SPRING 2016

SPLASH! AT CORNELL

SPECIAL POLYNOMIALS

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スペシャルポリノミアル

坂本ひかる

“Mathematics is a foreign language.”

–Chirag Bharadwaj, *et. al*.

BRIDGING THE GAP

- How can we overcome an overwhelming sea of symbols and extract meaning?
- How can we use these abstract notions to model real-life phenomena and make meaningful connections?
- How can we help change the world by developing new insights and conducting original research?

ABOUT ME

- From Flushing, NY
- Went to high school in Princeton, NJ
- Junior at Cornell University
  - B.Sc., *Computer Science*
  - B.Sc., *Electrical and Computer Engineering*
- Mathematics is one of my side interests!
- Other than that, just like you:
  - 19 years old
  - Interested in learning
  - Love helping others and teaching (probably)

ABOUT ME

- From Flushing, NY
- Went to high school in Princeton, NJ
- Relevant math classes I took in high school:
  - AP Calculus-AB, BC
  - Multivariable calculus, linear algebra
  - Differential equations, complex analysis
  - Real analysis (two semesters)
  - *Abstract algebra**
- What we will cover today is related to differential equations

I only scratched a* ***very* *small surface of this*

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OVERVIEW

200+ slides

- **120** minutes to get a quick introduction to some "interesting" polynomials in mathematics
- If time permits, we may discuss applications of these polynomials in physics
- Pace: very rapid!
  - We only have 120 minutes (and that's pushing it...)
- Holism vs reductionism: age-old question/answer

BACKGROUND

- I will assume ***complete*** familiarity with a few things:
  - How to solve single-variable differential equations
  - Separation of variables
  - Initial conditions and initial-value problems
  - Calculus of a single variable (power series, etc.)
- Things I do not expect you to have seen before:
  - Multivariable calculus
  - Other coordinate systems (**cylindrical**, **spherical**, **generalized**)
  - Kronecker delta, Levi-Civita symbol, group theory
  - The gamma function

BACKGROUND

- Don’t worry if you don’t understand everything!
  - The idea is to gain exposure to unfamiliar math
- Not everything you see here will be immediately useful in your high school mathematics
  - Unless you take a second-year differential equations class (beyond the introductory material)
- This is a **challenge**—get ready to be splashed!

GOALS

- What we will cover today:
    - Special polynomials that are orthogonal in nature
    - Polynomials in the solution space of differential equations that allow complex modeling with a relatively simple closed form
    - Formalisms and higher-order ideas
    - Applications to standard models (physics, etc.)

GOALS

- What we will cover today:

?

GOALS

- What we will cover today:

$$y = mx + b$$

GOALS

- What we will cover today:

$$y(x) = \sum_{j=0}^{\infty} c_j x^j$$

GOALS

- What we will cover today:

$$Y_\ell^m(\theta,\varphi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}P_\ell^m(cos\theta)e^{jm\varphi}$$

THE BEGINNING

- What is a differential equation?
  - Mathematical equation relating a function with its derivative(s)
  - Function from complex numbers to complex numbers: $\mathtt{fun} : \mathbb{C} \to \mathbb{C}$
  - Notation clarification: $\mathtt{fun} : \mathrm{dom}(\mathtt{fun}) \to \mathrm{cod}(\mathtt{fun})$
    - Here, $\mathrm{dom}(\mathtt{fun})$ and $\mathrm{cod}(\mathtt{fun})$ represent the domain and codomain, respectively

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CODOMAIN VS. RANGE

$$\text{range}(\texttt{fun}) \subseteq \text{cod}(\texttt{fun})$$

EXAMPLES

- Some examples of differential equations that you may or may not have encountered:
- Easy:

$$\frac{dy}{dx} = x$$

- Medium:

$$\frac{d^2y}{dx^2} + ky = \sin x$$

- Hard:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

SOLVING A SIMPLE ONE

- It is usually not too bad to solve the easy ones:

$$\frac{dy}{dx} = x$$

$$\int dy = \int x dx$$

$$y + c_1 = \frac{x^2}{2} + c_2$$

$$y = \frac{x^2}{2} + (c_2 - c_1)$$

$$\boxed{y = \frac{x^2}{2} + c}$$

SOLVING A SIMPLE ONE

- More formally, we state the solution as follows:
- *The solution to the first-order differential equation*

$$\frac{dy}{dx} = x$$

over the real numbers is the function $y : \mathbb{R} \to \mathbb{R}$, *where*

y is defined as follows:

$$y_c : x \mapsto \frac{x^2}{2} + c$$

$$y = \bigcup_{c \in \mathbb{R}} y_c$$

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over the real numbers *is the function* $y : \mathbb{R} \to \mathbb{R}$*, where*

y is defined as follows:

$$\boxed{y_c : x \mapsto \frac{x^2}{2} + c}$$ *mapping*

$$\boxed{y = \bigcup_{c \in \mathbb{R}} y_c}$$ *set of all possible y_c's*

SOLVING A SIMPLE ONE

- More precisely, note that y is actually a **function class**:

$$\boxed{y_c : x \mapsto \frac{x^2}{2} + c}$$

$$\boxed{y : \mathbb{R} \to \mathbb{R}}$$

$$\boxed{y = \bigcup_{c \in \mathbb{R}} y_c}$$

SOLVING A SIMPLE ONE

- In general, solutions to elementary differential equations will be function classes, not just functions
  - In other words, they form solution spaces
- Initial conditions or boundary conditions will specify which function within this class is the exact solution
  - "Textbook terminology": *general* v *specific* solution

$$\boxed{y = \bigcup_{c \in \mathbb{R}} y_c}$$

A SLIGHTLY HARDER ONE

- What about a harder differential equation, like this?

$$\frac{d^2y}{dx^2} + ky = \sin x$$

- This one requires us to make some assumptions. Let us first solve a simpler problem, i.e. *homogeneous*:

$$\frac{d^2y}{dx^2} + ky = 0$$

- We note that this seems to be some relationship that suggests that the **second derivative** is related to the **original function**

A SLIGHTLY HARDER ONE

- Functions that have somewhat similar properties:

$$y = e^x = \exp(x)$$

$$y = \sin x$$

$$y = \cos x$$

- Conjecture: the solution is of the general form $y = Ae^{sx}$
- Then, we can rewrite the differential equation as a simple quadratic equation to solve:

$$As^2e^{sx} + kAe^{sx} = 0$$

$$Ae^{sx}(s^2 + k) = 0$$

$$\boxed{s^2 + k = 0}$$

A SLIGHTLY HARDER ONE

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can rewrite in terms of exp

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$$As^2e^{sx} + kAe^{sx} = 0$$

$$Ae^{sx}(s^2 + k) = 0$$

exp is non-zero

use Zero Factor Property

reduction

$$\boxed{s^2 + k = 0}$$

A SLIGHTLY HARDER ONE

- Solving this equation when the contracted wave-number k is positive requires complex numbers:

$$s^2 + k = 0$$

$$s = \pm j\sqrt{k}$$

- Note that we use j to denote the *imaginary unit* here
- Since there are **two** values of s, there are two fundamental solutions that span the solution space:

$$y_1 = Ae^{j\sqrt{k}} \qquad y_2 = Be^{-j\sqrt{k}}$$

A SLIGHTLY HARDER ONE

- Any *linear combination* of these fundamental solutions should also work (i.e. superposition), so:

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 A e^{j\sqrt{k}} + c_2 B e^{-j\sqrt{k}} \end{aligned}$$

- Interestingly, we can rewrite this as the following:

$$\begin{aligned} c_1 A e^{j\sqrt{k}} + c_2 B e^{-j\sqrt{k}} &= (c_1 A + c_2 B)\left(\frac{e^{j\sqrt{k}} + e^{-j\sqrt{k}}}{2}\right) - (c_1 A - c_2 B)\left(\frac{e^{j\sqrt{k}} + e^{-j\sqrt{k}}}{2j}\right) \\ &= (c_1 A + c_2 B)\cos\sqrt{k} - (c_1 A - c_2 B)\sin\sqrt{k} && \textit{Euler's identity} \\ &= a_1 \cos\sqrt{k} + a_2 \sin\sqrt{k} && \textit{Renaming constants} \end{aligned}$$

A SLIGHTLY HARDER ONE

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Euler's identity

Renaming constants

alpha-equivalence

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Euler's identity

Renaming constants

homomorphisms

A SLIGHTLY HARDER ONE

- We now use this easier solution to solve the HARDER one by finding the solution that incorporates the RHS
- We assume that there is a "nonhomogeneous" solution of the form $Y = A\sin x + B\cos x$ (why?)
- Then, we can rewrite the differential equation:

$$\frac{d^2Y}{dx^2} + kY = \sin x$$

$$-A\sin x - B\cos x + kA\sin x + kB\cos x = \sin x$$

$$(kA - A - 1)\sin x + (kB - B)\cos x = 0$$

A SLIGHTLY HARDER ONE

- We need each coefficient to vanish, so we have that B vanishes, and $A = \dfrac{1}{k-1}$
- Then, we can put together everything to get
$$Y = \frac{\sin x}{k-1}$$
- Then, the homogeneous and nonhomogeneous parts can be superimposed to get the *particular sol.*:
$$y = y_{\text{hom}} + Y$$
$$\boxed{= a_1 \cos \sqrt{k} + a_2 \sin \sqrt{k} + \frac{\sin x}{k-1}}$$

A SLIGHTLY HARDER ONE

- This method can be generalized to harder problems
- The idea of homogeneous and nonhomogeneous components is related to the Superposition Theorem
- Solving the nonhomogeneous part the way we did is through **the method of undetermined coefficients**
- Other methods: *variation of parameters*, *guessing*, *the calculus of variations*, etc.

THE REAL QUESTION…

- What happens when we have a differential equation of the harder form shown below?

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

- Multivariable, so the solution must be a function of *several* variables, not just **one** like before!

$$u(x, y, z)$$

THE REAL QUESTION…

- The differential equation shown below is actually VERY important in physical and dynamical systems

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

- Known as ***Laplace's equation*** in three dimensions
    - Example of an *elliptic partial differential equation*
    - General theory of solutions = potential theory
    - Solution set applicability: *fluid dynamics*, *statics*, etc.

WHAT WE ARE DOING TODAY

- This leads us to the featured topic for today: **solving three very important differential equations**
- These differential equations are as follows:

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left(A - \frac{m^2}{1-x^2}\right)y = 0$$

$$x\frac{d^2y}{dx^2} + (\alpha + 1 - x)\frac{dy}{dx} + ny = 0$$

$$k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + s \cdot u(x, y, z) = 0$$

WHAT WE ARE DOING TODAY

- The canonical solutions to these differential equation generalizations are orthogonal, hypergeometric, and spatially-symmetric functions… ***special polynomials***

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HYPERGEOMETRIC

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left(A - \frac{m^2}{1-x^2}\right)y = 0$$

GENERALIZED LEGENDRE EQUATION

ORTHOGONAL

$$x\frac{d^2y}{dx^2} + (\alpha + 1 - x)\frac{dy}{dx} + ny = 0$$

GENERALIZED LAGUERRE EQUATION

SPHERICAL

$$k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + s \cdot u(x,y,z) = 0$$

GENERALIZED LAPLACE EQUATION

WHAT WE ARE DOING TODAY

- It turns out that the generalized Laplace equation is the form that the Schrödinger equation takes on
- i.e. we can solve physics problems using math!
- Interestingly, we can solve the differential equation **directly** for certain cases
  - More incredibly, the solutions are all ***polynomials***!

more on this soon

SPHERICAL

$$k\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+s\cdot u(x,y,z)=0$$

GENERALIZED LAPLACE EQUATION

A NOTE

- *Nobody seems to cover these results in sufficient detail* ***anywhere*** *online*

(relative to what they are used for)

- The results themselves are not exciting, and the process is quite tedious (but fun and engaging)
- Why publish "well-known" results? Just cite them!
- Poor mentality for initial understanding… assumes students can fill in the gaps
- Today: Helping everybody fill in those "easy" gaps

FUNCTIONS OF SEVERAL VARIABLES

- But how do functions of several variables even *work*?
  - **Just like** single-variable functions:

$$f(x, y) = x^2 + y^2$$

$$f(3, 4) = 3^2 + 4^2 = 9 + 16 = 25$$

    - *Total application*, sometimes called *application*
  - **Not like** single-variable functions:

$$f(x, 0) = x^2 + 0^2 = x^2 \qquad f(1, y) = 1^2 + y^2 = 1 + y^2$$

    - *Partial application*

FUNCTIONS OF SEVERAL VARIABLES

- Concept naturally extends to three-dimensions:

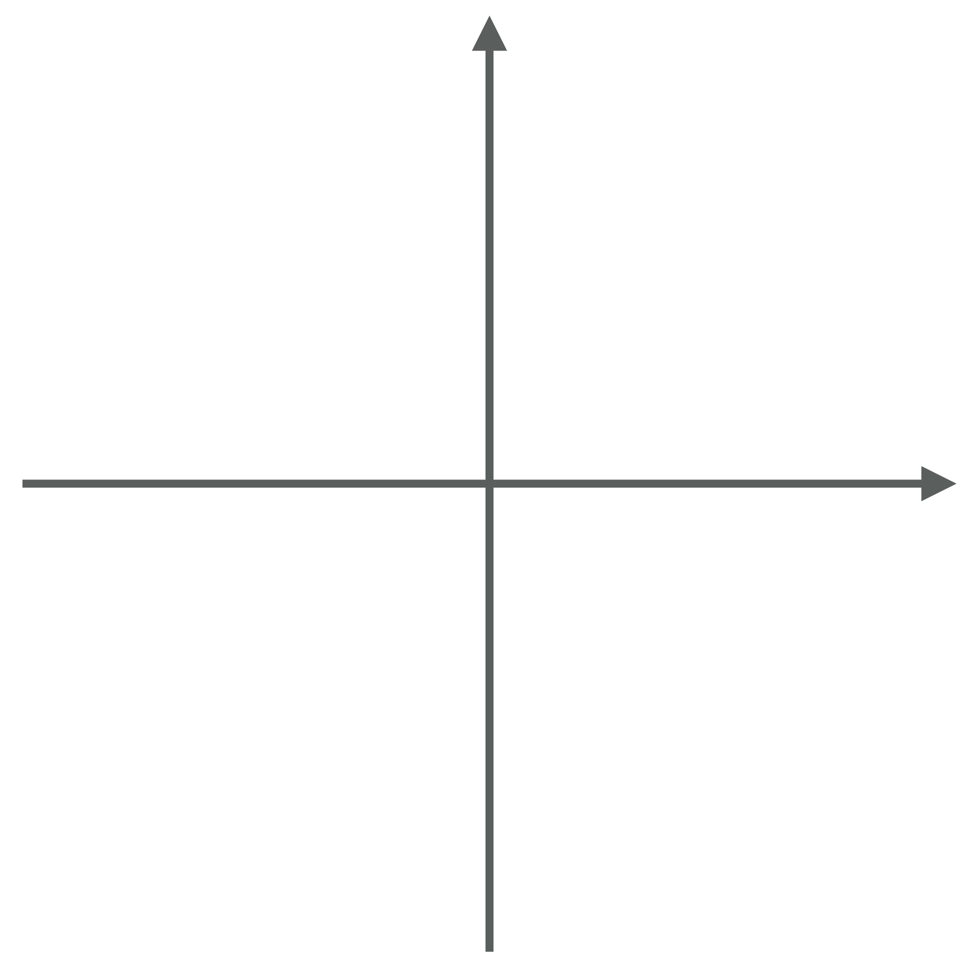
$$u(x, y, z) = \sin x + 3\cos(y^2 - z)$$

- But what about solving 2D/3D differential equations?
  - Need some new techniques
  - We will explore them as we proceed
  - Keep an open mind, and stay critical!
- Traces:

$$f(x) = u(x, 0, 0) = \sin x \qquad g(y) = u(0, y, 0) = 3\cos(y^2) \qquad h(z) = u(0, 0, z) = 3\cos z$$

DETOUR: SPHERICAL COORDINATES

- You are probably used to a *linear* coordinate system:



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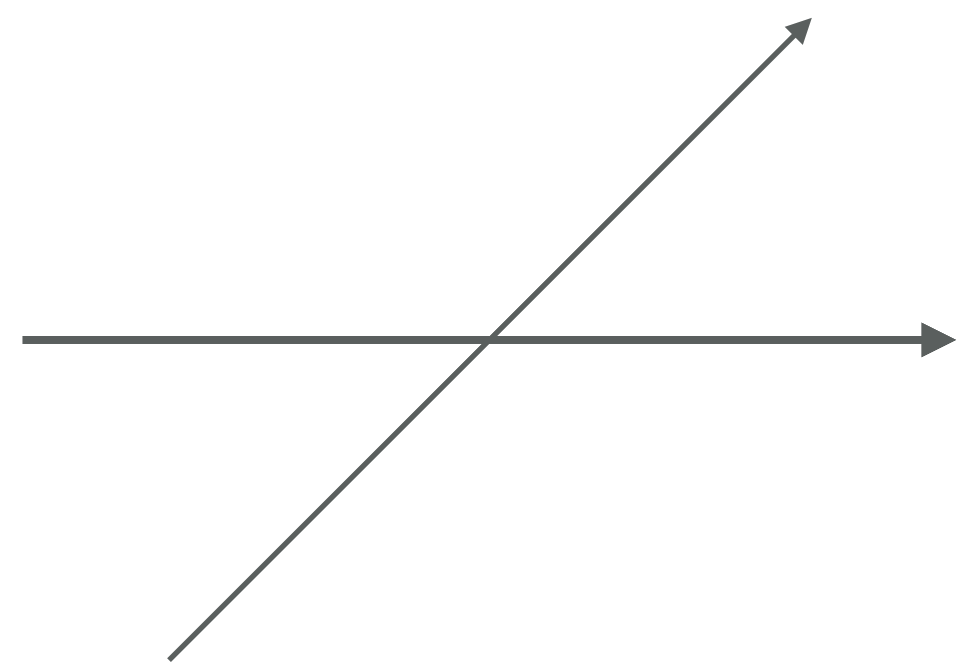

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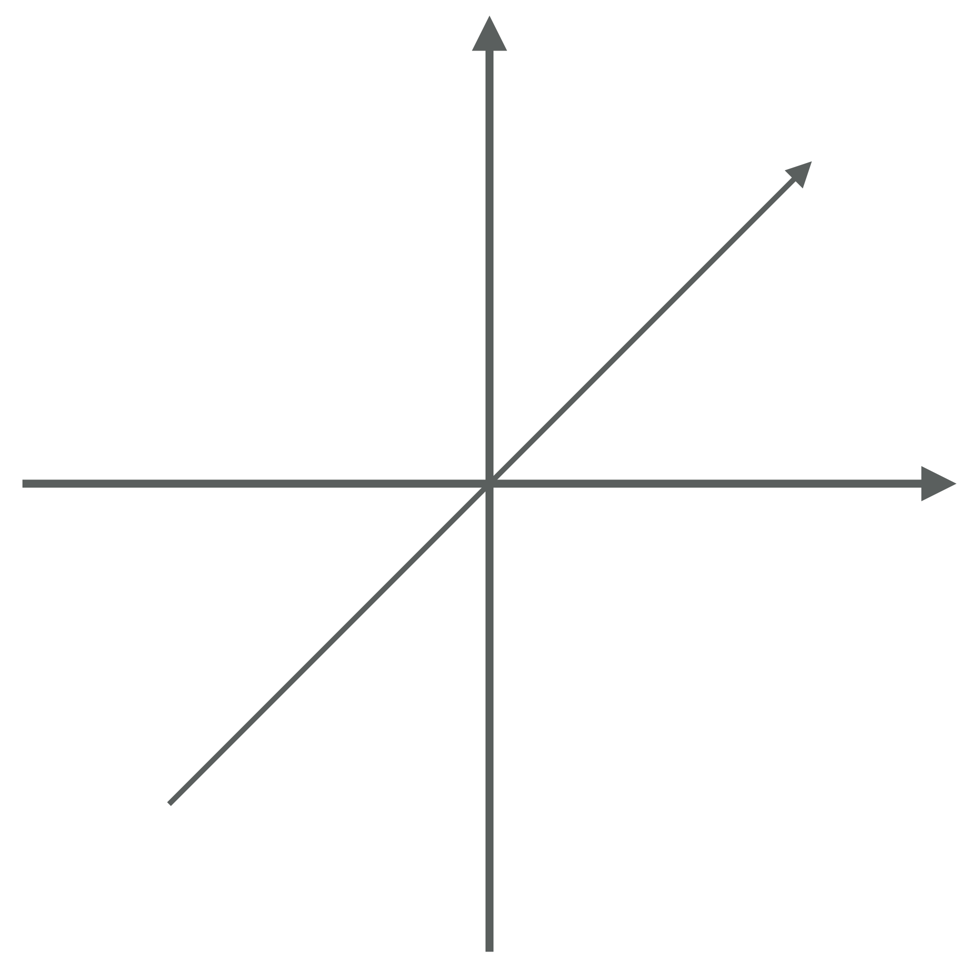

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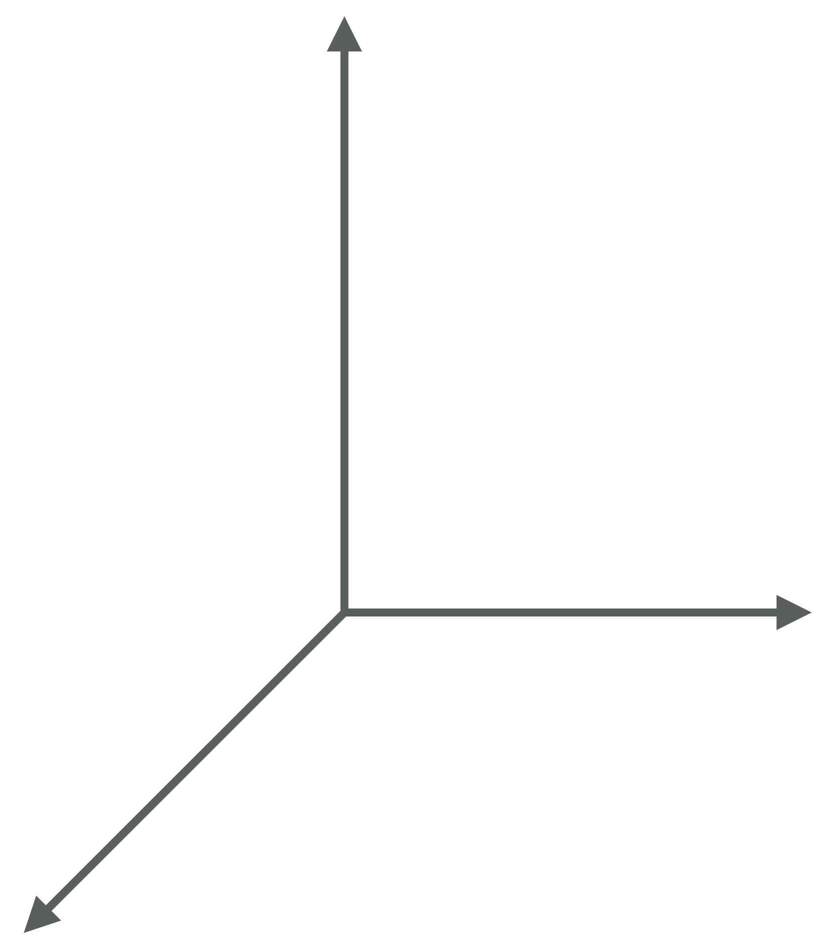

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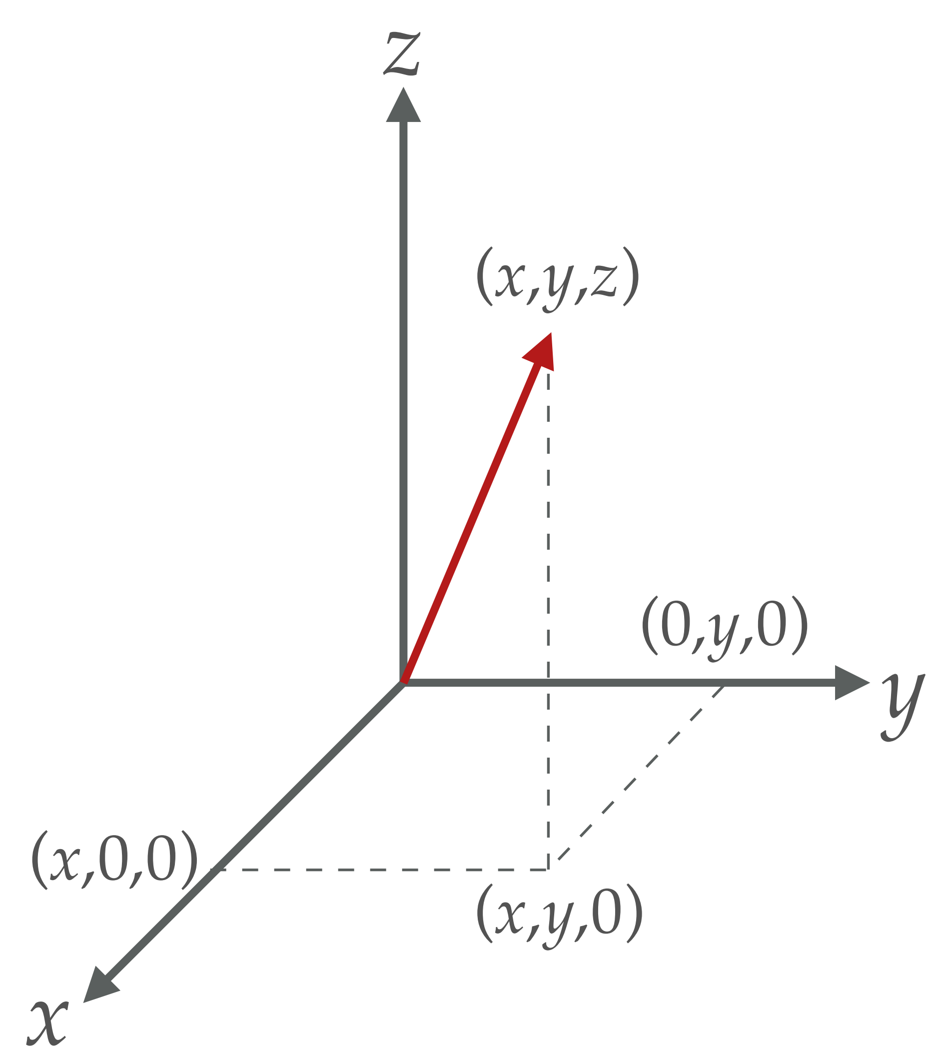
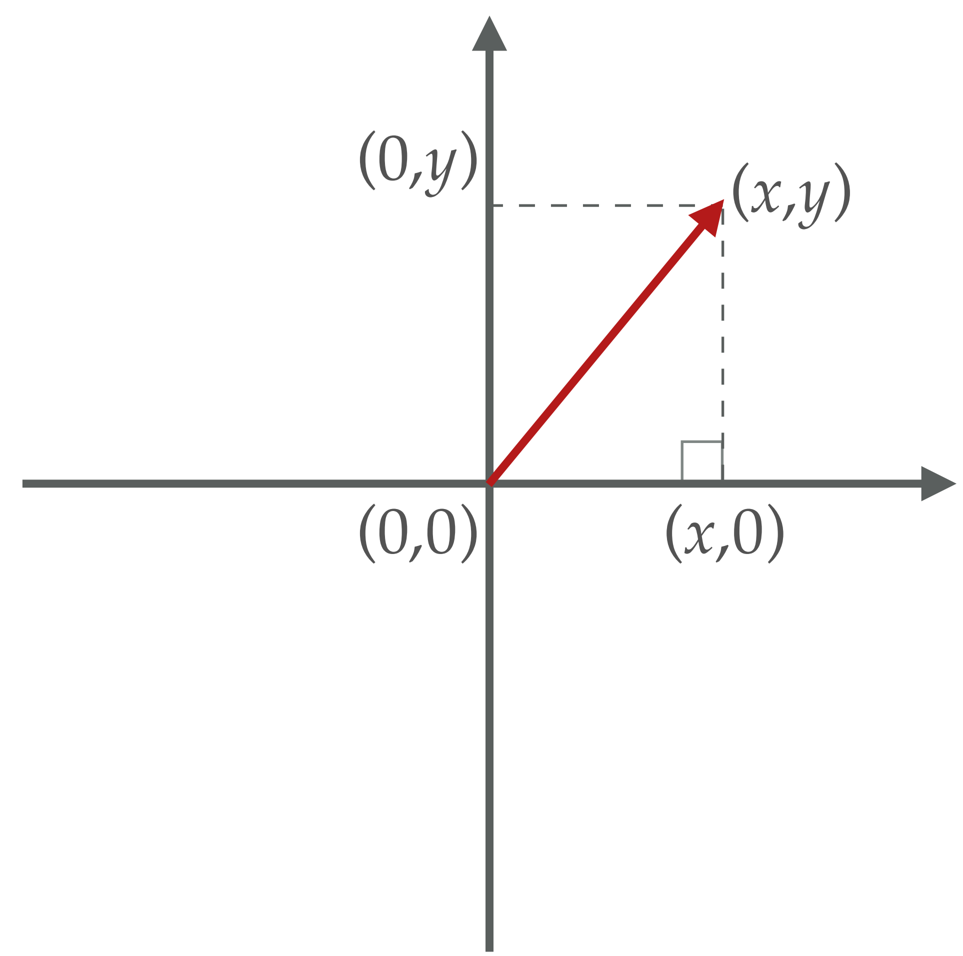


DETOUR: SPHERICAL COORDINATES

- But what about *curvilinear* coordinate systems?



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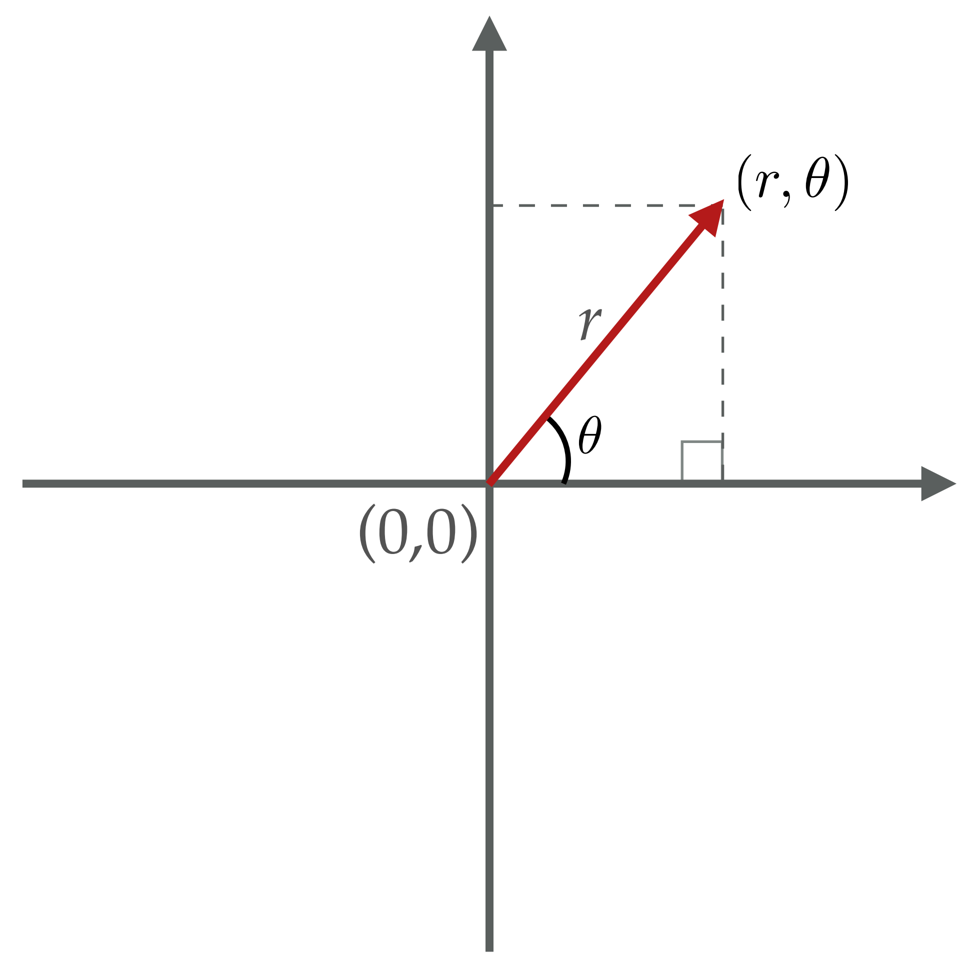


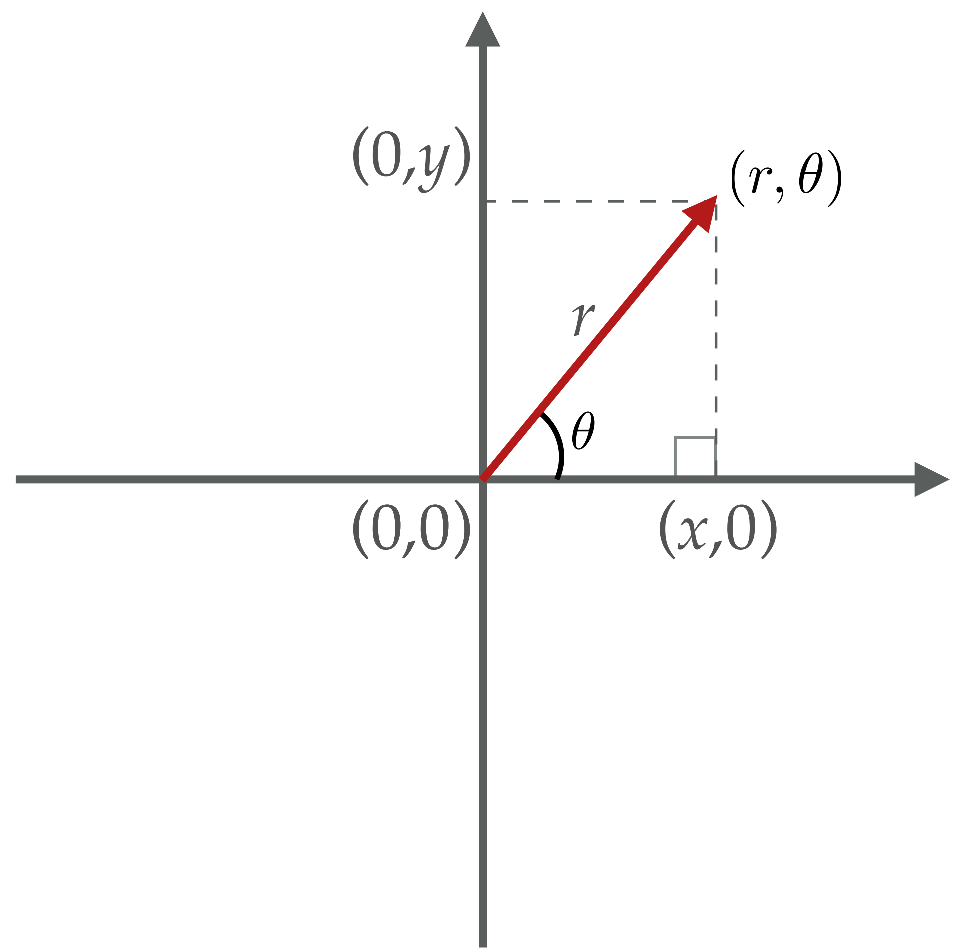
DETOUR: SPHERICAL COORDINATES

- Converting between the systems:

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

polar angle



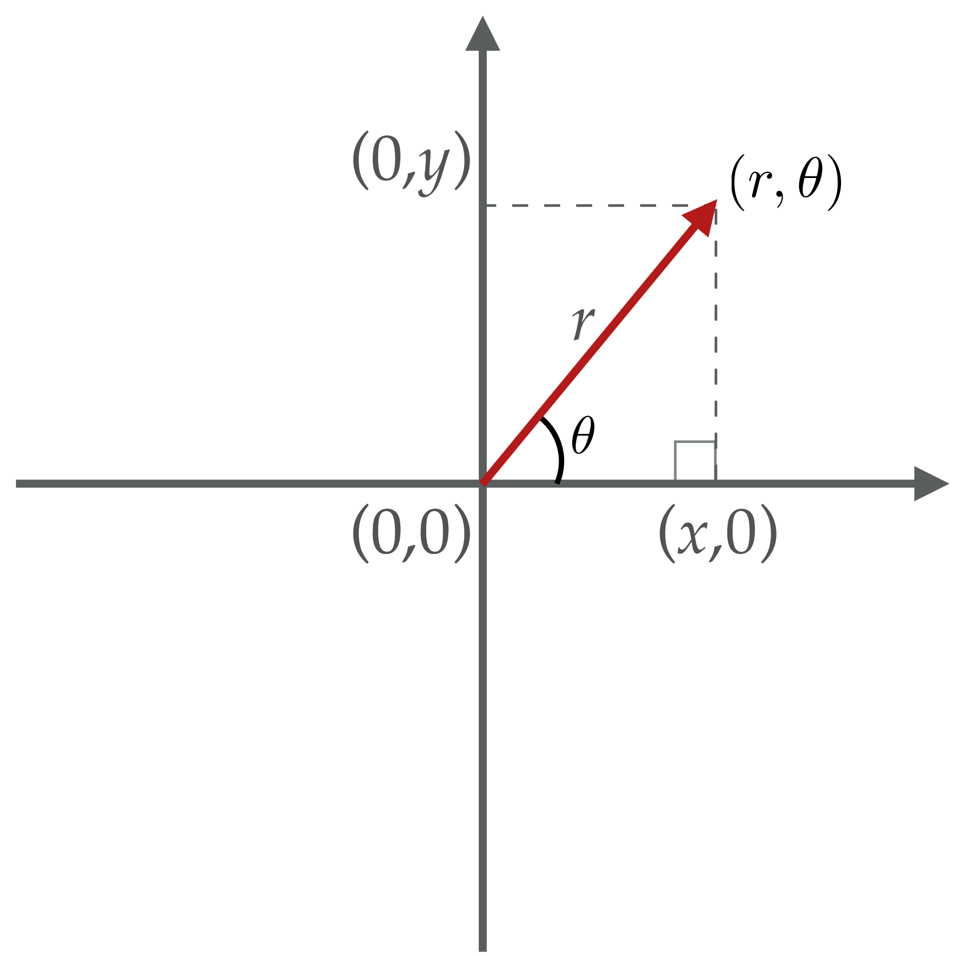
$$\cos\theta = \frac{x}{r} \qquad x = r\cos\theta$$

$$\sin\theta = \frac{y}{r} \qquad y = r\sin\theta$$

POLAR COORDINATES

DETOUR: SPHERICAL COORDINATES

- Converting between the systems:



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DETOUR: SPHERICAL COORDINATES

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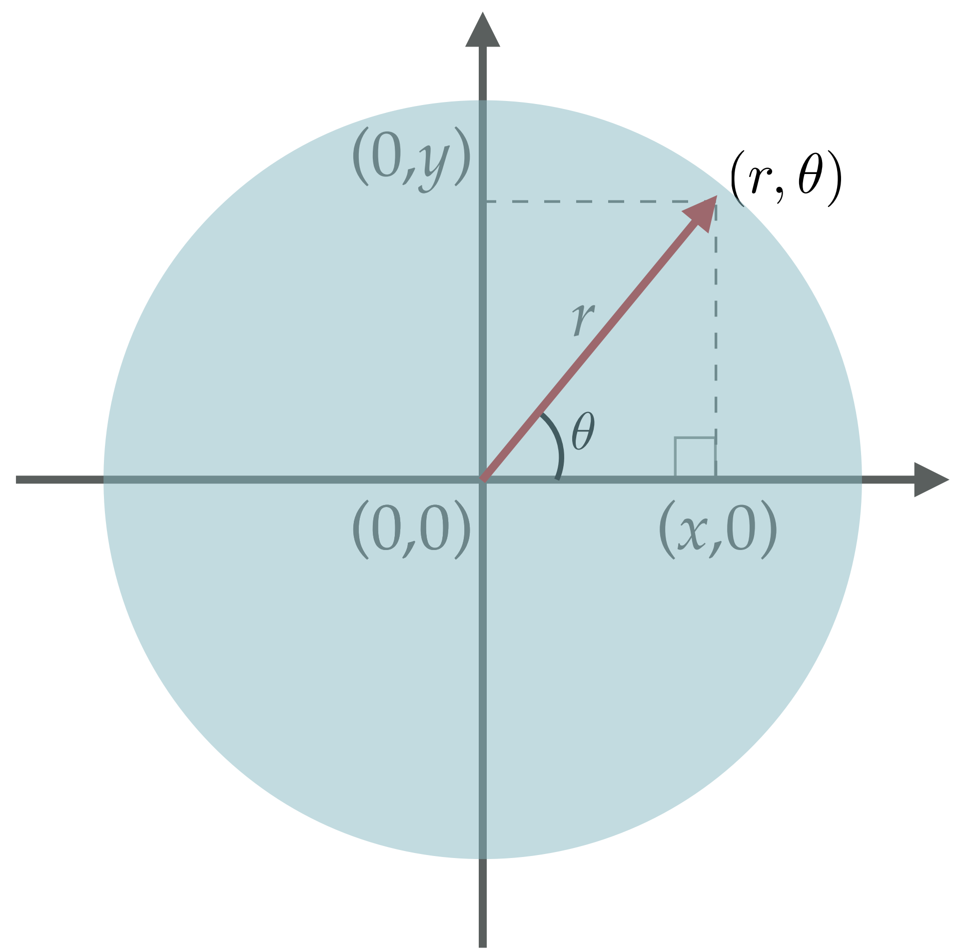

POLAR COORDINATES

DETOUR: SPHERICAL COORDINATES

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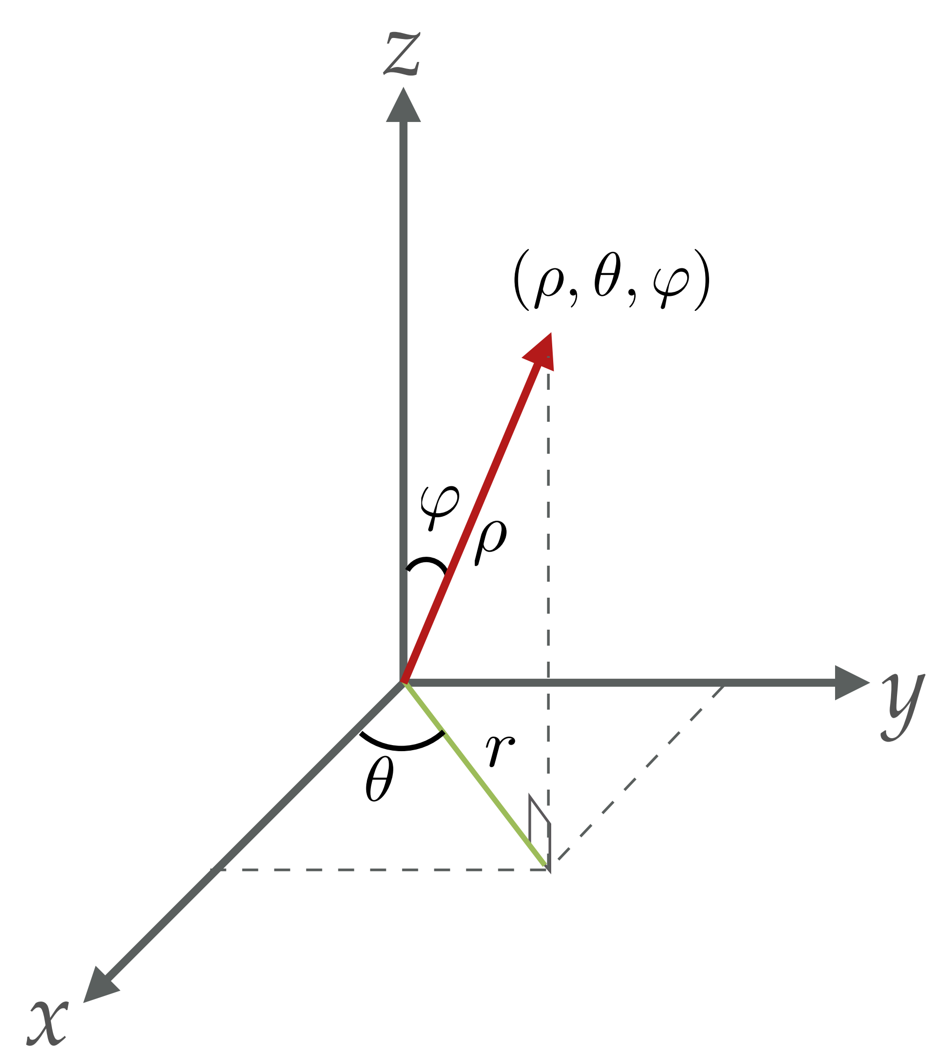
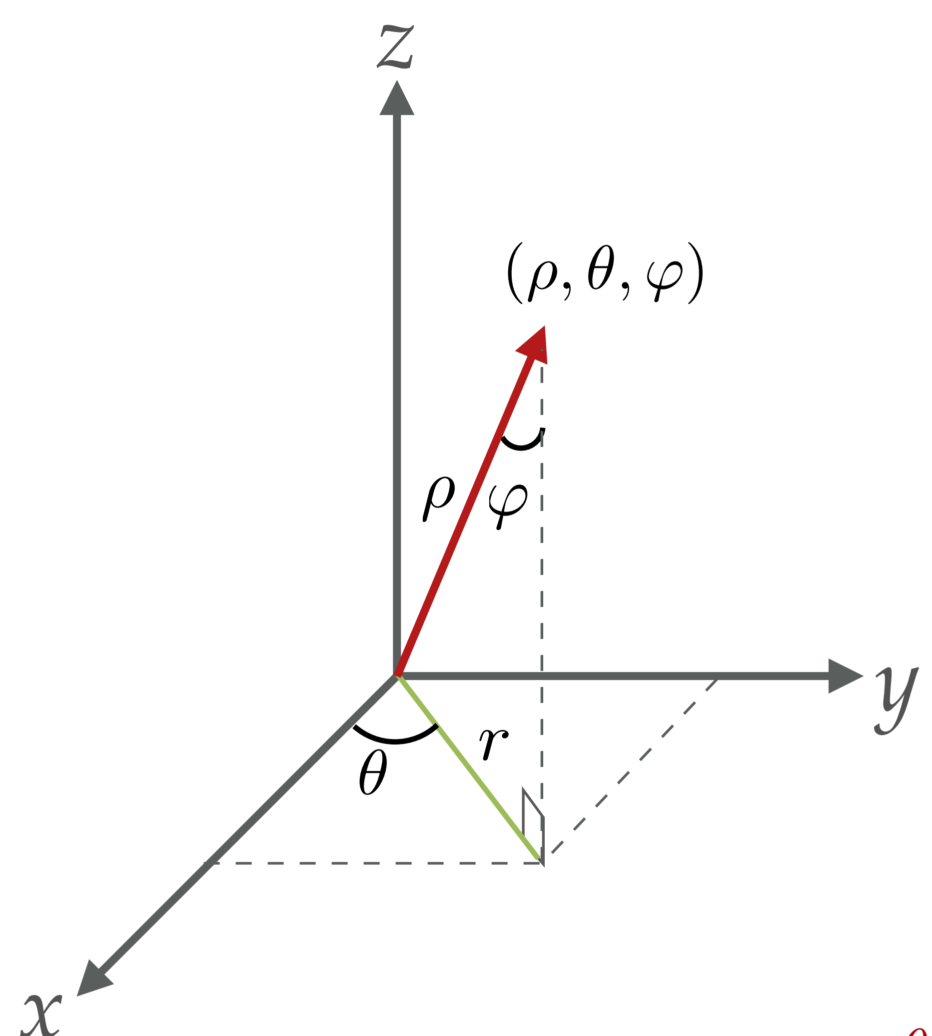


DETOUR: SPHERICAL COORDINATES

- But what about *curvilinear* coordinate systems?

Alternative way to draw this



$$z = \rho \cos \varphi$$

$$r = \rho \sin \varphi$$

$$x = r \cos \theta = \rho \sin \varphi \cos \theta$$

$$y = r \sin \theta = \rho \sin \varphi \sin \theta$$

$$\rho = \sqrt{r^2 + z^2} = \sqrt{x^2 + y^2 + x^2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \qquad \varphi = \cos^{-1}\left(\frac{z}{\rho}\right)$$

azimuthal angle

SPHERICAL COORDINATES

DETOUR: SPHERICAL COORDINATES

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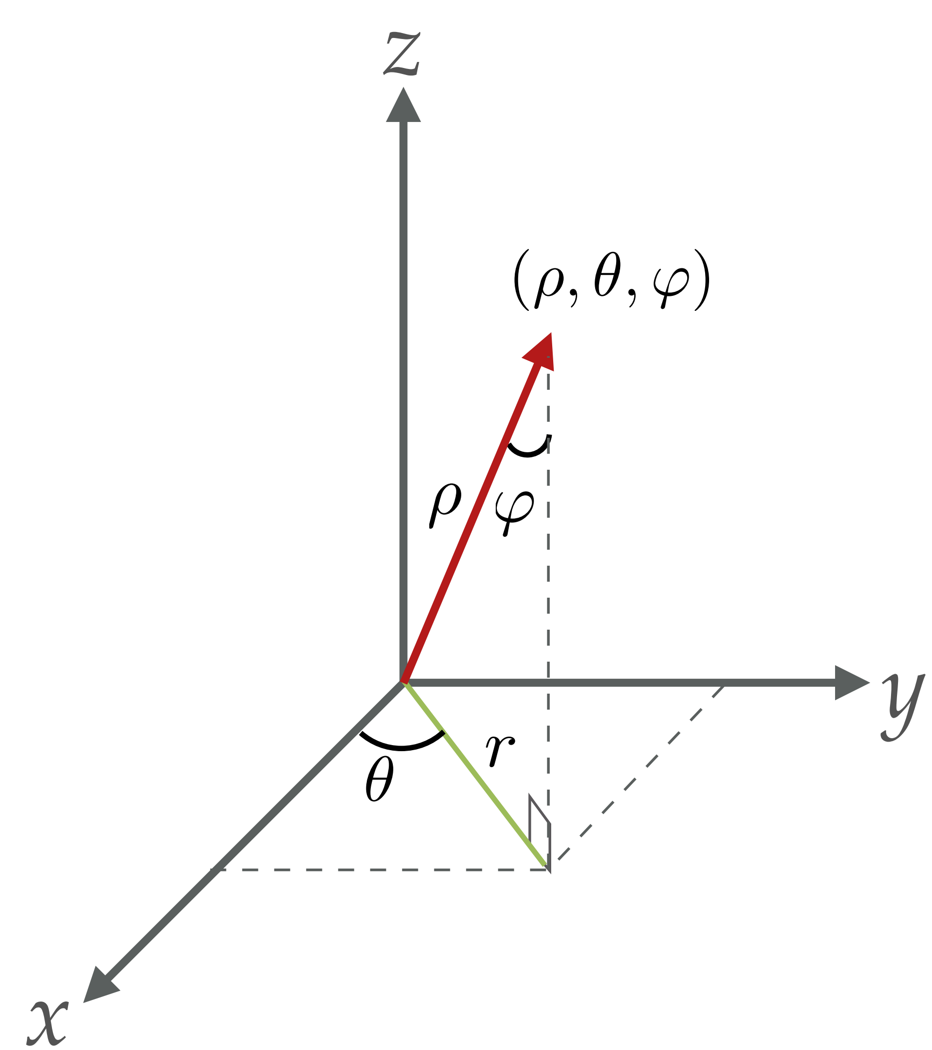

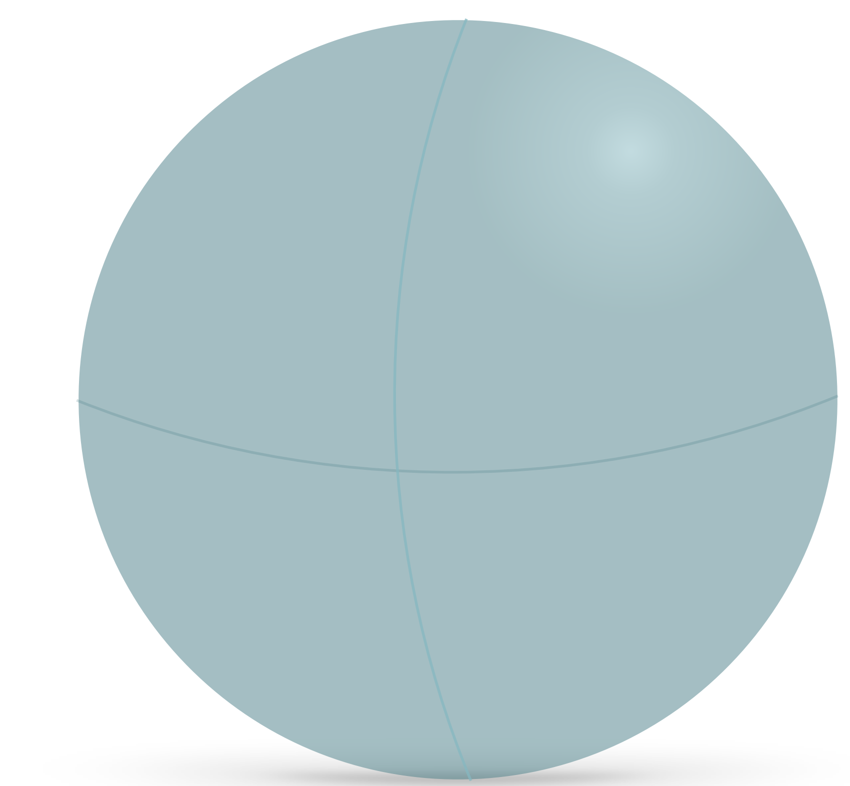

SPHERICAL COORDINATES

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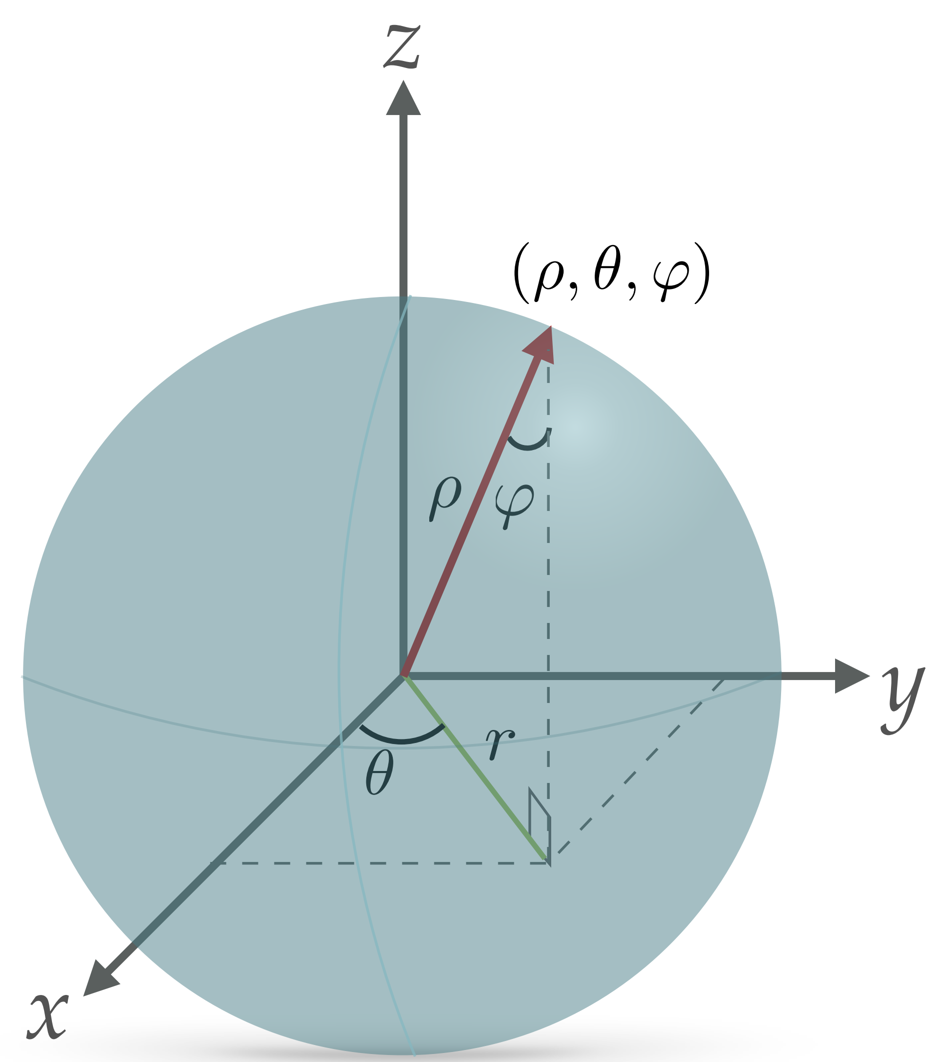

SPHERICAL COORDINATES

SOME SYMBOLS

- We define the *scalar* Laplacian operator as follows:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

- Thus, when applied to a function, the Laplacian is

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

- We can then write differential equations like this:

$$\nabla^2 u = 0$$

- This is a *partial differential equation* in three variables, whose solutions are *functions* $u(x,y,z)$

WARM-UP

- Let’s now try to solve a differential equation with a different technique: *separation of many variables*
- We will warm-up with a 2D example, in which we solve a partial differential equation
  - Twist: The solution is a function of *two variables*

$$\frac{\partial u}{\partial t} = k \frac{\partial u}{\partial x}$$

$$u(x,t) = ?$$

WARM-UP

- Problem: no *simple* integration technique like earlier
- Solution: *Assume a form* that simplifies life for us
- For example, let us assume that the variables of u are *separable*, i.e. there are functions $X(x)$ and $T(t)$ so:

$$u(x, t) = X(x)T(t)$$

- If we use this, we can *reduce* the partial differential equation down to the following expression:

$$X(x)T'(t) = kX'(x)T(t)$$

WARM-UP

- We can now actually separate the variables to both sides to "factor" the expression, as shown below:

$$k\frac{X'(x)}{X(x)} = \frac{T'(t)}{T(t)}$$

- In other words, the LHS is a function of x and the RHS is a function of t
- The only way this is possible for two independent types is if both functions are equivalent to a constant

$$k\frac{X'(x)}{X(x)} = \frac{T'(t)}{T(t)} = const. = -c$$

the negative sign is ***intentional*** *here*

WARM-UP

- This leads to **two** *single-variable* differential equations to solve:

$$kX'(x) = -cX(x) \qquad X'(x) + \frac{c}{k}X(x) = 0$$

$$T'(t) = -cT(t) \qquad T'(t) + cT(t) = 0$$

- These have some pretty basic solutions:

$$X(x) = Ae^{-\frac{c}{k}x} \qquad T(t) = Be^{-ct}$$

- We can put these together to get the overall one:

$$u(x,t) = ABe^{-c(\frac{x}{k}+t)} = Ke^{-c(\frac{x}{k}+t)}$$

constants c, K determined by initial conditions

THE CRUX

- We will use a slightly more complicated version of this concept to solve a 3D version of separation
- We begin our lesson by examining the following DE:

$$j\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi$$

- This is a partial differential equation that is quadratic in space and linear in time*
  - The solution Ψ has *complex* components in general
  - Don't worry about other variables/constants: $\hbar, m, V$

this phrase has a **very different meaning in computer science*

THE CRUX

$$j\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi$$

- This is called the Schrödinger wave equation
- The solution Ψ is called a wave function
- We will separate this into *two* differential equations and then generalize this
- First step: *Assume* the separable form $\Psi(x,t) = \psi(x)T(t)$

$$j\hbar\psi(x)\frac{dT}{dt} = -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}T(t) + V\psi(x)T(t)$$

ASIDE: NOTATION

- We will start omitting the arguments to the functions, since in reality, we are describing functions **without application** (*strictly speaking*, of course)
- In other words, we will write this:

$$j\hbar\psi\frac{dT}{dt} = -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}T + V\psi T$$

it's just cleaner!

- Rather than this:

$$j\hbar\psi(x)\frac{dT}{dt} = -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}T(t) + V\psi(x)T(t)$$

THE CRUX

- We can now divide through by ψT to get:

$$j\hbar \frac{1}{T}\frac{dT}{dt} = -\frac{\hbar^2}{2m}\frac{1}{\psi}\frac{d^2\psi}{dx^2} + V$$

- As before, this must be equal to a constant due to independence in variables, so we call this constant E
- We end up with the following:

$$j\hbar \frac{1}{T}\frac{dT}{dt} = -\frac{\hbar^2}{2m}\frac{1}{\psi}\frac{d^2\psi}{dx^2} + V = E$$

$$\frac{dT}{dt} = \frac{E}{j\hbar}T = -\frac{jE}{\hbar}T \qquad -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi$$

THE CRUX

- The T one is quite simple to solve (proof omitted):

$$\frac{dT}{dt} = \frac{E}{j\hbar}T = -\frac{jE}{\hbar}T$$

$$T(t) = Ae^{-jEt/\hbar}$$

- The ψ one is not so easy
- Before we proceed, let's **extend** it to a 3D-form:

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right) + V\psi = E\psi$$

spacetime

- This implies a solution $\psi(x,y,z)$, and overall $\Psi(x,y,z,t)$

IN OTHER WORDS…

- Our essential goal is to solve the PDE shown below:

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}\right)+(V-E)\psi=0$$

- Which is of the *form*…

$$k\cdot\nabla^2\psi+s\cdot\psi=0$$

$$k\cdot\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}\right)+s\cdot\psi=0$$

- So this is nothing more than a **glorified** *generalized Laplace equation*! But how do we solve *this*?

IN OTHER WORDS…

- Tempting: Separation of variables right away!
  - The wise and experienced differential-equation solvers say "**no!**"
  - It is actually better to convert everything into spherical coordinates first and *then* separate all the resulting variables

*the **derivatives** are the **ugly** part*

- Time to do the mapping:

$$x = \rho \sin\varphi \cos\theta \qquad y = \rho \sin\varphi \sin\theta \qquad z = \rho \cos\varphi$$

SPHERICAL MAPPING

- The general form of the differential equation is

$$\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}\right)+\frac{s}{k}\psi=0$$

- We want to convert this to spherical coordinates; let us first compute *first derivatives* via the Chain Rule:

$$\boxed{\frac{\partial\psi}{\partial x}=\frac{\partial\psi}{\partial\rho}\frac{\partial\rho}{\partial x}+\frac{\partial\psi}{\partial\varphi}\frac{\partial\varphi}{\partial x}+\frac{\partial\psi}{\partial\theta}\frac{\partial\theta}{\partial x}}$$

$$\boxed{\frac{\partial\psi}{\partial y}=\frac{\partial\psi}{\partial\rho}\frac{\partial\rho}{\partial y}+\frac{\partial\psi}{\partial\varphi}\frac{\partial\varphi}{\partial y}+\frac{\partial\psi}{\partial\theta}\frac{\partial\theta}{\partial y}}$$

$$\boxed{\frac{\partial\psi}{\partial z}=\frac{\partial\psi}{\partial\rho}\frac{\partial\rho}{\partial z}+\frac{\partial\psi}{\partial\varphi}\frac{\partial\varphi}{\partial z}+\frac{\partial\psi}{\partial\theta}\frac{\partial\theta}{\partial z}}$$

SPHERICAL MAPPING

- Let us develop explicit values for these derivatives:

$$\frac{\partial\rho}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2+z^2}} = \frac{\rho\sin\varphi\cos\theta}{\rho} = \boxed{\sin\varphi\cos\theta}$$

$$\sin\varphi\frac{\partial\varphi}{\partial x} = -\frac{z}{\rho^2}\frac{\partial\rho}{\partial x} = -\frac{\rho\cos\varphi}{\rho^2}\sin\varphi\cos\theta = -\frac{\cos\varphi}{\rho}\sin\varphi\cos\theta$$

$$\sec^2\theta\frac{\partial\theta}{\partial x} = -\frac{y}{x^2} = -\frac{\rho\sin\varphi\sin\theta}{\rho^2\sin^2\varphi\cos^2\theta} = -\frac{\sin\theta}{\rho\sin\varphi\cos^2\theta}$$

$$\frac{\partial\varphi}{\partial x} = \boxed{-\frac{\cos\varphi}{\rho}\cos\theta} \qquad \frac{\partial\theta}{\partial x} = \boxed{-\frac{\sin\theta}{\rho\sin\varphi}}$$

SPHERICAL MAPPING

- Let us develop explicit values for these derivatives:

$$\frac{\partial \rho}{\partial y} = \sin\varphi \sin\theta$$

$$\frac{\partial \varphi}{\partial y} = -\frac{\cos\varphi}{\rho}\sin\theta$$

$$\frac{\partial \theta}{\partial y} = \frac{\cos\theta}{\rho\sin\varphi}$$

SPHERICAL MAPPING

- Let us develop explicit values for these derivatives:

$$\frac{\partial \rho}{\partial z} = \cos \varphi$$

$$\frac{\partial \varphi}{\partial z} = \frac{\cot \varphi}{\rho}$$

$$\frac{\partial \theta}{\partial z} = 0$$

SPHERICAL MAPPING

- Let us develop explicit values for these derivatives:

$$\frac{\partial\psi}{\partial x} = \sin\varphi\cos\theta\frac{\partial\psi}{\partial\rho} - \frac{\cos\varphi}{\rho}\cos\theta\frac{\partial\psi}{\partial\varphi} - \frac{\sin\theta}{\rho\sin\varphi}\frac{\partial\psi}{\partial\theta}$$

$$\frac{\partial\psi}{\partial y} = \sin\varphi\sin\theta\frac{\partial\psi}{\partial\rho} - \frac{\cos\varphi}{\rho}\sin\theta\frac{\partial\psi}{\partial\varphi} + \frac{\cos\theta}{\rho\sin\varphi}\frac{\partial\psi}{\partial\theta}$$

$$\frac{\partial\psi}{\partial z} = \cos\varphi\frac{\partial\psi}{\partial\rho} + \frac{\cot\varphi}{\rho}\frac{\partial\psi}{\partial\varphi}$$

SPHERICAL MAPPING

- We find the second derivatives similarly (omitted):

$$
\begin{aligned}
\nabla^2\psi &= \frac{\partial^2\psi}{\partial^2 x} + \frac{\partial^2\psi}{\partial^2 y} + \frac{\partial^2\psi}{\partial^2 z} \\
&= \frac{\partial^2\psi}{\partial\rho^2} + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\varphi^2} + \frac{1}{\rho^2\sin^2\varphi}\frac{\partial^2\psi}{\partial\theta^2} + \frac{2}{\rho}\frac{\partial\psi}{\partial\rho} + \frac{1}{\rho^2}\cot\varphi\frac{\partial\psi}{\partial\varphi} \\
&= \boxed{\frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial\psi}{\partial\varphi}\right) + \frac{1}{\rho^2\sin^2\varphi}\frac{\partial^2\psi}{\partial\theta^2}}
\end{aligned}
$$

SPHERICAL MAPPING

- We can now rewrite the time-independent equation:

$$\frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial\psi}{\partial\rho}\right)+\frac{1}{\rho^2\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial\psi}{\partial\varphi}\right)+\frac{1}{\rho^2\sin^2\varphi}\frac{\partial^2\psi}{\partial\theta^2}+\frac{s}{k}\psi=0$$

- We use the technique of separation of variables:

$$\psi(\rho,\theta,\varphi)=R(\rho)Y(\theta,\varphi)$$

- Here, $R(\rho)$ is the *radial* solution, and $Y(\theta,\phi)$ defines a class of functions called the *spherical harmonics*
- We will split up $Y(\theta,\phi)$ further soon!

SPHERICAL MAPPING

- We can now rewrite the time-independent equation:

$$\frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial\psi}{\partial\rho}\right)+\frac{1}{\rho^2\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial\psi}{\partial\varphi}\right)+\frac{1}{\rho^2\sin^2\varphi}\frac{\partial^2\psi}{\partial\theta^2}+\frac{s}{k}\psi=0$$

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- We will split up $Y(\theta,\phi)$ further soon!

any solution in this space has a linear sum of angular and radial component structures!

SEPARATION OF VARIABLES

- We now plug in $\psi(\rho, \theta, \phi) = R(\rho)Y(\theta, \phi)$:

$$\frac{Y}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right) + \frac{R}{\rho^2}\left(\frac{1}{\sin\varphi}\frac{d}{d\varphi}\left(\sin\varphi\frac{\partial Y}{\partial\varphi}\right) + \frac{1}{\sin^2\varphi}\frac{\partial^2 Y}{\partial\theta^2}\right) + \frac{s}{k}RY = 0$$

- Pedantry: Note the correct usage of d versus ∂ above
- We can now expand this equation and separate all the R terms from all the Y terms
- We will do this on the next slide (buy more space)

SEPARATION OF VARIABLES

- Multiply through by ρ^2 and divide by RY:

$$\frac{1}{R}\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right)+\frac{1}{Y}\left(\frac{1}{\sin\varphi}\frac{d}{d\varphi}\left(\sin\varphi\frac{\partial Y}{\partial\varphi}\right)+\frac{1}{\sin^2\varphi}\frac{\partial^2 Y}{\partial\theta^2}\right)+\frac{s}{k}\rho^2=0$$

- We now rearrange so that R- and Y-terms separate:

$$\frac{1}{R}\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right)+\frac{s}{k}\rho^2=-\frac{1}{Y}\left(\frac{1}{\sin\varphi}\frac{d}{d\varphi}\left(\sin\varphi\frac{\partial Y}{\partial\varphi}\right)+\frac{1}{\sin^2\varphi}\frac{\partial^2 Y}{\partial\theta^2}\right)$$

- The LHS is a function of ρ, and the RHS is a function of θ and ϕ… how can they even be equal?

SEPARATION OF VARIABLES

- They are both constants! Let's call this constant as A:

$$\frac{1}{R}\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right)+\frac{s}{k}\rho^2=-\frac{1}{Y}\left(\frac{1}{\sin\varphi}\frac{d}{d\varphi}\left(\sin\varphi\frac{\partial Y}{\partial\varphi}\right)+\frac{1}{\sin^2\varphi}\frac{\partial^2 Y}{\partial\theta^2}\right)=A$$

function of ρ function of θ and ϕ const.

- We now have two separate equations to solve:

$$\frac{1}{Y}\left(\frac{1}{\sin\varphi}\frac{d}{d\varphi}\left(\sin\varphi\frac{\partial Y}{\partial\varphi}\right)+\frac{1}{\sin^2\varphi}\frac{\partial^2 Y}{\partial\theta^2}\right)=-A$$

$$\frac{1}{R}\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right)+\frac{s}{k}\rho^2=A$$

SEPARATION OF VARIABLES

- They are both constants! Let's call this constant as A:

$$\underbrace{\frac{1}{R}\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right)+\frac{s}{k}\rho^2}_{\text{function of }\rho}=\underbrace{-\frac{1}{Y}\left(\frac{1}{\sin\varphi}\frac{d}{d\varphi}\left(\sin\varphi\frac{\partial Y}{\partial\varphi}\right)+\frac{1}{\sin^2\varphi}\frac{\partial^2 Y}{\partial\theta^2}\right)}_{\text{function of }\theta\text{ and }\phi}=\underbrace{A}_{\text{const.}}$$

- We now have two separate equations to solve:

We'll tackle this…
$$\frac{1}{Y}\left(\frac{1}{\sin\varphi}\frac{d}{d\varphi}\left(\sin\varphi\frac{\partial Y}{\partial\varphi}\right)+\frac{1}{\sin^2\varphi}\frac{\partial^2 Y}{\partial\theta^2}\right)=-A$$

$$\frac{1}{R}\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right)+\frac{s}{k}\rho^2=A$$
Put aside until later (too tedious)…

SEPARATION OF VARIABLES

- Solving the first one isn't easy either!
- It's a partial differential equation in two variables…
  - Time to whip out separation of variables AGAIN
  - We'll call the functions $\Theta(\theta)$ and $\Phi(\phi)$ this time
- Let's first bring the equation to a canonical form:

$$Y(\theta, \varphi) = \Theta(\theta)\Phi(\varphi)$$

$$\sin\varphi \frac{d}{d\varphi}\left(\sin\varphi \frac{\partial Y}{\partial \varphi}\right) + \frac{\partial^2 Y}{\partial \theta^2} = -AY\sin^2\varphi$$

SEPARATION OF VARIABLES

- We can now separate the Θ terms from the Φ terms:

$$\sin\varphi \frac{d}{d\varphi}\left(\Theta \sin\varphi \frac{d\Phi}{d\varphi}\right) + \Phi \frac{d^2\Theta}{d\theta^2} = -A\Theta\Phi \sin^2\varphi$$

$$\Theta \sin\varphi \frac{d}{d\varphi}\left(\sin\varphi \frac{d\Phi}{d\varphi}\right) + \Phi \frac{d^2\Theta}{d\theta^2} = -A\Theta\Phi \sin^2\varphi$$

$$\frac{1}{\Phi} \sin\varphi \frac{d}{d\varphi}\left(\sin\varphi \frac{d\Phi}{d\varphi}\right) + \frac{1}{\Theta} \frac{d^2\Theta}{d\theta^2} = -A \sin^2\varphi$$

some *other* constant

$$\frac{1}{\Phi} \sin\varphi \frac{d}{d\varphi}\left(\sin\varphi \frac{d\Phi}{d\varphi}\right) + A \sin^2\varphi = -\frac{1}{\Theta} \frac{d^2\Theta}{d\theta^2} = B$$

some constant

SEPARATION OF VARIABLES

- Again, we end up with two equations to solve:

$$\frac{1}{\Phi}\sin\varphi\frac{d}{d\varphi}\left(\sin\varphi\frac{d\Phi}{d\varphi}\right)+A\sin^2\varphi=B$$

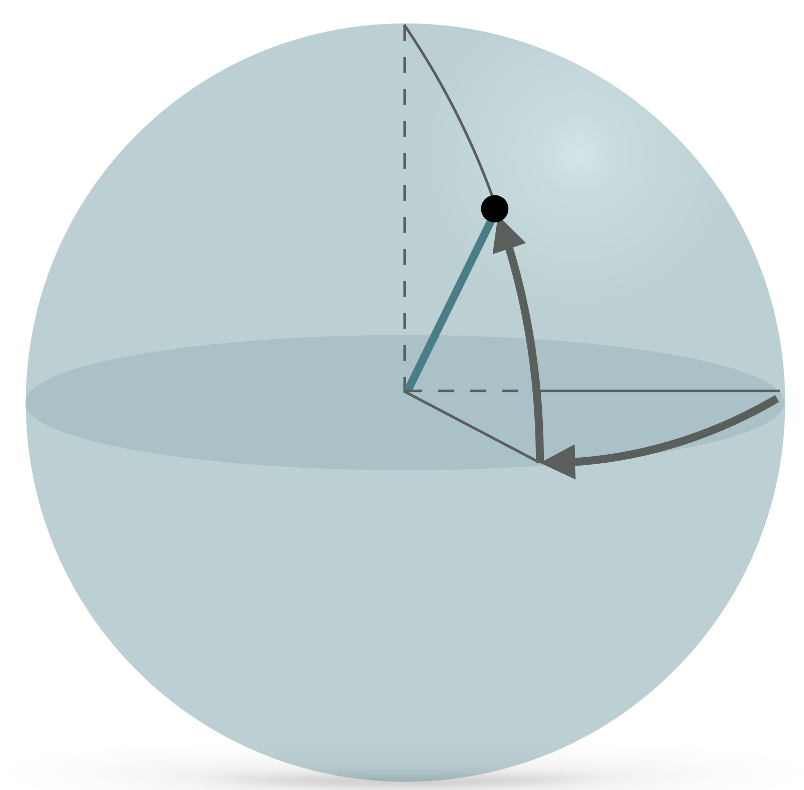
$$-\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}=B \longleftarrow$$

eigenvalue problem

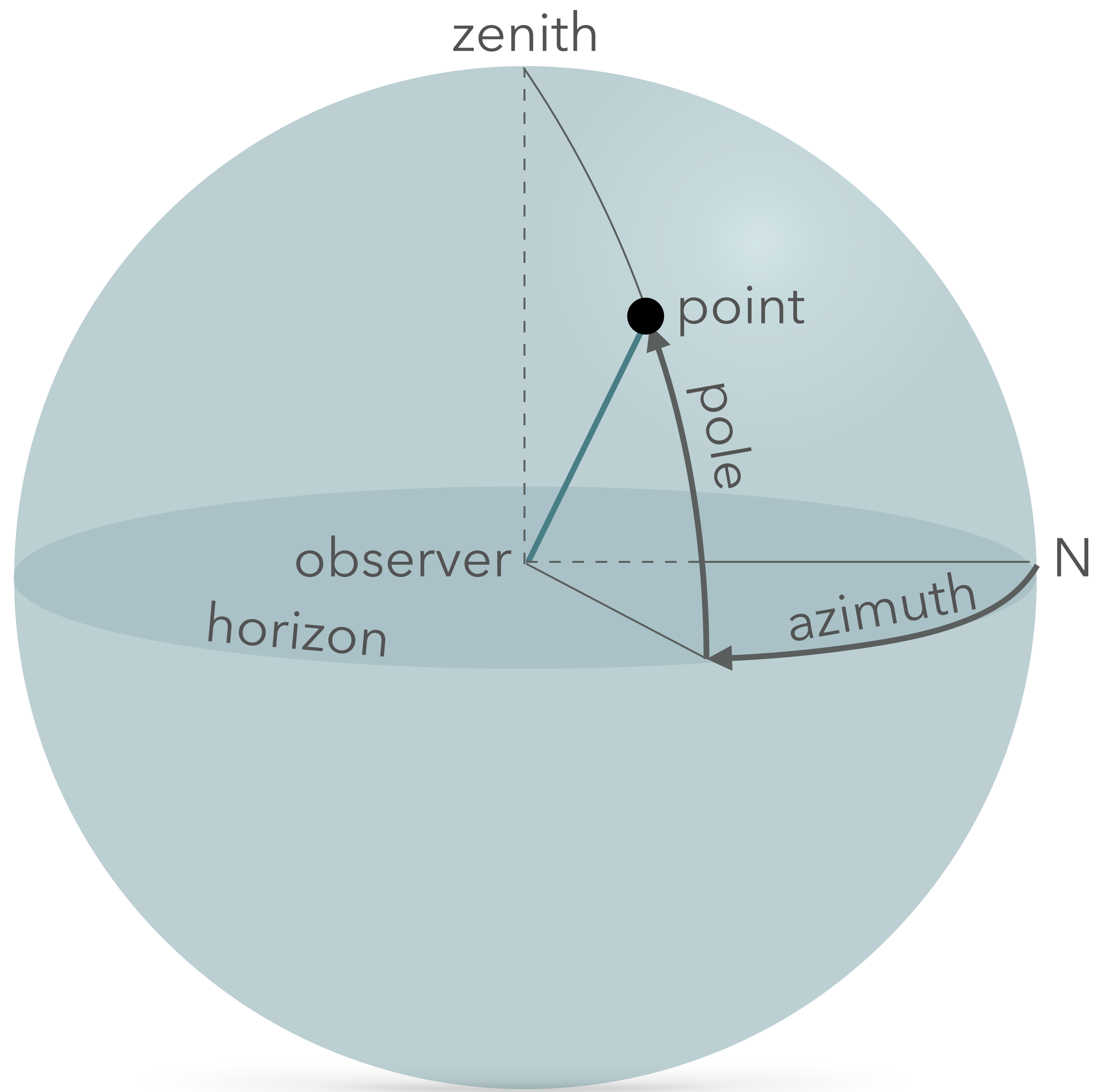
- The solution $\Phi(\phi)$ to the first equation is called the *polar* solution, and the solution $\Theta(\theta)$ to the second equation is called the *azimuthal* solution

NOMENCLATURE

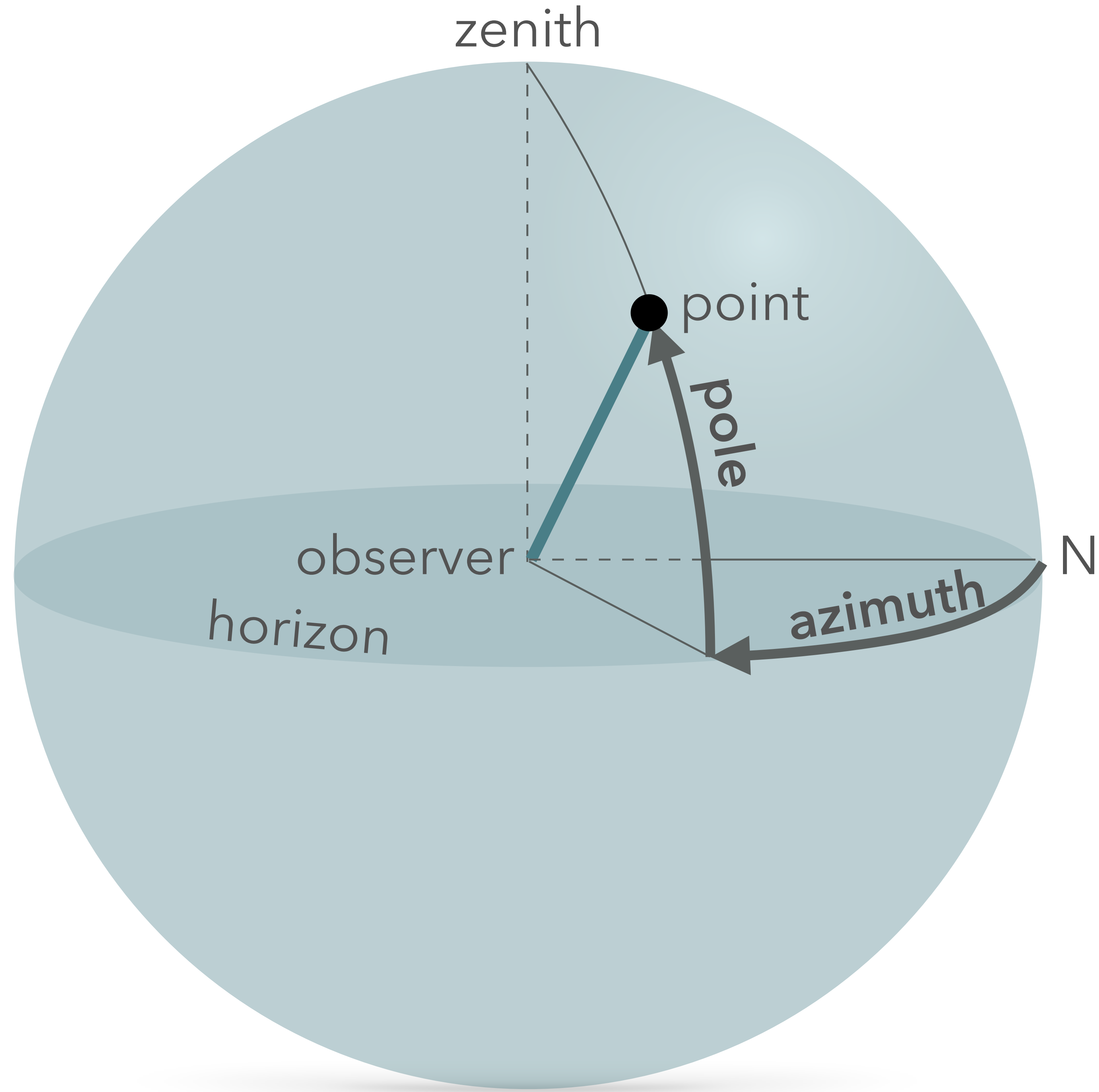
- That might have been confusing!
- Isn't *polar* supposed to be using θ, like in *polar* coordinates? The trick is in the measurement point:



NOMENCLATURE



NOMENCLATURE



THE AZIMUTHAL EQUATION

- We can solve the azimuthal equation pretty easily by assuming a solution of the form $\Theta(\theta) = Ce^{s\theta}$
- This is just our standard assumption for any second-order ordinary differential equation

$$\frac{d^2\Theta}{d\theta^2} + B\Theta = 0$$

$$Cs^2e^{s\theta} + BCe^{s\theta} = 0$$

$$Ce^{s\theta}(s^2 + B) = 0$$

$$\boxed{s^2 + B = 0}$$

***exp* > 0, so we can divide through by it**

THE AZIMUTHAL EQUATION

- The solution now depends on the nature of B:
  - **Case 1:** $B > 0$
  - **Case 2:** $B = 0$
  - **Case 3:** $B < 0$
- Are these all possible? Are the solutions different if we assume different values of B?
  - As it turns out, ***no***
  - Let's prove that now!

CASE 1

- In this case, we have $B > 0$, i.e. we can write $B = m^2$ for some m, as $m^2 > 0$ by definition (subtlety…)
- Then, we can proceed as follows:

$$s^2 + m^2 = 0$$

$$s = \pm jm$$

$$\Theta_1 = e^{jm\theta}$$

$$\Theta_2 = e^{-jm\theta}$$

$$\Theta_1 = c_1\Theta_1 + c_2\Theta_2$$

$$\Theta = c_1 e^{jm\theta} + c_2 e^{-jm\theta}$$

Interesting: We have a **boundary** condition on *Θ*

We cannot wrap around the horizon twice, i.e. we can traverse at most $\boldsymbol{2\pi}$ radians

CASE 1

- In this case, we have $B > 0$, i.e. we can write $B = m^2$ for some m, as $m^2 > 0$ by definition (subtlety…)
- Then, we can proceed as follows:

$$s^2 + m^2 = 0$$
$$s = \pm jm$$

$$\Theta_1 = e^{jm\theta}$$
$$\Theta_2 = e^{-jm\theta}$$

$$\Theta_1 = c_1\Theta_1 + c_2\Theta_2$$

$$\Theta = c_1 e^{jm\theta} + c_2 e^{-jm\theta}$$

Interesting: We have a **boundary** condition on Θ

$$\Theta(\theta + 2\pi) = \Theta(\theta)$$

pedantry: $\theta \bmod 2\pi \equiv \theta$

CASE 1

- Let's use this boundary condition to simplify things:

$$\begin{aligned}\Theta(\theta + 2\pi) &= c_1 e^{jm\theta + 2\pi mj} + c_2 e^{-jm\theta - 2\pi mj} \\ &= c_1 e^{2\pi mj} e^{jm\theta} + c_2 e^{-2\pi mj} e^{-jm\theta}\end{aligned}$$

$$\Theta(\theta) = c_1 e^{jm\theta} + c_2 e^{-jm\theta}$$

- The **only** way that we can achieve equality is like this:

$$e^{2\pi mj} = e^{-2\pi mj} = 1$$

- This essentially gives us conditions on the value of m
- We now whip out Euler's identity to help us here

CASE 1

- Recall: $e^{j\theta} = \cos\theta + j\sin\theta$

$$e^{2\pi mj} = \cos 2\pi m + j\sin 2\pi m = 1$$
$$e^{-2\pi mj} = \cos 2\pi m - j\sin 2\pi m = 1$$

- Add the two equations together to get the condition:

$$2\cos 2\pi m = 1 + 1 = 2$$
$$\cos 2\pi m = 1$$

- However, the unit circle tells us that $\cos\theta = 1$ if and only if $\theta = 0$, 2π, -2π, 4π, -4π, etc. Thus, m **is an integer!**

CASE 1

- The solution is then just $\Theta(\theta) = Ce^{jm\theta}$ for some C
  - We can let m range over **both** positive and negative numbers ($m > 0$, $m < 0$)
  - Note that we do NOT allow $m = 0$ (as $B \neq 0$!)
  - We still keep a constant C out front, as it is some combination of c_1 and c_2

CASE 2

- This case is quite simple, really (and special):

$$s^2 = 0$$
$$s = 0$$

- From here, the solution set is just $\Theta(\theta) = Ce^0 = C$
- However, if we look at this carefully, this is just the same as the solution from Case 1, but with $m = 0$
- Thus, we can **loop** the solutions together into one:

$$\Theta(\theta) = Ce^{jm\theta} \qquad m \in \mathbb{R}$$

CASE 3

- This case actually cannot happen, but let's show why
- Assume $B < 0$, i.e. $B = -m^2$ for some m
- Then, the differential equation is pretty easy to solve:

$$s^2 - m^2 = 0$$
$$s = \pm m$$

$$\Theta = c_1 e^{m\theta} + c_2 e^{-m\theta}$$

$$\Theta_1 = e^{m\theta}$$
$$\Theta_2 = e^{-m\theta}$$

$$\Theta(\theta + 2\pi) = c_1 e^{2\pi m} e^{m\theta} + c_2 e^{-2\pi m} e^{-m\theta}$$
$$\Theta(\theta) = c_1 e^{m\theta} + c_2 e^{-m\theta}$$

$$\Theta_1 = c_1 \Theta_1 + c_2 \Theta_2$$

$$e^{2\pi m} = e^{-2\pi m} = 1$$

$$\boxed{m = 0}$$

CASE 3

- However, $m = 0$ is a contradiction, as then $B = 0$, but we know that $B < 0$!
- It follows that this case cannot happen, i.e. $B \geq 0$

THE AZIMUTHAL EQUATION

- Having accounted for the trichotomy, we actually have a very interesting solution to this equation:

$$\Theta(\theta) = Ce^{jm\theta}$$

- Moreover, m must be an integer!
  - We say that "m is quantized"
  - That is, m takes on integer multiples of some principal value, which in this case is 1

THE AZIMUTHAL EQUATION

- Having accounted for the trichotomy, we actually have a very interesting solution to this equation:

$$\Theta(\theta) = Ce^{jm\theta}$$

- Moreover, m must be an integer!
  - Because this is the azimuthal equation and m is quantized, we can call m a *quantum number* of ψ
  - In fact, m is the ***azimuthal quantum number***

THE POLAR EQUATION

- Now, time to solve the harder of the two equations:

$$\frac{1}{\Phi}\sin\varphi\frac{d}{d\varphi}\left(\sin\varphi\frac{d\Phi}{d\varphi}\right)+A\sin^2\varphi=B$$

- We know that $B=m^2$, where m is an integer, from solving the azimuthal equation, so we have:

$$\frac{1}{\Phi}\sin\varphi\frac{d}{d\varphi}\left(\sin\varphi\frac{d\Phi}{d\varphi}\right)+A\sin^2\varphi=m^2$$

THE POLAR EQUATION

- Bringing this equation to canonical form, we have:

$$\sin\varphi \frac{d}{d\varphi}\left(\sin\varphi \frac{d\Phi}{d\varphi}\right) + A\sin^2\varphi\Phi = m^2\Phi$$

$$\sin^2\varphi \frac{d^2\Phi}{d\varphi^2} + \sin\varphi\cos\varphi \frac{d\Phi}{d\varphi} + (A\sin^2\varphi - m^2)\Phi = 0$$

$$\frac{d^2\Phi}{d\varphi^2} + \frac{\cos\varphi}{\sin\varphi}\frac{d\Phi}{d\varphi} + \left(A - \frac{m^2}{\sin^2\varphi}\right)\Phi = 0$$

- This looks painful. Introduce a change of variable:

$$x = \cos\varphi$$

THE POLAR EQUATION

- Let's rewrite the equation, noting that the change of variable gives some *new function* $y(x)$, not $\Phi(\phi)$:

$$\frac{d\Phi}{d\varphi} = \frac{d\Phi}{dx}\frac{dx}{d\varphi} = -\sin\varphi\frac{d\Phi}{dx}$$

$$\frac{d^2\Phi}{d\varphi^2} = \frac{d}{d\varphi}\left(\frac{d\Phi}{d\varphi}\right) = \frac{d}{d\varphi}\left(-\sin\varphi\frac{d\Phi}{dx}\right) = \sin^2\varphi\frac{d^2\Phi}{dx^2} - \cos\varphi\frac{d\Phi}{dx}$$

$$\sin^2\varphi\frac{d^2\Phi}{dx^2} - \cos\varphi\frac{d\Phi}{dx} + \frac{\cos\varphi}{\sin\varphi}\left(-\sin\varphi\frac{d\Phi}{dx}\right) + \left(A - \frac{m^2}{\sin^2\varphi}\right)\Phi = 0$$

THE POLAR EQUATION

- Another step and we are at the canonical form:

$$\sin^2\varphi \frac{d^2\Phi}{dx^2} - 2\cos\varphi \frac{d\Phi}{dx} + \left(A - \frac{m^2}{\sin^2\varphi}\right)\Phi = 0$$

- However, recall that $\sin^2\phi + \cos^2\phi = 1$, so:

$$\sin\varphi = \sqrt{1-\cos^2\varphi} = \sqrt{1-x^2}$$

- Thus, the overall differential equation becomes this:

subtle:
transform
$\Phi(\phi)$ to $y(x)$

$$\boxed{(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left(A - \frac{m^2}{1-x^2}\right)y = 0}$$

note:
$\Phi = y(\cos\phi)$

THE POLAR EQUATION

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left(A - \frac{m^2}{1-x^2}\right)y = 0$$

- But this is precisely the *generalized Legendre equation* that we discussed earlier!
  - It is actually called ***associated Legendre equation***

when $A = l(l+1)$

- We need to find solutions to this equation (hopefully they are polynomial, so things are kept simple)
- How can we find solutions to such a difficult ODE?

ASSOCIATED LEGENDRE EQUATION

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left(A - \frac{m^2}{1-x^2}\right)y = 0$$

- It is not trivial to solve this, so let us instead solve an "easier" version of this problem first: $m = 0$

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + Ay = 0$$

- This is called the ***Legendre equation***
    - Less intimidating, but still not easy to solve

LEGENDRE EQUATION

- We want polynomial solutions, so let us assume one:

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \sum_{n=0}^{\infty} a_n x^n$$

- We will use this form to develop find the necessary constants a_n so that we can find the polynomial
- We need to plug this series solution back into the differential equation and develop a condition
  - This condition will actually be a *recurrence relation*

LEGENDRE EQUATION

- We first need to find the derivatives as series:

$$\frac{dy}{dx} = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

- We can now plug this in to get a simpler form:

$$(1-x^2)\left(\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}\right) - 2x\left(\sum_{n=0}^{\infty} n a_n x^{n-1}\right) + A\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0$$

LEGENDRE EQUATION

- Now comes a round of algebra:

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} n(n-1)a_n x^n - 2\sum_{n=0}^{\infty} n a_n x^n + A\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} ((-n^2+n)a_n x^n - 2n a_n x^n + A a_n x^n) = 0$$

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} (A - n(n+1))a_n x^n = 0$$

- We need a way to bring that exponent down in the left-most term to make things easier here

LEGENDRE EQUATION

- Some clever trick that helps us: *re-indexing*!

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^{n}$$

- Thus, the equation ends up becoming this:

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^{n} + \sum_{n=0}^{\infty} (A - n(n+1))a_n x^n = 0$$

LEGENDRE EQUATION

trivial solution

- Grouping everything together, we have:

$$\sum_{n=0}^{\infty}((A-n(n+1))a_n+(n+2)(n+1)a_{n+2})x^n=0$$

- However, we cannot have $x=0$, as this leads to $y=0$
- Thus, we must have that the inner terms vanish!

$$(A-n(n+1))a_n+(n+2)(n+1)a_{n+2}=0$$

- This leads to the following recurrence relation:

$$\boxed{a_{n+2}=\frac{n(n+1)-A}{(n+1)(n+2)}a_n}$$

LEGENDRE EQUATION

- We need the solution to be a **finite** polynomial
- To achieve this, we need to ensure that the series terminates, i.e. there is some number l such that y is at most an l^{th}-degree polynomial
  - Moreover, we want the *least* such l, i.e. the least-upper bound of the polynomial (tightest-fitting)
- Formally, we can state this as such: There exists an l such that $a_l = 0$ (and thus $a_{l+2} = 0$, $a_{l+4} = 0$, etc.)

LEGENDRE EQUATION

- Then, because $a_l = 0$, we see that the following holds:

$$a_\ell = \frac{\ell(\ell+1) - A}{(\ell+1)(\ell+2)} a_{\ell-2} = 0$$

- This is **only** possible if the following is true: $A = l(l+1)$
- This is exciting! It tell us a lot of things:
  - A is also an integer (the product of two integers)
  - Is it true that the solution y ends up quantized on l?
  - What is the nature of y?

(*captures rotational symmetry, so magnetic quantum number*)

LEGENDRE EQUATION

- We need to ensure that the series actually does converge, so we will apply the ratio test here:

$$\lim_{n\to\infty}\left|\frac{s_{n+2}}{s_n}\right| < 1$$

- Note that s_n denotes the n^{th} term in the sequence:

$$\lim_{n\to\infty}\left|\frac{a_{n+2}x^{n+2}}{a_n x^n}\right| = \lim_{n\to\infty}\left|\frac{n(n+1)-\ell(\ell+1)}{(n+1)(n+2)}x^2\right| = \lim_{n\to\infty}\left|\frac{n}{n+2} - \frac{\ell(\ell+1)}{(n+1)(n+2)}\right|x^2$$

$$= |1-0|\,x^2 = x^2 < 1$$

LEGENDRE EQUATION

- However, recall that $x = \cos\phi$, so we end up with:

$$\cos^2\varphi < 1$$

- This is equivalent to saying the following:

$$-1 < \cos\varphi < 1$$

- Then, because -1 $\leq \cos\phi \leq 1$ usually, we see that we essentially permit all values of ϕ except for 0 and π
- These are the "poles" in the spherical coordinates:
  - This has to do with complex analysis: **singularities**!

LEGENDRE EQUATION

- The solution to the Legendre equation is going to depend on l, so let us denote it as $y = P_l(x)$
- We want to have an orthonormal basis for the fundamental solution space spanned by $P_l(x)$
  - Translation: $P_l(1) = 1$ to make it of unit size
- Also, it is important to denote the degree of $P_l(x)$:
  - $P_l(x)$ is at most an l^{th}-degree polynomial!

NORMALIZATION

- Why normalize the solutions?
  - The solutions $Y(\theta, \phi)$ are the spherical harmonics
  - Characterize the direction/shape of solutions
  - Radial solution characterizes length/outward projection of solutions
  - Direction vectors should be of unit length

RECURRENCE RELATION

- We need to re-factorize the recurrence relation:

$$
\begin{aligned}
a_{n+2} &= \frac{n(n+1) - \ell(\ell+1)}{(n+1)(n+2)} \\
&= \frac{n^2 + n - \ell^2 - \ell}{(n+1)(n+2)} \\
&= \frac{n^2 - \ell^2 + n - \ell}{(n+1)(n+2)} \\
&= \frac{(n+\ell)(n-\ell) + n - \ell}{(n+1)(n+2)} \\
&= \frac{(n+\ell+1)(n-\ell)}{(n+1)(n+2)}
\end{aligned}
$$

RECURRENCE RELATION

- Then, we know that the following is true:

$$a_{\ell+2} = a_{\ell+4} = \cdots = 0$$

- It follows that $P_l(x)$ is of the following form:

$$P_\ell(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_\ell x^\ell + a_{\ell+1} x^{\ell+1} + a_{\ell+3} x^{\ell+3} + \cdots$$

- However, we know that $P_l(x)$ is at most an l^{th}-degree polynomial, so we must have the following!

$$a_{\ell+1} = a_{\ell+3} = \cdots = 0$$

- This, in turn, implies predecession: $a_{\ell-1} = a_{\ell-3} = \cdots = 0$

RECURRENCE RELATION

- There are two possible cases:
  - Either $P_l(x)$ has zeroes for all *even*-subscript coefficients, or it has them for all *odd*-subscripts
- Mathematically speaking:
  - $P_l(x)$ is **either** an *even* or *odd*-degree polynomial!
- Another concise way of saying this:

$$\begin{cases} a_0 = 0, & \ell \text{ is odd} \\ a_1 = 0, & \ell \text{ is even} \end{cases}$$

Cascade the base-case through the rest of the coefficients

CASE ANALYSIS

- We want to break this up into two arguments
- We will consider odd-degree and even-degree polynomials separately:
  - l is odd
  - l is even

CASE 1: ℓ IS ODD

- We have that $a_0 = 0$ by implication
- As a result, half the terms drop out!
- We end up with the following result:

$$P_\ell(x) = a_1 x + a_3 x^3 + \cdots + a_\ell x^\ell = \sum_{n=0}^{\frac{\ell-1}{2}} a_{2n+1} x^{2n+1}$$

- We see here that $2n+1$ is guaranteed to be odd
- We also know that $(l–1)/2$ has to be an integer
  - l is odd, so $l = 2k+1$, i.e. $(2k+1–1)/2 = 2k/2 = k$ $(\in \mathbb{Z})$

CASE 1: ℓ IS ODD

- Let us step through the recursion:

$$a_{2n+1} = \frac{(2n-1+\ell+1)(2n-1-\ell)}{(2n-1+1)(2n-1+2)} a_{2n-1} = \frac{(2n+\ell)(2n-1-\ell)}{2n(2n+1)} a_{2n-1}$$

$$a_{2n-1} = \frac{(2n-3+\ell+1)(2n-3-\ell)}{(2n-3+1)(2n-3+2)} a_{2n-3} = \frac{(2n+\ell-2)(2n-3-\ell)}{(2n-1)(2n-2)} a_{2n-3}$$

etc., etc., etc.

$$a_{2n+1} = \frac{(2n+\ell)(2n-\ell-1)}{2n(2n+1)} \frac{(2n+\ell-2)(2n-\ell-3)}{(2n-1)(2n-3)} \cdots \frac{(2+\ell)(1-\ell)}{(3)(2)} a_1$$

$$a_{2n+1} = \frac{\prod_{k=0}^{n-1} (2n+\ell-2k)(2n-\ell-(2k+1))}{\prod_{k=0}^{2n} (k+1)} a_1$$

CASE 1: ℓ IS ODD

- Simplification:

$$a_{2n+1} = \frac{\prod_{k=0}^{n-1} (2n+\ell-2k)(2n-\ell-(2k+1))}{\prod_{k=0}^{2n} (k+1)} a_1$$

$$= \frac{\prod_{k=0}^{n-1} (2n-2k+\ell)(2n-2k-\ell-1)}{\prod_{k=0}^{2n} (k+1)} a_1$$

$$= \frac{a_1}{(2n+1)!} \prod_{k=0}^{n-1} (2n-2k+\ell)(2n-2k-\ell-1)$$

CASE 1: ℓ IS ODD

- Solving the subproblem:

$$\begin{aligned}&\prod_{k=0}^{n-1}(2n-2k+\ell)(2n-2k-\ell-1)\\ &\quad=\prod_{k=0}^{n-1}(2n-2k+\ell)\prod_{k=0}^{n-1}(2n-2k-\ell-1)\\ &\quad=R\cdot S\end{aligned}$$

- We now separately figure out what R and S are
  - i.e. this is actually algebraic separation of variables

CASE 1: ℓ IS ODD

- Let's first find R and then do a similar process for S:
  - This will actually be quite challenging (i.e. tedious)
  - We will need to cover some *extra* things first
  - Exponential distributions and the gamma function
  - Having this setup will be useful when figuring out how to simplify R

THE GAMMA FUNCTION

- This is a detour!
- Consider the following definition:

$$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} dx$$

- This is actually a *complete* definition, in that we have no simpler form in general!
- But there are cases to consider!
  - t is discrete!
  - t is continuous?

THE GAMMA FUNCTION

- The case of t being discrete (i.e. an integer):

$$\begin{aligned}\Gamma(1) = \int_0^\infty x^{1-1}e^{-x}dx &= \int_0^\infty e^{-x}dx \\ &= \left[-e^{-x}\right]_0^\infty \\ &= 0-(-1) = 1\end{aligned}$$

integration by parts

$$\begin{aligned}\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}dx &= -x^{t-1}e^{-x}|_0^\infty + (t-1)\int_0^\infty x^{t-2}e^{-x}dx \\ &= (-0+0) + (t-1)\int_0^\infty x^{(t-1)-1}e^{-x}dx \\ &= (t-1)\Gamma(t-1)\end{aligned}$$

THE GAMMA FUNCTION

- The case of t being discrete (i.e. an integer)
- We now have an *inductive definition*:
  - $\Gamma(1) = 1$
  - $\Gamma(t) = (t-1)\Gamma(t-1)$
  - As a result, we really see that $\Gamma(t) = (t-1)!$
  - This is exciting!
- Natural extensions?
  - Does this extend to continuous t, negative t, etc.?

yes, by a process called analytic continuation

THE GAMMA FUNCTION

- Some more cases that we will need to make use of:
  - The case of $t = 1/2$ will be particularly important
  - We use it as a basis to define the Gamma function over *half*-integer intervals!
  - Turns out that $\Gamma(t) = \sqrt{\pi}$
  - Proof?

THE GAMMA FUNCTION

$$
\begin{aligned}
\Gamma\left(\frac{1}{2}\right) = \int_0^\infty x^{\frac{1}{2}-1}e^{-x}dx &= \int_0^\infty x^{-\frac{1}{2}}e^{-x}dx \\
&= -x^{-\frac{1}{2}}e^{-x}|_0^\infty - \frac{1}{2}\int_0^\infty x^{-\frac{3}{2}}e^{-x}dx \\
&= -\frac{1}{2}\int_0^\infty x^{-\frac{3}{2}}e^{-x}dx \\
&= -\frac{1}{2}\int_0^\infty x^{\left(-\frac{1}{2}\right)-1}e^{-x}dx \\
&= -\frac{1}{2}\Gamma\left(-\frac{1}{2}\right)
\end{aligned}
$$

THE GAMMA FUNCTION

- We can iteratively repeat this process:

$$\Gamma\left(\frac{1}{2}\right) = -\frac{1}{2}\Gamma\left(-\frac{1}{2}\right)$$

$$\Gamma\left(-\frac{1}{2}\right) = -\frac{3}{2}\Gamma\left(-\frac{3}{2}\right)$$

$$\Gamma\left(-\frac{3}{2}\right) = -\frac{5}{2}\Gamma\left(-\frac{5}{2}\right)$$

$$\vdots$$

THE GAMMA FUNCTION

- We can iteratively repeat this process:

$$\Gamma\left(\frac{1}{2}\right) = ???$$

$$\Gamma\left(-\frac{1}{2}\right) = -2\Gamma\left(\frac{1}{2}\right)$$

$$\Gamma\left(-\frac{3}{2}\right) = -\frac{2}{3}\Gamma\left(-\frac{1}{2}\right) = \frac{4}{3}\Gamma\left(\frac{1}{2}\right)$$

$$\Gamma\left(-\frac{5}{2}\right) = -\frac{2}{5}\Gamma\left(-\frac{3}{2}\right) = -\frac{8}{15}\Gamma\left(\frac{1}{2}\right)$$

$$\Gamma\left(-\frac{7}{2}\right) = -\frac{2}{7}\Gamma\left(-\frac{5}{2}\right) = \frac{16}{105}\Gamma\left(\frac{1}{2}\right)$$

$$\vdots$$

THE GAMMA FUNCTION

- Similarly for *positive* half-integers:

$$\boxed{\Gamma\left(\frac{1}{2}\right) = ???}$$

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{4}\Gamma\left(\frac{1}{2}\right)$$

$$\Gamma\left(\frac{7}{2}\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \frac{15}{8}\Gamma\left(\frac{1}{2}\right)$$

$$\vdots$$

THE GAMMA FUNCTION

- The *basis* value of half-integer Gamma function:

$$\begin{aligned}
\Gamma\left(\frac{1}{2}\right) &= \int_0^\infty x^{-\frac{1}{2}} e^{-x} dx \\
&= \int_0^\infty 2e^{-u^2} du \\
&= 2\int_0^\infty e^{-u^2} du \\
&= \int_{-\infty}^\infty e^{-u^2} du \\
&= I
\end{aligned}$$

THE GAMMA FUNCTION

$$
\begin{aligned}
I^2 &= \int_{-\infty}^{\infty} e^{-u^2} du \int_{-\infty}^{\infty} e^{-v^2} dv \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-u^2-v^2} du dv \\
&= \int_{0}^{2\pi} \int_{0}^{\infty} e^{-r^2} r dr d\theta \\
&= \int_{0}^{2\pi} d\theta \int_{0}^{\infty} r e^{-r^2} dr \\
&= 2\pi \left(0 - \left(\frac{1}{2}\right)\right) \\
&= \pi
\end{aligned}
$$

$$\boxed{\Gamma\left(\frac{1}{2}\right) = I = \sqrt{\pi}}$$

THE GAMMA FUNCTION

- We can use this basis to find a general formula:

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\vdots$$

$$\Gamma\left(-\frac{7}{2}\right) = \frac{16}{105}\sqrt{\pi}$$

$$\Gamma\left(-\frac{5}{2}\right) = -\frac{8}{15}\sqrt{\pi}$$

$$\Gamma\left(-\frac{3}{2}\right) = \frac{4}{3}\sqrt{\pi}$$

$$\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$$

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}$$

$$\Gamma\left(\frac{5}{2}\right) = \frac{3}{4}\sqrt{\pi}$$

$$\Gamma\left(\frac{7}{2}\right) = \frac{15}{8}\sqrt{\pi}$$

$$\vdots$$

THE GAMMA FUNCTION

$$
\begin{aligned}
\Gamma\left(\frac{1}{2}-n\right) &= \frac{(-2)^n}{\prod_{k=0}^{n-1}(2k+1)}\sqrt{\pi} \\
&= \frac{(-2)^n \prod_{k=0}^{n}(2k)}{\prod_{k=0}^{n} k}\sqrt{\pi} \\
&= \frac{(-2)^n 2^n \prod_{k=0}^{n-1} k}{(2n)!}\sqrt{\pi} \\
&= \frac{(-4)^n n!}{(2n)!}\sqrt{\pi}
\end{aligned}
$$

THE GAMMA FUNCTION

$$\Gamma\left(\frac{1}{2}+n\right) = \frac{\prod_{k=0}^{n-1}(2k+1)}{2^n}\sqrt{\pi}$$

$$= \frac{\prod_{k=0}^{n} k}{2^n \prod_{k=0}^{n}(2k)}\sqrt{\pi}$$

$$= \frac{(2n)!}{2^n 2^n \prod_{k=0}^{n-1} k}\sqrt{\pi}$$

$$= \frac{(2n)!}{4^n n!}\sqrt{\pi}$$

ADEQUATELY PREPARED

- We can now find R and S using the gamma function:

$$R = \prod_{k=0}^{n-1} (2n - 2k + \ell)$$
$$= 2^n \prod_{k=0}^{n-1} \left(n - k + \frac{\ell}{2}\right)$$
$$= \frac{(-2)^n \Gamma\left(-\frac{\ell}{2}\right)}{\Gamma\left(-\frac{\ell}{2} - n\right)}$$

$$S = \prod_{k=0}^{n-1} (2n - 2k - \ell - 1)$$
$$= 2^n \prod_{k=0}^{n-1} \left(n - k - \frac{\ell + 1}{2}\right)$$
$$= \frac{(-2)^n \Gamma\left(\frac{\ell+1}{2}\right)}{\Gamma\left(\frac{\ell+1}{2} - n\right)}$$

ADEQUATELY PREPARED

- We can now find R and S using the gamma function:

$$R = \prod_{k=0}^{n-1} (2n - 2k + \ell)$$

$$= 2^n \prod_{k=0}^{n-1} \left(n - k + \frac{\ell}{2}\right)$$

$$= \frac{(-2)^n \Gamma\left(-\frac{\ell}{2}\right)}{\Gamma\left(-\frac{\ell}{2} - n\right)}$$

$$S = \prod_{k=0}^{n-1} (2n - 2k - \ell - 1)$$

$$= 2^n \prod_{k=0}^{n-1} \left(n - k - \frac{\ell + 1}{2}\right)$$

$$= \frac{(-2)^n \Gamma\left(\frac{\ell+1}{2}\right)}{\Gamma\left(\frac{\ell+1}{2} - n\right)}$$

$$\boxed{R \cdot S = \frac{4^n \Gamma\left(-\frac{\ell}{2}\right) \Gamma\left(\frac{\ell+1}{2}\right)}{\Gamma\left(\frac{\ell+1}{2} - n\right) \Gamma\left(-\frac{\ell}{2} - n\right)}}$$

CASE 1: ℓ IS ODD

- And now, we return to our original problem:

$$a_{2n+1} = \frac{a_1}{(2n+1)!} R \cdot S$$
$$= \frac{4^n \Gamma\left(-\frac{\ell}{2}\right) \Gamma\left(\frac{\ell+1}{2}\right)}{\Gamma\left(\frac{\ell+1}{2} - n\right) \Gamma\left(-\frac{\ell}{2} - n\right) (2n+1)!} a_1$$

- It turns out that this can be rearranged as factorials:

$$P_\ell(x) = \frac{1}{2^\ell} \sum_{n=0}^{\frac{\ell-1}{2}} \frac{(2\ell - 2n)!}{n!(\ell - n)!(\ell - 2n)!} x^{\ell - 2n}$$

$a_1 = 2^l$ is the normalization constant here

CASE 2: ℓ IS EVEN

- We can actually model this case using the other one!
  - Note that if l is even, then instead of $(l-1)/2$, we need to *raise* it to the next integer: $\lceil \mathbf{(l-1)/2} \rceil$
- We then combine both into a *single* case:

normalization factor

$$P_\ell(x) = \boxed{\frac{1}{2^\ell}} \sum_{n=0}^{\lceil \frac{\ell-1}{2} \rceil} \frac{(2\ell-2n)!}{n!(\ell-n)!(\ell-2n)!} x^{\ell-2n}$$

- We now rewrite this as a series of the form $\sum_{n=0}^{\ell}$

LEGENDRE EQUATION

- Notice the following *boundary condition*: $n = \left\lceil \frac{\ell - 1}{2} \right\rceil$
- We notice that there are two possible cases:

$$\begin{aligned} x^{\ell-2n} &= x^{\ell-2\left\lceil \frac{\ell-1}{2} \right\rceil} \\ &= x^{\ell-(\ell-1)} \\ &= x \end{aligned}$$

odd terms

$$\begin{aligned} x^{\ell-2n} &= x^{\ell-2\left\lceil \frac{\ell-1}{2} \right\rceil} \\ &= x^{\ell-\ell} \\ &= 1 \end{aligned}$$

even terms

- We should also take note of the following tricks:

$$\frac{\ell!}{n!(\ell-n)!} = \binom{\ell}{n} \qquad \frac{(2\ell-2n)!}{\ell!(\ell-2n)!} = \binom{2\ell-2n}{\ell}$$

LEGENDRE EQUATION

- We can now introduce the following:

$$
\begin{aligned}
\frac{(2\ell-2n)!}{n!(\ell-n)!(\ell-2n)!} &= \frac{1}{n!(\ell-n)!}\,\frac{(2\ell-2n)!}{(\ell-2n)!} \\
&= \frac{\ell!}{n!(\ell-n)!}\,\frac{\ell!(2\ell-2n)!}{(\ell-2n)!} \\
&= \binom{\ell}{n}\binom{2\ell-2n}{\ell}
\end{aligned}
$$

LEGENDRE EQUATION

- And now, let's rewrite the actual polynomial solution:

$$P_\ell(x) = \frac{1}{2^\ell} \sum_{n=0}^{\lceil \frac{\ell-1}{2} \rceil} \frac{(2\ell-2n)!}{n!(\ell-n)!(\ell-2n)!} x^{\ell-2n}$$

$$= \frac{1}{2^\ell} \sum_{n=0}^{\lceil \frac{\ell-1}{2} \rceil} \binom{\ell}{n} \binom{2\ell-2n}{\ell} x^{\ell-2n}$$

factoring the binomial coefficient

$$= \frac{2^\ell 2^\ell}{2^\ell} \sum_{n=0}^{\ell} \binom{\ell}{n} \binom{\frac{\ell+n-1}{2}}{\ell} x^n$$

due to limits on $n = \lceil (l-1)/2 \rceil$

$$= 2^\ell \sum_{n=0}^{\ell} \binom{\ell}{n} \binom{\frac{\ell+n-1}{2}}{\ell} x^n$$

LEGENDRE EQUATION

- And now, let's rewrite the actual polynomial solution:

$$P_\ell(x) = \frac{1}{2^\ell} \sum_{n=0}^{\lceil \frac{\ell-1}{2} \rceil} \frac{(2\ell-2n)!}{n!(\ell-n)!(\ell-2n)!} x^{\ell-2n}$$

factoring the binomial coefficient

$$= \frac{1}{2^\ell} \sum_{n=0}^{\lceil \frac{\ell-1}{2} \rceil} \binom{\ell}{n} \binom{2\ell-2n}{\ell} x^{\ell-2n}$$

due to limits on $n = \lceil(l\text{-}1)/2\rceil$

$$= \frac{2^\ell 2^\ell}{2^\ell} \sum_{n=0}^{\ell} \binom{\ell}{n} \binom{\frac{\ell+n-1}{2}}{\ell} x^n$$

transform

$$= 2^\ell \sum_{n=0}^{\ell} \binom{\ell}{n} \binom{\frac{\ell+n-1}{2}}{\ell} x^n$$

LEGENDRE POLYNOMIALS

- We now have a solution to the Legendre equation!

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \ell(\ell+1)y = 0$$

weights

$$y(x) = \sum_{\ell=0}^{\infty} c_\ell P_\ell(x)$$

any linear combination of Legendre polynomials

$$\boxed{P_\ell(x) = 2^\ell \sum_{n=0}^{\ell} \binom{\ell}{n} \binom{\frac{\ell+n-1}{2}}{\ell} x^n}$$

LEGENDRE POLYNOMIALS

- What do these actually ***evaluate*** to?

$$P_0(x) = 2^0 \left[\binom{0}{0} \binom{-\frac{1}{2}}{0} x^0 \right] = (1) \frac{0!}{0! \cdot 0!} \frac{\left(-\frac{1}{2}\right)!}{0! \cdot \left(-\frac{1}{2}\right)!} (1) = 1$$

$$P_1(x) = 2^1 \left[\binom{1}{0} \binom{0}{1} x^0 + \binom{1}{1} \binom{\frac{1}{2}}{1} x^1 \right] = 2 \left[\frac{1!}{0! \cdot 1!} \frac{0!}{1! \cdot (-1)!} (1) + \frac{1!}{1! \cdot 0!} \frac{\left(\frac{1}{2}\right)!}{1! \cdot \left(-\frac{1}{2}\right)!} (x) \right]$$

$$= 2(0) + \frac{\left(\frac{1}{2}\right)!}{\left(-\frac{1}{2}\right)!} (2x) = 2 \left[\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \right] x$$

$$= 2 \left[\frac{\left(\frac{2!}{4^1 \cdot 1!} \right) \sqrt{\pi}}{\sqrt{\pi}} \right] x = x$$

...ad nauseam, ad infinitum...

Is there a <u>cleaner</u> way?

normalization is <u>built-in</u>

this is why we introduced it!

LEGENDRE POLYNOMIALS

- What do these actually ***evaluate*** to?

$(-1)! = \infty$

$$P_0(x) = 2^0\left[\binom{0}{0}\binom{-\frac{1}{2}}{0}x^0\right] = (1)\frac{0!}{0!\cdot 0!}\frac{\left(-\frac{1}{2}\right)!}{0!\cdot\left(-\frac{1}{2}\right)!}(1) = 1$$

$$P_1(x) = 2^1\left[\binom{1}{0}\binom{0}{1}x^0 + \binom{1}{1}\binom{\frac{1}{2}}{1}x^1\right] = 2\left[\frac{1!}{0!\cdot 1!}\frac{0!}{1!\cdot(-1)!}(1) + \frac{1!}{1!\cdot 0!}\frac{\left(\frac{1}{2}\right)!}{1!\cdot\left(-\frac{1}{2}\right)!}(x)\right]$$

$$= 2(0) + \frac{\left(\frac{1}{2}\right)!}{\left(-\frac{1}{2}\right)!}(2x) = 2\left[\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\right]x$$

$$= 2\left[\frac{\left(\frac{2!}{4^1\cdot 1!}\right)\sqrt{\pi}}{\sqrt{\pi}}\right]x = x$$

...ad nauseam, ad infinitum...

Is there a <u>cleaner</u> way?

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LEGENDRE POLYNOMIALS

- Another method of finding these Legendre polynomials is by using a *generating function*
- Requires use of power series and Taylor expansions
- It turns out that the following is true:

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{\ell=0}^{\infty} P_\ell(x)t^\ell$$

$$(|t| < 1)$$

- That is, *the coefficients* of the Taylor expansion of this function about 0 *form the Legendre polynomials*

LEGENDRE POLYNOMIALS

- We can use the generating function to develop a general recursive method for finding polynomials!

$$\frac{\partial}{\partial t}\left[\frac{1}{\sqrt{1-2xt+t^2}}\right] = \frac{\partial}{\partial t}\sum_{\ell=0}^{\infty} P_\ell(x)t^\ell$$

$$\frac{x-t}{(1-2xt+t^2)^{3/2}} = \sum_{\ell=1}^{\infty} \ell P_\ell(x)t^{n-1}$$

$$\frac{x-t}{\sqrt{1-2xt+t^2}} = (1-2xt+t^2)\sum_{\ell=1}^{\infty} \ell P_\ell(x)t^{\ell-1}$$

LEGENDRE POLYNOMIALS

- We can use the generating function to develop a general recursive method for finding polynomials!

$$(x-t)\sum_{\ell=0}^{\infty} P_\ell(x)t^\ell = (1-2xt+t^2)\sum_{\ell=1}^{\infty} \ell P_\ell(x)t^{\ell-1}$$

$$\sum_{\ell=0}^{\infty} xP_\ell(x)t^\ell - \sum_{\ell=0}^{\infty} P_\ell(x)t^{\ell+1} = \sum_{\ell=1}^{\infty} \ell P_\ell(x)t^{\ell-1} - 2x\sum_{\ell=1}^{\infty} \ell P_\ell(x)t^\ell + \sum_{\ell=1}^{\infty} \ell P_\ell(x)t^{\ell+1}$$

$$\sum_{\ell=0}^{\infty}(2\ell+1)xP_\ell(x)t^\ell = \sum_{\ell=1}^{\infty} \ell P_\ell(x)t^{\ell-1} + \sum_{\ell=1}^{\infty}(\ell+1)P_\ell(x)t^{\ell+1}$$

$$(2\ell+1)xP_\ell(x) = \ell P_{\ell-1}(x) + (\ell+1)P_{\ell+1}(x)$$

$$\boxed{(\ell+1)P_{\ell+1}(x) = (2\ell+1)xP_\ell(x) - \ell P_{\ell-1}(x)}$$

LEGENDRE POLYNOMIALS

- We can generate a list of the Legendre polynomials:

$$L_0(x) = 1$$

$$L_1(x) = x$$

$$L_2(x) = -\frac{1}{2} + \frac{3}{2}x^2$$

$$L_3(x) = -\frac{3}{2}x + \frac{5}{2}x^3$$

$$L_4(x) = \frac{3}{8} - \frac{15}{4}x^2 + \frac{35}{8}x^4$$

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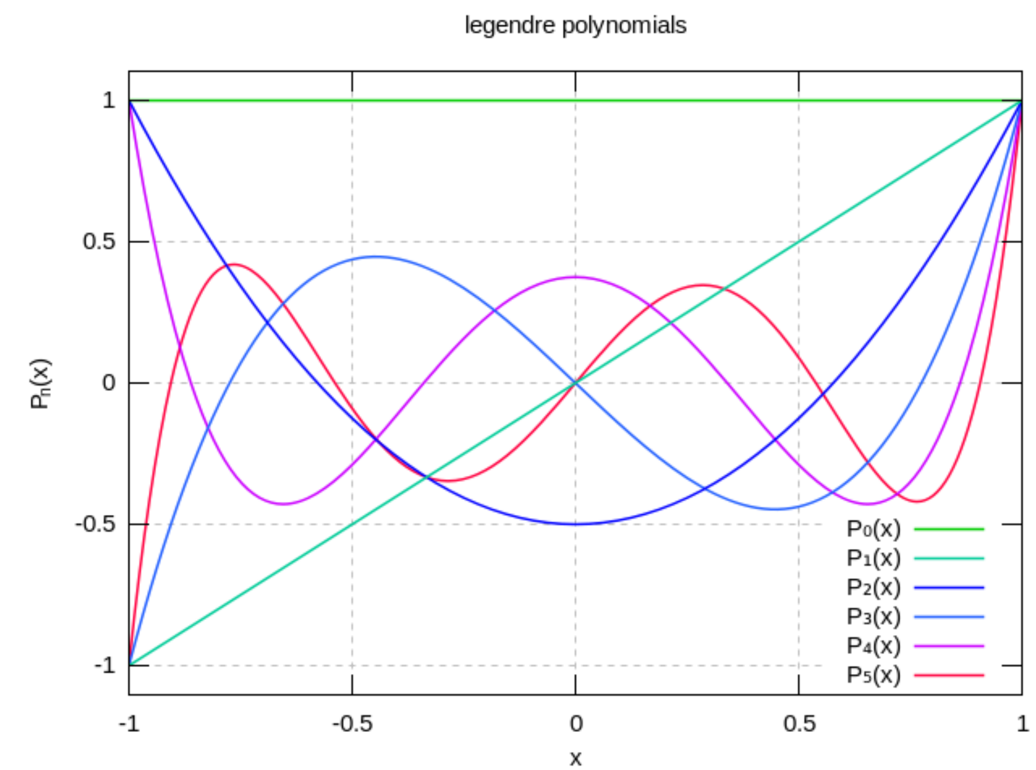

THE ASSOCIATED EQUATION

- All of that lovely machinery…
- …we still never ended up solving the actual polar equation, as we still need to solve the associated Legendre equation!
- How can we apply what we know already?
  - Some type of "undetermined coefficients"?
  - Varying the parameters?

THE ASSOCIATED EQUATION

- All of that lovely machinery…
- …we still never ended up solving the actual polar equation, as we still need to solve the associated Legendre equation! $(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left(A - \frac{m^2}{1-x^2}\right)y = 0$
- How can we apply what we know already?
  - Some type of "undetermined coefficients"?
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THE ASSOCIATED EQUATION

- We have a term of the form $\dfrac{m^2 y}{1-x^2}$ … perhaps we could get rid of the pesky denominator?
  - One idea is to have (1– x^2) built into y
  - Would need enough (1– x^2) terms to survive up to m differentiation rounds with respect to x
  - Thus, we would need $m/2$ of these terms:

$$y = (1 - x^2)^{m/2}$$

undetermined as of yet

  - In ***general***, there could be more: $y = (1 - x^2)^{m/2} w(x)$

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random reminder: we had $x = \cos\phi$

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- In ***general***, there could be more: $y = (1 - x^2)^{m/2} w(x)$

THE ASSOCIATED EQUATION

- Once we introduce this substitution for a possible solution, we want to actually *determine* the $w(x)$ term
- We can do this by rewriting the differential equation:

$$
\begin{aligned}
y' &= \frac{d}{dx}\left((1-x^2)^{m/2}w\right) \\
&= (1-x^2)^{m/2}w' + \frac{m}{2}(1-x^2)^{\frac{m}{2}-1}w \cdot (-2x) \\
&= (1-x^2)^{\frac{m}{2}-1}((1-x^2)w' - mxw)
\end{aligned}
$$

THE ASSOCIATED EQUATION

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- We can do this by rewriting the differential equation:

$$
\begin{aligned}
y'' &= \frac{d}{dx}\left(\frac{dy}{dx}\right) \\
&= \left(\frac{m}{2}-1\right)(1-x^2)^{\frac{m}{2}-2}\left[(1-x^2)w' - mxw\right] + (1-x^2)^{\frac{m}{2}-1}\left[(1-x^2)w'' - 2xw' - mxw' - mw\right] \\
&= (1-x^2)^{\frac{m}{2}-2}\left[\left(\frac{m}{2}-1\right)(1-x^2)w' - \left(\frac{m}{2}-1\right)mxw + (1-x^2)^2 w'' - (1-x^2)(m+2)xw' - (1-x^2)mw\right]
\end{aligned}
$$

- We can now convert the associated Legendre equation into one we can solve to find $w(x)$

THE ASSOCIATED EQUATION

- Note: Because $x = \cos\phi$, and we said that $\phi \neq 0$ or π, we can see that we are not allowing $x = \pm 1$
- This is good, because it means that $y = (1 - x^2)^{m/2}$ can never be 0, i.e. we have a non-trivial solution
- This is why we had that restriction imposed earlier!

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left(A - \frac{m^2}{1-x^2}\right)y = 0$$

reduces to

$$(1-x^2)w'' - 2(m+1)xw' + (\ell(\ell+1) - m(m+1))w = 0$$

THE ASSOCIATED EQUATION

- How can we solve this *new* equation for w?
- Let us return to the original Legendre equation!

$$(1-x^2)\xi'' - 2x\xi' + \ell(\ell+1)\xi = 0$$

- Note that $\xi(x)$ is a solution to the Legendre equation
- Let us now differentiate m times, as discussed earlier:

$$(1-x^2)\xi^{(m+2)} - 2x\xi^{(m+1)} + \ell(\ell+1)\xi^{(m)} - 2mx\xi^{(m+1)} - m^2\xi^{(m)} - m\xi^{(m)} = 0$$

$$(1-x^2)\xi^{(m+2)} - 2(m+1)x\xi^{(m+1)} + (\ell(\ell+1) - m(m+1))\xi^{(m)} = 0$$

THE ASSOCIATED EQUATION

- But this is precisely what we wanted, with $w = \xi^m \ldots$
- Thus, we see the following relationship plays out:
  - The solution to the associated Legendre equation involves one part of a solution to the regular one
  - …and one part of an adjustment term $(1 - x^2)^{m/2}$

$$w(x) = \frac{d^m}{dx^m}\left[P_\ell(x)\right]$$

THE ASSOCIATED EQUATION

- And so we have the aptly-named associated Legendre polynomials!
- We see that the solutions depend on both m and l, so we will denote these as $y(x) = P_l{}^m(x)$

$$P_l^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}\left[P_l(x)\right]$$

- Recall that the original differential equation had a m^2 term in it… thus, ***sign*** of m makes no difference:

$$P_l^m(x) = P_l^{-m}(x)$$

this symmetry in m is really helpful!

THE ASSOCIATED EQUATION

- To properly extend this idea, we realize that we cannot differentiate a negative number of times if m is negative, so we impose a constraint:

$$P_l^m(x) = (1-x^2)^{\frac{|m|}{2}} \frac{d^{|m|}}{dx^{|m|}} [P_l(x)]$$

- We now also realize that differentiation loses the "phase" of the solution, i.e. the relative sign:

$$P_l^m(x) = (-1)^m (1-x^2)^{\frac{|m|}{2}} \frac{d^{|m|}}{dx^{|m|}} [P_l(x)]$$

THE ASSOCIATED EQUATION

- Notice that if $|m| > l$, then the derivative goes to 0, as $P_l(x)$ is at most an l^{th}-degree polynomial!
- Thus, to prevent this "trivial solution", we take $|m| \leq l$, or $-l \leq m \leq l$
- This is a restriction you may have seen before!
    - Physics
    - Chemistry
    - Exciting!!

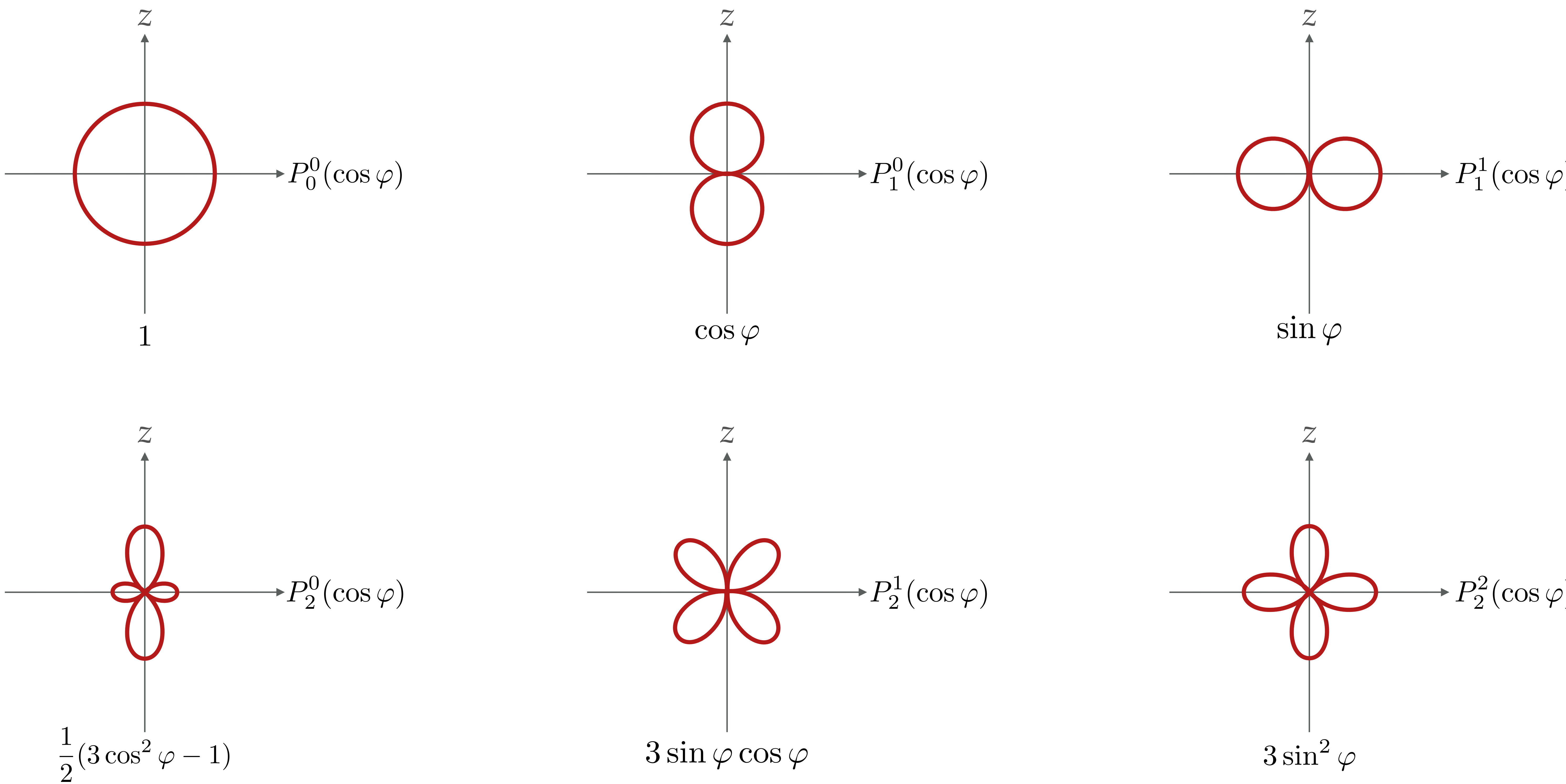
THE POLAR EQUATION

- We thus have our complete solution to the polar equation that we had initially set out to solve
  - Encountered many obstacles along the way
  - Learned about many *other* functions :)

$$\boxed{\Phi(\varphi) = P_\ell^m(x) = P_\ell^m(\cos\varphi)}$$

  - What does this even look like?
    - 2 degrees of freedom in l and m, so 2D graph?

THE POLAR EQUATION



THE SPHERICAL HARMONICS

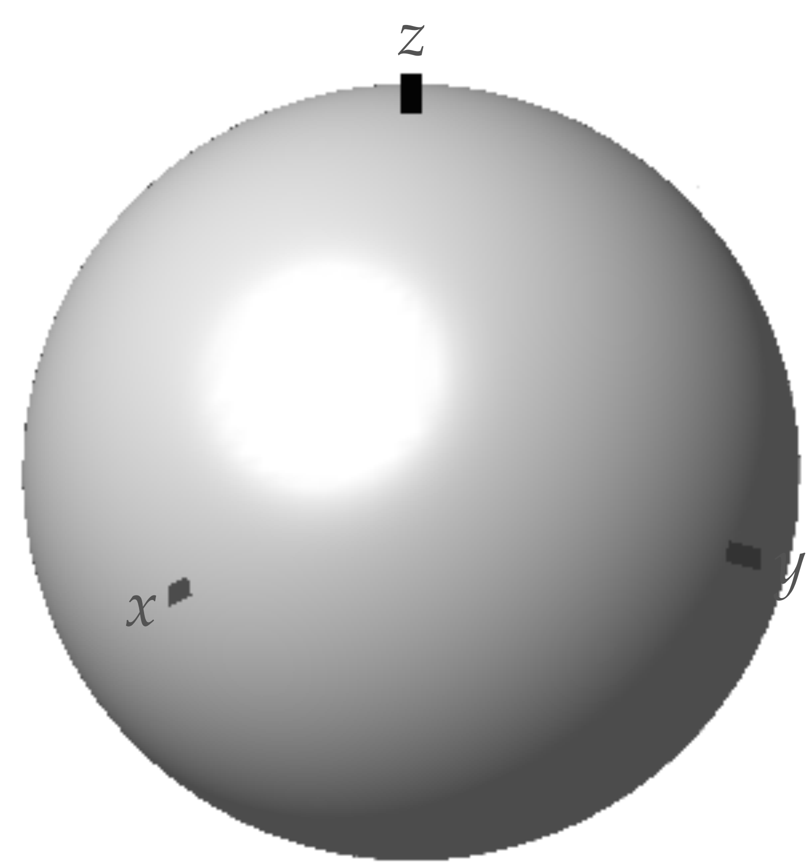
- The polar equation's solutions seem to describe 2D symmetry quite well
- We just need to tack on the azimuthal equation's rotational symmetry in the third dimension now:

$$Y_\ell^m(\theta, \varphi) = ce^{jm\theta} P_\ell^m(\cos\varphi)$$

quantized solutions

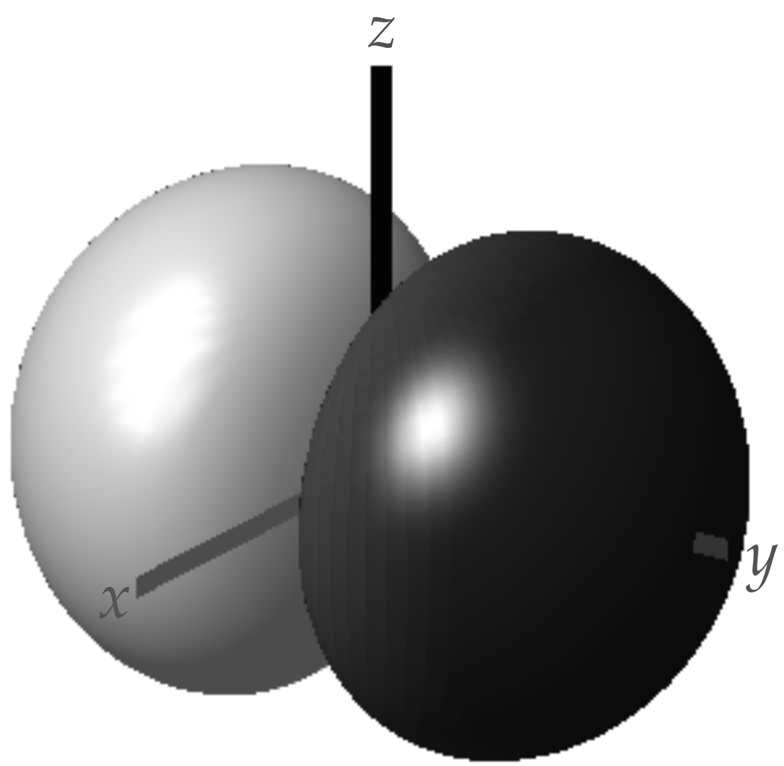
- These are called the **spherical harmonics**
- Note that this is true up to a normalization constant, which we leave undetermined, as it just scales radii

THE SPHERICAL HARMONICS

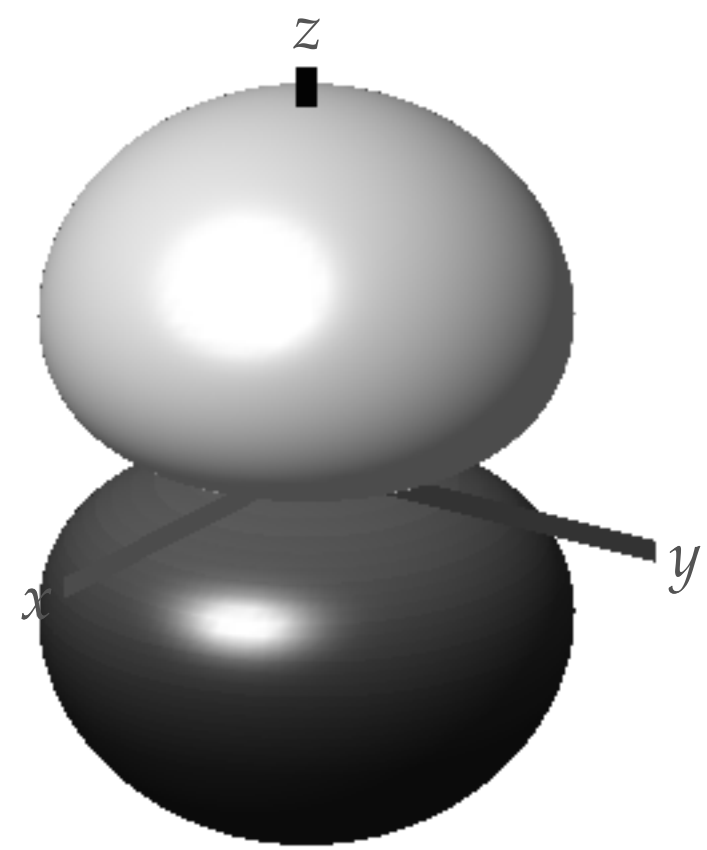


$Y_0^0(\theta, \varphi)$

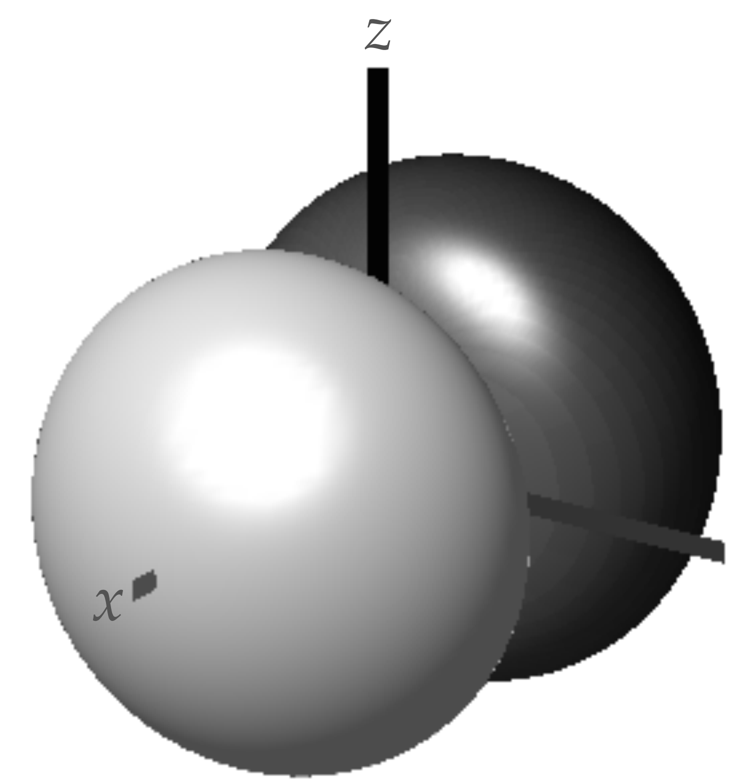
THE SPHERICAL HARMONICS



$Y_1^{-1}(\theta, \varphi)$

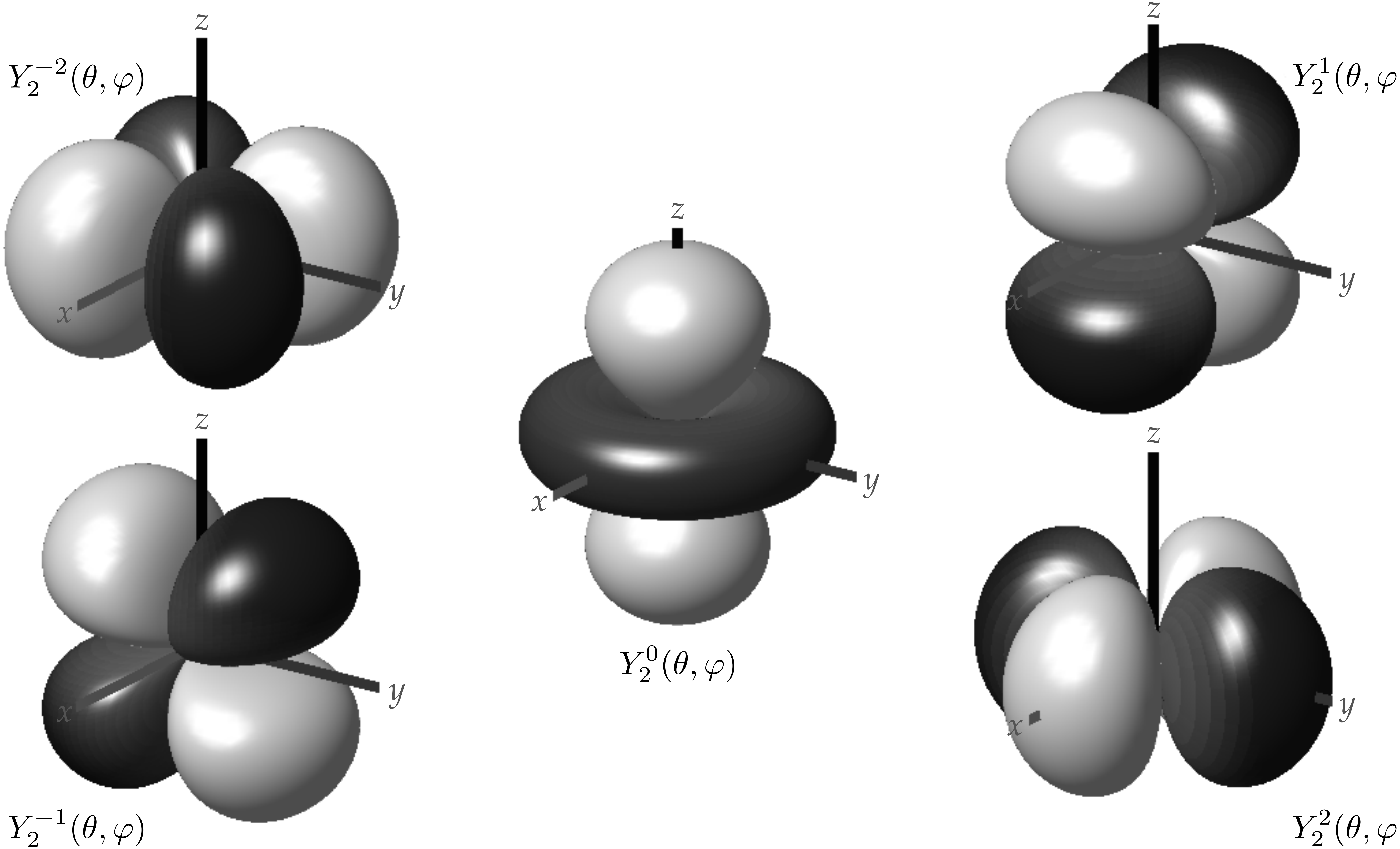


$Y_1^{0}(\theta, \varphi)$



$Y_1^{1}(\theta, \varphi)$

THE SPHERICAL HARMONICS



THE RADIAL EQUATION

- So… we never ended up solving the radial equation!
  - Many possible solutions/polynomial functions
  - Each one varies with nature of $V(r)$
    - a.k.a. potential function
    - hence "potential theory" uses of this material
  - Requires some more foresight, Frobenius method is not enough by itself
  - Advanced polynomials… *Laguerre*, *Chebyshev*

THE RADIAL EQUATION

- So… we never ended up solving the radial equation!
  - It turns out that the spherical harmonics themselves characterize wide variety of problems
  - The mark that changes the problem is the potential function $V(r)$
  - For example, we could set $V(r) = \dfrac{-Ze^2}{4\pi\varepsilon_0 r}$ for *H*-atoms
  - That is, we have a *precise** model of a hydrogen atom using the functions we learned about today!!

**up to the perturbation theory and fine structure of H-atoms…*

KEY POINTS

- Takeaway:
  - How to use mathematical tools you already have to solve new and interesting problems
  - Using mathematical ideas to model physical phenomena and shape/adapt to the situation
  - Being clever when it is necessary

QUESTIONS?

- If you have any questions, feel free to reach out!
- My email address is `cb625@cornell.edu`
  - May take some time to reply though
  - We are all busy students!
  - Check out my website: **`chiragbharadwaj.com`**
- I wish you the best of luck with future math studies!
  - ***Did this class interest you?*** Consider a **math major or minor** at university! Talk to mathematicians and join research community
  - Take a related class at high school through local colleges
  - Think ahead, plan for a little while, even if you're still young

THE FUTURE

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FALL 2016

SPLASH! AT CORNELL

ADVANCED POLYNOMIALS

CHIRAG BHARADWAJ