SPLASH! AT CORNELL

TECHNIQUES IN MATHEMATICS

Chirag Bharadwaj

コーネル大学でスプラッシュ

すうがく　ぎほう
数学の技法

坂本ひかる

"Mathematics is a **foreign language**."

–Chirag Bharadwaj, et. al.

BRIDGING THE GAP

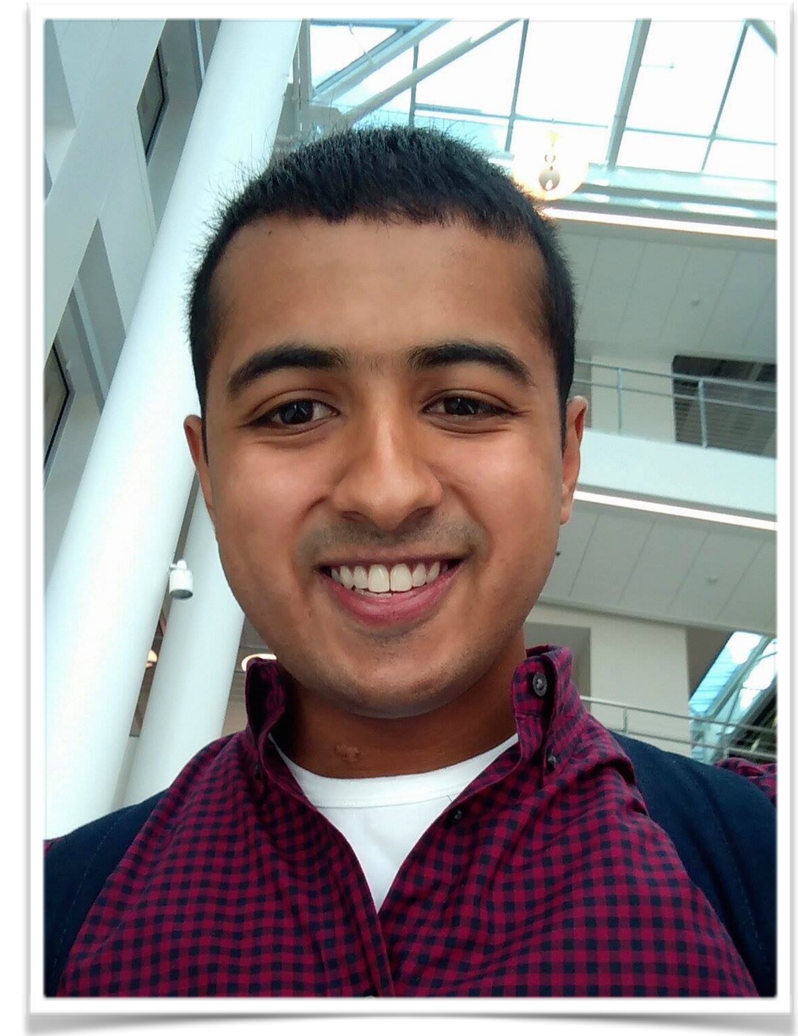
- Just what is this practice of mathematics?
- What are our goals as mathematicians?
- What are our goals as *scientists*?

- Some ideas:
  - Overcoming fear of an overwhelming sea of symbols
  - Extracting intuitive understanding of complex ideas
  - Developing simple models of real-life phenomena

COLOR SCHEME

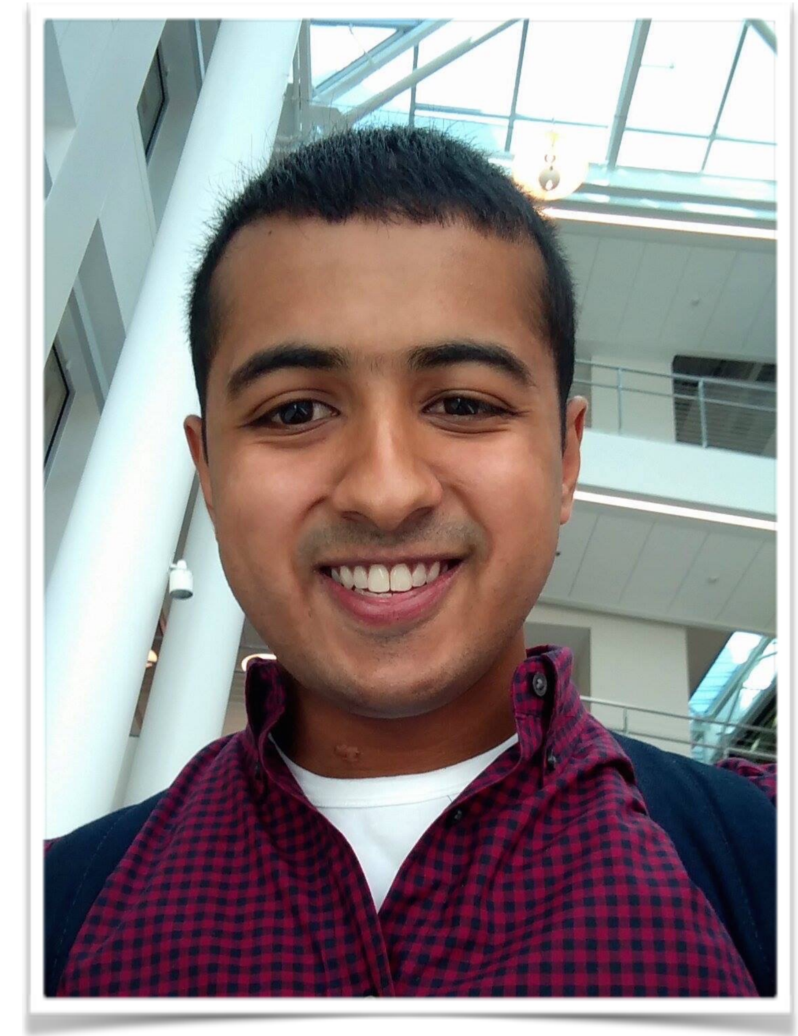
- In this presentation…
  - *Green text* refers to things I think you should know already given your past experiences
  - *Blue text* refers to things you may know or be able to reason about given enough time
  - *Red text* refers to things that you most likely do not know yet and will hopefully learn sometime soon!
  - *Magenta text* mostly refers to definitions/emphasis

ABOUT ME



- From Flushing, NY
- Went to high school in Princeton, NJ
- Senior at Cornell University
  - B.Sc., *Computer Science*
  - Minor in Electrical Engineering
- Future: *MSE student* at **Princeton University**
- Mathematics is one of my side interests!
- Other than that, just like you (except maybe a little older):
  - 20 years old
  - Interested in self-learning and teaching others

ABOUT ME



- From Flushing, NY
- Went to high school in Princeton, NJ
- Things I've taught at Splash before:
  - *Spring 2015:* Modern Complex Analysis
  - *Fall 2015:* Introduction to Japanese
  - *Spring 2016:* Special Polynomials
  - *Fall 2016:* Basic Group Theory in Music
  - *Spring 2017: Techniques in Mathematics*
- Pattern in my teaching?
  - *Spring* = more technical material
  - *Fall* = more accessible material

OVERVIEW

- **110** minutes
- Smörgåsbord of topics:
  - Reductionistic approach
  - Draws *heavily* from material in calculus
  - Idea: build up mathematical toolbox
- Focus of the precept:
  - Developing intuition for existing knowledge
- Pace: **reasonably fast**

200 slides

DISCLAIMER

- Don’t worry if you don’t understand *everything*!
  - The idea is to gain exposure to unfamiliar concepts
- Not everything you see here will be immediately useful in your high school mathematics
  - But it will teach you a little bit about how to think for yourself and teach yourself new things from old
- This is a **challenge**—get ready to be splashed!

INCREMENTAL ANALYSIS

FUNCTIONS

- Just what is a function?

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  - A formula
  - A relationship between independent parameters and dependent parameters
  - A mapping between a set of inputs and outputs

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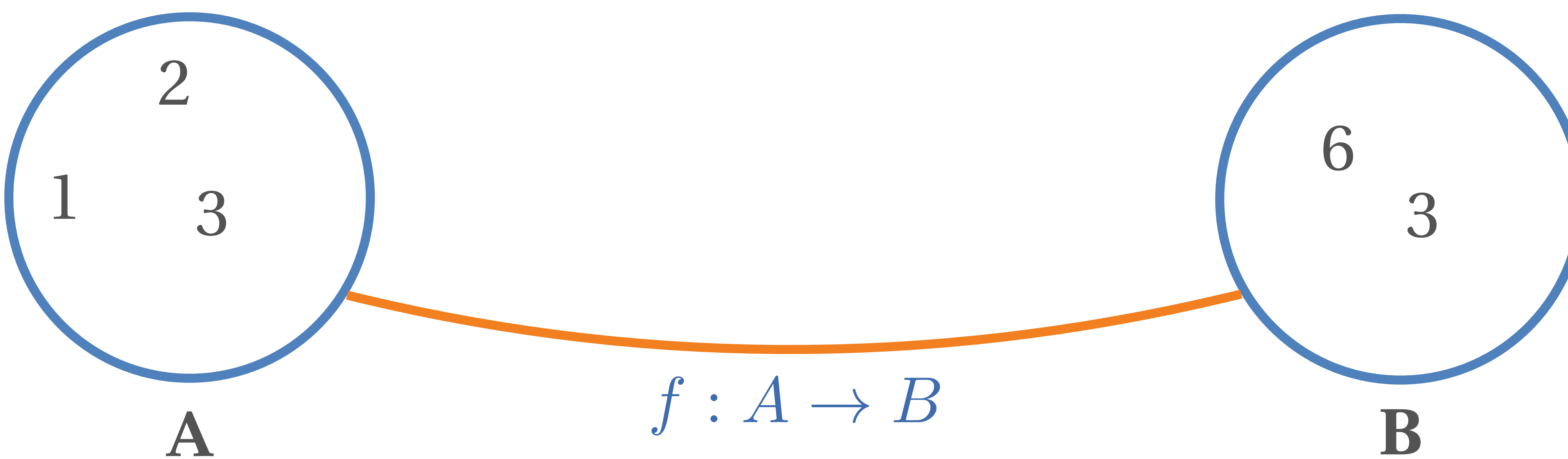

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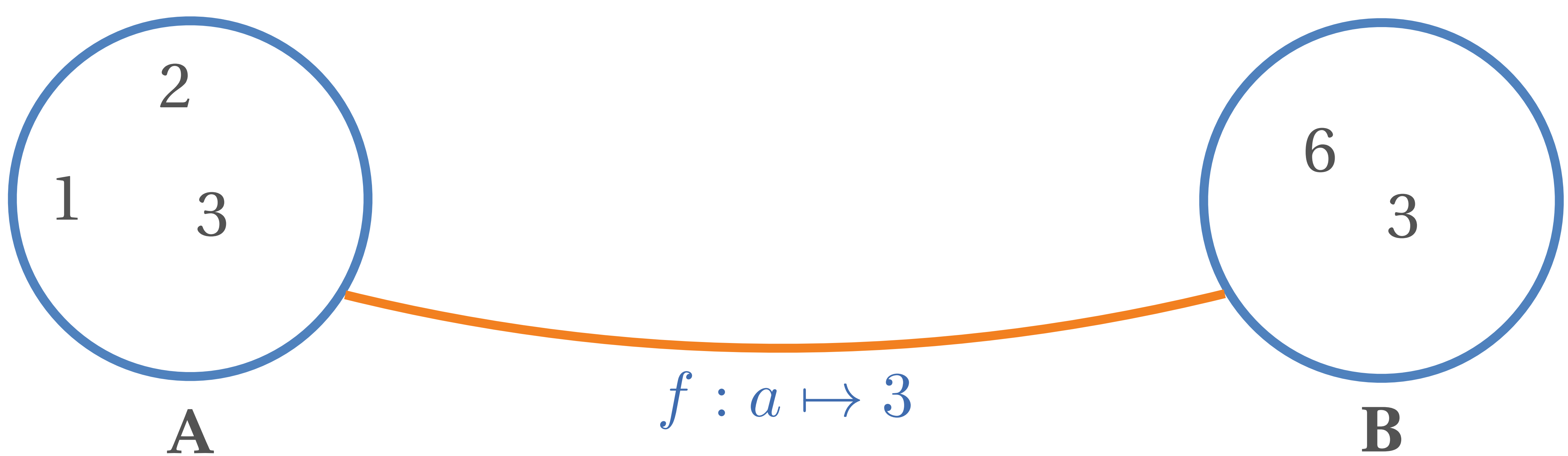
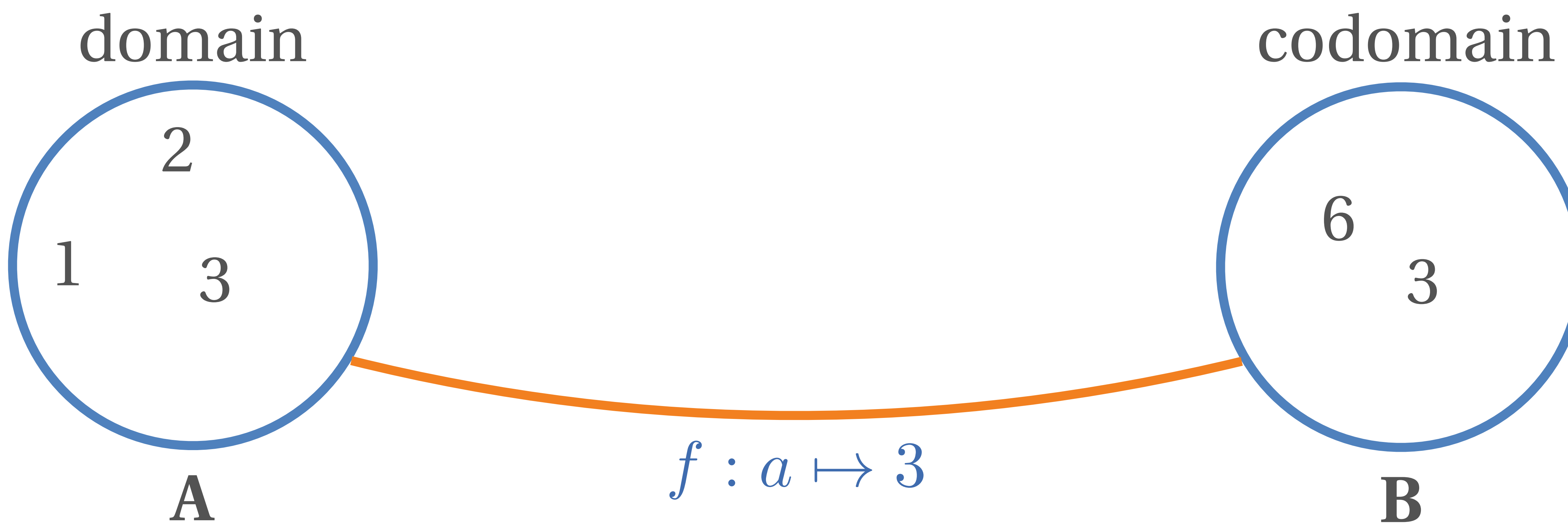


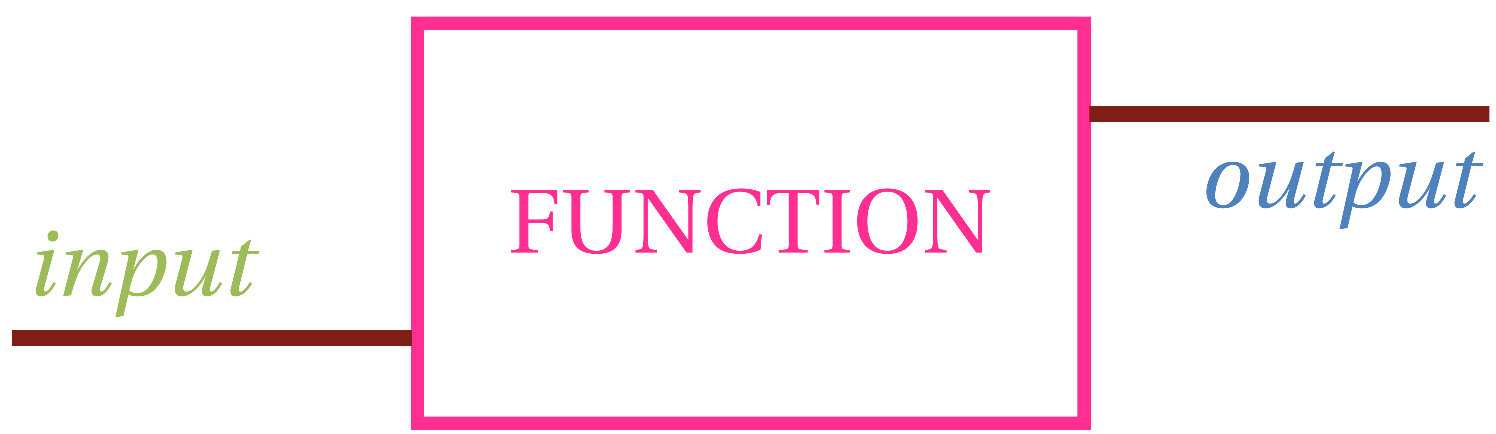
FUNCTIONS



FUNCTIONS

$$\mathrm{range}(\texttt{fun}) \subseteq \mathrm{cod}(\texttt{fun})$$

BLACK-BOX MODEL



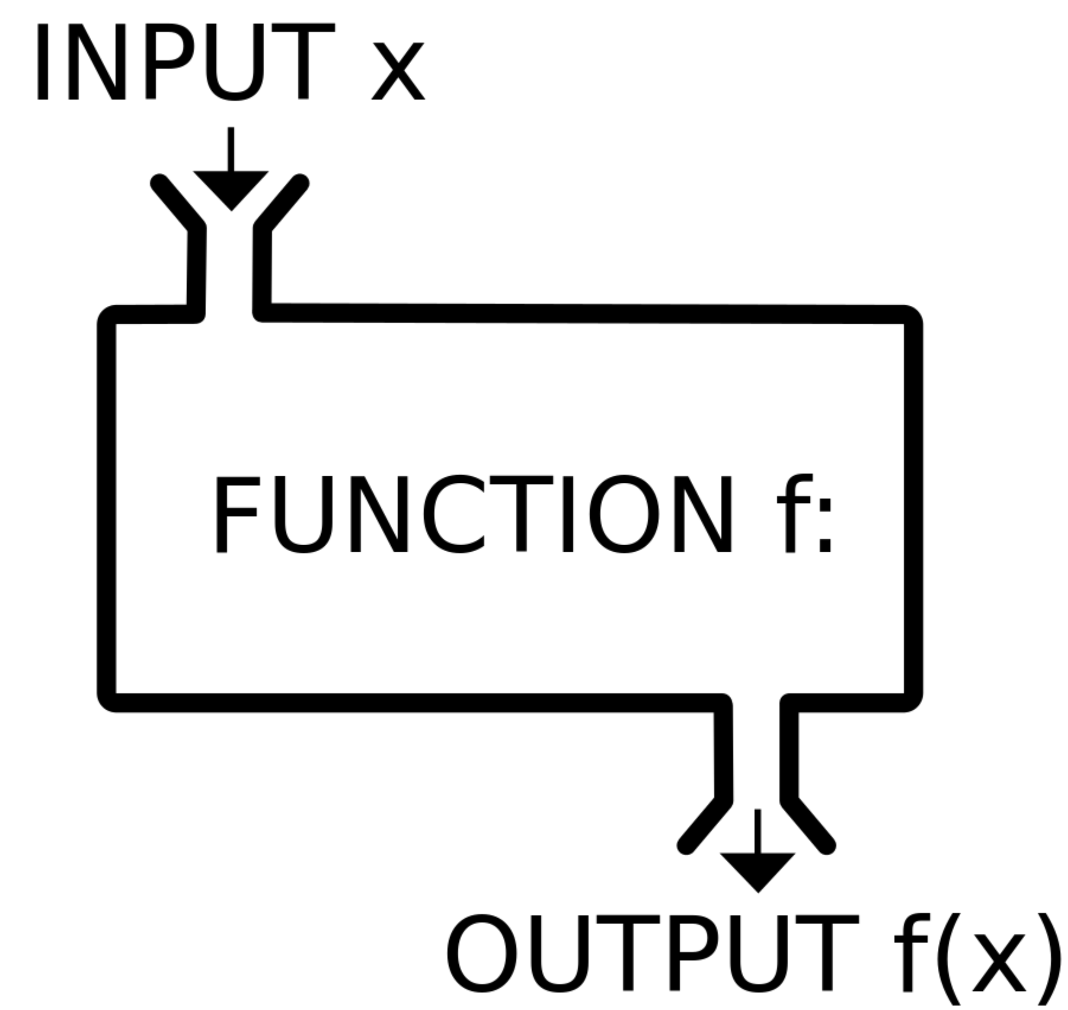
BLACK-BOX MODEL

BLACK-BOX MODELLING

- General technique
- Can be used when implementation is unknown
  - Also when implementation is too complicated…
- For our purposes:
  - Method of incremental analysis
  - How *sensitive* is a black box?
  - Controlled parameters: *inputs*
  - Observed parameters: *outputs*

INCREMENTAL ANALYSIS

- Imagine we have a black box representation of some function f
- Assumptions:
  - Too complicated to measure f
    - *Expensive computation*
  - Can only approximate



INCREMENTAL ANALYSIS

- One way to estimate values of f:
  - Local-linear approximations
- Assume a fixed *operating point* of function
  - a.k.a. bias point
  - One-time pre-computed exact answer
  - Operating point **input**: x_{BIAS}
  - Operating point **output**: $y_{\text{BIAS}} = f(x_{\text{BIAS}})$
- How *sensitive* is f to small changes about its operating point?

INCREMENTAL ANALYSIS

- Measuring sensitivity:
  - Change input by a small increment Δx
  - Observe how much output changes: Δy
- Semi-formalism:
  - Observed change in black box relative to the change in the supplied control
- Mathematical representation:

$$s = \frac{\Delta y}{\Delta x}$$

INCREMENTAL ANALYSIS

how small is "small"? 10^{-6} ?

- Measuring sensitivity:
  - Change input by a small increment Δx
  - Observe how much output changes: Δy
- Semi-formalism:
  - Observed change in black box relative to the change in the supplied control
- Mathematical representation:

$$s = \frac{\Delta y}{\Delta x}$$

we will revisit this

INCREMENTAL ANALYSIS

- So how do we actually approximate f?
  - Start from bias point
  - Need to account for sensitivity of black box
- Desired: value of f at a distance h from the bias point
  - That is, what is the value of $f(x_{\text{BIAS}} + h)$?

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poor approximation

- Want *proportionality* to sensitivity parameter

INCREMENTAL ANALYSIS

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$$f(x_{\text{BIAS}} + h) \approx f(x_{\text{BIAS}}) + s \cdot h$$

INCREMENTAL ANALYSIS

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$$f(x_{\text{BIAS}} + h) \approx f(x_{\text{BIAS}}) + \frac{\Delta y}{\Delta x} \cdot h$$

INCREMENTAL ANALYSIS

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$$f(x_{\text{BIAS}} + h) \approx f(x_{\text{BIAS}}) + \frac{\Delta y}{\Delta x} \cdot h$$

$$x = x_{\text{BIAS}} + h$$

INCREMENTAL ANALYSIS

- So how do we actually approximate f?
  - Start from bias point
  - Need to account for sensitivity of black box

$$f(x) \approx f(x_{\text{BIAS}}) + \frac{\Delta y}{\Delta x} \cdot (x - x_{\text{BIAS}})$$

independent of h

INCREMENTAL ANALYSIS

- So how do we actually approximate f ?
  - Start from bias point
  - Need to account for sensitivity of black box

$$f(x) \approx f(x_{\text{BIAS}}) + \frac{\Delta y}{\Delta x} \cdot (x - x_{\text{BIAS}})$$

independent of h

- Local-linear approximation about x_{BIAS}

INCREMENTAL ANALYSIS

- Recall:

$$f(x_{\text{BIAS}} + h) \approx f(x_{\text{BIAS}}) + \frac{\Delta y}{\Delta x} \cdot h$$

- What happens if $h = \Delta x$?

$$f(x_{\text{BIAS}} + \Delta x) \approx f(x_{\text{BIAS}}) + \frac{\Delta y}{\Delta x} \cdot \Delta x$$

$$\Delta y \approx f(x_{\text{BIAS}} + \Delta x) - f(x_{\text{BIAS}})$$

- Using this, we can revisit sensitivity:

$$s = \frac{\Delta y}{\Delta x} \approx \frac{f(x_{\text{BIAS}} + \Delta x) - f(x_{\text{BIAS}})}{\Delta x}$$

INCREMENTAL ANALYSIS

- This is just an approximation of the sensitivity!
- How can we make it more accurate?
  - Recall: *How small is "small"?*
  - Our sensitivity is **more accurate** if we make a **smaller change** in the supplied control

  - What if we make effectively no change in x?
  - That is: $\Delta x \approx 0$

INCREMENTAL ANALYSIS

- **True** sensitivity:

$$s = \lim_{\Delta x \to 0} \frac{f(x_{\text{BIAS}} + \Delta x) - f(x_{\text{BIAS}})}{\Delta x}$$

$$\boxed{s = f'(x_{\text{BIAS}})}$$

- **New intuition:**
  - Derivative of a function is a measure of sensitivity
  - How do outputs change as inputs change?
  - i.e. slope on a graph!

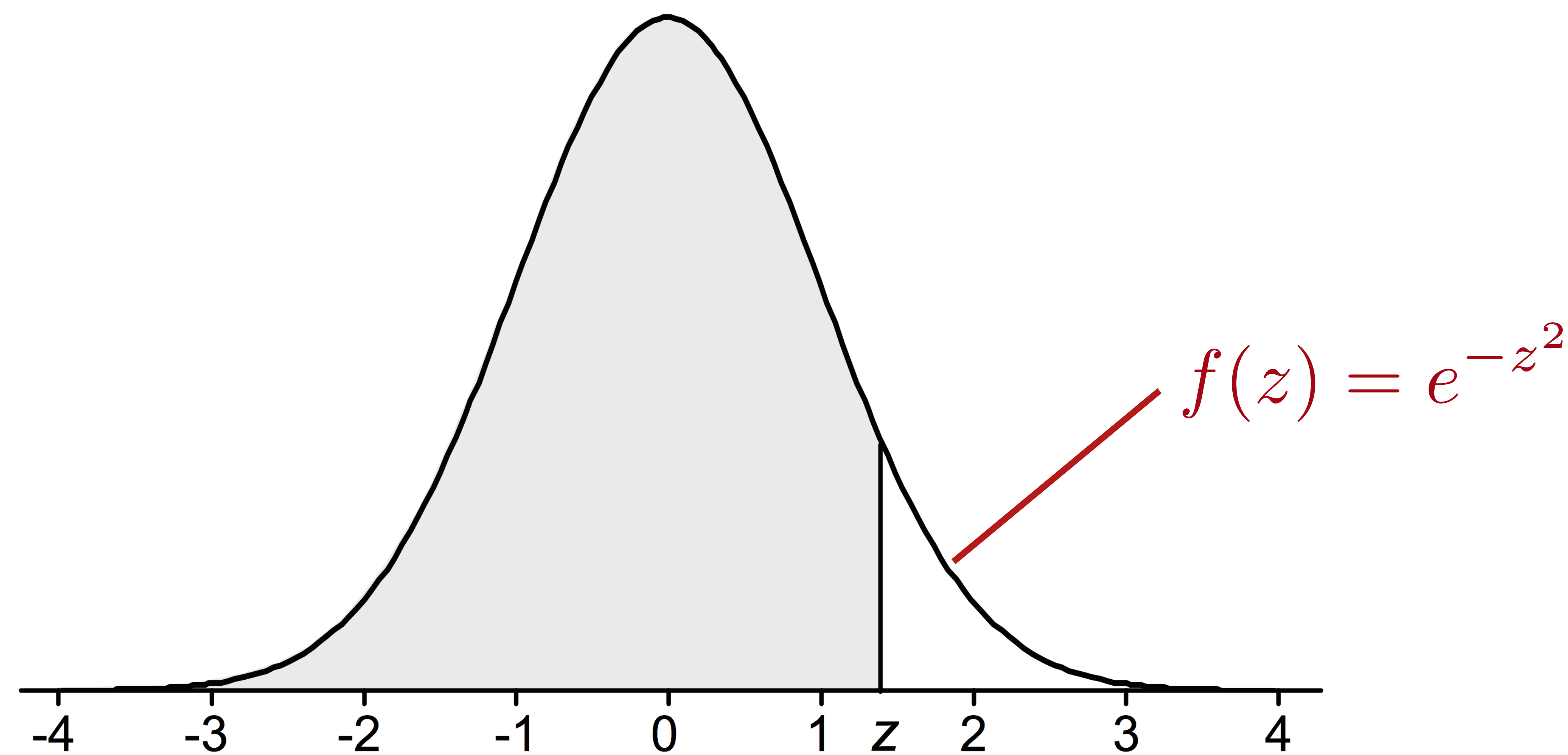
APPLICATIONS OF INCREMENTAL ANALYSIS

ISSUES

- Problem:
  - Explicit operation point is required
  - i.e. need to know black box's behavior at one exact point prior to analysis
- Solution:
  - Manufacturer's tables
  - i.e. a published table of values of f at various operation points
  - User is free to choose a specific biasing regimen

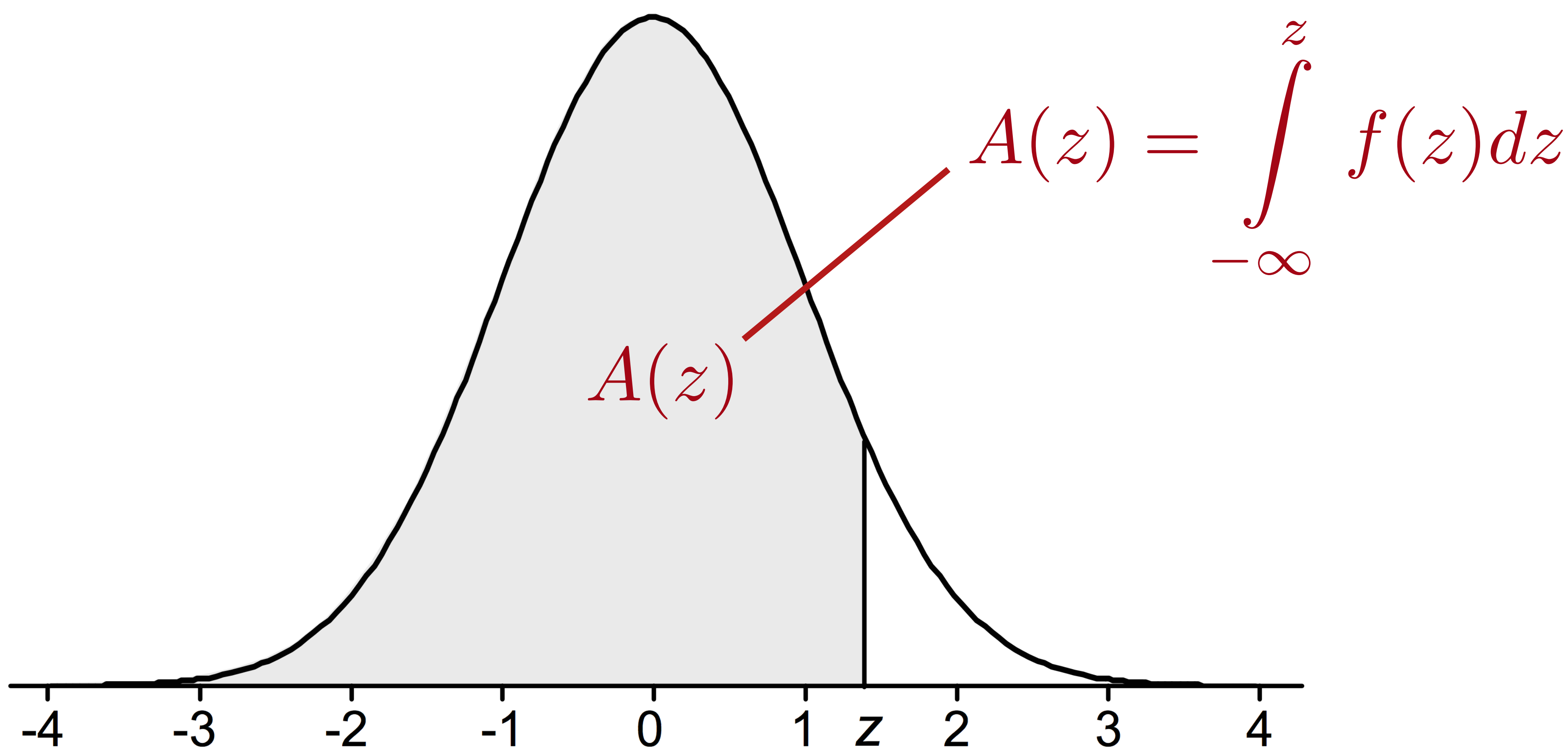
MANUFACTURER'S TABLES

- Example: accumulation functions



MANUFACTURER'S TABLES

- Example: **accumulation functions**



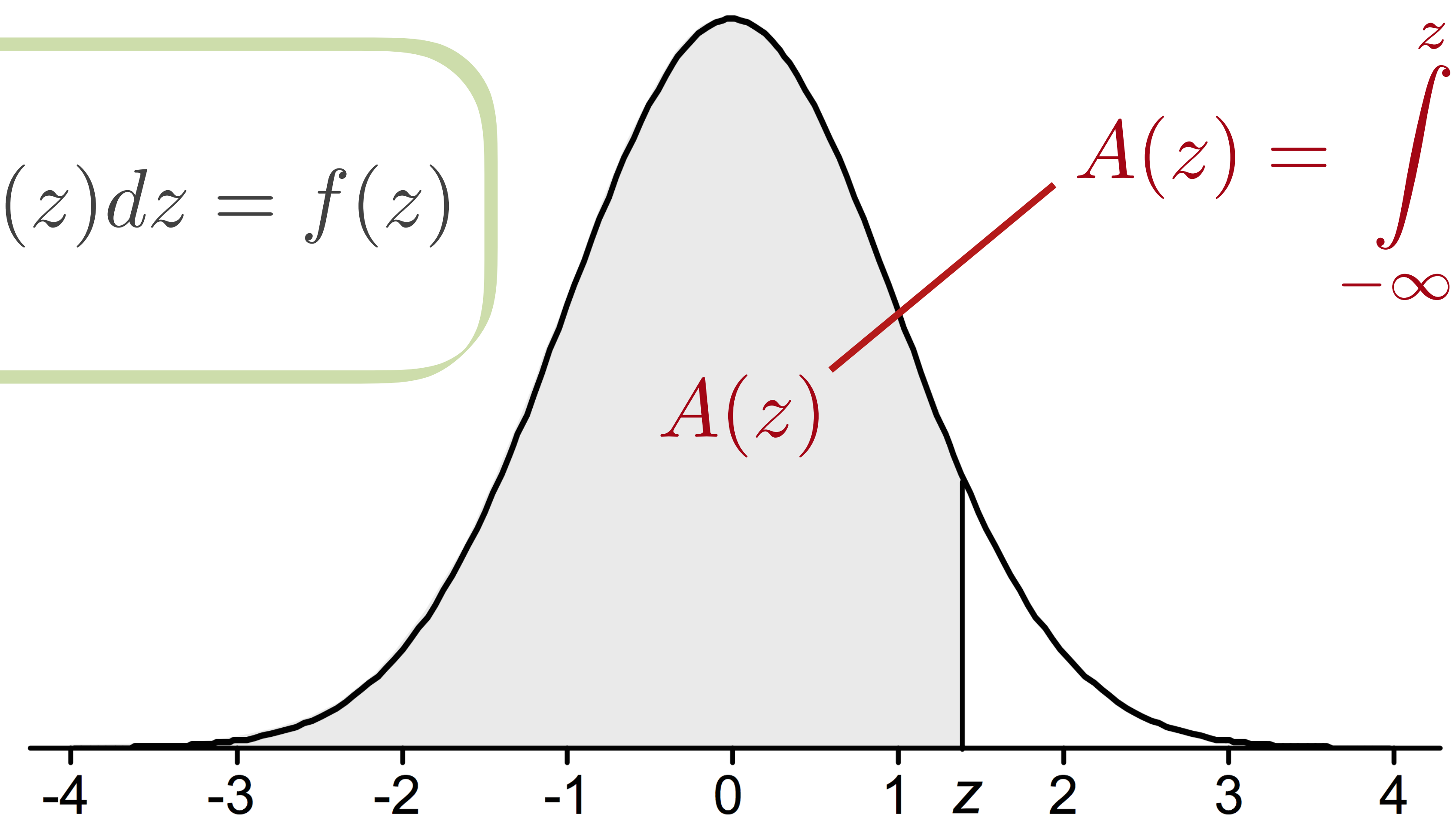
MANUFACTURER'S TABLES

- Example: **accumulation functions**

2nd FTC:

$$\frac{dA}{dz} = \frac{d}{dz}\int_{-\infty}^{z} f(z)dz = f(z)$$

$$A(z) = \int_{-\infty}^{z} f(z)dz$$



MANUFACTURER'S TABLES

- However, there is no closed-form answer for

$$\int_{-\infty}^{z} e^{-z^2}\, dz$$

- That is, we cannot easily compute $A(z)$!
  - Publish a table of various bias points
  - Bias points should sample the input space well
- Application: *statistics*
  - Also: thermodynamics, quantum physics, etc.

MANUFACTURER’S TABLES

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999							

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1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
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2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
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2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999							

$$z_{\text{BIAS}} = 1.45$$

$$A(1.45) \approx 0.9265$$

MANUFACTURER’S TABLES

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999							

$$A(-z) = 1 - A(z)$$

symmetry property

RANGE OF SENSITIVITY

- Recall:
  - Sensitivity is measured *at a bias point* x_{BIAS}
  - So we should call it s_{BIAS}
- Range of sensitivity:
  - What values are considered "very sensitive"?
  - What values are considered "very resistive"?
  - Breakeven point?

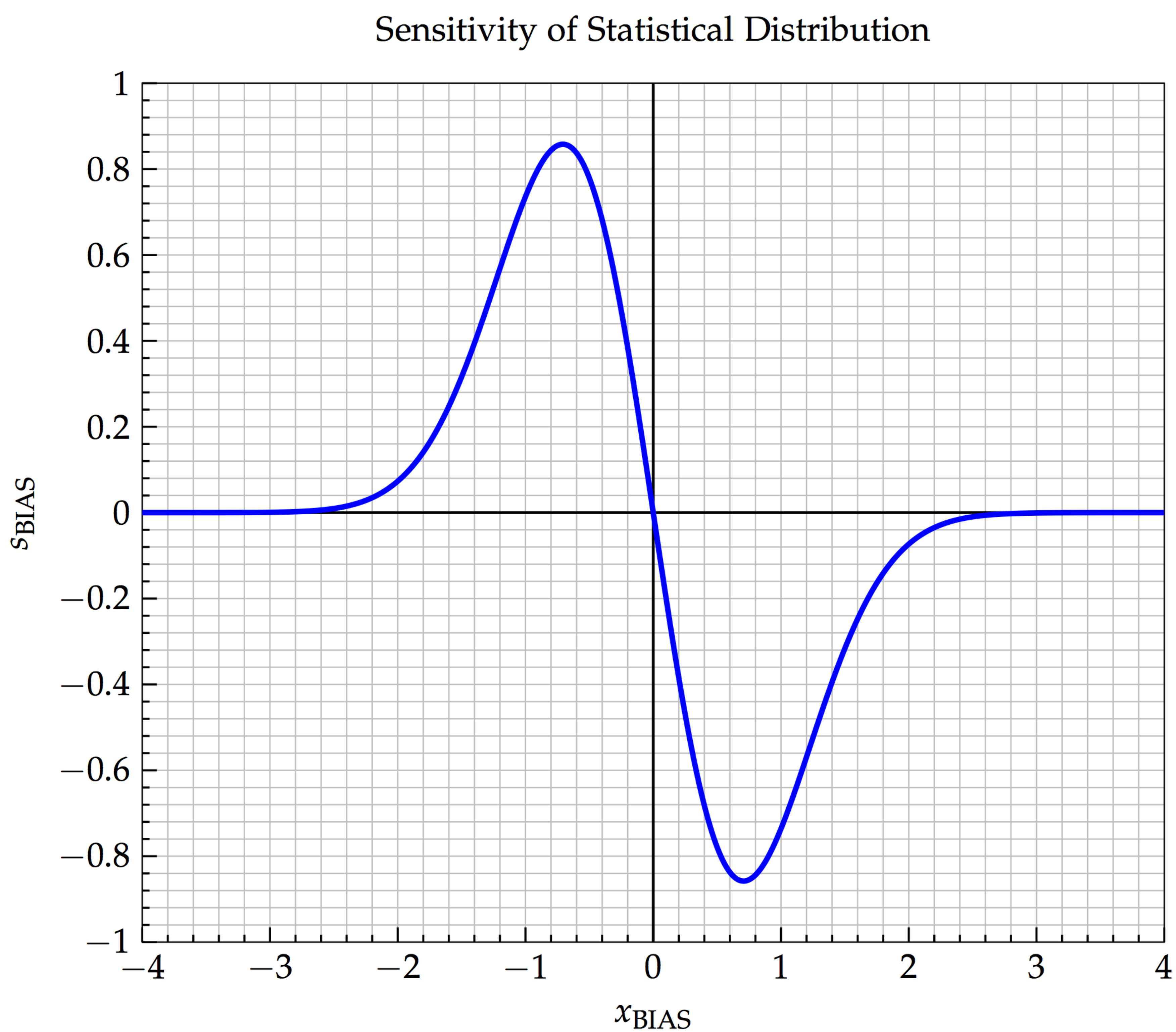
RANGE OF SENSITIVITY

- Recall:
  - Sensitivity is measured *at a bias point* x_{BIAS}
  - So we should call it s_{BIAS}
- Range of sensitivity:
  - Extremely sensitive: $|s_{\text{BIAS}}| = \infty$
  - Very sensitive: $|s_{\text{BIAS}}| > 1$
  - Breakeven: $|s_{\text{BIAS}}| = 1$
  - Very resistive: $0 < |s_{\text{BIAS}}| < 1$
  - Perfectly resistive: $|s_{\text{BIAS}}| = 0$

RANGE OF SENSITIVITY

- Recall:
  - We can measure *changes* in the black-box model but not direct quantities
- Sensitivity is based on input-driven changes
- Can we graph the sensitivity/resistivity of a function?
  - Dependent axis: sensitivity
  - Independent axis: bias point

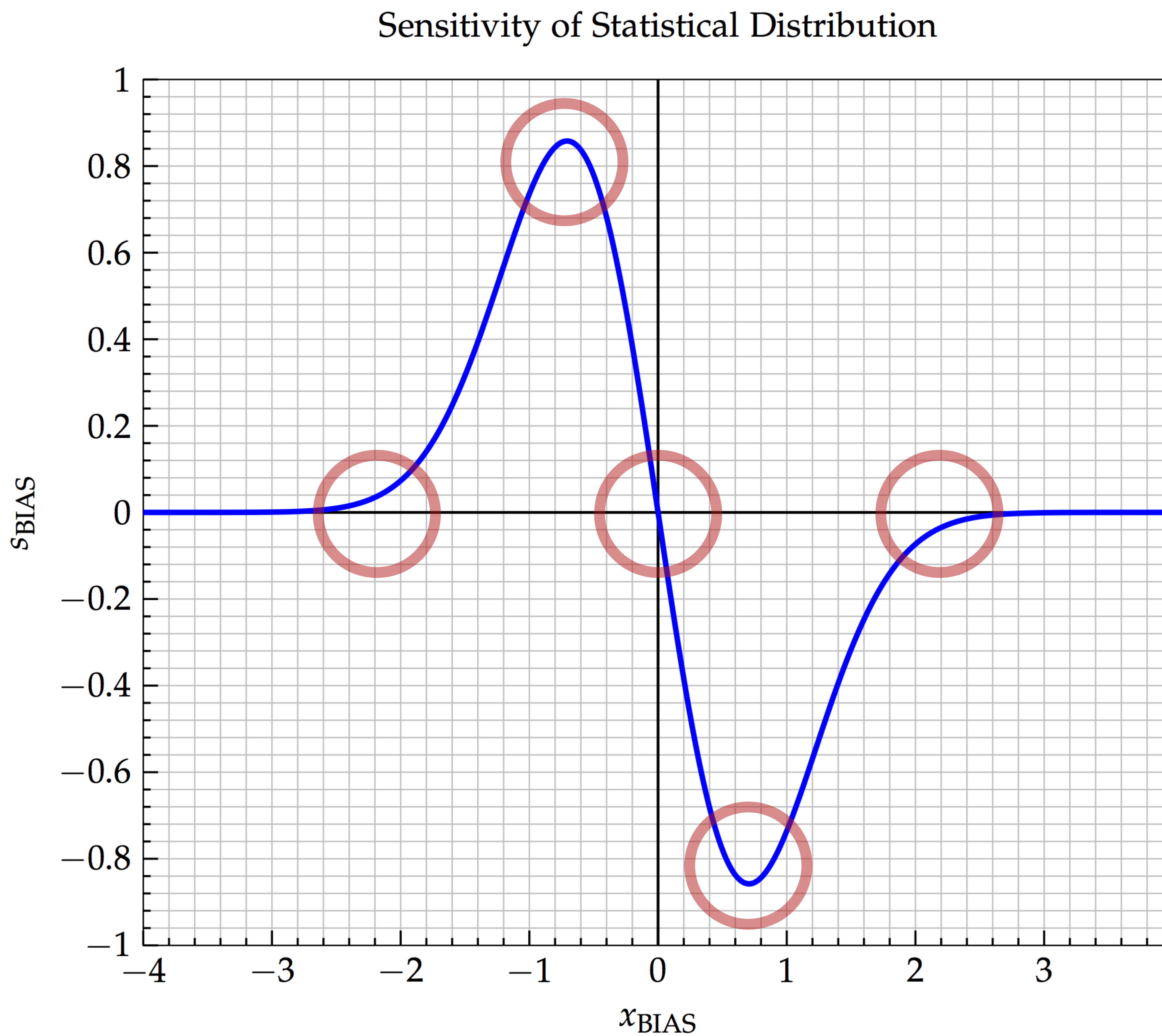
RANGE OF SENSITIVITY



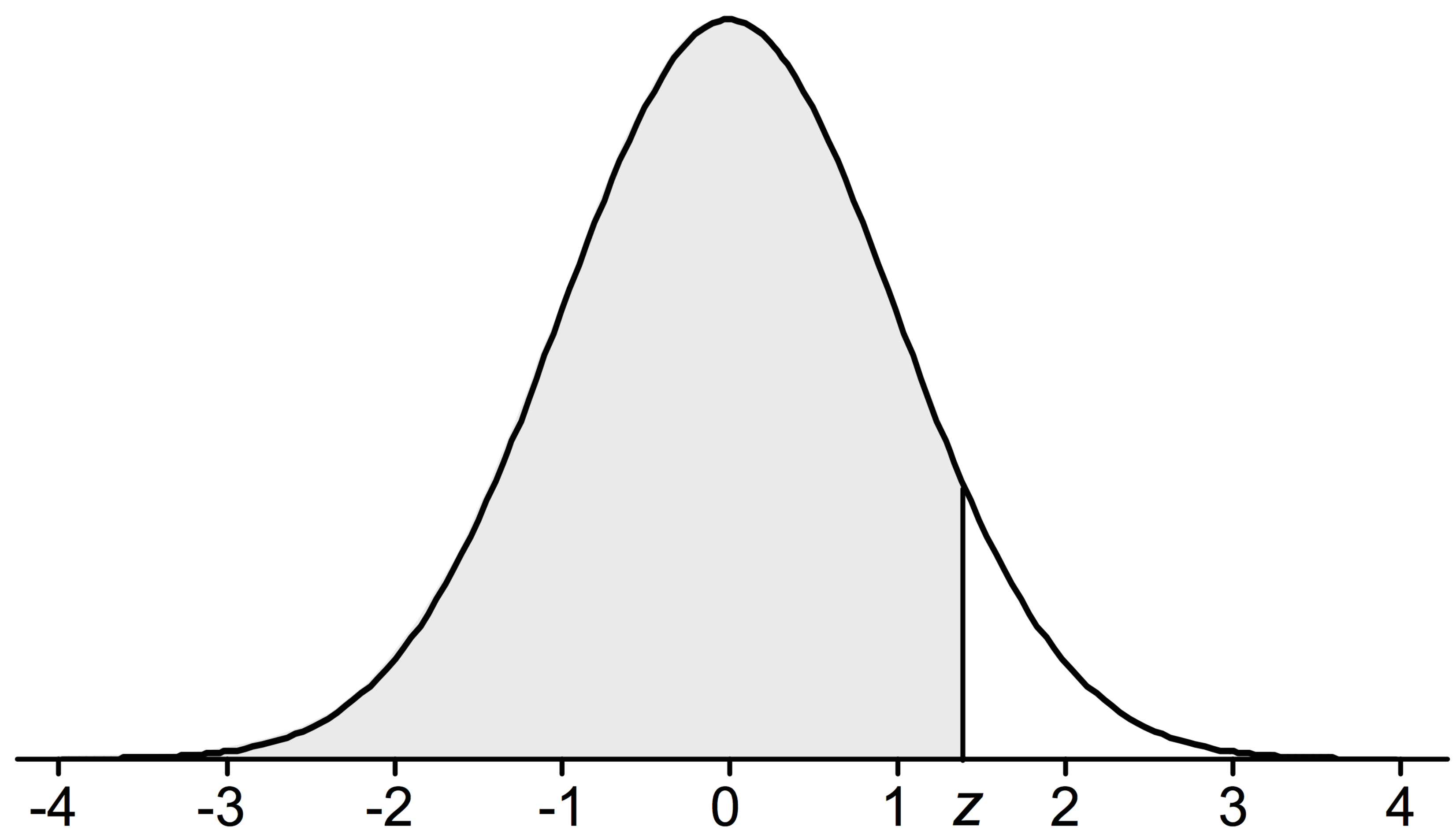
RANGE OF SENSITIVITY

- Over the entire range of the statistical distribution, we see that $0 < |s_{\text{BIAS}}| < 1$
- This suggests that the distribution is fairly resistive to any change
- Idea: local-linear approximation should work well in regions where sensitivity does not change "quickly"
  - Does this intuition hold water?

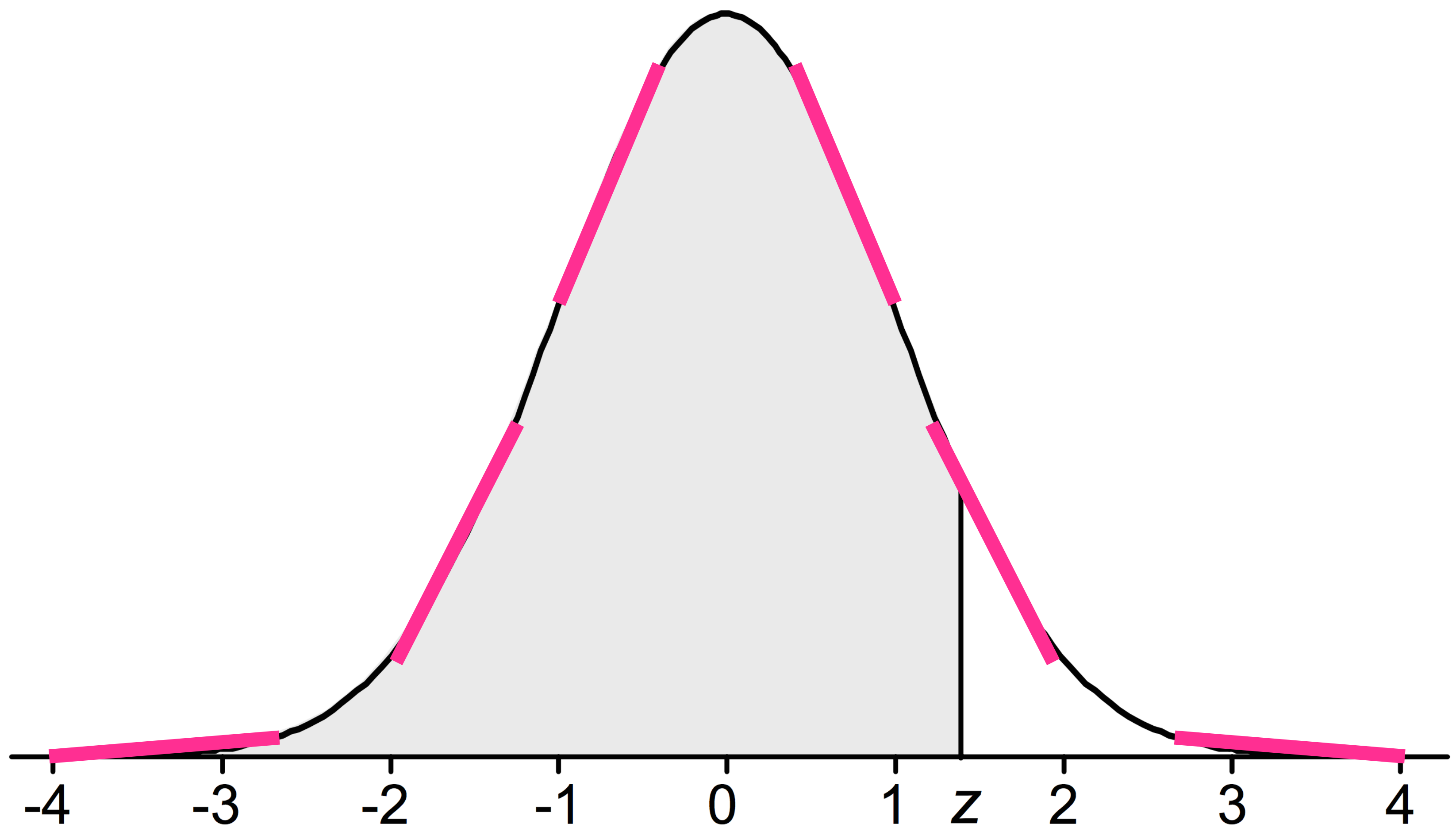
RANGE OF SENSITIVITY



RANGE OF SENSITIVITY



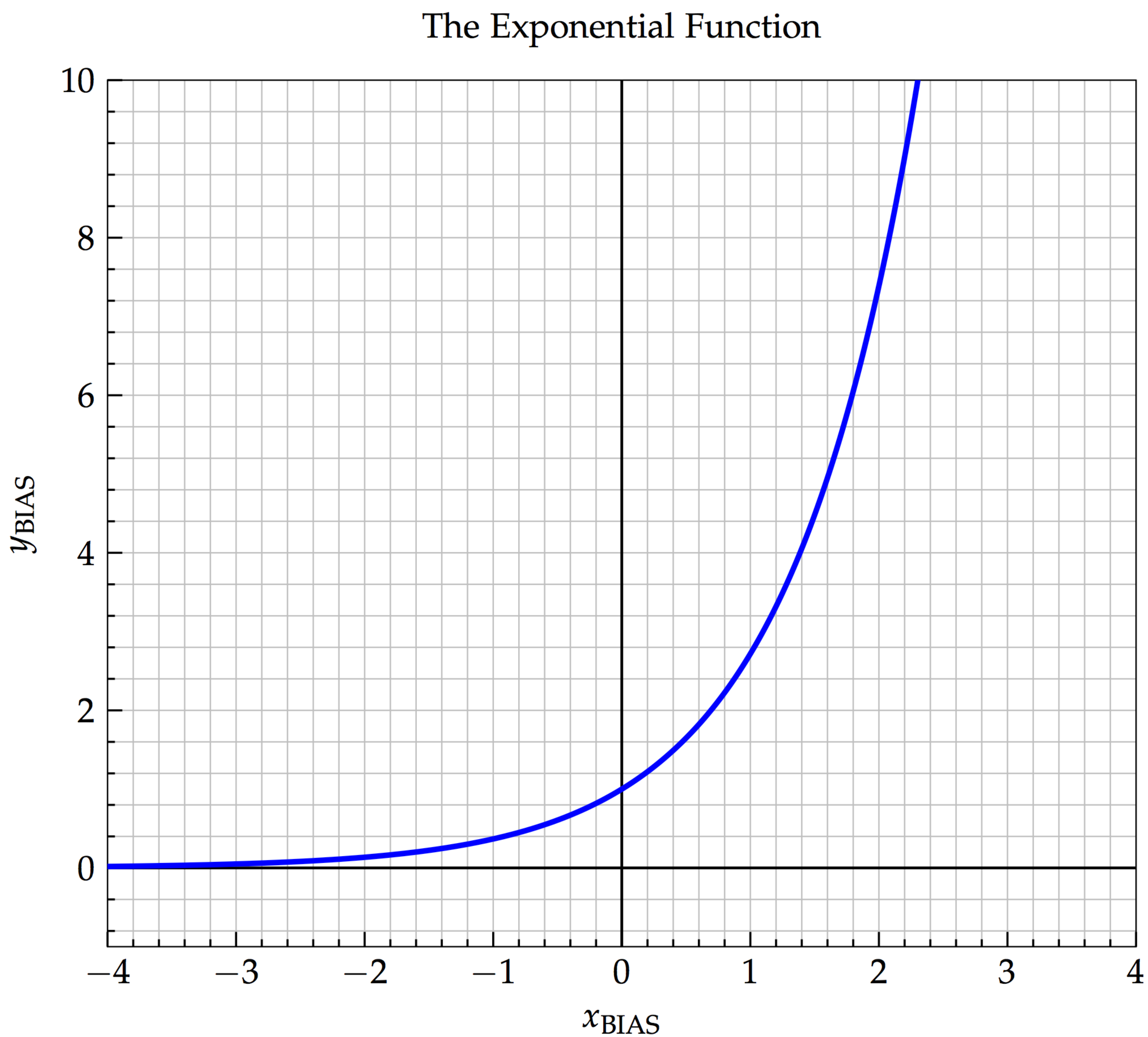
RANGE OF SENSITIVITY



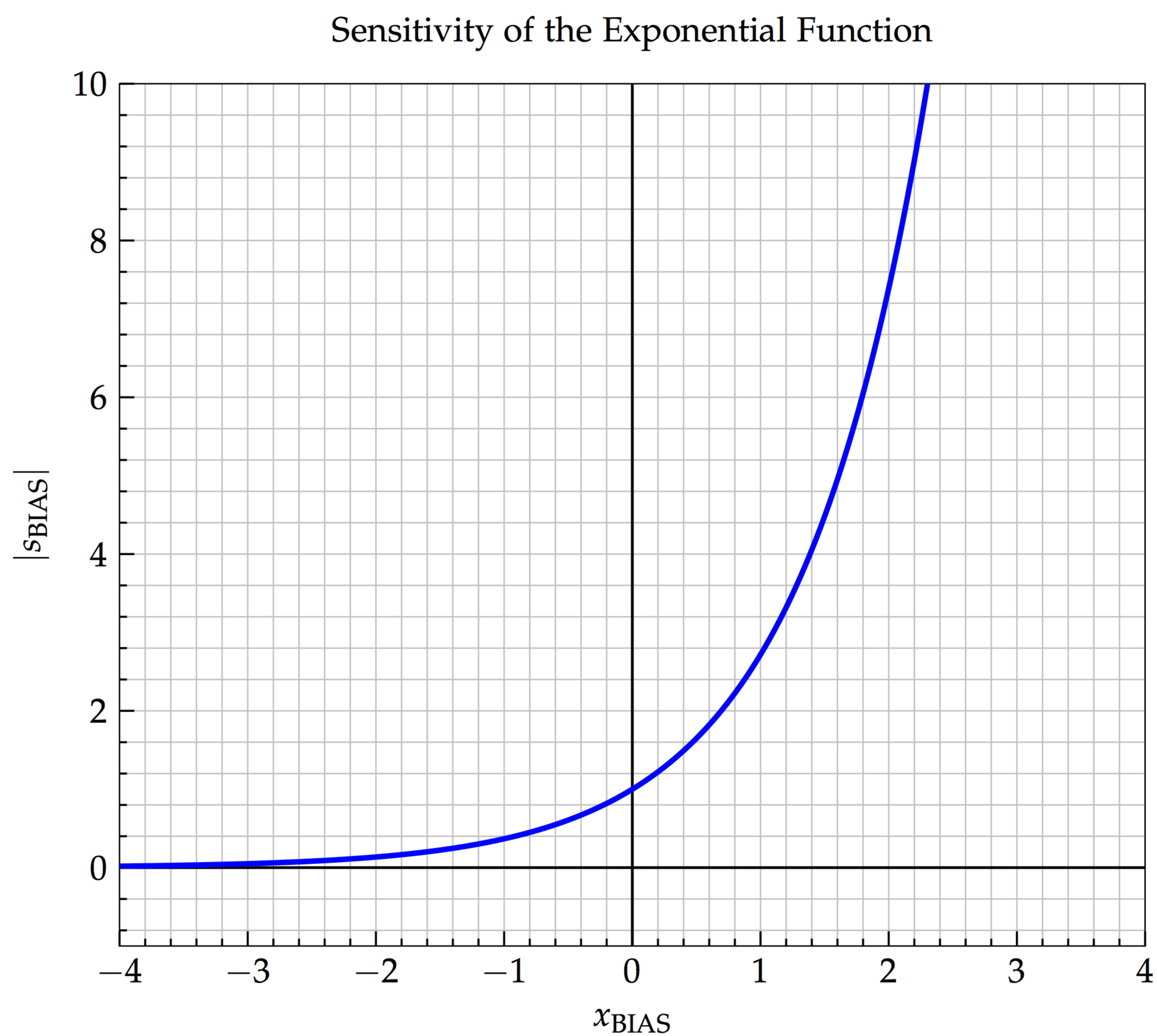
HIGH SENSITIVITY

- Are there examples of highly-sensitive functions?
  - Intuition: The exponential function responds *exponentially* to small changes in input signal
  - It should be highly sensitive!
- Can we confirm this graphically?

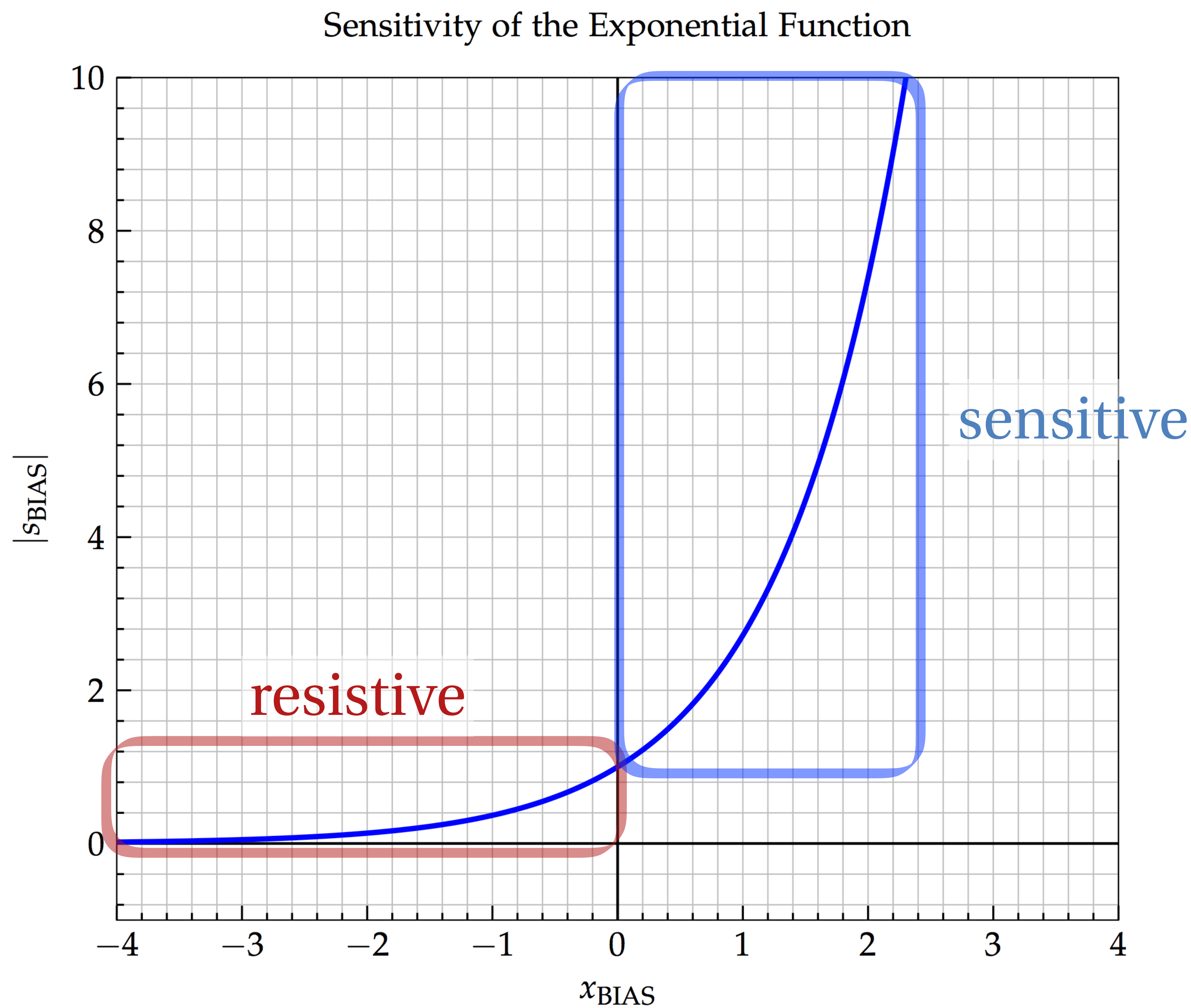
EXPONENTIAL FUNCTION



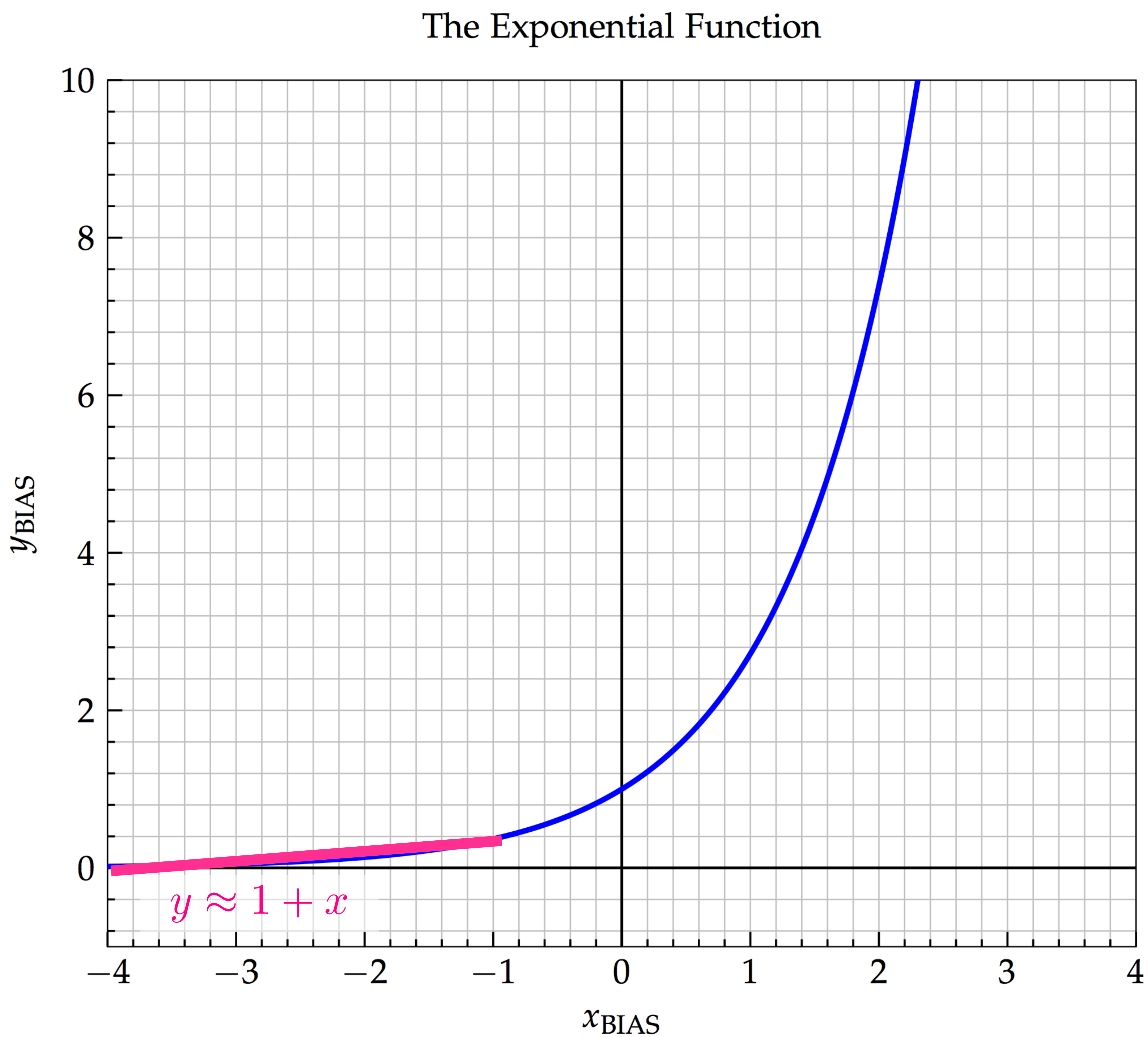
SENSITIVITY GRAPH



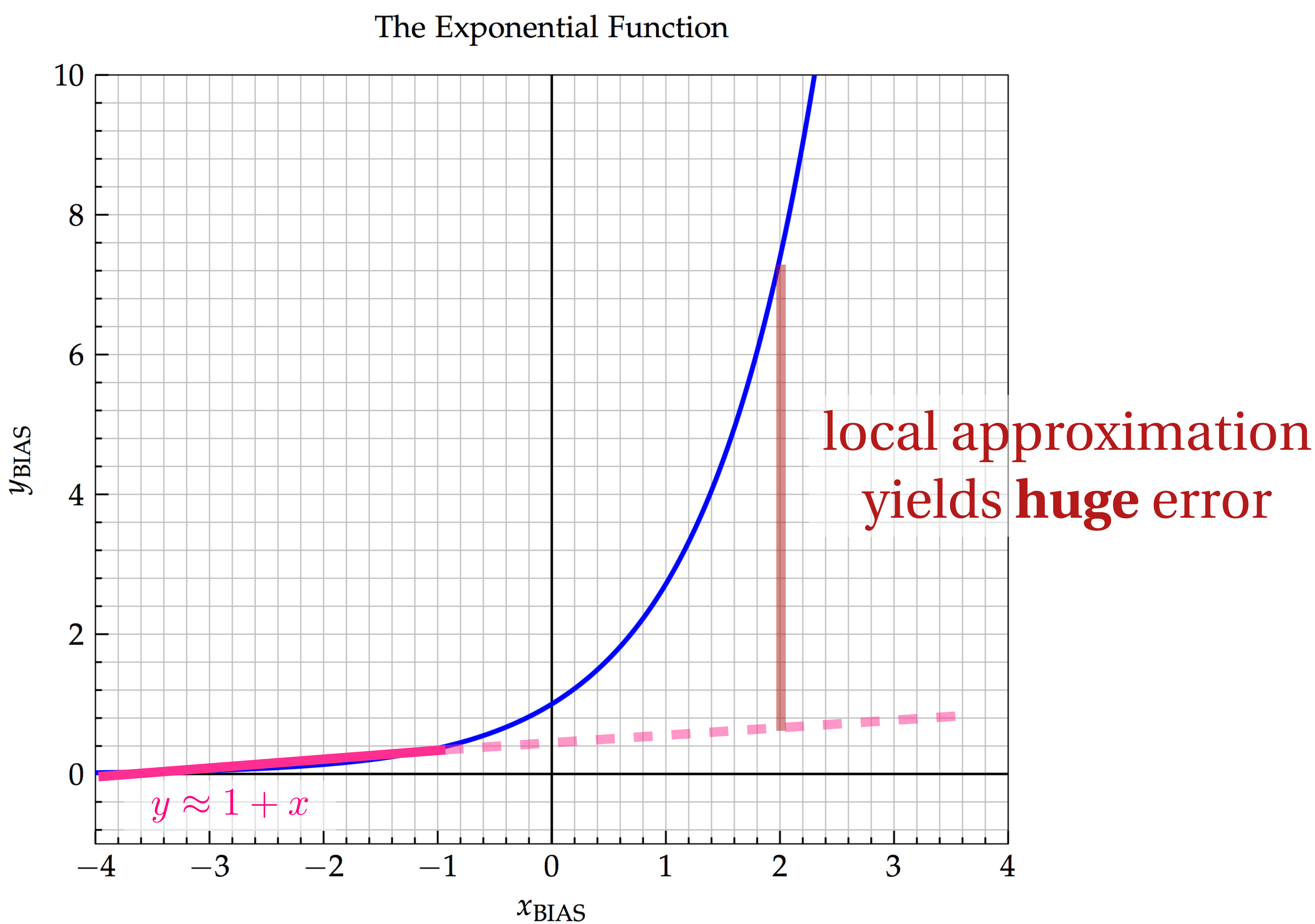
SENSITIVITY GRAPH



EXPONENTIAL FUNCTION



EXPONENTIAL FUNCTION



INTUITOR’S GUIDE

- Confirmation of intuition:
  - Derivatives are a measure of sensitivity
  - Statistical distribution derivative shown earlier
- What other applications of this are there?
  - First-order perturbation theory
  - Large- and small-signal modelling
  - Useful when impossible to model phenomena
    - Need heuristic for approximation
    - Measurement/data-based approach will work

INTERLUDE

SWITCHING GEARS

- Just a reminder!
- Content will **not** flow smoothly here
  - Idea: Presenting a bunch of topics
  - Synthesis: At some point in your academic career
  - Technical school, STEM majors, etc.

RECURSIVE CALCULUS

FUNCTION FAMILIES

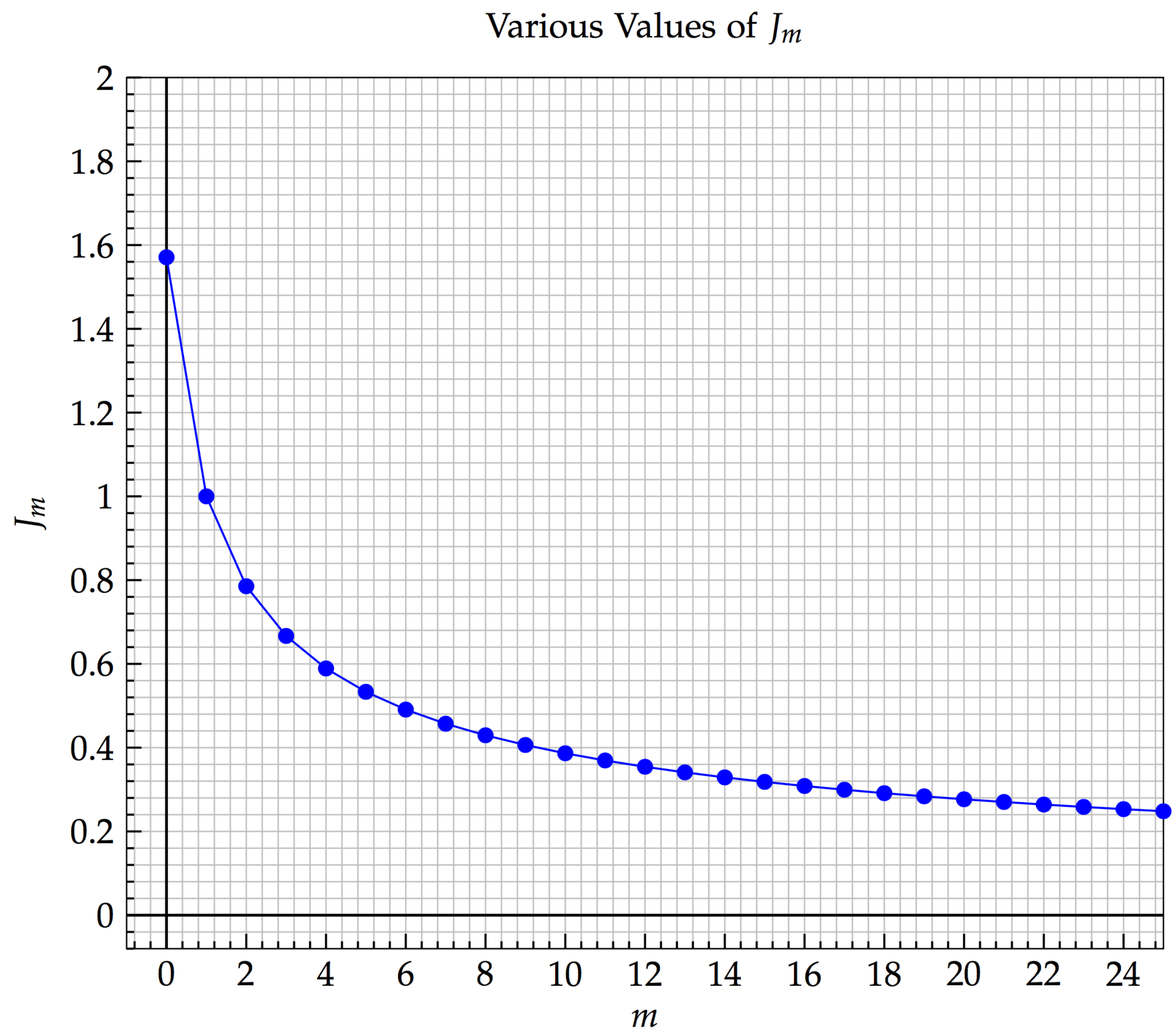
- Let's consider a family of values:

$$J_m = \int_0^{\pi/2} \sin^m \theta d\theta$$

- For each value of m, a separate value of J_m exists
  - Note well that m must be* an integer here!
  - We can graph them all to see some patterns
  - There should be some sort of general trend

*not exactly true, as we will see later

FUNCTION FAMILIES



DERIVATION

- We now formally determine the value of J_m
- It will end up being defined in terms of itself
    - A recursive function!
    - These occur a lot in mathematics

DERIVATION

$$
\begin{aligned}
J_m &= \int_0^{\pi/2} \sin^m \theta d\theta \\
&= \int_0^{\pi/2} \sin\theta \sin^{m-1} \theta d\theta \\
&= (\sin^{m-1}\theta)(-\cos\theta)\Big|_0^{\pi/2} + \int_0^{\pi/2} \cos\theta \left[(m-1)\sin^{m-2}\theta\right] \cos\theta d\theta \\
&= 0 + \int_0^{\pi/2} \left[(m-1)\sin^{m-2}\theta\right] \cos^2\theta d\theta \\
&= (m-1) \int_0^{\pi/2} \sin^{m-2}\theta \left(1 - \sin^2\theta\right) d\theta
\end{aligned}
$$

DERIVATION

$$J_m = \int_0^{\pi/2} \sin^m \theta d\theta$$

integration by parts

$$= \int_0^{\pi/2} \underset{dv}{\sin\theta}\, \overset{u}{\sin^{m-1}}\theta d\theta$$

$$= (\overset{u}{\sin^{m-1}}\theta)(\overset{v}{-\cos\theta})\Big|_0^{\pi/2} + \int_0^{\pi/2} \overset{v}{\cos\theta}\left[(m-1)\overset{du}{\sin^{m-2}}\theta\right]\cos\theta d\theta$$

$$= 0 + \int_0^{\pi/2} \left[(m-1)\sin^{m-2}\theta\right]\cos^2\theta d\theta$$

Fund. Identity of Trigonometry

$$= (m-1)\int_0^{\pi/2} \sin^{m-2}\theta\left(1-\sin^2\theta\right)d\theta$$

DERIVATION

$$
\begin{aligned}
J_m &= \int\limits_0^{\pi/2} \sin^m \theta d\theta \\
&= \vdots \\
&= (m-1) \int\limits_0^{\pi/2} \sin^{m-2} \theta - (m-1) \int\limits_0^{\pi/2} \sin^m \theta d\theta \\
&= (m-1) J_{m-2} - (m-1) J_m
\end{aligned}
$$

DERIVATION

$$
\begin{aligned}
J_m &= \int_0^{\pi/2} \sin^m \theta d\theta \\
&= \vdots \\
&= (m-1) \int_0^{\pi/2} \sin^{m-2} \theta - (m-1) \int_0^{\pi/2} \sin^m \theta d\theta \\
&= (m-1)J_{m-2} - (m-1)J_m
\end{aligned}
$$

collapsing the
definition

RECURSION

$$
\begin{aligned}
J_m &= (m-1)\cdot J_{m-2} - (m-1)\cdot J_m \\
J_m \cdot (1 + (m-1)) &= (m-1)\cdot J_{m-2} \\
J_m \cdot m &= (m-1)\cdot J_{m-2}
\end{aligned}
$$

$$\boxed{J_m = \frac{m-1}{m}\cdot J_{m-2}}$$

CASE 1: m IS ODD

$$
\begin{aligned}
J_m &= \frac{m-1}{m} \cdot J_{m-2} \\
&= \frac{(m-1)\cdot(m-3)\cdots 2}{m\cdot(m-2)\cdots 1} \cdot J_1 \\
&= \frac{[(m-1)\cdot(m-3)\cdots 2]^2}{m\cdot(m-1)\cdot(m-2)\cdots 2\cdot 1} \cdot J_1 \\
&= \frac{[(m-1)\cdot(m-3)\cdots 2]^2}{m!} \cdot J_1 \\
&= \frac{\left[2^{\frac{m-1}{2}}\left(\frac{m-1}{2}\right)\cdot\left(\frac{m-3}{2}\right)\cdots 1\right]^2}{m!} \cdot J_1 \\
&= \frac{2^{m-1}}{m!}\left[\left(\frac{m-1}{2}\right)\cdot\left(\frac{m-1}{2}-1\right)\cdots 1\right]^2 \cdot J_1 \\
&= \frac{2^{m-1}}{m!}\left[\left(\frac{m-1}{2}\right)!\right]^2 \cdot J_1
\end{aligned}
$$

CASE 2: m IS EVEN

$$
\begin{aligned}
J_m &= \frac{m-1}{m} \cdot J_{m-2} \\
&= \frac{(m-1)\cdot(m-3)\cdots 1}{m\cdot(m-2)\cdots 2} \cdot J_0 \\
&= \frac{m\cdot(m-1)\cdot(m-2)\cdots 2\cdot 1}{[m\cdot(m-2)\cdots 2]^2} \cdot J_0 \\
&= \frac{m!}{[m\cdot(m-2)\cdots 2]^2} \cdot J_0 \\
&= \frac{m!}{\left[2^{\frac{m}{2}}\left(\frac{m}{2}\right)\cdot\left(\frac{m}{2}-1\right)\cdots 1\right]^2} \cdot J_0 \\
&= \frac{m!}{2^m} \frac{1}{\left[\left(\frac{m}{2}\right)\cdot\left(\frac{m}{2}-1\right)\cdots 1\right]^2} \cdot J_0 \\
&= \frac{\frac{m!}{2^m}}{\left[\left(\frac{m}{2}\right)!\right]^2} \cdot J_0
\end{aligned}
$$

BASE CASES

- But we still can't compute anything!
- We need basis values of J_0 and J_1
  - Can construct rest out of these values
  - Will need to consider odd- and even- cases separately, since J_m depends only on J_{m-2} and the basis values may be **different**!

BASE CASES

$$J_0 = \int\limits_0^{\pi/2} \sin^0 \theta d\theta = \int\limits_0^{\pi/2} d\theta = \theta \Big|_0^{\pi/2} = \frac{\pi}{2}$$

$$J_1 = \int\limits_0^{\pi/2} \sin^1 \theta d\theta = \int\limits_0^{\pi/2} \sin \theta d\theta = -\cos \theta \Big|_0^{\pi/2} = 1$$

PUTTING EVERYTHING TOGETHER

$$\boxed{J_m = \frac{\frac{\pi}{2} \cdot \frac{m!}{2^m}}{\left[\left(\frac{m}{2}\right)!\right]^2}}$$

even

$$\boxed{J_m = \frac{2^{m-1}}{m!}\left[\left(\frac{m-1}{2}\right)!\right]^2}$$

odd

RELATED FAMILIES

- Consider the related family shown below:

$$K_m = \int_0^{\pi/2} \cos^m \theta d\theta$$

- We don't need to completely redo the derivations!
  - Can instead do a clever trick
  - Use previous results to generate new ones

DERIVATION

$$
\begin{aligned}
K_m &= \int_0^{\pi/2} \cos^m \theta d\theta \\
&= \int_0^{\pi/2} \underset{dv}{\cos\theta}\, \overset{u}{\cos^{m-1}\theta}\, d\theta \\
&= \left.(\overset{u}{\cos^{m-1}\theta})(\overset{v}{\sin\theta})\right|_0^{\pi/2} + \int_0^{\pi/2} \overset{v}{\sin\theta}\, \overset{du}{\left[(m-1)\cos^{m-2}\theta\right]} \sin\theta d\theta \\
&= 0 + \int_0^{\pi/2} \left[(m-1)\cos^{m-2}\theta\right] \sin^2\theta d\theta \\
&= (m-1)\int_0^{\pi/2} \cos^{m-2}\theta\left(1-\cos^2\theta\right) d\theta
\end{aligned}
$$

integration by parts

Fund. Identity of Trigonometry

DERIVATION

$$
\begin{aligned}
K_m &= \int_0^{\pi/2} \cos^m \theta d\theta \\
&= \vdots \\
&= (m-1) \int_0^{\pi/2} \cos^{m-2} \theta - (m-1) \int_0^{\pi/2} \cos^m \theta d\theta \\
&= (m-1)K_{m-2} - (m-1)K_m
\end{aligned}
$$

RECURSION

$$\begin{aligned} K_m &= (m-1)\cdot K_{m-2} - (m-1)\cdot K_m \\ K_m \cdot (1 + (m-1)) &= (m-1)\cdot K_{m-2} \\ K_m \cdot m &= (m-1)\cdot K_{m-2} \end{aligned}$$

$$\boxed{K_m = \frac{m-1}{m}\cdot K_{m-2}}$$

RECURSION

- In other words, this is the exact same recursion!
- That means we need to compute the same basis
  - Can cut down on re-derivation time as a result

$$K_0 = \int_0^{\pi/2} \cos^0 \theta d\theta = \int_0^{\pi/2} d\theta = \theta\Big|_0^{\pi/2} = \frac{\pi}{2}$$

$$K_1 = \int_0^{\pi/2} \cos^1 \theta d\theta = \int_0^{\pi/2} \cos \theta d\theta = \sin \theta\Big|_0^{\pi/2} = 1$$

SAVING TIME

- Same recursion, same basis:

$$\boxed{K_m = J_m}$$

- This type of time-saving analysis is very useful!
  - Akin to "pattern-recognition"
  - Keep a cache of well-known existing results

APPLICATIONS OF RECURSIVE CALCULUS

PURPOSE?

- So why did we learn this technique?
- Consider this:

$$\int_0^\infty \frac{x^2}{(1+kx^2)^4}\,dx$$

- How would we integrate something like this, where k is just any arbitrary constant?
  - Doesn't even have to be an integer!
  - Hard to do this with just integration by parts
  - Hint: ***u*-substitution**…

ADVANCED INTEGRALS

- Idea: Perform a variable substitution as a method of transforming the problem into an easier one:

$$u = 1 + kx^2$$

$$\frac{du}{dx} = 2kx$$

- Making these substitutions leads to the following…

ADVANCED INTEGRALS

$$\int_0^\infty \frac{x^2}{(1+kx^2)^4}dx = \int_1^\infty \frac{x^2}{u^4}\frac{1}{2kx}du$$

$$= \frac{1}{2k}\int_1^\infty \frac{x}{u^4}du$$

$$= \frac{1}{2k}\int_1^\infty \frac{\sqrt{\dfrac{u-1}{k}}}{u^4}du$$

$$= \frac{1}{2k^{3/2}}\int_1^\infty \frac{\sqrt{u-1}}{u^4}du$$

ADVANCED INTEGRALS

- This leads to another problem…

$$\int_{1}^{\infty} \frac{\sqrt{u-1}}{u^4} du$$

- To determine the value of this integral, we need yet another variable substitution
  - Let's try a trigonometric one this time
  - But which one works best here?

TRIGONOMETRY REVIEW

- What is the natural substitution? Want $\sqrt{u-1}$ form!

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 + \cot^2\theta = \csc^2\theta$$

$$\tan^2\theta + 1 = \sec^2\theta$$

TRIGONOMETRY REVIEW

- What is the natural substitution? Want $\sqrt{u-1}$ form!

$$\sin^2\theta - 1 = -\cos^2\theta$$

$$\csc^2\theta - 1 = \cot^2\theta$$

$$\sec^2\theta - 1 = \tan^2\theta$$

correct forms

SUBSTITUTIONS

- There are **two** natural choices:

$$u = \csc^2 \theta$$

$$u = \sec^2 \theta$$

- Both of these give reasonable answers
  - Should agree, though
  - Proof?

SINUSOID

$$u = \csc^2\theta$$

$$\frac{du}{d\theta} = 2\csc\theta\left(-\csc\theta\cot\theta\right) = -2\csc^2\theta\cot\theta = -\frac{2\cos\theta}{\sin^3\theta}$$

SINUSOID

$$\int_{1}^{\infty} \frac{\sqrt{u-1}}{u^4} du = \int_{\pi/2}^{0} \frac{\sqrt{\csc^2\theta - 1}}{\left(\csc^2\theta\right)^4} \left(-\frac{2\cos\theta}{\sin^3\theta}\right) d\theta$$

$$= \int_{0}^{\pi/2} \frac{\cot\theta}{\csc^8\theta} \cdot \frac{2\cos\theta}{\sin^3\theta} d\theta$$

$$= \int_{0}^{\pi/2} \frac{2\cos^2\theta}{\csc^4\theta} d\theta = \int_{0}^{\pi/2} 2\cos^2\theta \sin^4\theta d\theta$$

$$= \int_{0}^{\pi/2} 2\left(1 - \sin^2\theta\right)\sin^4\theta d\theta$$

SINUSOID

$$\int_1^\infty \frac{\sqrt{u-1}}{u^4}\,du = \vdots$$

$$= 2\int_0^{\pi/2} \sin^4\theta\, d\theta - 2\int_0^{\pi/2} \sin^6\theta\, d\theta$$

$$= 2J_4 - 2J_6$$

$$= \boxed{2\left(J_4 - J_6\right)}$$

COSINUSOID

$$u = \sec^2 \theta$$

$$\frac{du}{d\theta} = 2 \sec \theta (\sec \theta \tan \theta) = 2 \sec^2 \theta \tan \theta = \frac{2 \sin \theta}{\cos^3 \theta}$$

COSINUSOID

$$\int_1^\infty \frac{\sqrt{u-1}}{u^4} du = \int_0^{\pi/2} \frac{\sqrt{\sec^2\theta - 1}}{\left(\sec^2\theta\right)^4} \left(\frac{2\sin\theta}{\cos^3\theta}\right) d\theta$$

$$= \int_0^{\pi/2} \frac{\tan\theta}{\sec^8\theta} \cdot \frac{2\sin\theta}{\cos^3\theta} d\theta$$

$$= \int_0^{\pi/2} \frac{2\sin^2\theta}{\sec^4\theta} d\theta = \int_0^{\pi/2} 2\sin^2\theta\cos^4\theta d\theta$$

$$= \int_0^{\pi/2} 2\left(1-\cos^2\theta\right)\cos^4\theta d\theta$$

COSINUSOID

$$\int_1^\infty \frac{\sqrt{u-1}}{u^4}du = \vdots$$

$$= 2\int_0^{\pi/2} \cos^4\theta d\theta - 2\int_0^{\pi/2} \cos^6\theta d\theta$$

$$= 2K_4 - 2K_6$$

$$= \boxed{2\left(K_4 - K_6\right)}$$

COMPARING FORMS

- Recall:
  - The families J_m and K_m are equivalent!
- So both substitutions yield the same value:

$$J_4 = K_4 = \frac{\frac{\pi}{2} \cdot \frac{4!}{2^4}}{\left[\left(\frac{4}{2}\right)!\right]^2}$$

$$= \frac{\pi}{2} \cdot \frac{24}{16} \cdot \frac{1}{[2]^2}$$

$$= \frac{3\pi}{16}$$

$$J_6 = K_6 = \frac{\frac{\pi}{2} \cdot \frac{6!}{2^6}}{\left[\left(\frac{6}{2}\right)!\right]^2}$$

$$= \frac{\pi}{2} \cdot \frac{720}{64} \cdot \frac{1}{[6]^2}$$

$$= \frac{5\pi}{32}$$

PUTTING IT TOGETHER

$$\begin{aligned}
2(J_4 - J_6) &= 2(K_4 - K_6) \\
&= 2\left(\frac{3\pi}{16} - \frac{5\pi}{32}\right) \\
&= 2\left(\frac{6\pi}{32} - \frac{5\pi}{32}\right) \\
&= 2\left(\frac{\pi}{32}\right) \\
&= \frac{\pi}{16}
\end{aligned}$$

PUTTING IT TOGETHER

$$\int_0^\infty \frac{x^2}{(1+kx^2)^4}dx = \frac{1}{2k^{3/2}}\int_1^\infty \frac{\sqrt{u-1}}{u^4}du$$

$$= \frac{1}{2k^{3/2}} \cdot \frac{\pi}{16}$$

$$= \boxed{\frac{\pi}{32}k^{-3/2}}$$

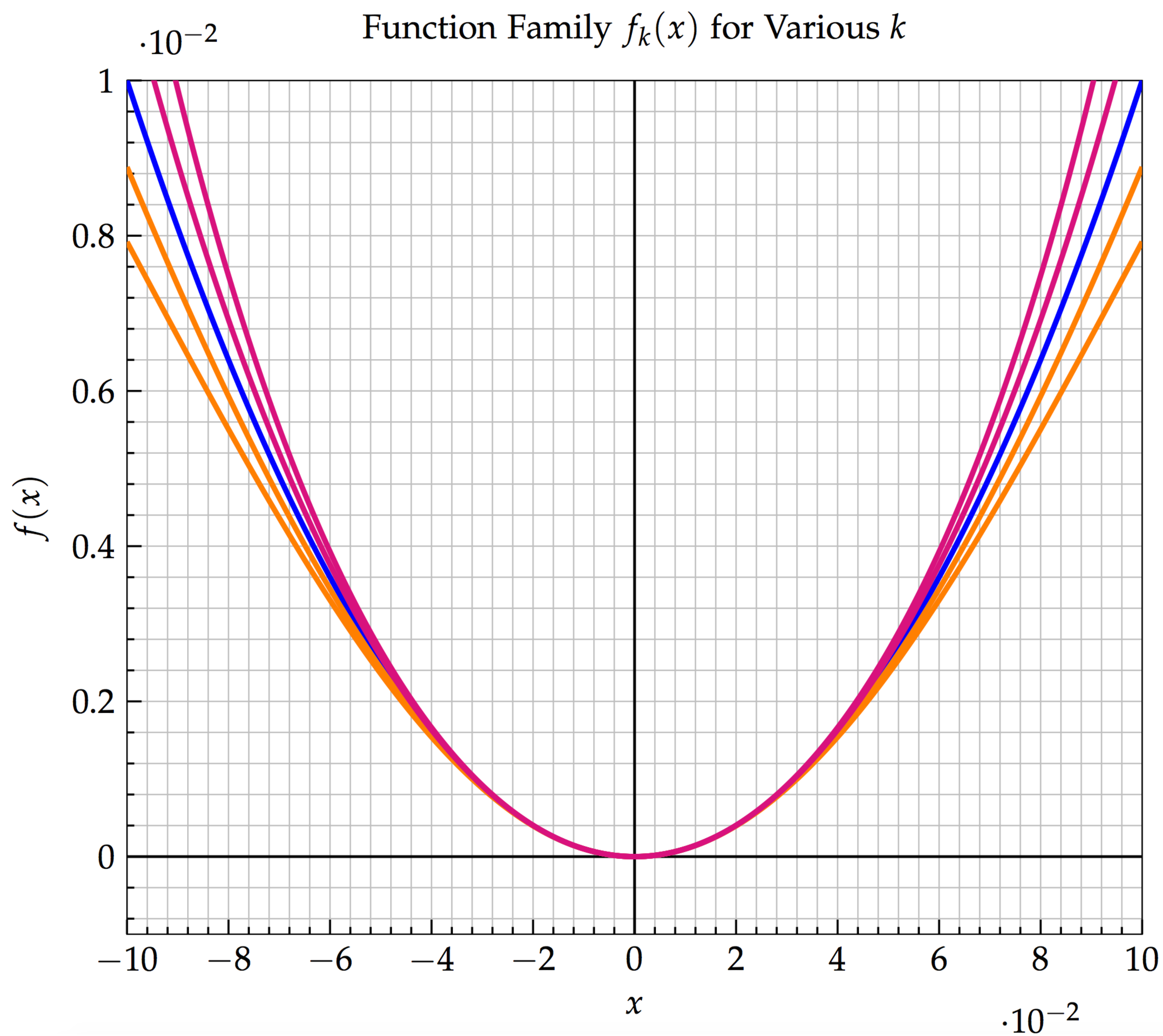
ACCUMULATION FUNCTION

- We can define a function family

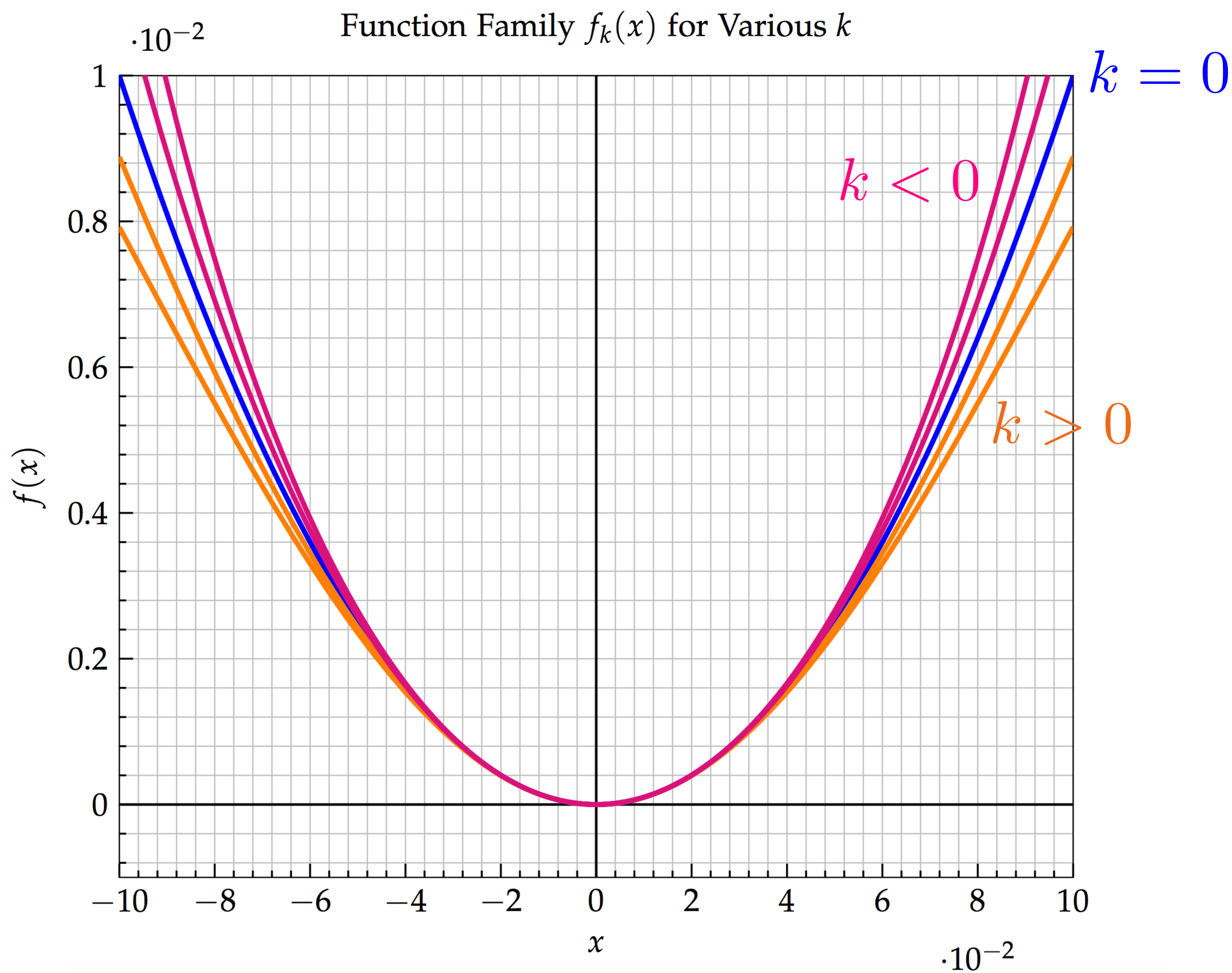
$$f_k(x) = \frac{x^2}{(1 + kx^2)^4}$$

- What are the properties of f_k ?
  - Should graph family for various values of k
  - i.e. various instances

FUNCTION FAMILY



FUNCTION FAMILY



SENSITIVITY REVISITED

- We can define a function

$$g(k) = \int_0^\infty \frac{x^2}{(1+kx^2)^4}\,dx = \frac{\pi}{32}k^{-3/2}$$

- How sensitive is g to changes in k?
  - Idea: Seems to be an inverse-power law
  - Would expect it to be highly sensitive in so-called "fractional region" (i.e. where $0 < k \leq 1$)

SENSITIVITY REVISITED

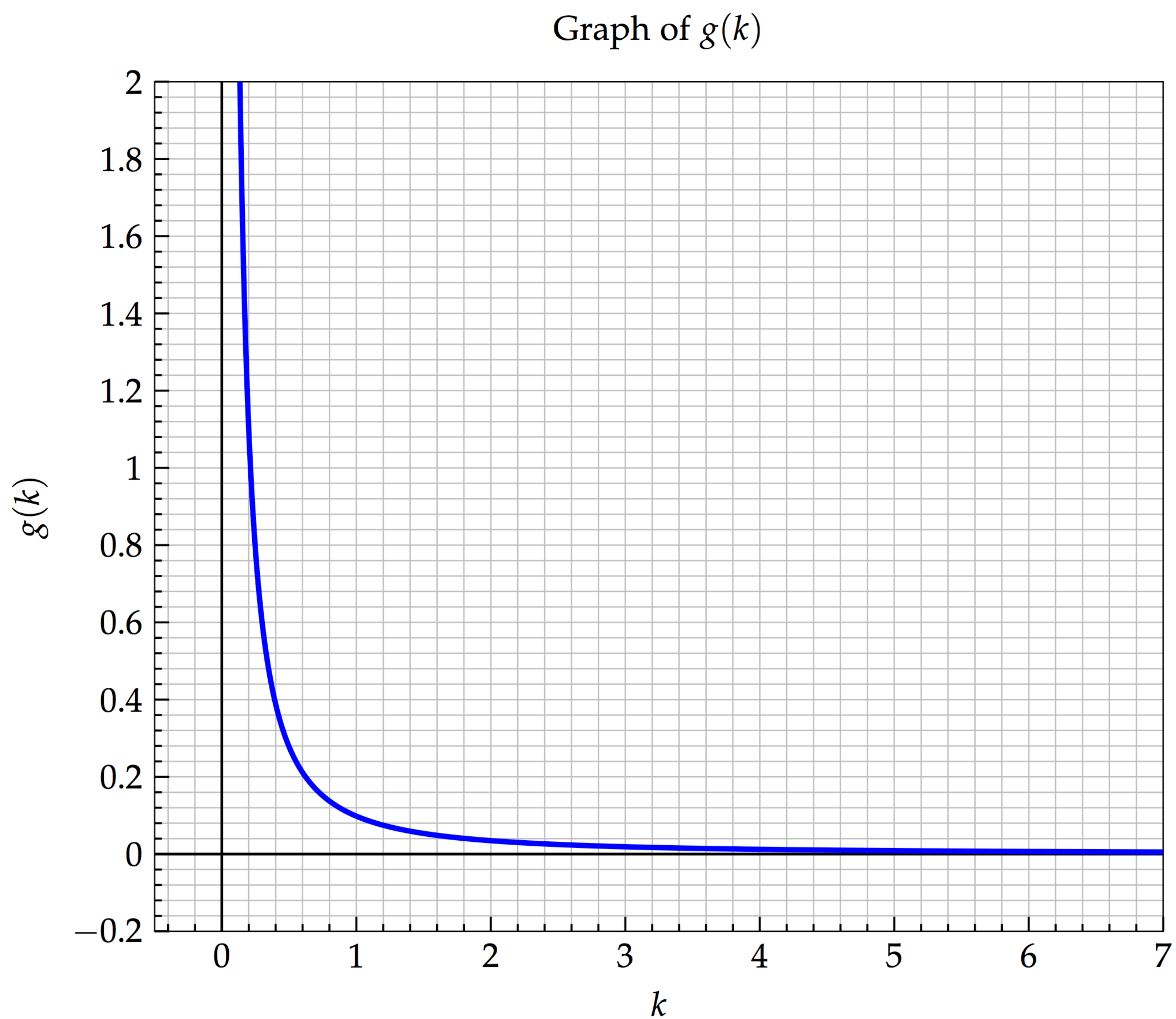
- We can define a function

very similar to an accumulation function

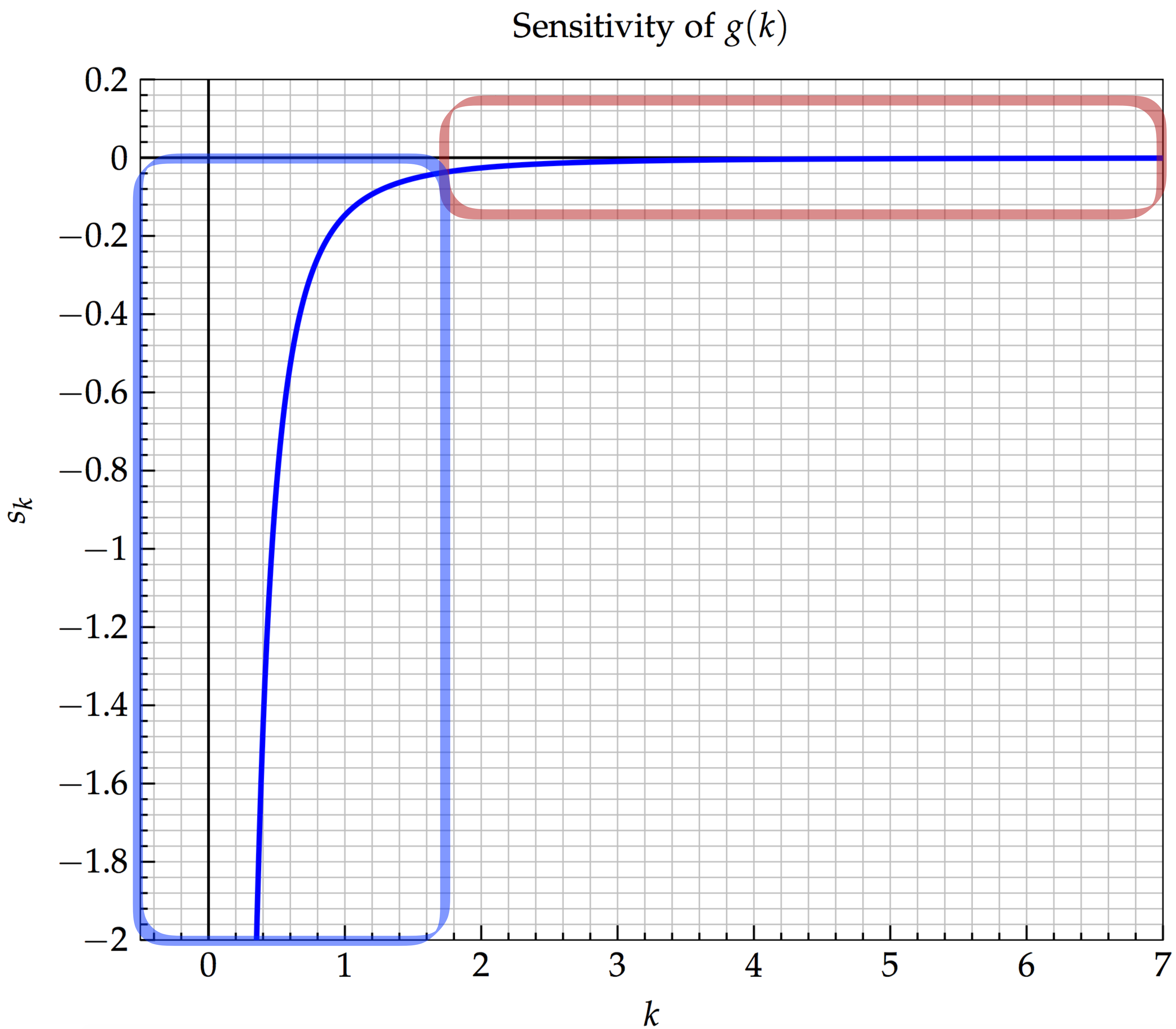
$$g(k) = \int_0^\infty \frac{x^2}{(1+kx^2)^4} dx = \frac{\pi}{32} k^{-3/2}$$

- How sensitive is g to changes in k?
  - Idea: Seems to be an inverse-power law
  - Would expect it to be highly sensitive in so-called "fractional region" (i.e. where $0 < k \leq 1$)

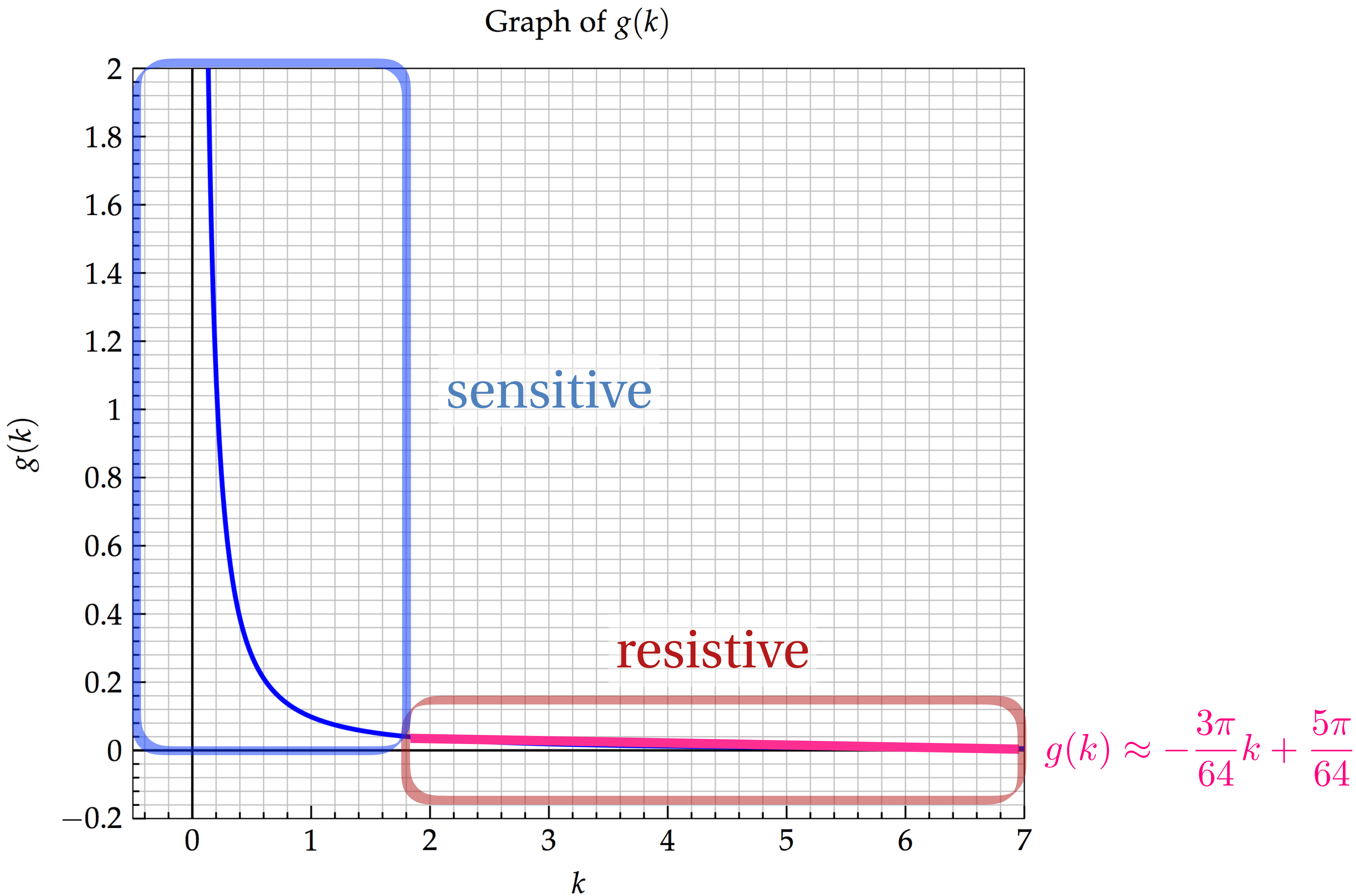
FUNCTION GRAPH



SENSITIVITY GRAPH



FUNCTION GRAPH



ANALYTIC EXTENSIONS

REVISITING FAMILIES

- Recall:

$$\boxed{J_m = \frac{\frac{\pi}{2} \cdot \frac{m!}{2^m}}{\left[\left(\frac{m}{2}\right)!\right]^2}}$$

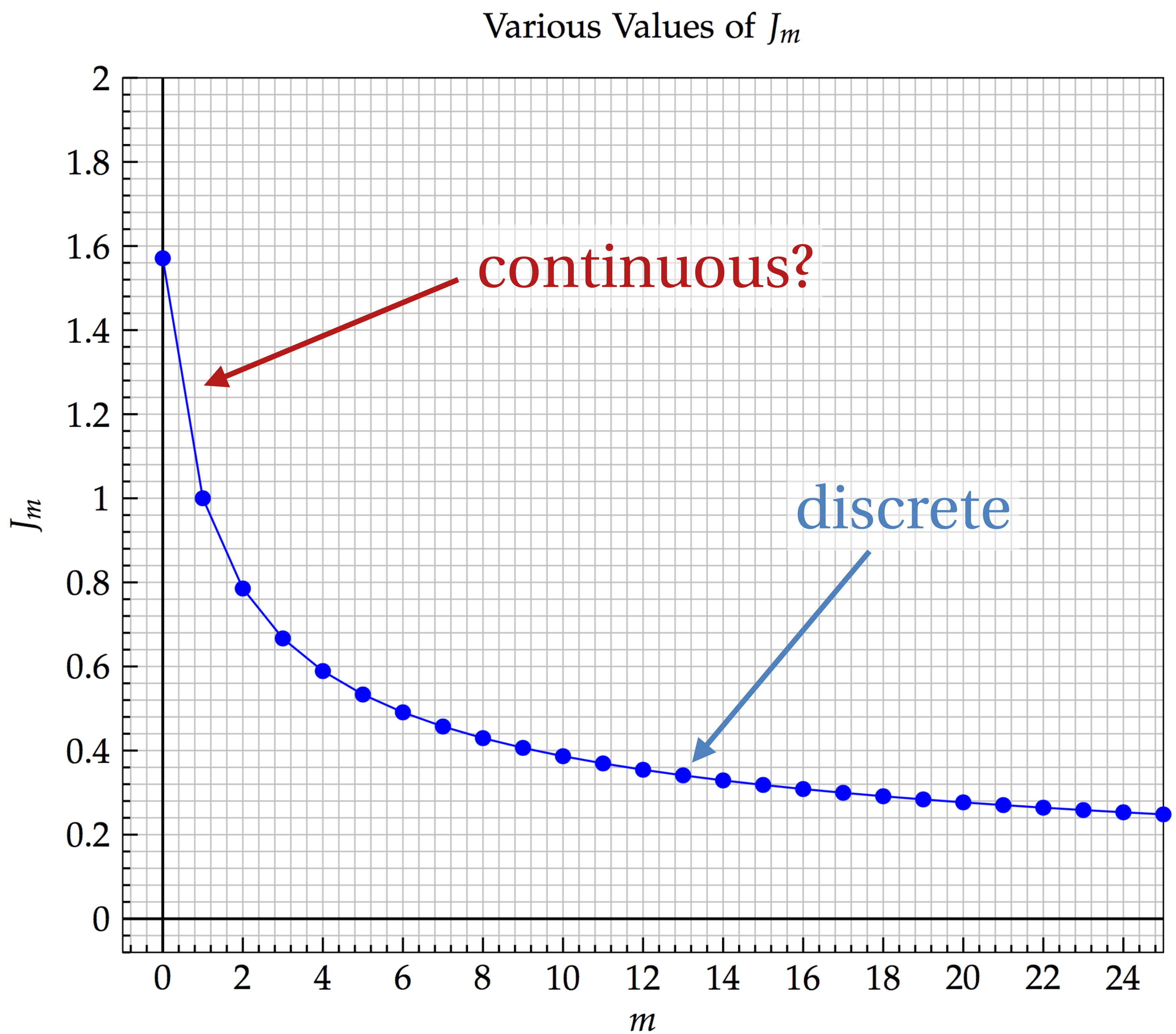
even

$$\boxed{J_m = \frac{2^{m-1}}{m!}\left[\left(\frac{m-1}{2}\right)!\right]^2}$$

odd

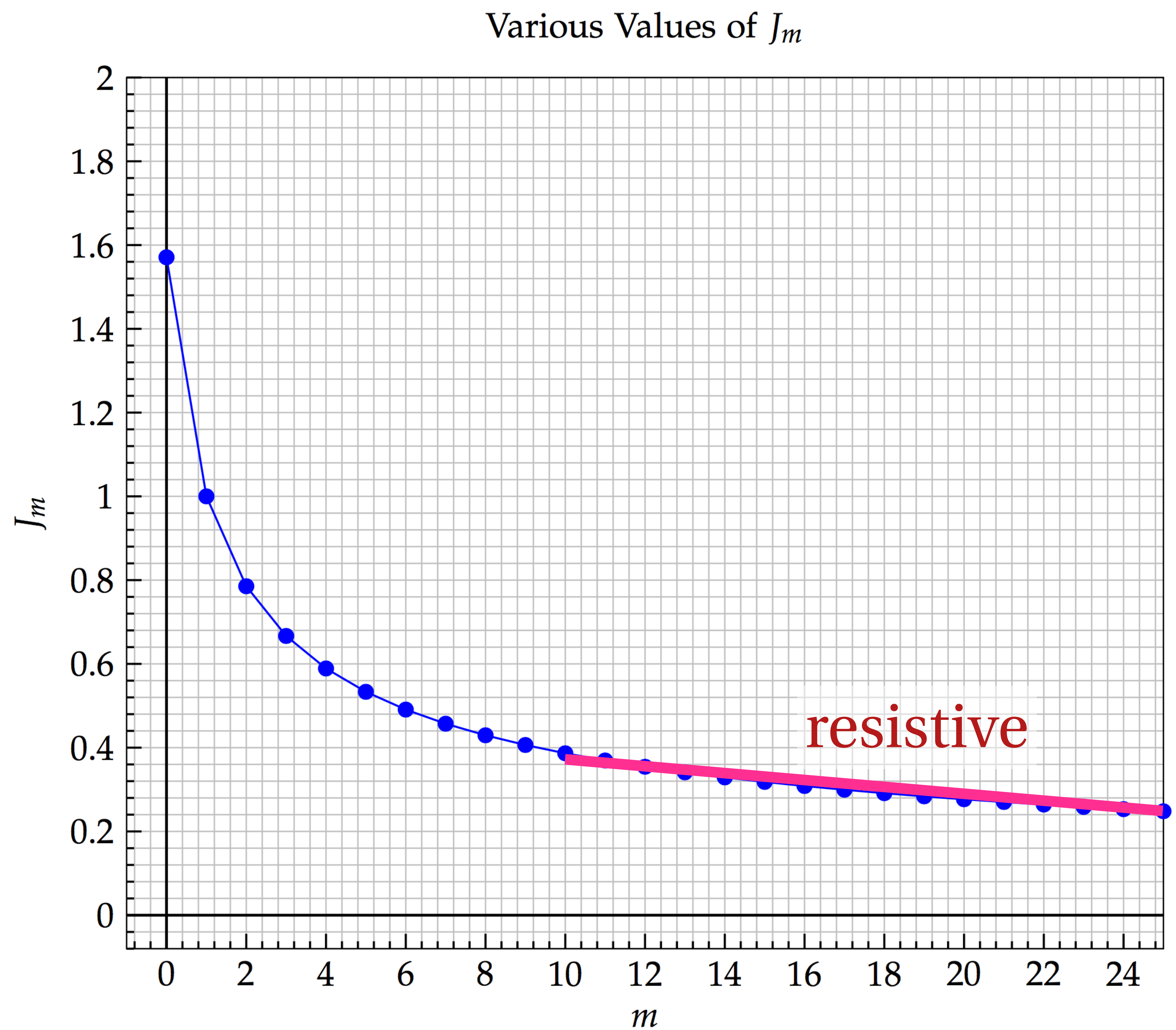
REVISITING FAMILIES

- Recall:

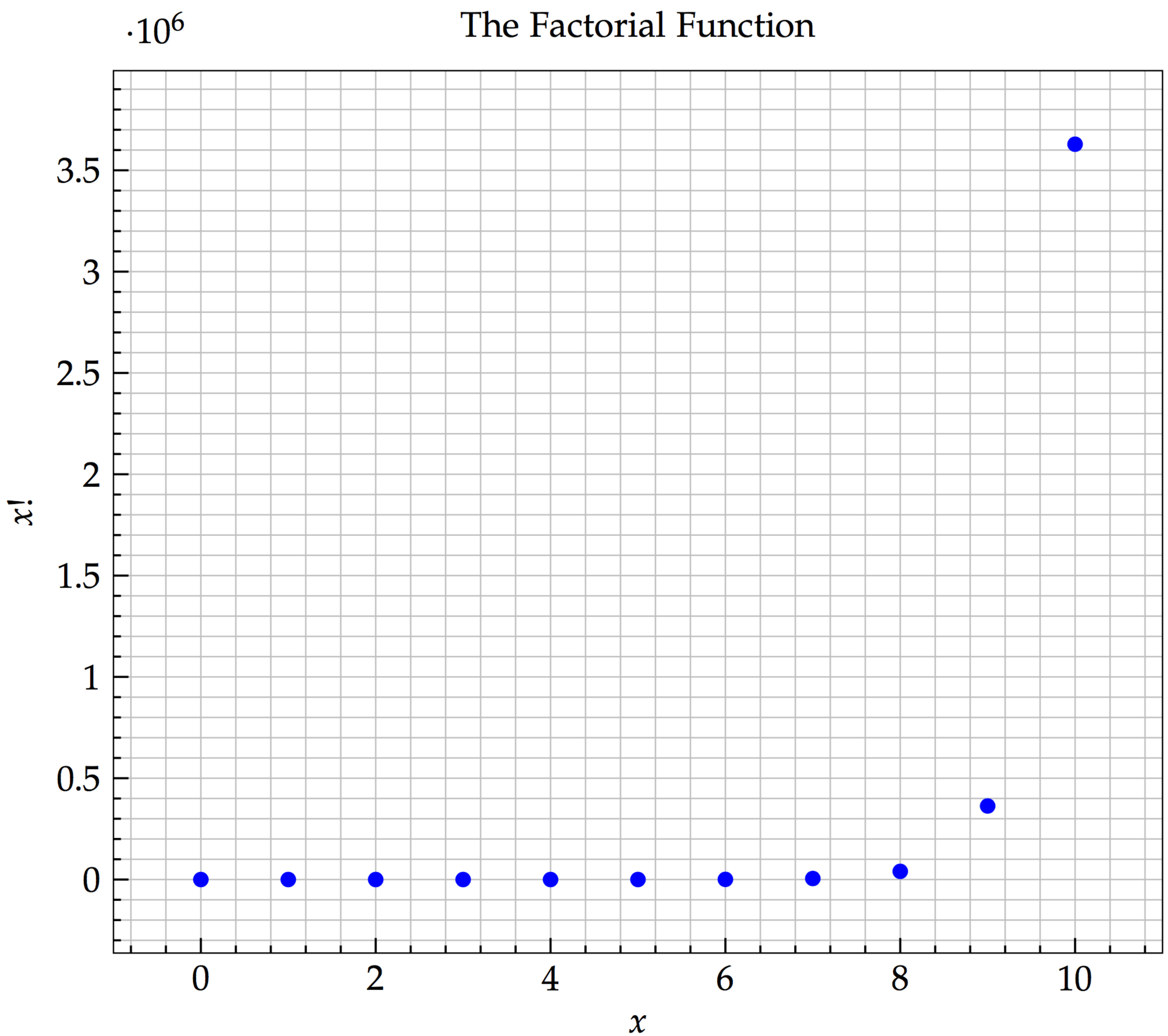


REVISITING FAMILIES

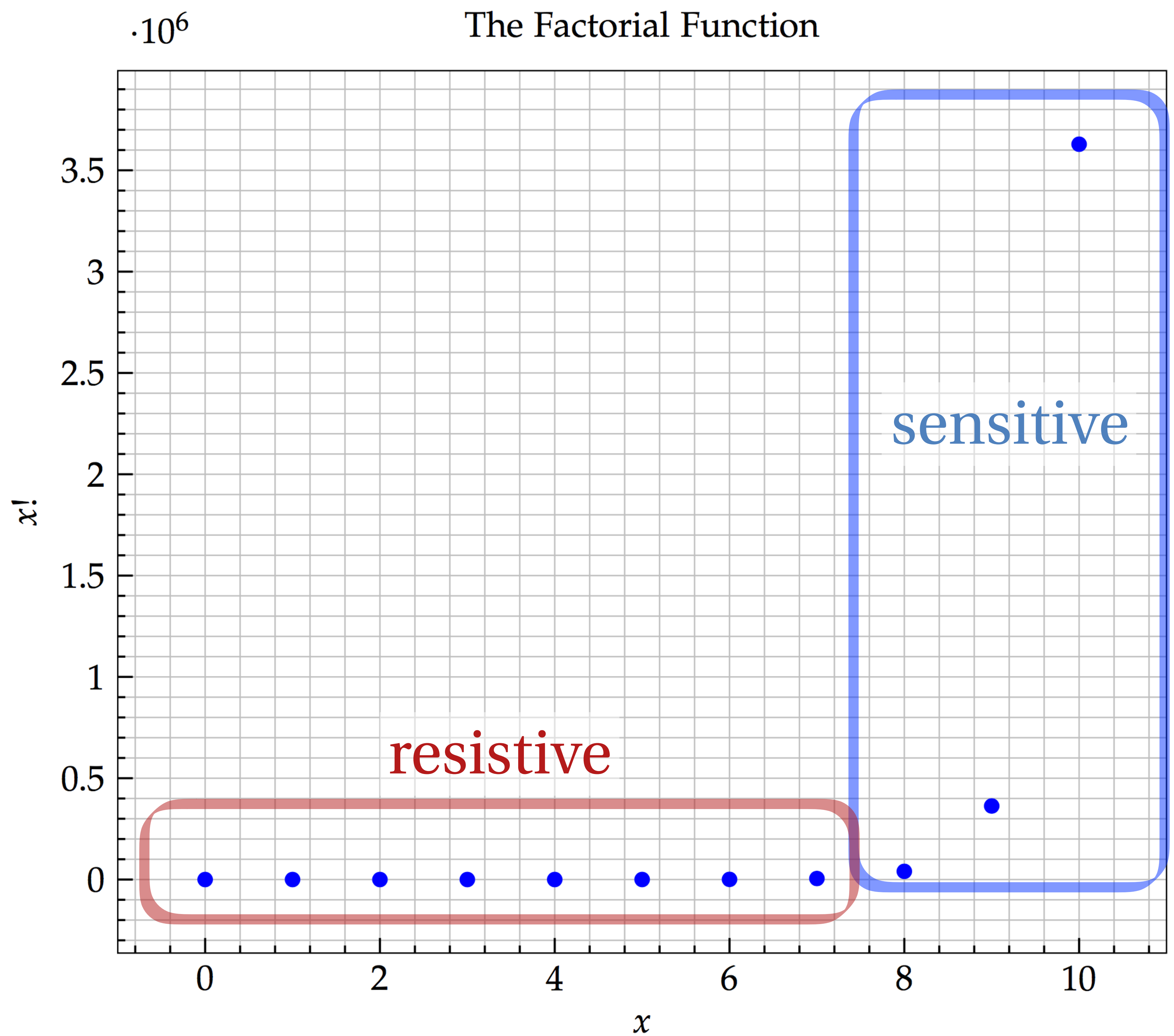
- BTW:



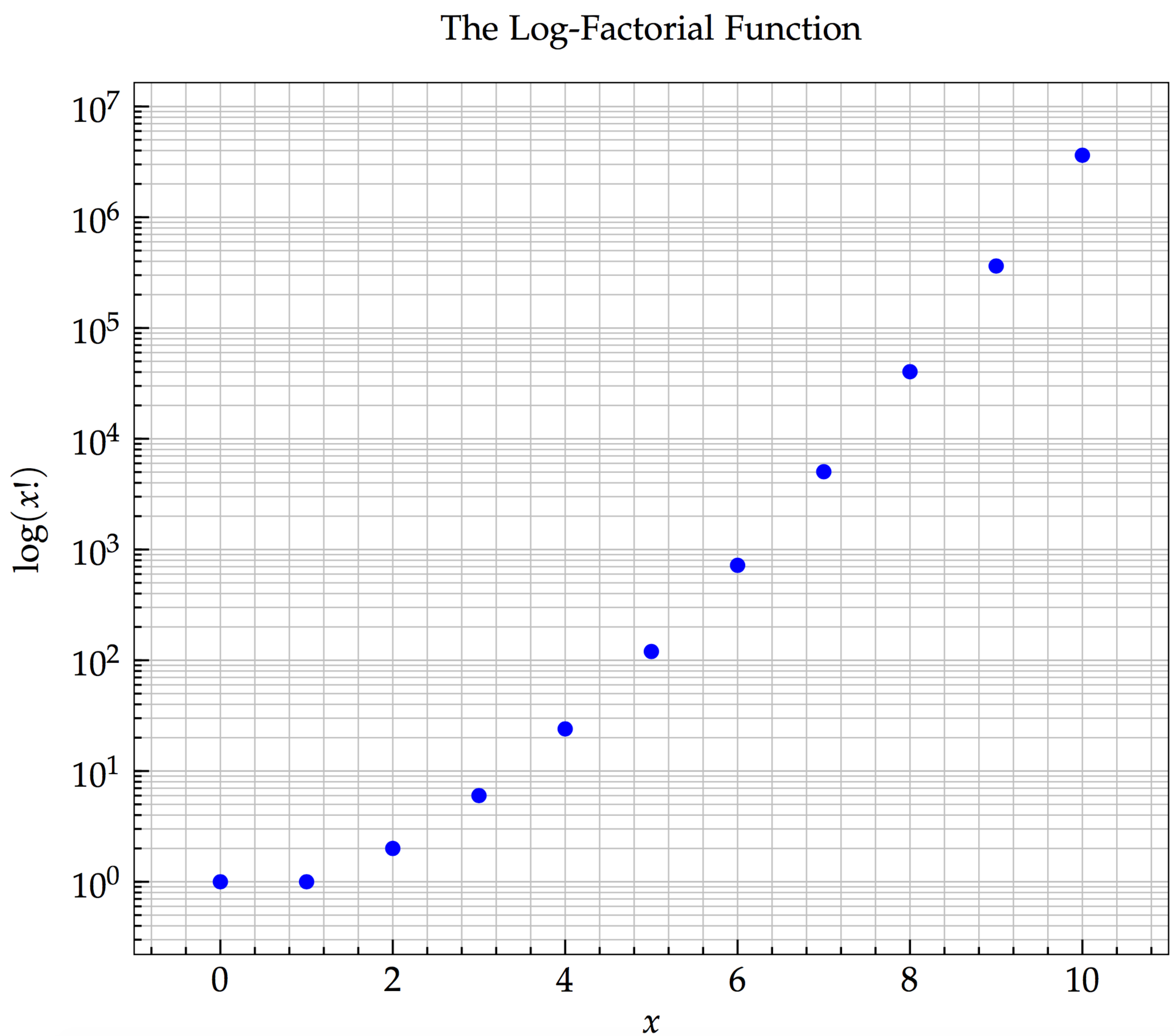
THE FACTORIAL FUNCTION



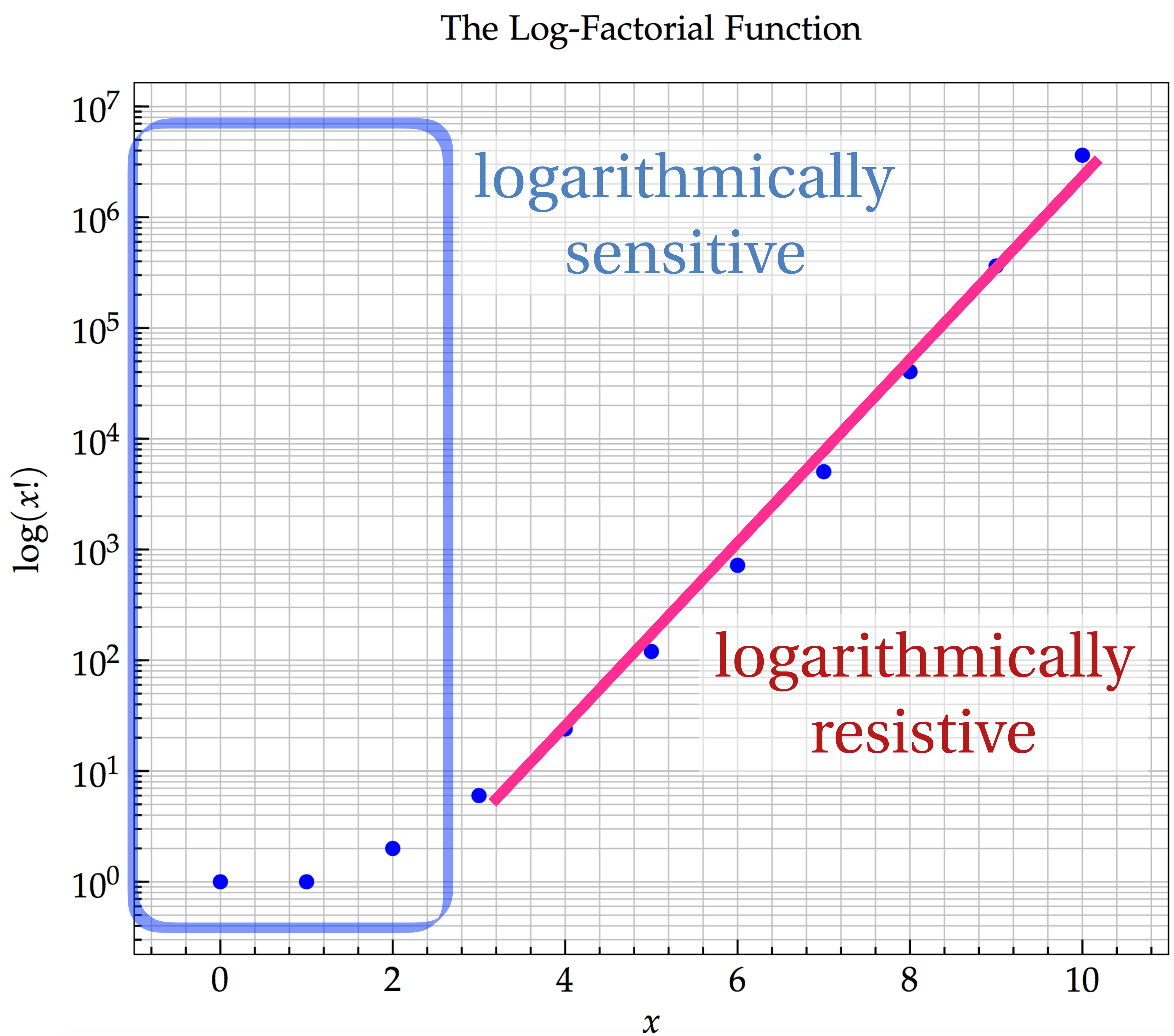
THE FACTORIAL FUNCTION



THE LOG-FACTORIAL FUNCTION



THE LOG-FACTORIAL FUNCTION



THE GAMMA FUNCTION

- Let us extend the idea of factorials!
- Consider the following definition:

$$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} dx$$

- This is actually a *complete* definition, in that we have no simpler form in general!
- But there are cases to consider!
  - t is discrete!
  - t is continuous?

THE GAMMA FUNCTION

- The case of t being discrete (i.e. an integer):

$$\Gamma(1) = \int_0^\infty x^{1-1} e^{-x} dx = \int_0^\infty e^{-x} dx$$
$$= \left[-e^{-x}\right]_0^\infty$$
$$= 0 - (-1) = 1$$

THE GAMMA FUNCTION

- The case of t being discrete (i.e. an integer):

$$\Gamma(1) = \int_0^\infty x^{1-1} e^{-x} dx = \int_0^\infty e^{-x} dx$$
$$= \left[-e^{-x}\right]_0^\infty$$
$$= 0 - (-1) = 1$$

integration by parts

$$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} dx = -x^{t-1} e^{-x}|_0^\infty + (t-1) \int_0^\infty x^{t-2} e^{-x} dx$$
$$= (-0 + 0) + (t-1) \int_0^\infty x^{(t-1)-1} e^{-x} dx$$
$$= (t-1)\Gamma(t-1)$$

THE GAMMA FUNCTION

- The case of t being discrete (i.e. an integer)
- We now have an *inductive definition*:
  - $\Gamma(1) = 1$
  - $\Gamma(t) = (t-1)\Gamma(t-1)$
  - As a result, we really see that $\Gamma(t) = (t-1)!$
  - This is exciting!
- Natural extensions?
  - Does this extend to continuous t, negative t, etc.?

yes, by a process called analytic continuation

THE GAMMA FUNCTION

- Some more cases that we will need to make use of:
  - The case of $t = 1/2$ will be particularly important
  - We use it as a basis to define the Gamma function over *half*-integer intervals!
  - Turns out that $\Gamma(t) = \sqrt{\pi}$
  - Proof?

THE GAMMA FUNCTION

$$
\begin{aligned}
\Gamma\left(\frac{1}{2}\right) = \int_0^\infty x^{\frac{1}{2}-1} e^{-x} dx &= \int_0^\infty x^{-\frac{1}{2}} e^{-x} dx \\
&= -x^{-\frac{1}{2}} e^{-x} \Big|_0^\infty - \frac{1}{2} \int_0^\infty x^{-\frac{3}{2}} e^{-x} dx \\
&= -\frac{1}{2} \int_0^\infty x^{-\frac{3}{2}} e^{-x} dx \\
&= -\frac{1}{2} \int_0^\infty x^{\left(-\frac{1}{2}\right)-1} e^{-x} dx \\
&= -\frac{1}{2} \Gamma\left(-\frac{1}{2}\right)
\end{aligned}
$$

THE GAMMA FUNCTION

- We can iteratively repeat this process:

$$
\begin{aligned}
\Gamma\left(\frac{1}{2}\right) &= -\frac{1}{2}\Gamma\left(-\frac{1}{2}\right) \\
\Gamma\left(-\frac{1}{2}\right) &= -\frac{3}{2}\Gamma\left(-\frac{3}{2}\right) \\
\Gamma\left(-\frac{3}{2}\right) &= -\frac{5}{2}\Gamma\left(-\frac{5}{2}\right) \\
&\vdots
\end{aligned}
$$

THE GAMMA FUNCTION

- We can *iteratively* repeat this process:

$$\boxed{\Gamma\left(\frac{1}{2}\right) = ???}$$

$$\Gamma\left(-\frac{1}{2}\right) = -2\Gamma\left(\frac{1}{2}\right)$$

$$\Gamma\left(-\frac{3}{2}\right) = -\frac{2}{3}\Gamma\left(-\frac{1}{2}\right) = \frac{4}{3}\Gamma\left(\frac{1}{2}\right)$$

$$\Gamma\left(-\frac{5}{2}\right) = -\frac{2}{5}\Gamma\left(-\frac{3}{2}\right) = -\frac{8}{15}\Gamma\left(\frac{1}{2}\right)$$

$$\Gamma\left(-\frac{7}{2}\right) = -\frac{2}{7}\Gamma\left(-\frac{5}{2}\right) = \frac{16}{105}\Gamma\left(\frac{1}{2}\right)$$

$$\vdots$$

THE GAMMA FUNCTION

- Similarly for *positive* half-integers:

$$\boxed{\Gamma\left(\frac{1}{2}\right) = ???}$$

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{4}\Gamma\left(\frac{1}{2}\right)$$

$$\Gamma\left(\frac{7}{2}\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \frac{15}{8}\Gamma\left(\frac{1}{2}\right)$$

$$\vdots$$

THE GAMMA FUNCTION

- The *basis* value of half-integer Gamma function:

$$\begin{aligned}
\Gamma\left(\frac{1}{2}\right) &= \int_0^\infty x^{-\frac{1}{2}} e^{-x} dx \\
&= \int_0^\infty 2e^{-u^2} du \\
&= 2\int_0^\infty e^{-u^2} du \\
&= \int_{-\infty}^\infty e^{-u^2} du \\
&= I
\end{aligned}$$

THE GAMMA FUNCTION

$$
\begin{aligned}
I^2 &= \int_{-\infty}^{\infty} e^{-u^2} du \int_{-\infty}^{\infty} e^{-v^2} dv \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-u^2-v^2} dudv \\
&= \int_{0}^{2\pi} \int_{0}^{\infty} e^{-r^2} rdrd\theta \\
&= \int_{0}^{2\pi} d\theta \int_{0}^{\infty} re^{-r^2} dr \\
&= 2\pi \left(0 - \left(\frac{1}{2}\right)\right) \\
&= \pi
\end{aligned}
$$

$$\boxed{\Gamma\left(\frac{1}{2}\right) = I = \sqrt{\pi}}$$

THE GAMMA FUNCTION

- We can use this basis to find a general formula:

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\vdots$$

$$\Gamma\left(-\frac{7}{2}\right) = \frac{16}{105}\sqrt{\pi}$$

$$\Gamma\left(-\frac{5}{2}\right) = -\frac{8}{15}\sqrt{\pi}$$

$$\Gamma\left(-\frac{3}{2}\right) = \frac{4}{3}\sqrt{\pi}$$

$$\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$$

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}$$

$$\Gamma\left(\frac{5}{2}\right) = \frac{3}{4}\sqrt{\pi}$$

$$\Gamma\left(\frac{7}{2}\right) = \frac{15}{8}\sqrt{\pi}$$

$$\vdots$$

THE GAMMA FUNCTION

$$\Gamma\left(\frac{1}{2}-n\right) = \frac{(-2)^n}{\prod\limits_{k=0}^{n-1}(2k+1)}\sqrt{\pi}$$

$$= \frac{(-2)^n \prod\limits_{k=0}^{n}(2k)}{\prod\limits_{k=0}^{n} k}\sqrt{\pi}$$

$$= \frac{(-2)^n 2^n \prod\limits_{k=0}^{n-1} k}{(2n)!}\sqrt{\pi}$$

$$= \frac{(-4)^n n!}{(2n)!}\sqrt{\pi}$$

THE GAMMA FUNCTION

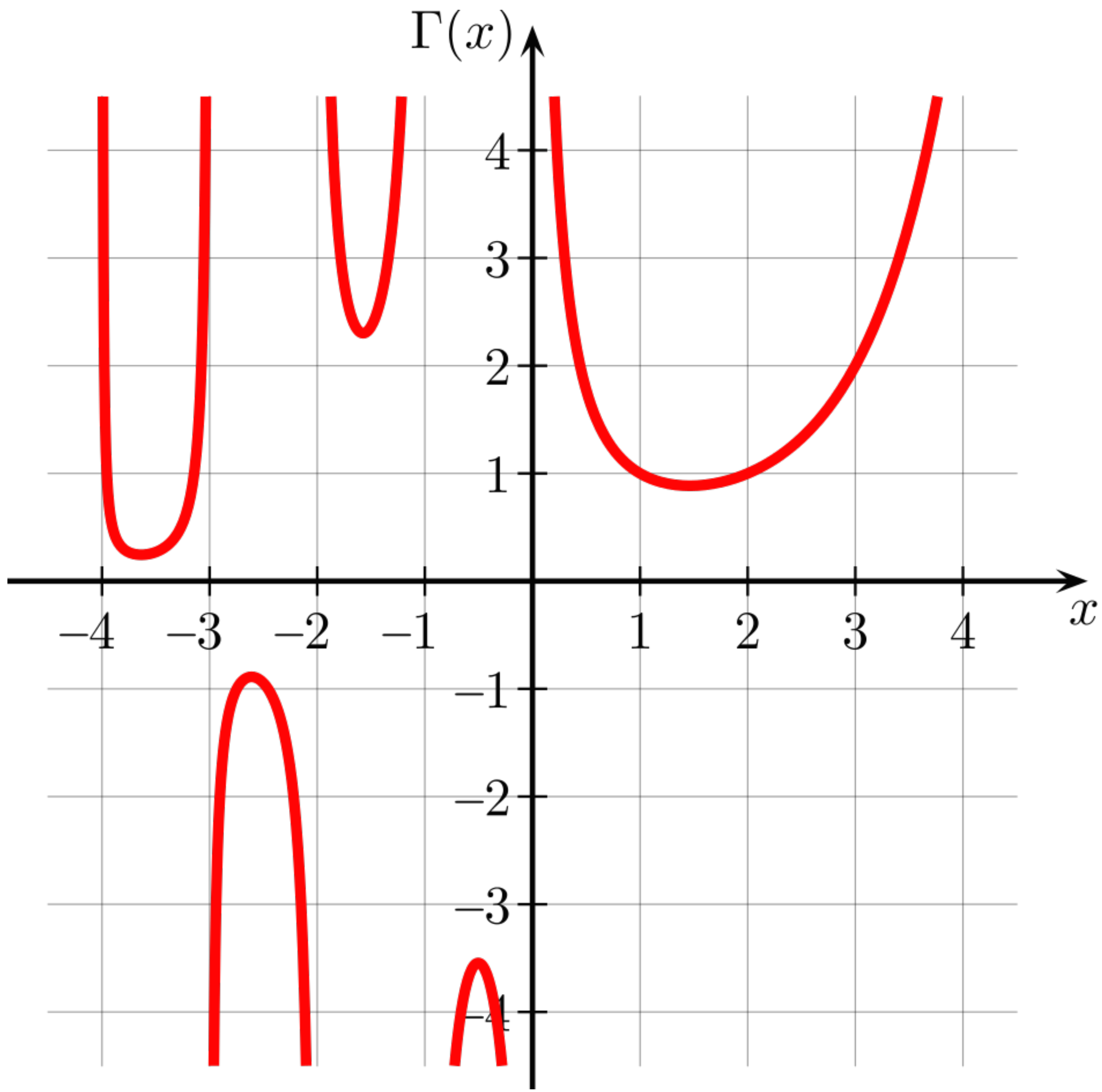
$$\Gamma\left(\frac{1}{2}+n\right) = \frac{\prod_{k=0}^{n-1}(2k+1)}{2^n}\sqrt{\pi}$$

$$= \frac{\prod_{k=0}^{n} k}{2^n \prod_{k=0}^{n}(2k)}\sqrt{\pi}$$

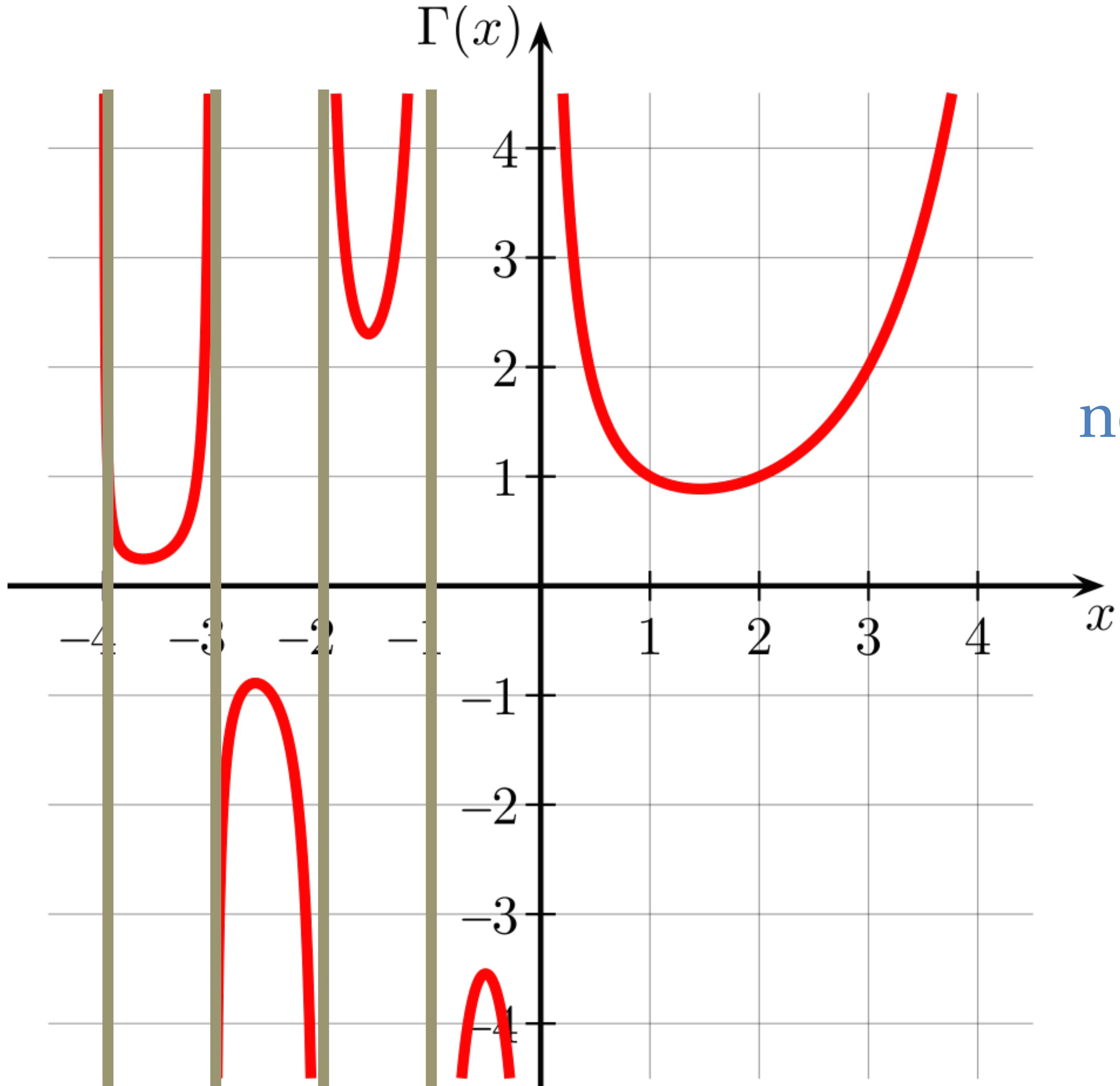
$$= \frac{(2n)!}{2^n 2^n \prod_{k=0}^{n-1} k}\sqrt{\pi}$$

$$= \frac{(2n)!}{4^n n!}\sqrt{\pi}$$

THE GAMMA FUNCTION

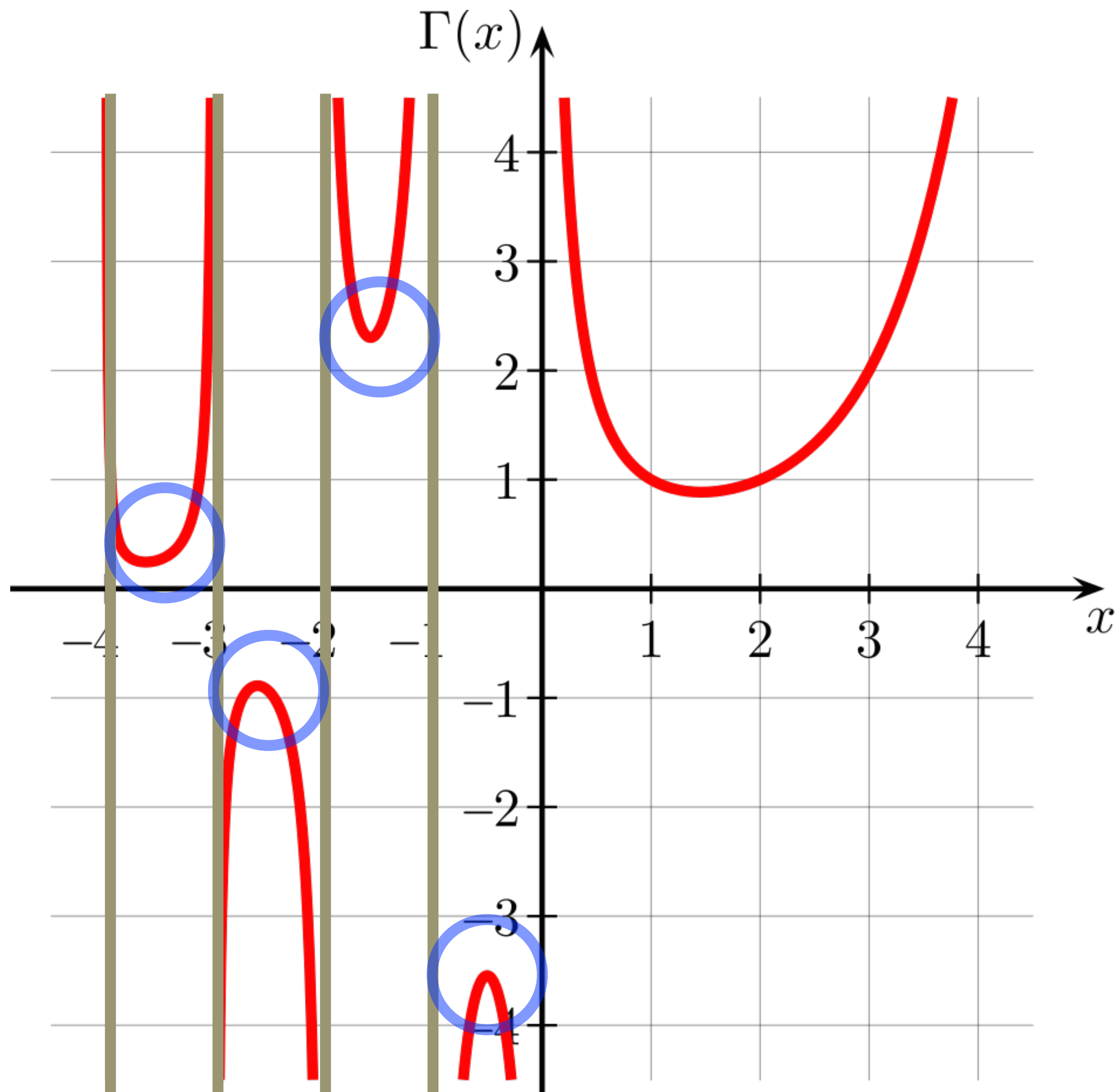


THE GAMMA FUNCTION



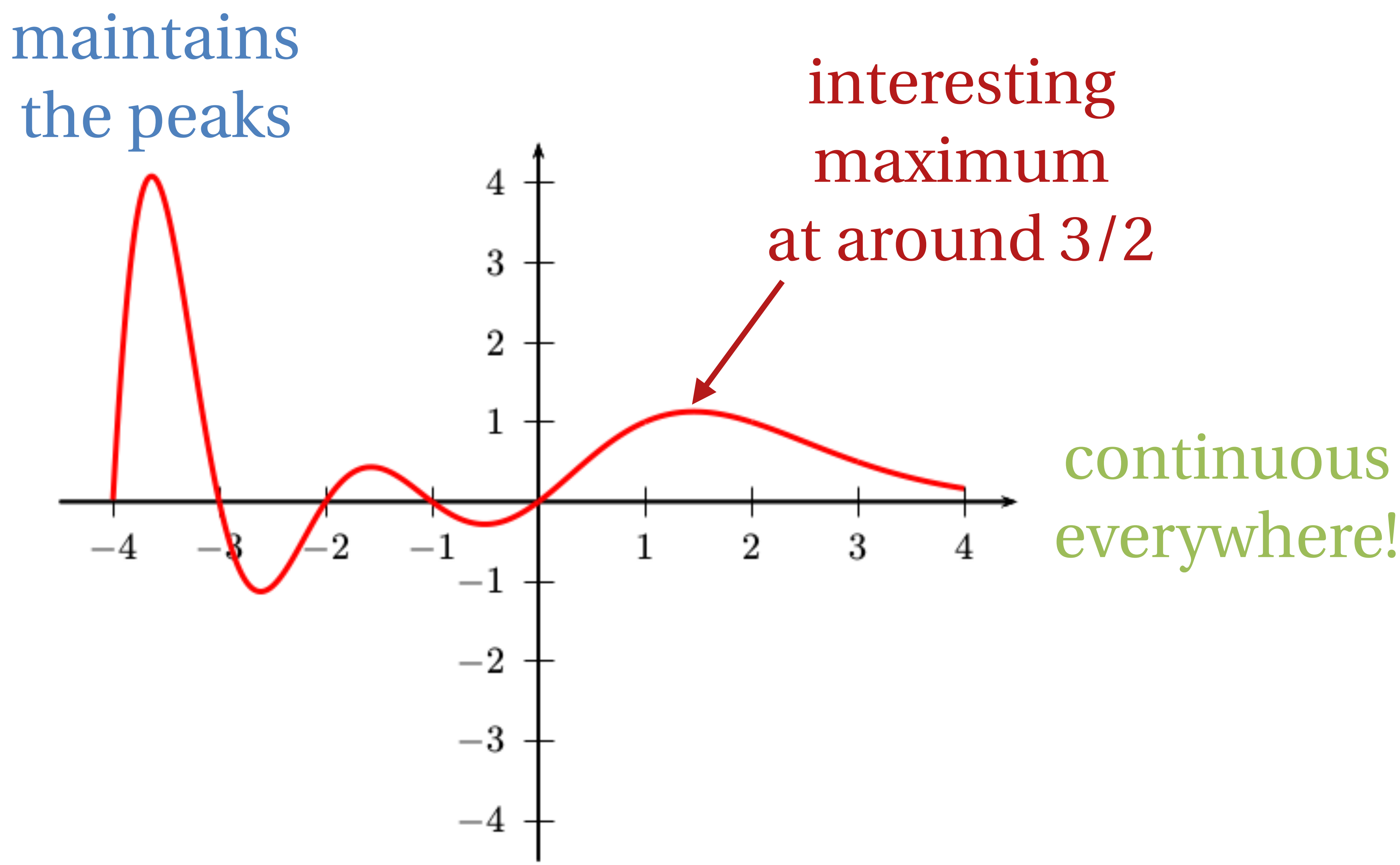
undefined at negative integers

THE GAMMA FUNCTION



"peaks" at negative half-integers

THE INVERSE GAMMA FUNCTION



THE GAMMA FUNCTION

- As we saw, the gamma function is simply a generalization of the factorial function!
  - Continuous factorials
  - Analytic continuation
- Highly sensitive to changes in input
  - Especially around integer boundaries
  - Near “peaks” in the negative bias region

INDICATOR FUNCTIONS

IVERSON BRACKET

- The Iverson bracket is a simple, yet general, function that indicates whether or not something is true:

$$[P] = \begin{cases} 1, & P \text{ is true} \\ 0, & \text{otherwise} \end{cases}$$

proposition

IVERSON BRACKET

- Properties of the Iverson bracket:
  - $[P \land Q] = \boxed{[P] \cdot [Q]}$
  - $[\neg P] = \boxed{1 - [P]}$
  - $$\begin{aligned} [P \lor Q] = [\neg(\neg P \land \neg Q)] &= 1 - [\neg P \land \neg Q] \\ &= [\neg P] \cdot [\neg Q] \\ &= (1 - [P])(1 - [Q]) \\ &= 1 + [P] \cdot [Q] - [P] - [Q] \\ &= \boxed{1 + [P \land Q] - [P] - [Q]} \end{aligned}$$

IVERSON BRACKET

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$$[P] = \begin{cases} 1, & P \text{ is true} \\ 0, & \text{otherwise} \end{cases}$$

proposition

too generalized
to use effectively

KRONECKER DELTA

- The Kronecker Delta function can be defined in terms of the Iverson bracket:

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & \text{otherwise} \end{cases} = \boxed{[i = j]}$$

KRONECKER DELTA

- The Kronecker Delta function can be defined in terms of the Iverson bracket:

$$\boxed{[P]} = \begin{cases} 1, & \boxed{P \text{ is true}} \\ 0, & \text{otherwise} \end{cases}$$

KRONECKER DELTA

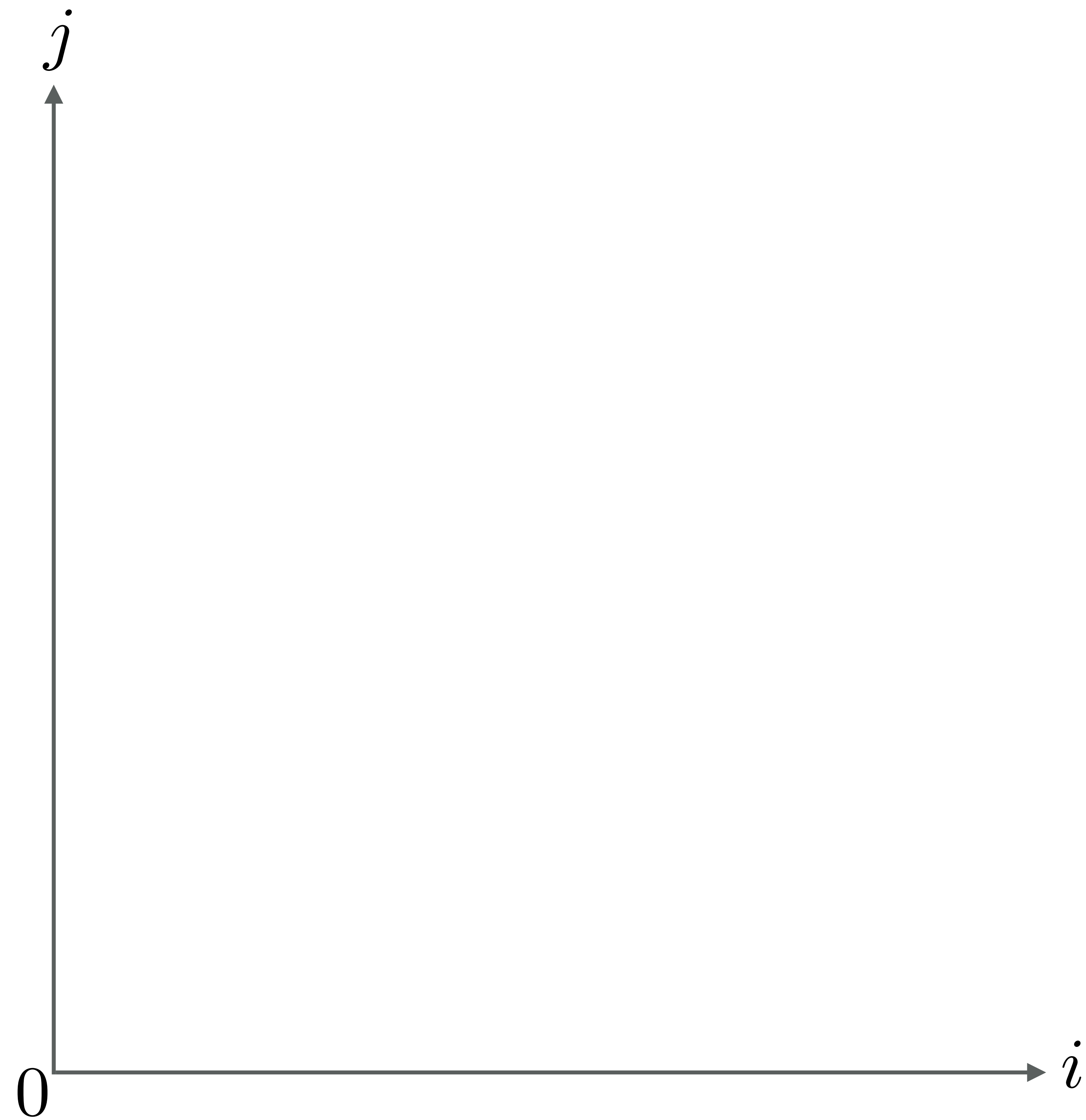
- The Kronecker Delta is **extremely** useful in mathematics and physics
    - Let us first analyze its sensitivity
    - …and then understand its importance

KRONECKER DELTA

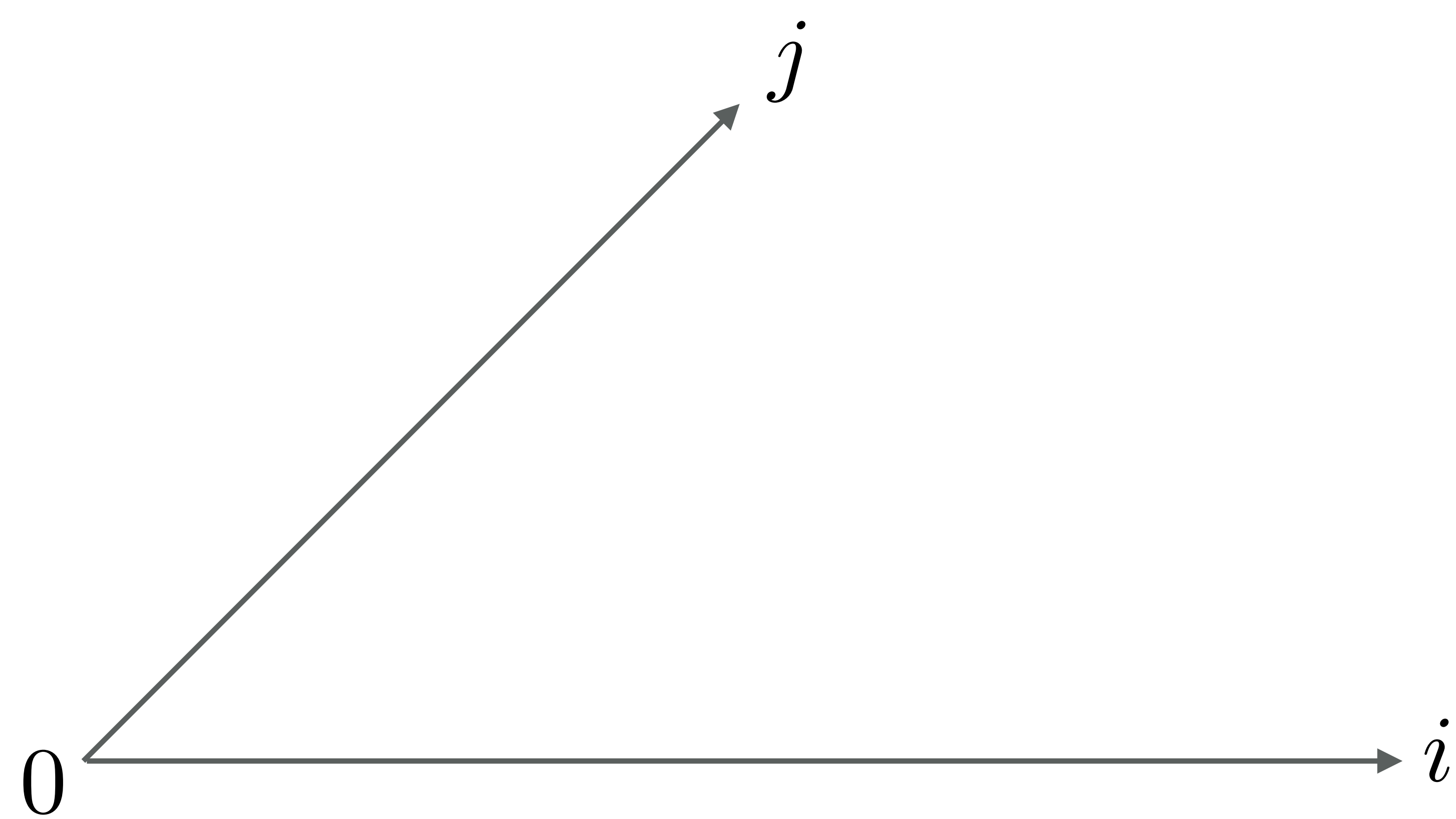
- The Kronecker Delta is **extremely** useful in mathematics and physics
  - Let us first analyze its sensitivity
  - …and then understand its importance

  - Problem: ***2 inputs*** *to the delta function…*

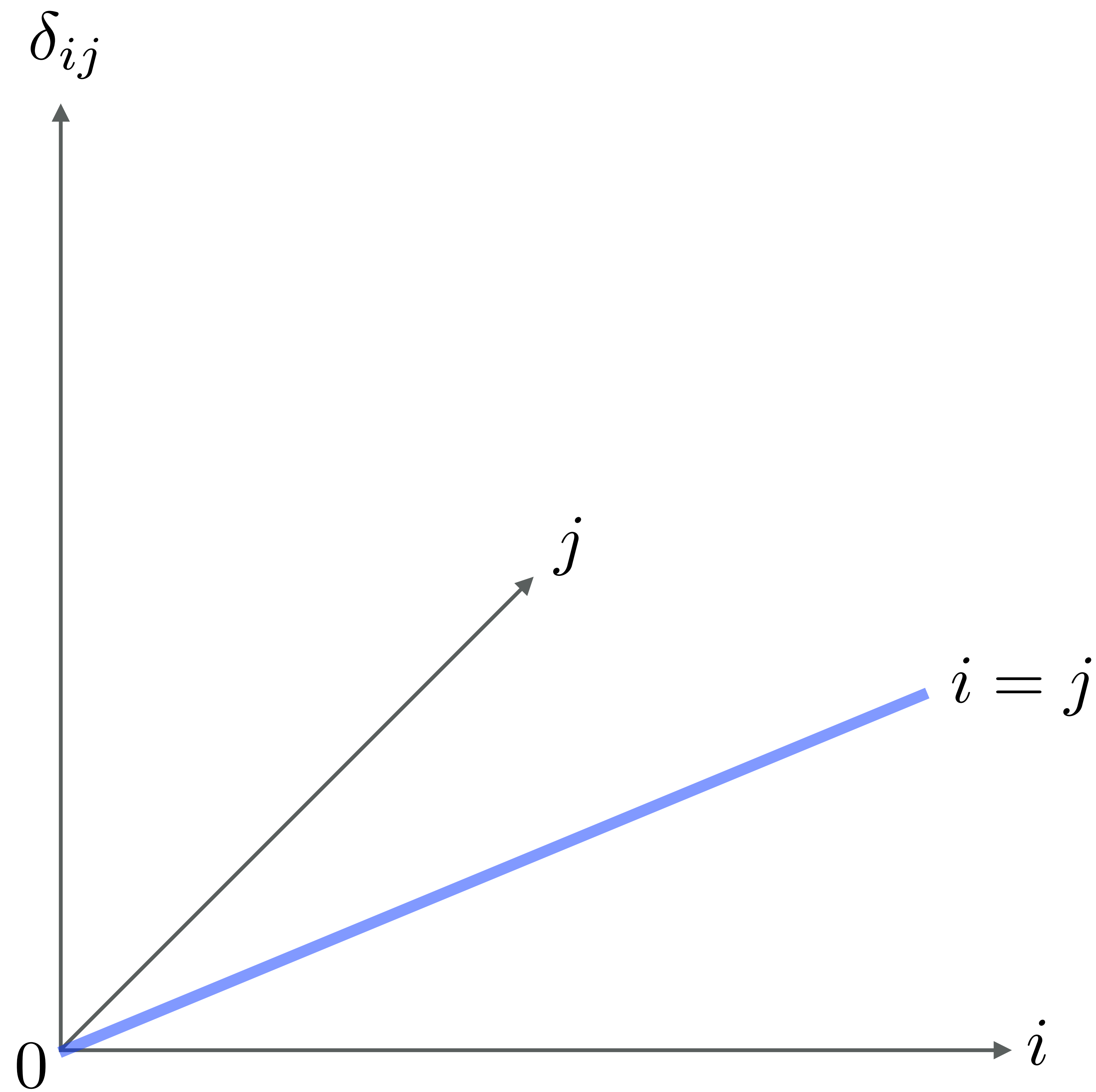
KRONECKER DELTA



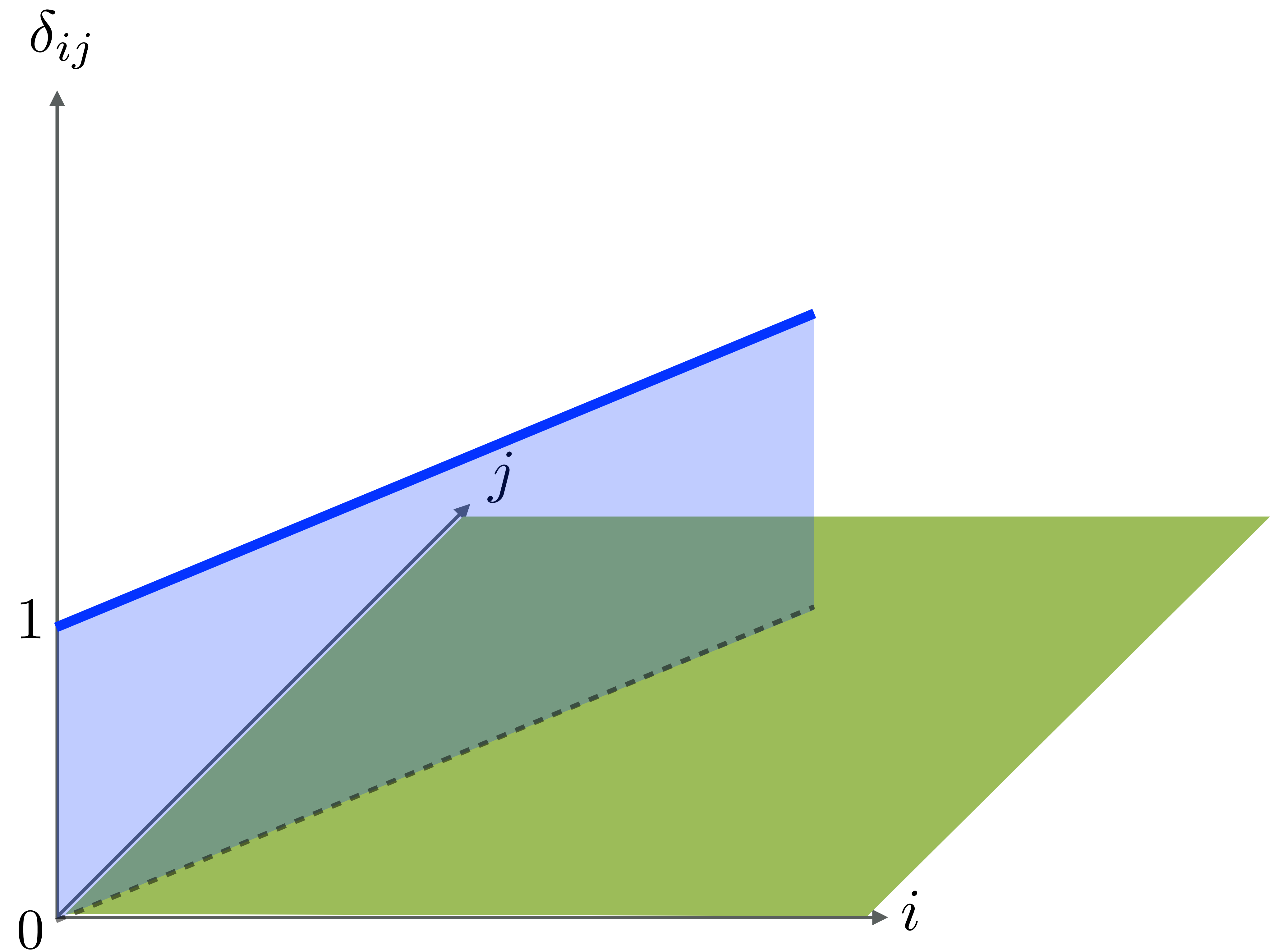
KRONECKER DELTA



KRONECKER DELTA



KRONECKER DELTA



KRONECKER DELTA

- So… is the Kronecker Delta sensitive?
  - Totally depends on what we're talking about!
  - Along the **line**, very insensitive (no changes)
  - On the **boundary** between the line and the surface, very sensitive (rapid change)
  - On the **surface**, very insensitive (no bumps)

- Conclusion: 3D sensitivity analysis is a lot harder!

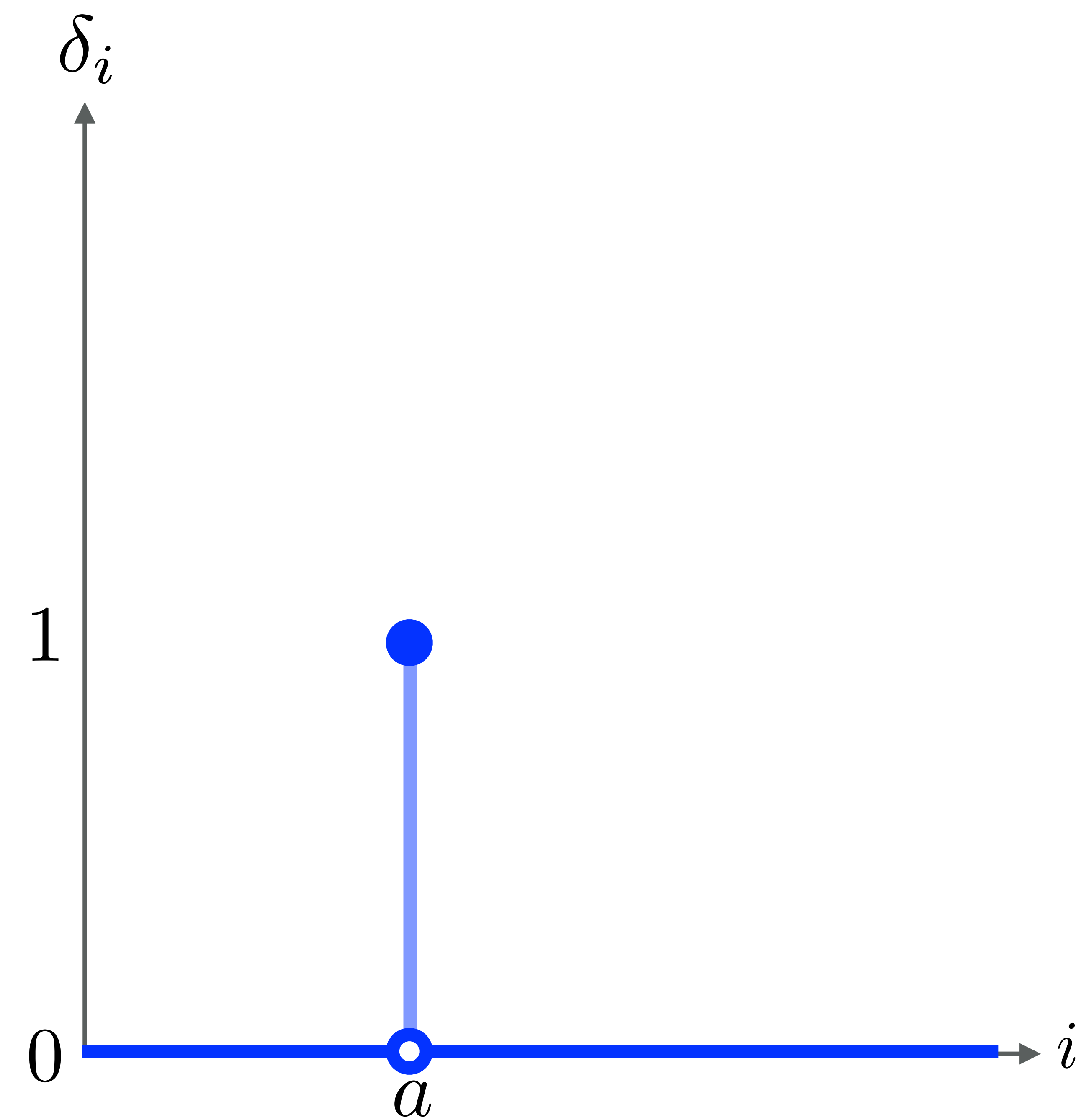
TRACE ANALYSIS

- Idea:
  - Hold one independent variable constant, vary the other independent variable
  - Repeat analysis for other independent variable
  - Superimpose the two, assuming a linear system

- These are the trace functions of the Kronecker Delta!

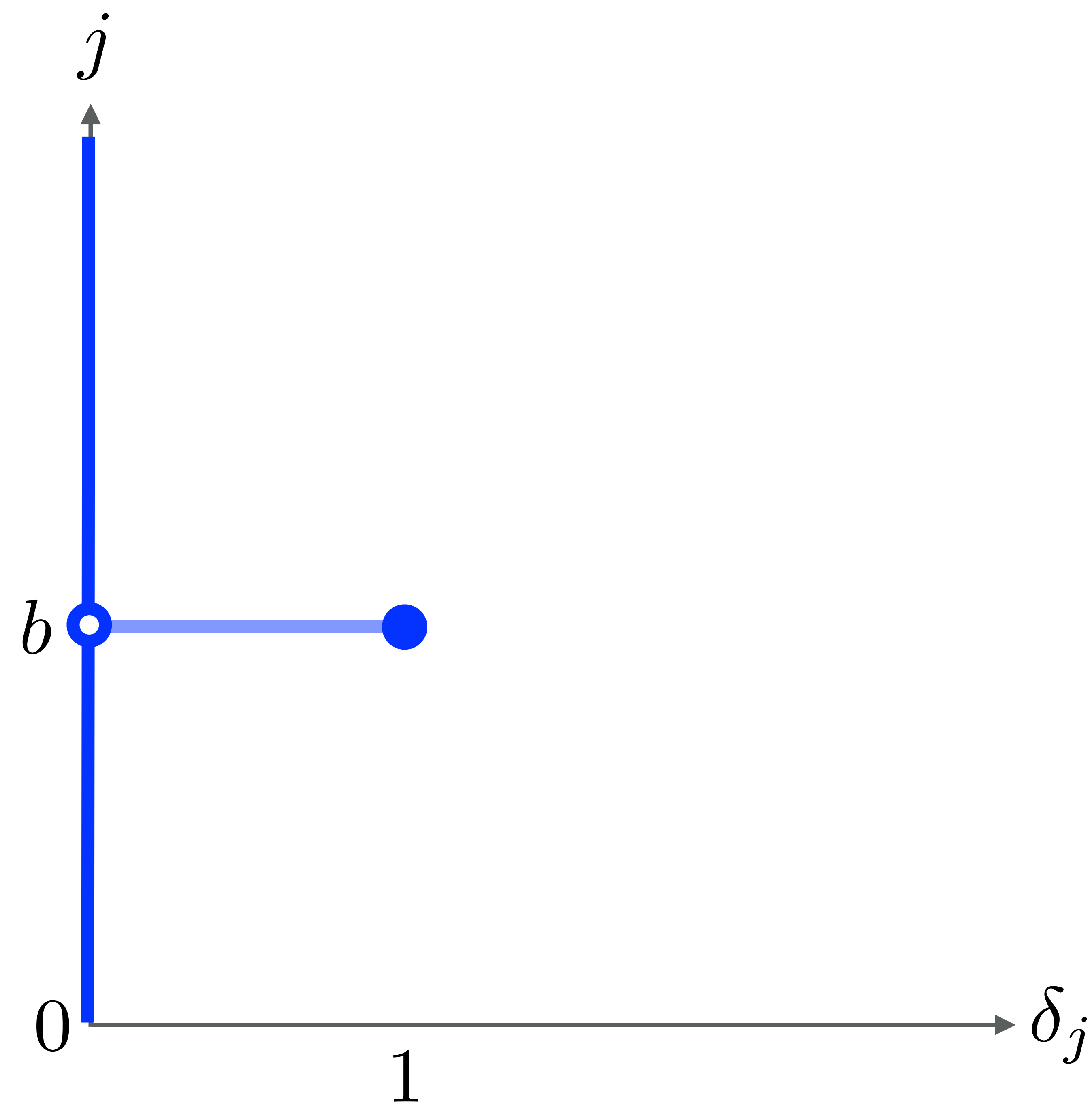
TRACE ANALYSIS

- Let's hold j constant at some value a:



TRACE ANALYSIS

- Let's hold i constant at some value b:



TRACE ANALYSIS

- Is δ_i sensitive?
  - Only at the boundary between the line and the point where $i = a$
- Is δ_j sensitive?
  - Only at the boundary between the line and the point where $j = b$
- Is δ_{ij} sensitive?
  - The two definitions above will only collide if $a = b$, i.e. if and only if $i = j$
  - So it is sensitive between the line $i = j$ and the plane

APPLICATIONS

- So why even use this Kronecker Delta?
  - Good application: matrices/vector notation
  - Simplified analysis
  - Generalized techniques

MATRICES

- The identity matrix in n dimensions:

$$I_{n \times n} = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}$$

- We can use the matrix index convention:
  - Implied matrices when using indices
  - Requires us to understand the context

MATRICES

- The identity matrix in n dimensions:

$$I_{n \times n} = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}$$

- We can use the matrix index convention:
    - I_{ij} is the entry in row i and column j of matrix $I_{n \times n}$

$$I_{ij} = \delta_{ij}$$

EINSTEIN CONVENTION

- The Einstein convention when doing sums:
  - Consider some bound expression within a sum under the following circumstances:
    - The bound expression uses the summation index (indices in general)
    - Some summation index is repeated at least twice in the bound expression
    - That summation index is not otherwise bound outside of the sum (that would be ambiguous)

EINSTEIN CONVENTION

- The Einstein convention when doing sums:
  - If all of those conditions hold
  - Can remove the sum and just keep the index
  - i.e. repeated index implies the sum over it!

$$\boxed{\sum_{k=1}^{n} c_k x_k \xrightarrow{\text{Ein.}} c_k x_k}$$

MATRICES

- Consider the following two vectors:

$$a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \qquad b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

- Their inner product is defined as:

$$\langle a, b \rangle = a_1 b_1 + \cdots + a_n b_n = \sum_{i=1}^{n} a_i b_i = \boxed{\sum_{i=1}^{n} \sum_{j=1}^{n} \delta_{ij} a_i b_j}$$

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MATRICES

- Consider the following two vectors:

$$a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \qquad b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

- The norm is defined as:

$$||a||^2 = \langle a, a \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} \delta_{ij} a_i a_j \boxed{= \sum_{i=1}^{n} a_i^2}$$

PROPERTIES

- Properties of the Kronecker Delta:
  - $\delta_{ij} = \delta_{ji}$
  - $$\sum_{k=1}^{n} \delta_{ik}\delta_{kj} = \delta_{ij}$$
  - $$\sum_{i=1}^{n} \delta_{ii} = n$$

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \delta_{ij} = \min\{m, n\}$$

$$\sum_{j=1}^{n} \delta_{ij} a_j = a_i$$

$$\sum_{i=1}^{n} \delta_{ij} a_i = a_j$$

PROPERTIES

- Properties of the Kronecker Delta (Ein. convention):
  - $\delta_{ij} = \delta_{ji}$
  - $\delta_{ik}\delta_{kj} = \delta_{ij}$ $\qquad$ $\delta_{ij}a_j = a_i$
  - $\delta_{ii} = n$ $\qquad$ $\delta_{ij}a_i = a_j$

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\delta_{ij} = \min\{m, n\}$$

THIRD-ORDER SYSTEMS

- The Kronecker Delta is an example of a second-order system in terms of subscript notation
- Can we have an equivalent version that works as a third-order system?
  - Need some measure of tri-directional equality
  - Levi-Civita symbol!

LEVI-CIVITA SYMBOL

- Let's start with another second-order symbolic system:
  - Now indices can only be "true" or "false"
  - Binary system of classification
  - 2D Levi-Civita symbol is as follows:

$$\varepsilon_{ij} = \begin{cases} 0, & i = j \\ 1, & (i, j) = (\vdash_1, \vdash_2) \\ -1, & (i, j) = (\vdash_2, \vdash_1) \end{cases}$$

  - Seems to be an extension/partition of the Kronecker Delta, albeit with some extra classification going on

SOME PROPERTIES

- The generalized product: $\varepsilon_{ij}\varepsilon_{mn}$
- Cases to consider:

(i,j,m,n)	$\varepsilon_{ij}\varepsilon_{mn}$	(i,j,m,n)	$\varepsilon_{ij}\varepsilon_{mn}$
$(\vdash_1,\vdash_1,\vdash_1,\vdash_1)$	0	$(\vdash_2,\vdash_1,\vdash_1,\vdash_1)$	0
$(\vdash_1,\vdash_1,\vdash_1,\vdash_2)$	0	$(\vdash_2,\vdash_1,\vdash_1,\vdash_2)$	-1
$(\vdash_1,\vdash_1,\vdash_2,\vdash_1)$	0	$(\vdash_2,\vdash_1,\vdash_2,\vdash_1)$	1
$(\vdash_1,\vdash_1,\vdash_2,\vdash_2)$	0	$(\vdash_2,\vdash_1,\vdash_2,\vdash_2)$	0
$(\vdash_1,\vdash_2,\vdash_1,\vdash_1)$	0	$(\vdash_2,\vdash_2,\vdash_1,\vdash_1)$	0
$(\vdash_1,\vdash_2,\vdash_1,\vdash_2)$	1	$(\vdash_2,\vdash_2,\vdash_1,\vdash_2)$	0
$(\vdash_1,\vdash_2,\vdash_2,\vdash_1)$	-1	$(\vdash_2,\vdash_2,\vdash_2,\vdash_1)$	0
$(\vdash_1,\vdash_2,\vdash_2,\vdash_2)$	0	$(\vdash_2,\vdash_2,\vdash_2,\vdash_2)$	0

- So the product is "1" if $i = m$ **and** $j = n$: $1 \cdot \delta_{im}\delta_{jn}$

SOME PROPERTIES

- The generalized product: $\varepsilon_{ij}\varepsilon_{mn}$
- Cases to consider:

(i,j,m,n)	$\varepsilon_{ij}\varepsilon_{mn}$	(i,j,m,n)	$\varepsilon_{ij}\varepsilon_{mn}$
$(\vdash_1,\vdash_1,\vdash_1,\vdash_1)$	0	$(\vdash_2,\vdash_1,\vdash_1,\vdash_1)$	0
$(\vdash_1,\vdash_1,\vdash_1,\vdash_2)$	0	$(\vdash_2,\vdash_1,\vdash_1,\vdash_2)$	-1
$(\vdash_1,\vdash_1,\vdash_2,\vdash_1)$	0	$(\vdash_2,\vdash_1,\vdash_2,\vdash_1)$	1
$(\vdash_1,\vdash_1,\vdash_2,\vdash_2)$	0	$(\vdash_2,\vdash_1,\vdash_2,\vdash_2)$	0
$(\vdash_1,\vdash_2,\vdash_1,\vdash_1)$	0	$(\vdash_2,\vdash_2,\vdash_1,\vdash_1)$	0
$(\vdash_1,\vdash_2,\vdash_1,\vdash_2)$	1	$(\vdash_2,\vdash_2,\vdash_1,\vdash_2)$	0
$(\vdash_1,\vdash_2,\vdash_2,\vdash_1)$	-1	$(\vdash_2,\vdash_2,\vdash_2,\vdash_1)$	0
$(\vdash_1,\vdash_2,\vdash_2,\vdash_2)$	0	$(\vdash_2,\vdash_2,\vdash_2,\vdash_2)$	0

- So the product is "–1" if $i = n$ **and** $j = m$: $-1 \cdot \delta_{in}\delta_{jm}$

SOME PROPERTIES

- The generalized product: $\varepsilon_{ij}\varepsilon_{mn}$
- Cases to consider:

(i,j,m,n)	$\varepsilon_{ij}\varepsilon_{mn}$	(i,j,m,n)	$\varepsilon_{ij}\varepsilon_{mn}$
$(\vdash_1,\vdash_1,\vdash_1,\vdash_1)$	0	$(\vdash_2,\vdash_1,\vdash_1,\vdash_1)$	0
$(\vdash_1,\vdash_1,\vdash_1,\vdash_2)$	0	$(\vdash_2,\vdash_1,\vdash_1,\vdash_2)$	-1
$(\vdash_1,\vdash_1,\vdash_2,\vdash_1)$	0	$(\vdash_2,\vdash_1,\vdash_2,\vdash_1)$	1
$(\vdash_1,\vdash_1,\vdash_2,\vdash_2)$	0	$(\vdash_2,\vdash_1,\vdash_2,\vdash_2)$	0
$(\vdash_1,\vdash_2,\vdash_1,\vdash_1)$	0	$(\vdash_2,\vdash_2,\vdash_1,\vdash_1)$	0
$(\vdash_1,\vdash_2,\vdash_1,\vdash_2)$	1	$(\vdash_2,\vdash_2,\vdash_1,\vdash_2)$	0
$(\vdash_1,\vdash_2,\vdash_2,\vdash_1)$	-1	$(\vdash_2,\vdash_2,\vdash_2,\vdash_1)$	0
$(\vdash_1,\vdash_2,\vdash_2,\vdash_2)$	0	$(\vdash_2,\vdash_2,\vdash_2,\vdash_2)$	0

- So the product is either –1 **or** 1: $(1 \cdot \delta_{im}\delta_{jn}) + (-1 \cdot \delta_{in}\delta_{jm})$

SOME PROPERTIES

- The generalized product: $\varepsilon_{ij}\varepsilon_{mn}$
- Cases to consider:

(i,j,m,n)	$\varepsilon_{ij}\varepsilon_{mn}$	(i,j,m,n)	$\varepsilon_{ij}\varepsilon_{mn}$
$(\vdash_1,\vdash_1,\vdash_1,\vdash_1)$	0	$(\vdash_2,\vdash_1,\vdash_1,\vdash_1)$	0
$(\vdash_1,\vdash_1,\vdash_1,\vdash_2)$	0	$(\vdash_2,\vdash_1,\vdash_1,\vdash_2)$	-1
$(\vdash_1,\vdash_1,\vdash_2,\vdash_1)$	0	$(\vdash_2,\vdash_1,\vdash_2,\vdash_1)$	1
$(\vdash_1,\vdash_1,\vdash_2,\vdash_2)$	0	$(\vdash_2,\vdash_1,\vdash_2,\vdash_2)$	0
$(\vdash_1,\vdash_2,\vdash_1,\vdash_1)$	0	$(\vdash_2,\vdash_2,\vdash_1,\vdash_1)$	0
$(\vdash_1,\vdash_2,\vdash_1,\vdash_2)$	1	$(\vdash_2,\vdash_2,\vdash_1,\vdash_2)$	0
$(\vdash_1,\vdash_2,\vdash_2,\vdash_1)$	-1	$(\vdash_2,\vdash_2,\vdash_2,\vdash_1)$	0
$(\vdash_1,\vdash_2,\vdash_2,\vdash_2)$	0	$(\vdash_2,\vdash_2,\vdash_2,\vdash_2)$	0

- So the product is either –1 **or** 1: $\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$

SOME PROPERTIES

$$\varepsilon_{ij}\varepsilon_{mn} = \begin{vmatrix} \delta_{im} & \delta_{in} \\ \delta_{jm} & \delta_{jn} \end{vmatrix} = \boxed{\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}}$$

$$\begin{aligned} \varepsilon_{ij}\varepsilon_{in} &= \delta_{ii}\delta_{jn} - \delta_{in}\delta_{ji} \\ &= 2\delta_{jn} - \delta_{ji}\delta_{in} \\ &= 2\delta_{jn} - \delta_{jn} \\ &= \boxed{\delta_{jn}} \end{aligned}$$

$$\begin{aligned} \varepsilon_{ij}\varepsilon_{ij} &= \delta_{ii}\delta_{jj} - \delta_{ij}\delta_{ji} \\ &= \delta_{ii}\delta_{jj} - \delta_{ii} \\ &= \delta_{ii}(\delta_{jj} - 1) \\ &= 2(2 - 1) \\ &= \boxed{2} \end{aligned}$$

SOME PROPERTIES

$$\varepsilon_{ij}\varepsilon_{mn} = \begin{vmatrix} \delta_{im} & \delta_{in} \\ \delta_{jm} & \delta_{jn} \end{vmatrix} = \boxed{\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}}$$

$$\begin{aligned} \varepsilon_{ij}\varepsilon_{in} &= \delta_{ii}\delta_{jn} - \delta_{in}\delta_{ji} \\ &= 2\delta_{jn} - \delta_{ji}\delta_{in} \\ &= 2\delta_{jn} - \delta_{jn} \\ &= \boxed{\delta_{jn}} \end{aligned}$$

$$\begin{aligned} \varepsilon_{ij}\varepsilon_{ij} &= \delta_{jj} \\ &= \boxed{2} \end{aligned}$$

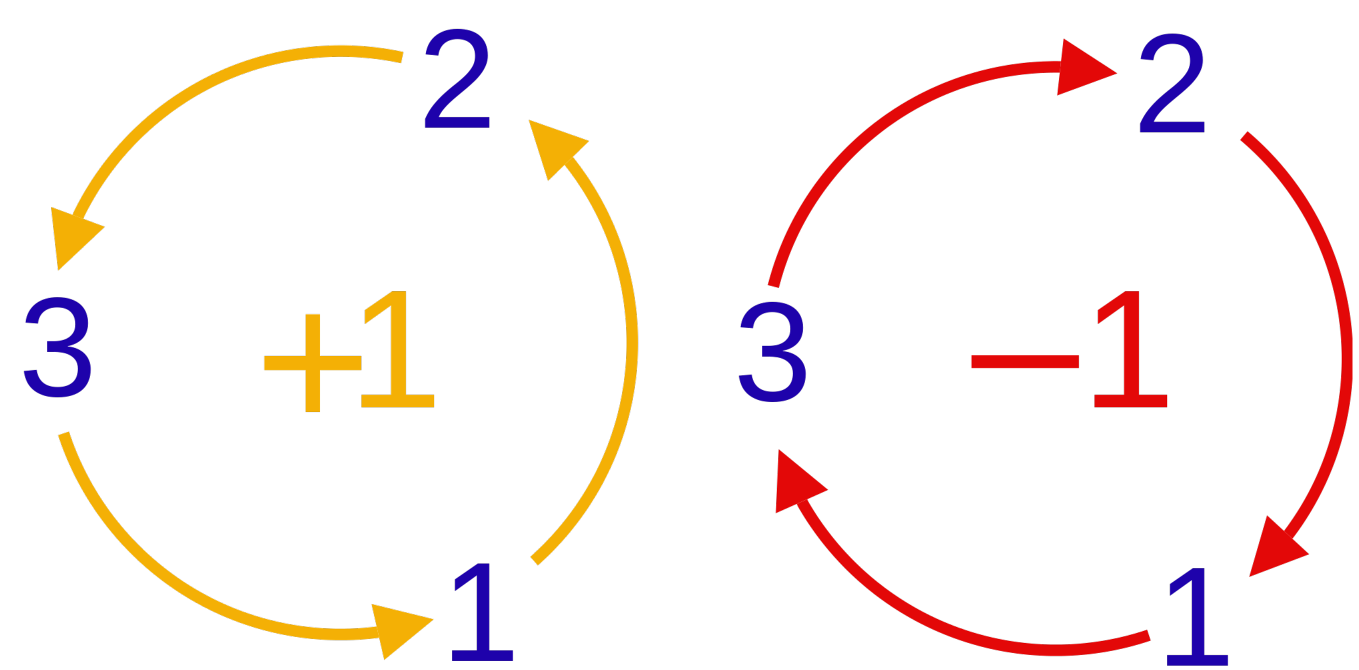
LEVI-CIVITA SYMBOL

- Now, let's consider the third-order symbolic system:
  - Indices can be "true" or "false" or "indeterminate"
  - Ternary system of classification
  - 3D Levi-Civita symbol is as follows:

$$\varepsilon_{ijk} = \begin{cases} 0, & i = j, j = k, \text{or } k = i \\ 1, & (i, j, k) = (\vdash_1, \vdash_2, \vdash_3), (\vdash_2, \vdash_3, \vdash_1), \text{or } (\vdash_3, \vdash_1, \vdash_2) \\ -1, & (i, j, k) = (\vdash_3, \vdash_2, \vdash_1), (\vdash_1, \vdash_3, \vdash_2), \text{or } (\vdash_2, \vdash_1, \vdash_3) \end{cases}$$

LEVI-CIVITA SYMBOL

- Now, let's consider the third-order symbolic system:
  - Indices can be "true" or "false" or "indeterminate"
  - Ternary system of classification
  - 3D Levi-Civita symbol mnemonic:



SOME PROPERTIES

$$\begin{aligned}
\varepsilon_{ijk}\varepsilon_{lmn} &= \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix} \\
&= \delta_{il} \begin{vmatrix} \delta_{jm} & \delta_{jn} \\ \delta_{km} & \delta_{kn} \end{vmatrix} - \delta_{im} \begin{vmatrix} \delta_{jl} & \delta_{jn} \\ \delta_{kl} & \delta_{kn} \end{vmatrix} + \delta_{in} \begin{vmatrix} \delta_{jl} & \delta_{jm} \\ \delta_{kl} & \delta_{km} \end{vmatrix} \\
&= \boxed{\delta_{il}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) - \delta_{im}(\delta_{jl}\delta_{kn} - \delta_{jn}\delta_{kl}) + \delta_{in}(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})}
\end{aligned}$$

SOME PROPERTIES

$$\begin{aligned}
\varepsilon_{ijk}\varepsilon_{lmn} &= \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix} \\
&= \delta_{il} \begin{vmatrix} \delta_{jm} & \delta_{jn} \\ \delta_{km} & \delta_{kn} \end{vmatrix} - \delta_{im} \begin{vmatrix} \delta_{jl} & \delta_{jn} \\ \delta_{kl} & \delta_{kn} \end{vmatrix} + \delta_{in} \begin{vmatrix} \delta_{jl} & \delta_{jm} \\ \delta_{kl} & \delta_{km} \end{vmatrix} \\
&= \delta_{il}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) - \delta_{im}(\delta_{jl}\delta_{kn} - \delta_{jn}\delta_{kl}) + \delta_{in}(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})
\end{aligned}$$

$$\begin{aligned}
\varepsilon_{ijk}\varepsilon_{imn} &= \delta_{ii}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) - \delta_{im}(\delta_{ji}\delta_{kn} - \delta_{jn}\delta_{ki}) + \delta_{in}(\delta_{ji}\delta_{km} - \delta_{jm}\delta_{ki}) \\
&= 3\delta_{jm}\delta_{kn} - 3\delta_{jn}\delta_{km} - \delta_{ji}\delta_{im}\delta_{kn} + \delta_{jn}\delta_{ki}\delta_{im} + \delta_{ji}\delta_{in}\delta_{km} - \delta_{jm}\delta_{ki}\delta_{in} \\
&= 3\delta_{jm}\delta_{kn} - 3\delta_{jn}\delta_{km} - \delta_{jm}\delta_{kn} + \delta_{jn}\delta_{km} + \delta_{jn}\delta_{km} - \delta_{jm}\delta_{kn} \\
&= \boxed{\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}}
\end{aligned}$$

SOME PROPERTIES

$$
\begin{aligned}
\varepsilon_{ijk}\varepsilon_{lmn} &= \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix} \\
&= \delta_{il} \begin{vmatrix} \delta_{jm} & \delta_{jn} \\ \delta_{km} & \delta_{kn} \end{vmatrix} - \delta_{im} \begin{vmatrix} \delta_{jl} & \delta_{jn} \\ \delta_{kl} & \delta_{kn} \end{vmatrix} + \delta_{in} \begin{vmatrix} \delta_{jl} & \delta_{jm} \\ \delta_{kl} & \delta_{km} \end{vmatrix} \\
&= \delta_{il}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) - \delta_{im}(\delta_{jl}\delta_{kn} - \delta_{jn}\delta_{kl}) + \delta_{in}(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon_{ijk}\varepsilon_{ijn} &= \delta_{ii}(\delta_{jj}\delta_{kn} - \delta_{jn}\delta_{kj}) - \delta_{ij}(\delta_{ji}\delta_{kn} - \delta_{jn}\delta_{ki}) + \delta_{in}(\delta_{ji}\delta_{kj} - \delta_{jj}\delta_{ki}) \\
&= 9\delta_{kn} - 3\delta_{kj}\delta_{jn} - \delta_{ij}\delta_{ji}\delta_{kn} + \delta_{ij}\delta_{jn}\delta_{ki} + \delta_{ji}\delta_{in}\delta_{kj} - 3\delta_{ki}\delta_{in} \\
&= 9\delta_{kn} - 3\delta_{kn} - \delta_{ii}\delta_{kn} + \delta_{in}\delta_{ki} + \delta_{jn}\delta_{kj} - 3\delta_{kn} \\
&= 9\delta_{kn} - 3\delta_{kn} - 3\delta_{kn} + \delta_{kn} + \delta_{kn} - 3\delta_{kn} \\
&= \boxed{2\delta_{kn}}
\end{aligned}
$$

SOME PROPERTIES

$$
\begin{aligned}
\varepsilon_{ijk}\varepsilon_{lmn} &= \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix} \\
&= \delta_{il} \begin{vmatrix} \delta_{jm} & \delta_{jn} \\ \delta_{km} & \delta_{kn} \end{vmatrix} - \delta_{im} \begin{vmatrix} \delta_{jl} & \delta_{jn} \\ \delta_{kl} & \delta_{kn} \end{vmatrix} + \delta_{in} \begin{vmatrix} \delta_{jl} & \delta_{jm} \\ \delta_{kl} & \delta_{km} \end{vmatrix} \\
&= \delta_{il}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) - \delta_{im}(\delta_{jl}\delta_{kn} - \delta_{jn}\delta_{kl}) + \delta_{in}(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon_{ijk}\varepsilon_{ijn} &= \delta_{jj}\delta_{kn} - \delta_{jn}\delta_{kj} \\
&= 3\delta_{kn} - \delta_{kj}\delta_{jn} \\
&= 3\delta_{kn} - \delta_{kn} \\
&= \boxed{2\delta_{kn}}
\end{aligned}
$$

SOME PROPERTIES

$$\begin{aligned}
\varepsilon_{ijk}\varepsilon_{lmn} &= \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix} \\
&= \delta_{il} \begin{vmatrix} \delta_{jm} & \delta_{jn} \\ \delta_{km} & \delta_{kn} \end{vmatrix} - \delta_{im} \begin{vmatrix} \delta_{jl} & \delta_{jn} \\ \delta_{kl} & \delta_{kn} \end{vmatrix} + \delta_{in} \begin{vmatrix} \delta_{jl} & \delta_{jm} \\ \delta_{kl} & \delta_{km} \end{vmatrix} \\
&= \delta_{il}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) - \delta_{im}(\delta_{jl}\delta_{kn} - \delta_{jn}\delta_{kl}) + \delta_{in}(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})
\end{aligned}$$

$$\begin{aligned}
\varepsilon_{ijk}\varepsilon_{ijk} &= \delta_{ii}(\delta_{jj}\delta_{kk} - \delta_{jk}\delta_{kj}) - \delta_{ij}(\delta_{ji}\delta_{kk} - \delta_{jk}\delta_{ki}) + \delta_{ik}(\delta_{ji}\delta_{kj} - \delta_{jj}\delta_{ki}) \\
&= 27 - 3\delta_{jj} - 3\delta_{ii} + \delta_{ij}\delta_{ji} + \delta_{ji}\delta_{ik}\delta_{kj} - 3\delta_{ii} \\
&= 27 - 9 - 9 + \delta_{ii} + \delta_{jk}\delta_{kj} - 9 \\
&= 27 - 9 - 9 + 3 + \delta_{jj} - 9 \\
&= 27 - 9 - 9 + 3 + 3 - 9 \\
&= \boxed{6}
\end{aligned}$$

SOME PROPERTIES

$$\begin{aligned}
\varepsilon_{ijk}\varepsilon_{lmn} &= \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix} \\
&= \delta_{il} \begin{vmatrix} \delta_{jm} & \delta_{jn} \\ \delta_{km} & \delta_{kn} \end{vmatrix} - \delta_{im} \begin{vmatrix} \delta_{jl} & \delta_{jn} \\ \delta_{kl} & \delta_{kn} \end{vmatrix} + \delta_{in} \begin{vmatrix} \delta_{jl} & \delta_{jm} \\ \delta_{kl} & \delta_{km} \end{vmatrix} \\
&= \delta_{il}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) - \delta_{im}(\delta_{jl}\delta_{kn} - \delta_{jn}\delta_{kl}) + \delta_{in}(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})
\end{aligned}$$

$$\begin{aligned}
\varepsilon_{ijk}\varepsilon_{ijk} &= 2\delta_{kk} \\
&= \boxed{6}
\end{aligned}$$

APPLICATIONS OF INDICATOR FUNCTIONS

FUNCTIONS OF SEVERAL VARIABLES

- But how do functions of several variables even *work*?
  - Kronecker Delta is an example!
  - So is the 2D Levi-Civita symbol

FUNCTIONS OF SEVERAL VARIABLES

- But how do functions of several variables even *work*?
  - **Just like** single-variable functions:

$$f(x, y) = x^2 + y^2$$

$$f(3, 4) = 3^2 + 4^2 = 9 + 16 = 25$$

    - Total application, sometimes called *application*
  - **Not like** single-variable functions:

$$f(x, 0) = x^2 + 0^2 = x^2 \qquad f(1, y) = 1^2 + y^2 = 1 + y^2$$

    - Partial application

FUNCTIONS OF SEVERAL VARIABLES

- Concept naturally extends to three-dimensions:

$$u(x,y,z) = \sin x + 3\cos(y^2 - z)$$

- What are the uses for this?
  - Need some new techniques
  - We will explore them as we proceed
  - Keep an open mind, and stay critical!
- Traces:

$$f(x) = u(x,0,0) = \sin x \qquad g(y) = u(0,y,0) = 3\cos(y^2) \qquad h(z) = u(0,0,z) = 3\cos z$$

PARTIAL DERIVATIVES

- Derivative of a function with respect to a variable
  - While holding the OTHER variables constant

$$u(x, y, z) = \sin x + 3\cos(y^2 - z)$$

$$\frac{\partial u}{\partial x} = \cos x$$

$$\frac{\partial u}{\partial y} = -3\sin(y^2 - z) \cdot 2y = -6y\sin(y^2 - z)$$

$$\frac{\partial u}{\partial z} = -3\sin(y^2 - z) \cdot (-1) = 3\sin(y^2 - z)$$

VECTORS

- Now let us consider 3D-vectors:

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1 \vec{e_1} + x_2 \vec{e_2} + x_3 \vec{e_3}$$

$$= \sum_{n=1}^{3} x_n \vec{e_n}$$

$$= x_n \vec{e_n}$$

GRADIENTS

- The gradient of a function is a measure of its sensitivity with respect to all parameters at once:

$$\nabla f(x, y, z) = \frac{\partial f}{\partial x}\vec{e_1} + \frac{\partial f}{\partial y}\vec{e_2} + \frac{\partial f}{\partial z}\vec{e_3}$$

"grad f"

- More generally:

$$\begin{aligned} \nabla f(x_1, \dots, x_n) &= \frac{\partial f}{\partial x_1}\vec{e_1} + \cdots + \frac{\partial f}{\partial x_n}\vec{e_n} \\ &= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\vec{e_i} \\ &= \frac{\partial f}{\partial x_i}\vec{e_i} \end{aligned}$$

$= \partial_i f$

GRADIENTS

- The gradient of a function is a measure of its sensitivity with respect to all parameters at once:

$$u(x, y, z) = \sin x + 3\cos(y^2 - z)$$

$$\nabla u = \cos(x)\vec{e_1} - 6y\sin(y^2 - z)\vec{e_2} + 3\sin(y^2 - z)\vec{e_3}$$
$$= \begin{pmatrix} \cos x \\ -6y\sin(y^2 - z) \\ 3\sin(y^2 - z) \end{pmatrix}$$

DIVERGENCE

- The divergence of a vector is a measure of the rate at which the "flow" of the vector is similar to a "source"
  - Sources generate values outward (as opposed to sinks, which pull them inward)
  - Keep in mind that this is multivariable stuff going on here, so things may not be functions in the sense of a 2D plane (i.e. vertical line test)

DIVERGENCE

- The divergence of a vector is a measure of the rate at which the "flow" of the vector is similar to a "source"

$$\vec{x} = U(x_1, x_2, x_3)\vec{e_1} + V(x_1, x_2, x_3)\vec{e_2} + W(x_1, x_2, x_3)\vec{e_3}$$

vector **field**, not just a vector

$$\nabla \cdot \vec{x} = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{pmatrix} \cdot \begin{pmatrix} U(x_1, x_2, x_3) \\ V(x_1, x_2, x_3) \\ W(x_1, x_2, x_3) \end{pmatrix}$$

$$= \frac{\partial U}{\partial x_1} + \frac{\partial V}{\partial x_2} + \frac{\partial W}{\partial x_3}$$ " div x "

DIVERGENCE

- The divergence of a **field** is a measure of the rate at which the "flow" of the **field** is similar to a "source"

$$\vec{x} = U(x_1, x_2, x_3)\vec{e_1} + V(x_1, x_2, x_3)\vec{e_2} + W(x_1, x_2, x_3)\vec{e_3}$$

$$\begin{aligned}\nabla \cdot \vec{x} &= \begin{pmatrix} \dfrac{\partial}{\partial x_1} \\ \dfrac{\partial}{\partial x_2} \\ \dfrac{\partial}{\partial x_3} \end{pmatrix} \cdot \begin{pmatrix} U(x_1, x_2, x_3) \\ V(x_1, x_2, x_3) \\ W(x_1, x_2, x_3) \end{pmatrix} \\ &= \frac{\partial U}{\partial x_1} + \frac{\partial V}{\partial x_2} + \frac{\partial W}{\partial x_3} \quad \text{" div } x \text{ "}\end{aligned}$$

CURL

- The curl of a field is a measure of the rate of rotation of the field with respect to each parameter

$$\vec{x} = U(x_1, x_2, x_3)\vec{e_1} + V(x_1, x_2, x_3)\vec{e_2} + W(x_1, x_2, x_3)\vec{e_3}$$

$$\nabla \times \vec{x} = \begin{vmatrix} \vec{e_1} & \vec{e_2} & \vec{e_3} \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ U & V & W \end{vmatrix}$$

" curl *x*"

$$= \left(\frac{\partial W}{\partial x_2} - \frac{\partial V}{\partial x_3}\right)\vec{e_1} - \left(\frac{\partial W}{\partial x_1} - \frac{\partial U}{\partial x_3}\right)\vec{e_2} + \left(\frac{\partial V}{\partial x_1} - \frac{\partial U}{\partial x_2}\right)\vec{e_3}$$

CONVENTION

- To discern between concepts we use simultaneously:
  - Functions use lowercase letters without an arrow:

$$f$$

  - Vectors use lowercase letters with an arrow:

$$\vec{x}$$

  - Fields use uppercase letters (without an arrow):

$$F$$

CONVENTION

- To discern between concepts we use simultaneously:
  - Functions use lowercase letters without an arrow:

$$\partial_i f \quad \text{“ grad } F \text{ ”}$$

  - Vectors use lowercase letters with an arrow:

$$\vec{x}$$

  - Fields use uppercase letters (without an arrow):

$$\partial_i F_i \qquad \varepsilon_{ijk}\partial_j F_k$$

“ div F ” “ curl F ”

SOME PROOFS

- Can prove complex/deep results using this "indicial notation" to make our lives easier
  - Extended proofs involve assuming a general form and then proving results by expanding out the definitions
  - "Cute proofs" follow

SOME PROOFS

$$
\begin{aligned}
\nabla \times (\nabla f) &= \varepsilon_{ijk}\partial_j(\nabla f)_k \\
&= \varepsilon_{ijk}\partial_j(\partial_m f)_k \\
&= \varepsilon_{ijk}\partial_j\partial_k f \\
&= \varepsilon_{ijk}\partial_k\partial_j f \\
&= -\varepsilon_{ikj}\partial_k\partial_j f \\
&= -\varepsilon_{ijk}\partial_j\partial_k f \\
&= 0
\end{aligned}
$$

SOME PROOFS

$$
\begin{aligned}
\nabla \times (\nabla f) &= \varepsilon_{ijk}\partial_j(\nabla f)_k && \text{“small model”} \\
&= \varepsilon_{ijk}\partial_j(\partial_m f)_k \\
&= \boxed{\varepsilon_{ijk}\partial_j\partial_k f} \\
&= \varepsilon_{ijk}\partial_k\partial_j f && \text{swap derivatives} \\
&= -\varepsilon_{ikj}\partial_k\partial_j f && \text{permute Levi-Civita} \\
&= \boxed{-\varepsilon_{ijk}\partial_j\partial_k f} && \text{swap } j \text{ and } k \\
&= 0 && \text{expr} = -\text{expr}
\end{aligned}
$$

SOME PROOFS

$$
\begin{aligned}
\nabla \cdot (\nabla \times F) &= \partial_m(\varepsilon_{ijk}\partial_j F_k)_m \\
&= \partial_m \varepsilon_{mjk}\partial_j F_k \\
&= \varepsilon_{mjk}\partial_m \partial_j F_k \\
&= \varepsilon_{mjk}\partial_j \partial_m F_k \\
&= -\varepsilon_{jmk}\partial_j \partial_m F_k \\
&= -\varepsilon_{mjk}\partial_m \partial_j F_k \\
&= 0
\end{aligned}
$$

SOME PROOFS

$$
\begin{aligned}
\nabla \cdot (\nabla \times F) &= \partial_m (\varepsilon_{ijk} \partial_j F_k)_m && \text{“small model”} \\
&= \partial_m \varepsilon_{mjk} \partial_j F_k && \text{collapse indices} \\
&= \boxed{\varepsilon_{mjk} \partial_m \partial_j F_k} && \\
&= \varepsilon_{mjk} \partial_j \partial_m F_k && \text{swap derivatives} \\
&= -\varepsilon_{jmk} \partial_j \partial_m F_k && \text{permute Levi-Civita} \\
&= \boxed{-\varepsilon_{mjk} \partial_m \partial_j F_k} && \text{swap } m \text{ and } j \\
&= 0 && \text{expr = –expr}
\end{aligned}
$$

SOME PROOFS

$$
\begin{aligned}
\nabla \times (\nabla \times F) &= \varepsilon_{ijk}\partial_j(\nabla \times F)_k \\
&= \varepsilon_{ijk}\partial_j(\varepsilon_{lmn}\partial_m F_n)_k \\
&= \varepsilon_{ijk}\partial_j\varepsilon_{kmn}\partial_m F_n \\
&= \varepsilon_{ijk}\varepsilon_{kmn}\partial_j\partial_m F_n \\
&= \varepsilon_{kij}\varepsilon_{kmn}\partial_j\partial_m F_n \\
&= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})\partial_j\partial_m F_n \\
&= \delta_{im}\delta_{jn}\partial_j\partial_m F_n - \delta_{in}\delta_{jm}\partial_j\partial_m F_n \\
&= \partial_n\partial_m F_n - \partial_j\partial_j F_n \\
&= \partial_m\partial_n F_n - \partial_j^2 F_n \\
&= \nabla(\nabla \cdot F) - \nabla^2 F
\end{aligned}
$$

SOME PROOFS

$$
\begin{aligned}
\nabla \times (\nabla \times F) &= \varepsilon_{ijk}\partial_j(\nabla \times F)_k \\
\text{“small model”} \quad &= \varepsilon_{ijk}\partial_j(\varepsilon_{lmn}\partial_m F_n)_k \\
\text{collapse indices} \quad &= \varepsilon_{ijk}\partial_j \varepsilon_{kmn}\partial_m F_n \\
&= \varepsilon_{ijk}\varepsilon_{kmn}\partial_j\partial_m F_n \\
\text{permute Levi-Civita} \quad &= \varepsilon_{kij}\varepsilon_{kmn}\partial_j\partial_m F_n \\
\text{expand Levi-Civita} \quad &= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})\partial_j\partial_m F_n \\
&= \delta_{im}\delta_{jn}\partial_j\partial_m F_n - \delta_{in}\delta_{jm}\partial_j\partial_m F_n \\
\text{collapse Kronecker Delta} \quad &= \partial_n\partial_m F_n - \partial_j\partial_j F_n \\
\text{swap derivatives} \quad &= \partial_m\partial_n F_n - \partial_j^2 F_n \\
\text{return to “big model”} \quad &= \nabla(\nabla \cdot F) - \nabla^2 F
\end{aligned}
$$

CLOSING REMARKS

CLOSING REMARKS

- Hodge-podge of topics
- Various tips/tricks to add to mathematical toolkit
- Material used here is immensely useful in mathematical physics, physical chemistry, etc.
  - More mathematical flavor
  - More in-depth analysis based on simple precepts
  - More derivations and proofs about behaviors
  - More **intuition**!

CLOSING REMARKS

- If you have any further questions, feel free to reach out!
- My email address is `cb625@cornell.edu`
- Starting Fall 2017, reach me at `chiragb@princeton.edu`
  - May take some time to reply—we are all busy students!
  - Check out my website: `chiragbharadwaj.com`
- I wish you the best of luck with future studies!
  - *Did this class interest you?* Consider a math major at university! Talk to people and join a research group.
  - Take a related class in high school through local colleges


Hope you
had fun!
Splash!
Spring 2017