NAME: %: 7 January 2013

DECA CHAPTER 6 TEST

RETURN THE TEST WITH YOUR ANSWERS!

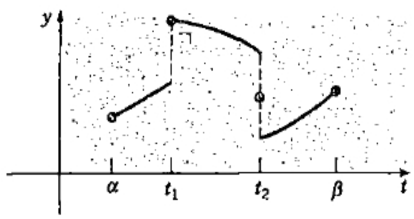
1. Evaluate the improper integral $\int_0^\infty e^{ct}\,dt$, where c is a constant.

A. For what values of c does the integral converge? What does the integral converge to?

B. For what values of c does the integral diverge?

2. A. What does it mean for a function to be *piecewise continuous* over an interval?

B. Is the function graphed below a piecewise continuous function? Why or why not?



3. An integral transform $F(s) = \int_\alpha^\beta K(s,t) f(t)\,dt$ is essentially a linear transformation in $\mathbb{R}^2$.

A. Using your knowledge from linear algebra, what is the name of the operator $K(s,t)$ of the linear transformation?

B. If the integral transform is a *Laplace transform*, then what does $K(s,t)$ equal?

4. Prove that the Laplace transform is a linear transform using your knowledge from linear algebra.

5. Find the Laplace transform of t^n, where n is a positive integer. Can you make a similar case if n is a negative integer? Explain why or why not.

6. The *gamma function* is denoted by $\Gamma(p)$ and is defined by $\Gamma(p+1) = \int_0^\infty e^{-x} x^p\,dx$. As $x \to \infty$, the integral converges for all values of p. The integral can also be shown to converge at $x = 0$ for $p > -1$ despite the fact that the integrand becomes unbounded as $x \to 0$. Answer the following questions about $\Gamma(p)$.

A. Show that if $p > 0$, $\Gamma(p+1) = p\Gamma(p)$.

B. Show that $\Gamma(n+1) = n!$ for a positive integer n.

In Questions 7-10, prove each of the functions $f(t)$ shown have the Laplace transform $\Im[f(t)]$ in the table.

$f(t)$	$F(s) = \Im[f(t)]$
$\sinh at$	$\dfrac{a}{s^2 - a^2}$
$\cosh at$	$\dfrac{s}{s^2 - a^2}$
$e^{at}\sin bt$	$\dfrac{b}{(s-a)^2 + b^2}$
$e^{at}\cos bt$	$\dfrac{s-a}{(s-a)^2 + b^2}$

7. See table.

8. See table.

9. See table.

10. See table.

11. Using the Maclaurin series for $\sin t$, find the Laplace transform for $f(t) = \sin t$ by applying the transform term-by-term to the Maclaurin series expansion.

12. Repeat Problem 11 for $f(t) = \cos t$.

13. Solve the differential equation $y'' + \omega^2 y = \cos 2t$ with initial conditions $y(0) = 1$ and $y'(0) = 0$ assuming that $\omega^2 \neq 4$.

14. Using the fact that a Laplace transform is a linear transformation and your knowledge from linear algebra, prove that the inverse Laplace transform of $G(s) = \dfrac{1}{s^2 - 4s + 5}$ is $g(t) = \Im^{-1}[G(s)] = e^{2t}\sin t$.

15. Find the inverse Laplace transform of $F(s) = \dfrac{2^{n+1} n!}{s^{n+1}}$.

16. Find the Laplace transform of the Dirac delta function, which is defined by the conditions $\delta(t - t_0) = 0;\quad t_0 \neq 0$ and $\int_{-\infty}^{\infty} \delta(t - t_0)\,dt = 1$. In other words, prove that $f(t_0) = \int_{-\infty}^{\infty} \delta(t - t_0) f(t)\,dt$.

Hint: You may have to use l'Hospital's rule as well as the Mean Value Theorem for integrals.

17. Suppose the *convolution* $h(t)$ of two functions f and g is $h(t) = \int_0^t f(t-\tau) g(\tau)\,d\tau = \int_0^t f(\tau) g(t-\tau)\,d\tau$. In common notation, this is written $h(t) = (f * g)(t)$.

A. Prove the commutative law, $f * g = g * f$.

B. Prove the distributive law, which is given by $f * (g_1 + g_2) = f * g_1 + f * g_2$.

C. Prove the associative law, which is given by $(f * g) * h = f * (g * h)$.

D. Prove the zero product law, $f * 0 = 0 * f = 0$.

Notice that they are **laws (theorems)**, not postulates.

18. Find the inverse Laplace transform of $F(s) = \dfrac{G(s)}{s^2 + 1}$ using the convolution theorem.

19. If $f(t) = t^m$ and $g(t) = t^n$, where m and n are positive integers, show that the convolution $f * g$ is $f * g = t^{m+n+1} \int_0^1 u^m (1-u)^n\,du$.

20. Use the convolution theorem to do the opposite of Problem 19: prove $\int_0^1 u^m (1-u)^n\,du = \dfrac{m!\,n!}{(m+n+1)!}$.