190100042 DS - 203 Assign - 2

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① PDF $f \rightarrow$ $P_X(X=x) = \begin{cases} A_1 e^{-\lambda_1 x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$

$P_Y(Y=y) = \begin{cases} A_2 e^{-\lambda_2 y} & y \geq 0 \\ 0 & \text{otherwise} \end{cases}$

X & Y are independent RV.

To find $\rightarrow$ dist of min(X, Y) & max(X, Y)

CDF for X & Y $\rightarrow$ $CDF_X = \begin{cases} 1 - e^{-\lambda_1 x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$

$CDF_Y = \begin{cases} 1 - e^{-\lambda_2 y} & Y \geq 0 \\ 0 & \text{otherwise} \end{cases}$

$\boxed{\because CDF_X = \int_{-\infty}^{x} f_X(x)\,dx}$

(a) let $Z = \min(X, Y)$ & $z > 0$

$$P(Z > z) = P(X > z \;\&\; Y > z)$$

$$\Rightarrow P(Z > z) = P(X > z) \cdot P(Y > z) \quad \{\because \text{independent}\}$$

$$= [1 - P(X \le z)] \cdot [1 - P(Y \le z)]$$

$$= [1 - (1 - e^{-\lambda_1 z})] \cdot [1 - (1 - e^{-\lambda_2 z})]$$

$$= e^{-(\lambda_1 + \lambda_2) z}$$

$\Rightarrow$ CDF for $Z \rightarrow F_Z(z) = P(Z \le z) = 1 - P(Z > z)$

$$\Rightarrow F_Z(z) = 1 - e^{-(\lambda_1 + \lambda_2) z}$$

PDF for Z $\quad P_Z(z) = \frac{d}{dz} F_Z(z) = (\lambda_1 + \lambda_2) e^{-(\lambda_1 + \lambda_2) z}$ for $\underline{\underline{z > 0}}$

$$\Rightarrow P_Z(z) = \begin{cases} (\lambda_1 + \lambda_2) e^{-(\lambda_1 + \lambda_2) z} & z > 0 \\ 0 & \text{otherwise} \end{cases}$$

(b). let $Z = \max(X, Y)$; $z > 0$

$$P(Z < z) = P(X < z \;\&\; Y < z) = P(X < z) \cdot P(Y < z)$$

($\because$ independent)

$$\Rightarrow P(Z < z) = (1 - e^{-\lambda_1 z}) \cdot (1 - e^{-\lambda_2 z})$$

$$= 1 - e^{-\lambda_1 z} - e^{-\lambda_2 z} + e^{-(\lambda_1 + \lambda_2) z}$$

$\Rightarrow$ CDF of Z, $\quad F_Z(z) = 1 - e^{-\lambda_1 z} - e^{-\lambda_2 z} - e^{-(\lambda_1 + \lambda_2) z}$

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$\Rightarrow$ PDF of Z, $P_Z(Z=z) = \frac{d}{dz} F_Z(z)$

$$\Rightarrow P_Z = \frac{d}{dz}\left(1 - e^{-\lambda_1 z} - e^{-\lambda_2 z} + e^{-(\lambda_1+\lambda_2) z}\right)$$

$$= \lambda_1 e^{-\lambda_1 z} + \lambda_2 e^{-\lambda_2 z} - (\lambda_1+\lambda_2) e^{-(\lambda_1+\lambda_2) z}$$

$$\Rightarrow \text{Dist for } Z = \max(x, y) = \begin{cases} \lambda_1 e^{-\lambda_1 x} + \lambda_2 e^{-\lambda_2 x} - (\lambda_1+\lambda_2) e^{-(\lambda_1+\lambda_2)x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

(2) 3 (W) 6 (R) 5 (B). 6 selections

Total (14)

X: white balls selected Y: blue balls selected.

$E(X|Y=3) \Rightarrow$ Expectation value of X given 3 blue balls selected.

$$E(X) = \sum_{i=1}^{\infty} x_i P_X(x_i)$$

$Y=3 \Rightarrow X=\{0,1,2,3\} \Rightarrow$ No. of red $= \underbrace{\{3,2,1,0\}}_{\text{resp.}}$

$P(\text{W selected}) = \frac{3}{14}$; $P(\text{R selected}) = \frac{6}{14}$; $P(\text{B selected}) = \frac{5}{14}$

$$E(X|Y=3) = \frac{1 \cdot \overbrace{{}^6C_3 \cdot \left(\frac{5}{14}\right)^3 \cdot {}^3C_1 \cdot \left(\frac{3}{14}\right)\left(\frac{6}{14}\right)^2}^{P(W=1,\,B=3,\,R=2)} + 2 \cdot \overbrace{{}^6C_3\left(\frac{5}{14}\right)^3 \cdot {}^3C_2\left(\frac{3}{14}\right)^2 \cdot \left(\frac{6}{14}\right)}^{P(W=2,\,B=3,\,R=1)} + 3 \cdot \underbrace{{}^6C_3\left(\frac{5}{14}\right)^3 \cdot \left(\frac{3}{14}\right)^3}_{P(W=3,\,B=3,\,R=0)}}{{}^6C_3\left(\frac{5}{14}\right)^3 \cdot \left(\frac{9}{14}\right)^3}$$

$$= \frac{\cancel{{}^6C_3\left(\frac{5}{14}\right)^3}\left[1 \cdot {}^3C_1 \cdot \frac{3}{14} \cdot \left(\frac{6}{14}\right)^2 + 2 \cdot {}^3C_2 \cdot \left(\frac{3}{14}\right)^2 \cdot \frac{6}{14} + 3 \cdot \left(\frac{3}{14}\right)^3\right]}{\cancel{{}^6C_3\left(\frac{5}{14}\right)^3} \cdot \left(\frac{9}{14}\right)^3}$$

$$= \frac{\cancel{3} \cdot \cancel{3} \cdot \cancel{36}^{\,4} + 2 \cdot \cancel{3} \cdot \cancel{9} \cdot \cancel{6}^{\,2} + \cancel{3} \cdot \cancel{27}^{\,3}}{\cancel{9} \cdot \cancel{9} \cdot 9}$$

$$\frac{4+4+1}{9} = \boxed{1}$$

(3) $X_1 = \{0, 1, \cdots n_1\}$ $X_2 = \{0, 1, \cdots n_2\}$

Condition $\rightarrow$ $X_1 + X_2 = m$

Distribution $\Rightarrow$ $P_{X_1}(X_1 = i) = {}^{n_1}C_i \, p^i (1-p)^{n_1 - i}$

$P_{X_2}(X_2 = i) = {}^{n_2}C_i \, p^i (1-p)^{n_2 - i}$

PMF of $X_1 \mid (X_1 + X_2 = m)$

$$p = P(X_1 = i \mid X_1 + X_2 = m) = \frac{P(X_1 = i \,\&\, X_1 + X_2 = m)}{P(X_1 + X_2 = m)}$$

$\because$ independent

$$= \frac{P(X_1 = i \,\&\, X_2 = m-i)}{P(X_1 + X_2 = m)} \qquad = \frac{P(X_1 = i) \cdot P(X_2 = m-i)}{P(X_1 + X_2 = m)}$$

$X_1 + X_2$ is binomial dist. with $(n_1 + n_2, p)$

$$\Rightarrow p = \frac{{}^{n_1}C_i \, p^i (1-p)^{n_1 - i} \cdot {}^{n_2}C_{m-i} \, p^{m-i} (1-p)^{n_2 - m + i}}{{}^{n_1+n_2}C_m \cdot p^m \cdot (1-p)^{n_1 + n_2 - m}}$$

$$= \frac{{}^{n_1}C_i \cdot {}^{n_2}C_{m-i} \, \cancel{p^m} \cdot \cancel{(1-p)^{n_1+n_2-m}}}{{}^{n_1+n_2}C_m \, \cancel{p^m} \, \cancel{(1-p)^{n_1+n_2-m}}}$$

$$\Rightarrow \boxed{P(X_1 = i \mid X_1 + X_2 = m) = \frac{{}^{n_1}C_i \cdot {}^{n_2}C_{m-i}}{{}^{n_1+n_2}C_m}}$$

(4). ~~We~~ Let (X, Y) be 2 R.V.

We require → ~~$Cov(X,Y) \neq 0$~~ & $P(X/Y) \neq P(X) \cdot P(Y)$

~~uncorrelated~~ — not independent

$$\boxed{Cov(X, Y) = 0}$$

↓

uncorrelated

let joint PDF, $P(X,Y) = \begin{cases} 1/4 & X=-1, Y=+1 \\ 1/2 & X=0, Y=0 \\ 1/4 & X=+1, Y=+1 \end{cases}$

$\Rightarrow$

$P_X(X=x) = \begin{cases} 1/4 & x=-1 \\ 1/2 & x=0 \\ 1/4 & x=1 \end{cases}$

$P_Y(Y=y) = \begin{cases} 1/2 & y=1 \\ 1/2 & y=0 \end{cases}$

$\Rightarrow E(XY) = (-1)\left(\frac{1}{4}\right) + 0\left(\frac{1}{2}\right) + 1\left(\frac{1}{4}\right) = 0$

$E(X) = \frac{1}{4}(-1) + 0(1/2) + \frac{1}{4}(1) = 0$

$E(Y) = 1/2(1) + 0(1/2) = 1/2$

$E(X,Y) = E(XY) - E(X)\cdot E(Y) = 0 - 0 = 0$

$\Rightarrow$ X, Y are uncorelated

Now check prob for $X=-1$ & $Y=1$

$P(X=-1, Y=1) = \frac{1}{4}$ $\quad P(X=-1) = 1/4 \quad P(Y=1) = \frac{1}{2}$

$P(X=-1)\cdot P(Y=1) = \frac{1}{8} \neq \frac{1}{4}$

$\Rightarrow$ X, Y are not independent

$\Rightarrow$ X, Y are RV which are uncorrelated but not independent.

Poisson

(5) Distribution for mean $\lambda \to P(X=i) = \frac{e^{-\lambda}\lambda^i}{i!}$

Exponential — $1 \to \begin{cases} 1 \cdot e^{-1 \cdot x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$

Since 'λ' has exp. distribution & $\lambda \geq 0$

$\Rightarrow$ Distribution for 'λ' is $\to$

$$P(X=\lambda_0) = e^{-\lambda_0}$$

~~So,~~ To find $\to P\{X=n\}$

We know that X has poisson distribution

$$P(X=n) = \int_0^\infty P(X=n \mid \lambda = x) \cdot e^{-x} \cdot dx$$

$\uparrow$ distribution for 'x'.

$$= \int_0^\infty e^{-x} \frac{x^n}{n!} \cdot e^{-x}\, dx$$

$$= \frac{1}{n!} \int_0^\infty x^n \cdot e^{-2x}\, dx \quad \text{let } 2x = t \Rightarrow x = \frac{t}{2} \text{ \& } dx = \frac{1}{2} dt.$$

$$\Rightarrow p = \frac{1}{n!} \int_0^\infty \frac{t^n}{2^n} \cdot e^{-t} \frac{dt}{2} = \frac{1}{2^{n+1} \cdot n!} \underbrace{\int_0^\infty e^{-t} \cdot t^n\, dt}_{\Gamma(n+1) = n!}$$

$$\Rightarrow p = \frac{1}{2^{n+1} \cdot n!} \cdot n! = \boxed{\frac{1}{2^{n+1}}}$$

Hence Proved

(6). $f_{X,Y}(x,y) = \begin{cases} c(1+xy) & 2 \le x \le 3 \ \& \ 1 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$

Joint Density fcn.

Now, we know that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY} \cdot dy \cdot dx = 1$

$\Rightarrow \int_2^3 \int_1^2 c(1+xy)\, dy \cdot dx = 1$

$\Rightarrow c \int_2^3 \left(\int_1^2 (1+xy)\, dy \right) dx = 1$

$\Rightarrow c \int_2^3 \left[y + \frac{xy^2}{2} \right]_1^2 dx = c \int_2^3 \left[1 + \frac{x}{2}(3) \right] dx = c \left[x + \frac{3x^2}{4} \right]_2^3$

$= c\left(1 + \frac{3}{4}(5)\right) = \frac{19c}{4} = 1 \Rightarrow \boxed{c = \frac{4}{19}}$

$f_X = \int_{-\infty}^{\infty} f_{XY}\, dy = \int_1^2 \frac{4}{19}(1+xy)\, dy = \frac{4}{19}\left[y + \frac{x}{2}y^2 \right]_1^2$

$= \boxed{\frac{4}{19}\left[1 + \frac{3x}{2} \right]}$

$f_Y = \int_{-\infty}^{\infty} f_{xy}\, dx = \int_2^3 \frac{4}{19}(1+xy)\, dx = \frac{4}{19}\left[x + \frac{y}{2}x^2 \right]_2^3$

$= \boxed{\frac{4}{19}\left[1 + \frac{y}{2}(5) \right]}$

$$\Rightarrow f_X = \begin{cases} 4/19\left(1+\frac{3x}{2}\right) & 2 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

$$\& \quad f_Y = \begin{cases} 4/19\left(1+5Y/2\right) & 1 \le Y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

(7) Prob. dist. for no. of acc. $\rightarrow \dfrac{e^{-\lambda} \cdot \lambda^{+i}}{i!}$ (λ = mean)

Prob. dist. for Poisson mean of each person $\rightarrow \lambda e^{-\lambda}$.

$$\Rightarrow P(\lambda = x) = x \cdot e^{-x}$$

Poisson mean

To find:

$$P(\underbrace{\text{no. of acc. of random person}}_{X} = n)$$

$$P(X=n) = \int_0^{\infty} P(X=n \mid \lambda = x) \cdot P(\lambda = x) \cdot dx$$

Poisson mean

$$= \int_0^{\infty} \frac{e^{-x} \cdot x^{+n}}{n!} \cdot x e^{-x} \cdot dx = \frac{1}{n!} \int_0^{\infty} e^{-2x} \cdot x^{1+n} \, dx$$

take $2x = t \Rightarrow dx = dt/2$

$$\Rightarrow p = \frac{1}{n!} \int_0^{\infty} e^{-t} \frac{t^{n+1}}{2^{n+1}} \frac{dt}{2} = \frac{1}{2^{n+2} \cdot n!} \underbrace{\int_0^{\infty} e^{-t} \cdot t^{n+1} \, dt}_{\Gamma(n+2) = (n+1)!}$$

$$\Rightarrow p = \frac{1}{2^{n+2}} \cdot \frac{(n+1)!}{n!} = \frac{n+1}{2^{n+2}}$$

$$\Rightarrow \boxed{p = \frac{n+1}{2^{n+2}}}$$

⑧ $P(\text{ppl visited} = i) = \dfrac{e^{-\lambda} \cdot \lambda^{i}}{i!}$

$P(\text{person visited is Female}) = p$
$P(\quad \text{—"—} \quad \text{male}) = 1-p$ → Similar to Binary (p)

We want prob. of exactly n women & m men.

So total ppl visited will be $m+n$ & out of them any n will be women each independently with prob. p & rem. 'm' will be men —"— $(1-p)$.
→ Similar to Binomial $(m+n, p)$

So To find: $P(\text{ppl visited} = m+n, \text{men} = m, \text{women} = n) = p$

$$p = \frac{e^{-\lambda} \cdot \lambda^{m+n}}{(m+n)!} \cdot \left({}^{m+n}C_n\right) \cdot p^n \cdot (1-p)^m$$

$$p = \frac{e^{-\lambda}\, \lambda^{m+n}}{\cancel{(m+n)!}} \cdot \frac{\cancel{(m+n)!}}{m!\, n!}\, p^n \cdot (1-p)^m$$

$$\Rightarrow \cancel{p = \frac{e^{-\lambda}\, p^n (1-p)^m}{m!\, n!}}$$

$$\Rightarrow \boxed{p = \frac{e^{-\lambda} \cdot \lambda^{m+n}\, p^n \cdot (1-p)^m}{m!\, n!}}$$

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9 (a) $Cov(X_1, X_2) = E(X_1 - E(X_1)) \cdot E(X_2 - E(X_2))$

$Cov(aX_1+b, cX_2+d) = E((aX_1+b) - E(aX_1+b)) \cdot E((cX_2+d) - E(cX_2+d))$

$= E((aX_1+\cancel{b}) - E(aX_1) - \cancel{b}) \cdot E(cX_2+\cancel{d} - E(cX_2) - \cancel{d})$

$= E(aX_1 - aE(X_1)) \cdot E(cX_2 - cE(X_2))$

$= aE(X_1 - E(X_1)) \cdot cE(X_2 - E(X_2))$

$= ac \cdot E(X_1 - E(X_1)) \cdot E(X_2 - E(X_2))$

$= ac \cdot Cov(X_1, X_2)$

$\Rightarrow \boxed{Cov(aX_1+b, cX_2+d) = ac \cdot Cov(X_1, X_2)}$

Hence Proved

(b) To prove $Cov(X_1+X_2, X_3) = Cov(X_1, X_3) + Cov(X_2, X_3)$

$Cov(X_1+X_2, X_3) = E(X_1+X_2 - E(X_1+X_2)) \cdot E(X_3 - E(X_3))$

$= E(X_1 - E(X_1) + X_2 - E(X_2)) \cdot E(X_3 - E(X_3))$

$= [E(X_1 - E(X_1)) + E(X_2 - E(X_2))] \cdot E(X_3 - E(X_3))$

$= E(X_1 - E(X_1)) \cdot E(X_3 - E(X_3)) + E(X_2 - E(X_2)) \cdot E(X_3 - E(X_3))$

$= Cov(X_1, X_3) + Cov(X_2, X_3)$

$\Rightarrow \boxed{Cov(X_1+X_2, X_3) = Cov(X_1, X_3) + Cov(X_2, X_3)}$

Hence Proved

(10) · $n = 100$ i.i.d samples.

$\hat{\mu} = 0.45$; we want μ to lie in an interval around $\hat{\mu}$ with prob ≥ 0.95

We have $P(|\hat{\mu}-\mu| > \epsilon) \leq \underbrace{2e^{-n\epsilon^2}}_{\delta = 0.05}$ (from given)

$\Rightarrow \cancel{P(|\hat{\mu}-\mu| \geq \epsilon)} \Rightarrow 2e^{-n\epsilon^2} = 0.05$ $(n = 100)$

$\Rightarrow e^{-n\epsilon^2} = \frac{0.05}{2}$ $\Rightarrow -n\epsilon^2 = \ln \frac{0.05}{2}$

$$\Rightarrow \epsilon = \sqrt{\frac{1}{n} \ln \frac{2}{0.05}} = \sqrt{\frac{\ln 40}{100}} = \frac{\sqrt{\ln 40}}{10} = \boxed{0.192}$$

$\boxed{\Rightarrow \text{Confidence interval: } (0.258, 0.642) \leftarrow}$

~~Confidence interval~~.

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(b) we want that ~~[illegible]~~ $\epsilon \to \epsilon/2$

$$\epsilon = \sqrt{\frac{1}{n} \left| \log\left(\frac{8}{2}\right) \right|} \Rightarrow n \to 4n$$

$\Rightarrow$ Total samples req. = 400

$\Rightarrow$ $\boxed{\text{300 more samples req.}}$