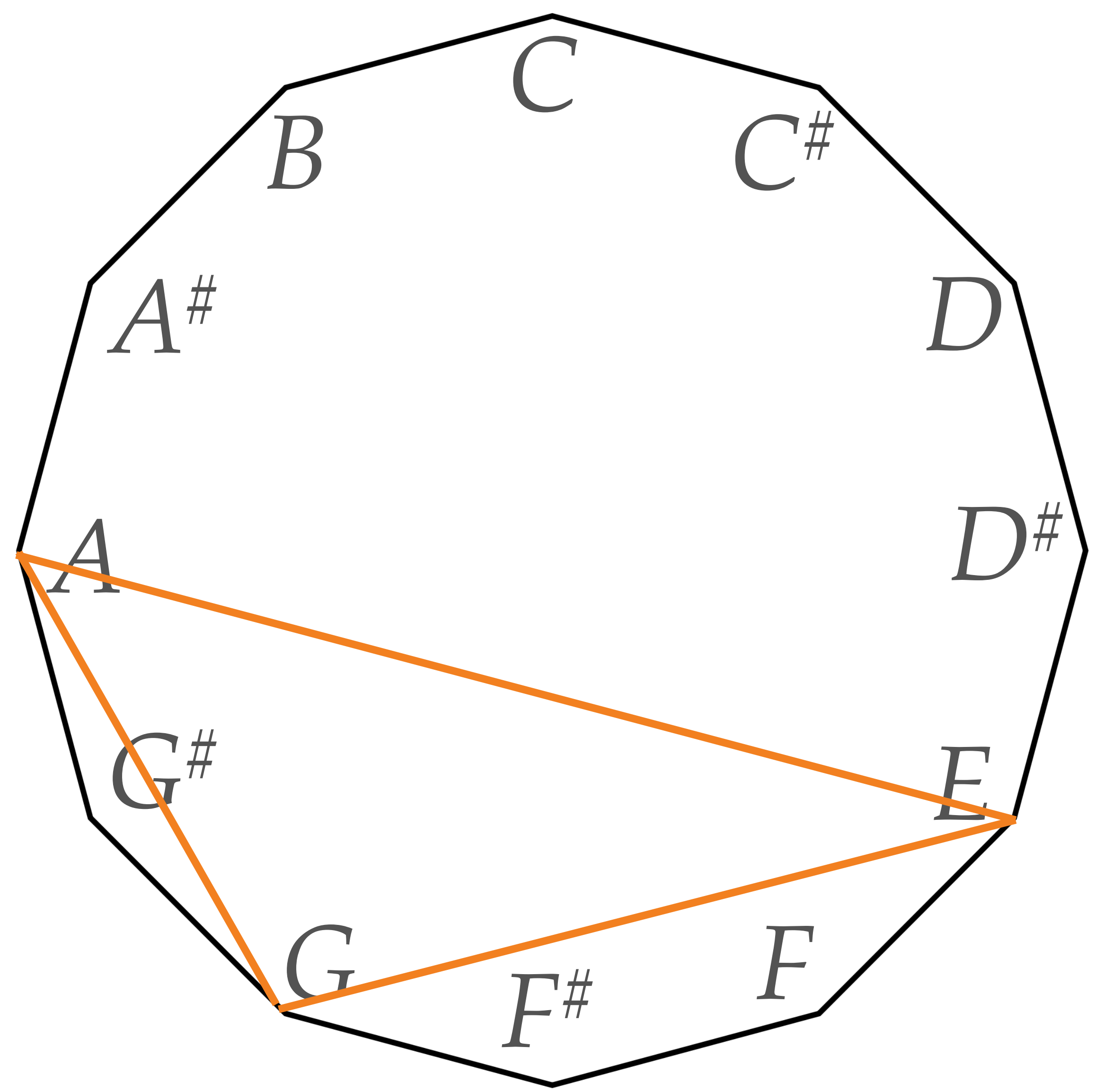
PERMUTATION GROUPS

- If we allow *dilations* in the permutation group, the triangle can cover all of the chords!

id = augmented triad in root position

α = 1st inversion

α^2 = 2nd inversion



CLOSING REMARKS

CLOSING REMARKS

- These notes are merely an *introduction* to the subject
  - We could easily talk more… but we only had 2 hours!
- The material discussed here could easily be the content of a masters' thesis or even of a small academic paper
  - More mathematical flavor
  - More in-depth analysis
  - Greater variety of applications
  - Structural analysis based on mathematical precepts

CLOSING REMARKS

- If you have any further questions, feel free to reach out!
- My email address is **cb625@cornell.edu**
    - May take some time to reply though
    - We are all busy students! (Especially potential grad students…)
    - Check out my website: **chiragbharadwaj.com**
- I wish you the best of luck with future studies!
    - *Did this class interest you?* Consider a math major or music major at university! Talk to people and join a research group.
    - Take a related class at high school through local colleges
    - Think ahead, plan for a little while, even if you're still young


Hope you had fun!
Splash! Fall 2016