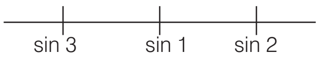
JEE Type Solved Examples :
Single Option Correct Type Questions

- **Ex. 1.** *In the inequality below, the value of the angle is expressed in radian measure. Which one of the inequalities below is true?*

(a) $\sin 1 < \sin 2 < \sin 3$ (b) $\sin 3 < \sin 2 < \sin 1$
(c) $\sin 2 < \sin 1 < \sin 3$ (d) $\sin 3 < \sin 1 < \sin 2$

Sol. (*d*) We have, $\sin 1 - \sin 2$

$$= -2\cos\frac{3}{2}\cdot\sin\frac{1}{2} < 0$$



$\therefore \quad \sin 1 < \sin 2$

Similarly $\quad \sin 1 - \sin 3 \quad$ ...(i)

$$= -2\cos 2\sin 1 > 0$$

$\Rightarrow \quad \sin 1 > \sin 3 \quad$ ...(ii)

From Eqs. (i) and (ii), we get

$$\sin 3 < \sin 1 < \sin 2$$

- **Ex. 2.** *In a ΔABC, $\angle B < \angle C$ and the values of B and C satisfy the equation $2\tan x - k(1+\tan^2 x) = 0$, where $(0 < k < 1)$. Then, the measure of $\angle A$ is*

(a) $\frac{\pi}{3}$ (b) $\frac{2\pi}{3}$
(c) $\frac{\pi}{2}$ (d) $\frac{3\pi}{4}$

Sol. (*c*) $\because k = \frac{2\tan x}{1+\tan^2 x} = \sin 2x$

$\Rightarrow \quad \sin 2C = \sin 2B$

But $\quad \angle C > \angle B$

$\Rightarrow \quad 2C = \pi - 2B \;\Rightarrow\; B + C = \frac{\pi}{2}$

$\therefore \quad \angle A = \frac{\pi}{2}$

- **Ex. 3.** *If M and m are maximum and minimum value of the function $f(x) = \frac{\tan^2 x + 4\tan x + 9}{1+\tan^2 x}$, then $(M+m)$ equals*

(a) 20 (b) 14
(c) 10 (d) 8

Sol. (*c*) Given, $f(x) = \frac{\tan^2 x + 4\tan x + 9}{1+\tan^2 x}$

$$= \frac{2(2\tan x)}{1+\tan^2 x} + 4\left(\frac{1-\tan^2 A}{1+\tan^2 A}\right) + 5$$

$$= 2\sin 2x + 4\cos 2x + 5$$

$\therefore \quad R_f = [\sqrt{5} - \sqrt{20}, 5 + \sqrt{20}]$

Hence, $(M+m) = 10$.

Alternate Method

$$f(x) = \frac{\tan^2 x}{1+\tan^2 x} + \frac{4\tan x}{1+\tan^2 x} + \frac{9}{1+\tan^2 x}$$

$$= \sin^2 x + 2\sin 2x + 9\cos^2 x$$

$$= 1 + 4(1+\cos 2x) + 2\sin 2x$$

$$= 5 + 2\sin 2x + 4\cos 2x$$

- **Ex. 4.** *The value of $4\cos\frac{\pi}{10} - 3\sec\frac{\pi}{10} - \tan\frac{\pi}{10}$ is equal to*

(a) 1
(b) $\sqrt{5} - 1$
(c) $\sqrt{5} + 1$
(d) zero

Sol. (*d*) We have, $4\cos 18^\circ - \frac{3}{\cos 18^\circ} - 2\tan 18^\circ$

$$= \frac{4\cos^2 18^\circ - 3 - 2\sin 18^\circ}{\cos 18^\circ}$$

$$= \frac{2(1+\cos 36^\circ) - 2\sin 18^\circ - 3}{\cos 18^\circ}$$

$$= \frac{2(1+\cos 36^\circ - \sin 18^\circ) - 3}{\cos 18^\circ}$$

$$= \frac{2\left(1+\frac{1}{2}\right) - 3}{\cos 18^\circ} = 0$$

- **Ex. 5.** *For $0 < A < \frac{\pi}{2}$, the value of*

$$\log_{\frac{1}{2}}\left(\frac{1}{1+2\cos^2 A} + \frac{2}{\sec^2 A + 2}\right) \textit{ is equal to}$$

(a) 1 (b) -1
(c) 2 (d) 0

Sol. (*d*) As, $\left(\frac{1}{1+2\cos^2 A} + \frac{2}{\sec^2 A + 2}\right)$

$$= \left(\frac{1}{1+2\cos^2 A} + \frac{2\cos^2 A}{1+2\cos^2 A}\right)$$

$$= \frac{(1+2\cos^2 A)}{(1+2\cos^2 A)} = 1$$

Hence, $\log_{\frac{1}{2}}(1) = 0$.

Ex. 6. *The sum* $\dfrac{1}{\sin 45^\circ \sin 46^\circ}+\dfrac{1}{\sin 47^\circ \sin 48^\circ}$ $+\dfrac{1}{\sin 49^\circ \sin 50^\circ}+\ldots+\dfrac{1}{\sin 133^\circ \sin 134^\circ}$ *is equal to*

(a) $\sec(1^\circ)$ (b) $\operatorname{cosec}(1^\circ)$
(c) $\cot(1^\circ)$ (d) None of these

Sol. (*b*) $T_1=\dfrac{1}{\sin 1^\circ}\left[\dfrac{\sin(46^\circ-45^\circ)}{\sin 45^\circ \sin 46^\circ}\right]=\dfrac{1}{\sin 1^\circ}[\cot 45^\circ-\cot 46^\circ]$

$$T_2=\frac{1}{\sin 1^\circ}\left[\frac{\sin(48^\circ-47^\circ)}{\sin 48^\circ \sin 47^\circ}\right]$$
$$=\frac{1}{\sin 1^\circ}[\cot 47^\circ-\cot 48^\circ]$$
$$T_l=\frac{1}{\sin 1^\circ}\left[\frac{\sin(133^\circ-134^\circ)}{\sin 133^\circ \sin 134^\circ}\right]$$
$$=\frac{1}{\sin 1^\circ}[\cot 133^\circ-\cot 134^\circ]$$

On adding

$$\sum_{r=1}^{l} T_r=\frac{1}{\sin 1^\circ}[\{\cot 45^\circ+\cot 47^\circ+\cot 49^\circ+\ldots+\cot 133^\circ\}-\{\cot 46^\circ+\cot 48^\circ+\cot 50^\circ+\ldots+\cot 134^\circ\}]$$
$$=\operatorname{cosec} 1^\circ$$

[all terms cancelled except $\cot 45^\circ$ remains]

Ex. 7. *The range of k for which the inequality* $k\cos^2 x-k\cos x+1\geq 0\ \forall\ x\in(-\infty,\infty)$, *is*

(a) $k<\dfrac{-1}{2}$ (b) $k<4$
(c) $\dfrac{-1}{2}\leq k\leq 4$ (d) $\dfrac{1}{2}\leq k\leq 5$

Sol. (*c*) We have

$$k\cos^2 x-k\cos x+1\geq 0\ \forall\ x\in(-\infty,\infty)$$
$$\Rightarrow\quad k(\cos^2 x-\cos x)+1\geq 0$$

But $\cos^2 x-\cos x=\left(\cos x-\dfrac{1}{2}\right)^2-\dfrac{1}{4}$

$$\Rightarrow\quad -\frac{1}{4}\leq\cos^2 x-\cos x\leq 2$$

$\therefore$ We have, $2k+1\geq 0$ and $-\dfrac{k}{4}+1\geq 0$

Hence, $-\dfrac{1}{2}\leq k\leq 4.$

Ex. 8. *If* $f(\theta)=\dfrac{1-\sin 2\theta+\cos 2\theta}{2\cos 2\theta}$, *then value of* $f(11^\circ)\cdot f(34^\circ)$ *equals*

(a) $\dfrac{1}{2}$ (b) $\dfrac{3}{4}$ (c) $\dfrac{1}{4}$ (d) 1

Sol. (*a*) $f(\theta)=\dfrac{1-\sin 2\theta+\cos 2\theta}{2\cos 2\theta}$

$$=\frac{(\cos\theta-\sin\theta)^2+(\cos^2\theta-\sin^2\theta)}{2(\cos\theta-\sin\theta)(\cos\theta+\sin\theta)}$$
$$=\frac{(\cos\theta-\sin\theta)+(\cos\theta+\sin\theta)}{2(\cos\theta+\sin\theta)}$$
$$=\frac{2\cos\theta}{2(\cos\theta+\sin\theta)}=\frac{1}{1+\tan\theta}$$
$$f(11^\circ)\cdot f(34^\circ)=\frac{1}{(1+\tan 11^\circ)}\cdot\frac{1}{(1+\tan 34^\circ)}$$
$$=\frac{1}{(1+\tan 11^\circ)}\cdot\frac{1}{(1+\tan(45^\circ-11^\circ))}$$
$$=\frac{1}{(1+\tan 11^\circ)}\cdot\frac{1}{1+\dfrac{1-\tan 11^\circ}{1+\tan 11^\circ}}$$
$$=\frac{1}{(1+\tan 11^\circ)}\cdot\frac{1+\tan 11^\circ}{2}=\frac{1}{2}$$

Ex. 9. *The variable 'x' satisfying the equation* $|\sin x\cos x|+\sqrt{2+\tan^2 x+\cot^2 x}=\sqrt{3}$, *belongs to the interval*

(a) $\left[0,\dfrac{\pi}{3}\right]$ (b) $\left[\dfrac{\pi}{3},\dfrac{\pi}{2}\right]$
(c) $\left[\dfrac{3\pi}{4},\pi\right]$ (d) Non-existent

Sol. (*d*) $|\sin x\cos x|+|\tan x+\cot x|=\sqrt{3}$

$$\Rightarrow\quad |\sin x\cos x|+\frac{1}{|\sin x\cos x|}=\sqrt{3}$$

but $|\sin x\cos x|+\dfrac{1}{|\sin x\cos x|}\geq 2$

Hence, no solution.

Ex. 10. *Let* α *be a real number such that* $0\leq\alpha\leq\pi$. *If* $f(x)=\cos x+\cos(x+\alpha)+\cos(x+2\alpha)$ *takes some constant number c for any* $x\in R$, *then the value of* $[c+\alpha]$ *is equal to*

Note $[y]$ denotes greatest integer less than or equal to y.

(a) 0 (b) 1 (c) −1 (d) 2

Sol. (*d*) $f(x)=\cos x+\cos(x+2\alpha)+\cos(x+\alpha)$

$$=2\cos(x+\alpha)\cos\alpha+\cos(x+\alpha)$$
$$=(2\cos\alpha+1)\cos(x+\alpha)$$

As $\cos(x+\alpha)$ can take any real value from -1 to 1, $\forall x\in R$

$\therefore$ $f(x)$ is constant, so $(2\cos\alpha+1)=0$ must hold.

$$\Rightarrow\quad \alpha=\frac{2\pi}{3}\text{ and }c=0$$

Hence, $[c+\alpha]=\left[0+\dfrac{2\pi}{3}\right]=2$

● **Ex. 11.** *In a ΔABC, if $4\cos A\cos B+\sin 2A+\sin 2B+\sin 2C=4$, then ΔABC is*

(a) right angle but not isosceles
(b) isosceles but not right angled
(c) right angle isosceles
(d) obtuse angled

Sol. (*c*) We have, $4\cos A\cos B+4\sin A\sin B\sin C=4$

$$\Rightarrow \quad \sin C=\frac{1-\cos A\cos B}{\sin A\cos B}\le 1$$
$$\Rightarrow \quad 1\le \sin A\sin B+\cos A\cos B$$
$$\Rightarrow \quad \cos(A-B)\ge 1$$
$$\Rightarrow \quad A=B \text{ and } \sin C=\frac{1-\cos^2 A}{\sin^2 A}=1$$
$$\therefore \quad C=90^\circ$$
$$\text{and} \quad A=B=\frac{\pi}{4}\text{ (each).}$$

Ex. 12. *For $\theta_1,\theta_2,\dots,\theta_n\in\left(0,\frac{\pi}{2}\right)$, if $\ln(\sec\theta_1-\tan\theta_1)+\ln(\sec\theta_2-\tan\theta_2)+\dots+\ln(\sec\theta_n-\tan\theta_n)+\ln\pi=0$, then the value of $\cos((\sec\theta_1+\tan\theta_1)(\sec\theta_2+\tan\theta_2)\dots\dots(\sec\theta_n+\tan\theta_n))$ is equal to*

(a) $\cos\left(\frac{1}{\pi}\right)$ (b) -1
(c) 1 (d) 0

Sol. (*b*) $\ln\{(\sec\theta_1-\tan\theta_1)(\sec\theta_2-\tan\theta_2)\dots(\sec\theta_n-\tan\theta_n)\}$
$$=\ln\left\{\frac{1}{\pi}\right\}$$

[**Note** If $0<x<\frac{\pi}{2}$, $\sec x-\tan x=\frac{1-\sin x}{\cos x}>0$]

$$\therefore (\sec\theta_1-\tan\theta_1)(\sec\theta_2-\tan\theta_2)\dots(\sec\theta_n-\tan\theta_n)=\frac{1}{\pi} \quad \dots\text{(i)}$$

Let $(\sec\theta_1+\tan\theta_1)(\sec\theta_2+\tan\theta_2)\dots(\sec\theta_n+\tan\theta_n)=x$ ...(ii)

On multiplying Eqs. (i) and (ii) we get

$$1=\frac{x}{\pi}$$
$$\therefore \quad x=\pi$$
$$\therefore \cos((\sec\theta_1+\tan\theta_1)(\sec\theta_2+\tan\theta_2)\dots(\sec\theta_n+\tan\theta_n))=\cos\pi=-1$$

● **Ex. 13.** *If A, B, C are interior angles of ΔABC such that $(\cos A+\cos B+\cos C)^2+(\sin A+\sin B+\sin C)^2=9$, then number of possible triangles is*

(a) 0 (b) 1
(c) 3 (d) infinite

Sol. (*d*) $(\Sigma\cos A)^2+(\Sigma\sin A)^2=9$

$\Sigma(\cos^2 A+\sin^2 A)+2(\Sigma\cos A\cos B+\sin A\sin B)$

$$3+2\Sigma\cos(A-B)\le 3+2(3)=9.$$

Equality holds if $A=B=C$

$\Rightarrow$ ΔABC is equilateral $\Rightarrow$ Infinite many equilateral

[**Note** We can vary side length of equilateral triangle]

● **Ex. 14.** *If $\operatorname{cosec}\frac{\pi}{32}+\operatorname{cosec}\frac{\pi}{16}+\operatorname{cosec}\frac{\pi}{8}+\operatorname{cosec}\frac{\pi}{4}+\operatorname{cosec}\frac{\pi}{2}=\cot\frac{\pi}{k}$, then the value of k is*

(a) 64 (b) 96 (c) 48 (d) 32

Sol. (*a*) $T_1=\operatorname{cosec}\theta=\dfrac{\sin\left(\theta-\frac{\theta}{2}\right)}{\sin\frac{\theta}{2}\sin\theta}$; $\theta=\dfrac{\pi}{32}$

$$T_1=\cot\frac{\theta}{2}-\cot\theta$$

$$\left.\begin{aligned}T_2&=\cot\theta-\cot 2\theta\\T_3&=\cot 2\theta-\cos 2^2\theta\\T_4&=\cot 2^2\theta-\cos 2^3\theta\\T_5&=1\end{aligned}\right]\text{sum}=1+\cot\frac{\theta}{2}-\cot 8\theta$$

$$\text{Sum }=1+\cot\frac{\theta}{2}-\cot 8\theta$$
$$=1+\cot\frac{\pi}{64}-\cot\frac{\pi}{4}=\cot\frac{\pi}{64}=\cot\frac{\pi}{k}\ \therefore\ k=64$$

● **Ex. 15.** *Let $S=\sum_{r=1}^{5}\cos(2r-1)\frac{\pi}{11}$ and $P=\prod_{r=1}^{4}\cos\left(2^r\frac{\pi}{15}\right)$, then*

(a) $\log_S P=-4$ (b) $P=3S$
(c) $\operatorname{cosec} S>\operatorname{cosec} P$ (d) $\tan^{-1}P<\tan^{-1}S$

Sol. (d) We have, $\sum_{r=1}^{5}\cos(2r-1)\frac{\pi}{11}$

$$=\underbrace{\cos\frac{\pi}{11}+\cos\frac{3\pi}{11}+\cos\frac{5\pi}{11}+\cos\frac{7\pi}{11}+\cos\frac{9\pi}{11}}_{\text{cosine series}}$$

$$=\frac{2\,.\cos\left(\frac{5\pi}{11}\right).\sin\left(\frac{5\pi}{11}\right)}{2\cdot\sin\frac{\pi}{11}}=\frac{\sin\left(\frac{10\pi}{11}\right)}{2\cdot\sin\left(\frac{\pi}{11}\right)}=\frac{1}{2}$$

Also, $\prod_{r=1}^{\pi}\cos\left(2^r\frac{\pi}{15}\right)=\cos\frac{2\pi}{15}.\cos\frac{4\pi}{15}.\cos\frac{8\pi}{15}.\cos\frac{16\pi}{15}$

$$=\frac{\sin\left(\frac{32\pi}{15}\right)}{2^4\cdot\sin\left(\frac{2\pi}{15}\right)}=\frac{\sin\left(2\pi+\frac{2\pi}{15}\right)}{16\cdot\sin\left(\frac{2\pi}{15}\right)}$$

$$= \frac{\sin\left(\frac{2\pi}{15}\right)}{16 \cdot \sin\left(\frac{2\pi}{15}\right)} = \frac{1}{16}$$

Therefore, $\tan^{-1} P < \tan^{-1} S$.

● **Ex. 16.** *Set of values of x lying in* $[0, 2\pi]$ *satisfying the inequality* $|\sin x| > 2 \sin^2 x$ *contains*

(a) $\left(0, \frac{\pi}{6}\right) \cup \left(\pi, \frac{7\pi}{6}\right)$ (b) $\left(0, \frac{7\pi}{6}\right)$

(c) $\frac{\pi}{6}$ (d) None of these

Sol. (*a*) $|\sin x| > 2\sin^2 x$

$\Rightarrow |\sin x|(2|\sin x| - 1) < 0$

$\Rightarrow 0 < |\sin x| < \frac{1}{2}$

$\Rightarrow x \in \left(0, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, \pi\right) \cup \left(\pi, \frac{7\pi}{6}\right) \cup \left(\frac{11\pi}{6}, 2\pi\right)$

● **Ex. 17.** *The number of ordered pairs* (x, y), *when* $x, y \in [0, 10]$ *satisfying* $\left(\sqrt{\sin^2 x - \sin x + \frac{1}{2}}\right) \cdot 2^{\sec^2 y} \le 1$ *is*

(a) 0 (b) 16

(c) infinite (d) 12

Sol. (*b*) $\sqrt{\sin^2 x - \sin x + \frac{1}{2}} = \sqrt{\left(\sin x - \frac{1}{2}\right)^2 + \frac{1}{2}} \ge \frac{1}{2}, \forall x$

and $\sec^2 y \ge 1, \forall y$, so $2^{\sec^2 y} \ge 2$. Hence, the above inequality holds only for those values of x and y for which $\sin x = \frac{1}{2}$

and $\sec^2 y = 1$.

Hence, $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$ and $y = 0, \pi, 2\pi, 3\pi$. Hence, required number of ordered pairs are 16.

● **Ex. 18.** *The least values of* $\operatorname{cosec}^2 x + 25 \sec^2 x$ *is*

(a) 0 (b) 26

(c) 28 (d) 36

Sol. (*d*) $\operatorname{cosec}^2 x + 25\sec^2 x = 26 + \cot^2 x + 25\tan^2 x$

$= 26 + 10 + (\cot x - 5\tan x)^2 \ge 36$

● **Ex. 19.** *If* $x \sin a + y \sin 2a + z \sin 3a = \sin 4a$

$x \sin b + y \sin 2b + z \sin 3b = \sin 4b$

$x \sin c + y \sin 2c + z \sin 3c = \sin 4c$

Then, the roots of the equation $t^3 - \left(\frac{z}{2}\right)t^2 - \left(\frac{y+2}{4}\right)t + \left(\frac{z-x}{8}\right) = 0$, *a, b, c* $\ne n\pi$, *are*

(a) $\sin a, \sin b, \sin c$

(b) $\cos a, \cos b, \cos c$

(c) $\sin 2a, \sin 2b, \sin 2c$

(d) $\cos 2a, \cos 2b, \cos 2c$

Sol. (*b*) Equation first can be written as

$x \sin a + y \times 2\sin a \cos a + z \times \sin a(3 - 4\sin^2 a)$

$= 2 \times 2\sin a \cos a \cos 2a$

$\Rightarrow \quad x + 2y\cos a + z(3 + 4\cos^2 a - 4)$

$= 4\cos a(2\cos^2 a - 1)$ as $\sin a \ne 0$

$\Rightarrow \quad 8\cos^3 a - 4z\cos^2 a - (2y + 4)\cos a + (z - x) = 0$

$\Rightarrow \quad \cos^3 a - \left(\frac{z}{2}\right)\cos^2 a - \left(\frac{y+2}{4}\right)\cos a + \left(\frac{z-x}{8}\right) = 0$

which shows that $\cos a$ is a root of the equation

$$t^3 = \left(\frac{z}{2}\right)t^2 - \left(\frac{y+z}{4}\right)t + \left(\frac{z-x}{8}\right) = 0$$

Similarly, from second and third equation we can verify that $\cos b$ and $\cos c$ are the roots of the given equation.

● **Ex. 20.** *Let* α *and* β *be any two positive values of x for which* $2\cos x, |\cos x|$ *and* $1 - 3\cos^2 x$ *are in GP. The minium value of* $|\alpha + \beta|$ *is*

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$

(c) $\frac{\pi}{2}$ (d) None of these

Sol. (*d*) $\because 2\cos x, |\cos x|, 1 - 3\cos^2 x$ are in GP.

$\therefore \quad \cos^2 x = 2\cos x \cdot (1 - 3\cos^2 x)$

$\Rightarrow \quad 6\cos^3 x + \cos^2 x - 2\cos x = 0$

$\therefore \quad \cos x = 0, \frac{1}{2}, -\frac{2}{3}$

$\therefore \quad x = \frac{\pi}{2}, \frac{\pi}{3}, \cos^{-1}\left(-\frac{2}{3}\right)$ [$\because \alpha, \beta$ are positive]

If $\quad \alpha = \frac{\pi}{2}, \beta = \frac{\pi}{3}$

Then, $|\alpha - \beta| = \frac{\pi}{6}$

● **Ex. 21.** *Let n be an odd integer. If* $\sin n\theta = \sum_{r=0}^{n} b_r \sin^r \theta$ *for all real* θ, *then*

(a) $b_0 = 1, b_1 = 3$ (b) $b_0 = 0, b_1 = n$

(c) $b_0 = -1, b_1 = n$ (d) $b_0 = 0, b_1 = n^2 - 3n - 3$

Sol. (*b*) Given, $\sin n\theta = \sum_{r=0}^{n} b_r \sin^r \theta = b_0 + b_1 \sin\theta + b_2\sin^2\theta$

$+ \ldots + b_n \sin^n \theta$ ...(i)

Putting $\theta = 0$ in Eq. (i), we get $0 = b_0$

Again, Eq. (i) can be written as $\sin n\theta = \sum_{r=0}^{n} b_r \sin^r \theta$

$$\frac{\sin n\theta}{\sin\theta} = \sum_{r=0}^{n} b_r \sin^{r-1}\theta$$

On taking limit as $\theta \to 0$, we get

$$\lim_{\theta\to 0} \frac{\sin n\theta}{\sin\theta} = b_1$$

$\Rightarrow \quad \lim_{\theta\to 0} n\left(\frac{\sin\theta}{n\theta}\right)\left(\frac{\theta}{\sin\theta}\right) = b_1$

$\Rightarrow \quad n = b_1$

Hence, $\quad b_0 = 0; b_1 = n$

● **Ex. 22.** *The minimum and maximum values of $ab\sin x + b\sqrt{(1-a^2)}\cos x + c\ (|a|<1, b>0)$ respectively are*

(a) $\{b-c, b+c\}$ (b) $\{b+c, b-c\}$

(c) $\{c-b, b+c\}$ (d) None of these

Sol. (c) $ab\sin x + b\sqrt{(1-a^2)}\cos x$

Now, $\sqrt{(ab)^2 + (b\sqrt{(1-a^2)})^2}$

$= \sqrt{a^2b^2 + b^2(1-a^2)}$

$= b\sqrt{(a^2 + 1 - a^2)} = b$

$\Rightarrow \quad b\{(a\sin x + \sqrt{(1-a^2)}\cos x)\}$

Let, $\quad a = \cos\alpha$,

$\therefore \quad \sqrt{(1-a^2)} = \sin\alpha$

$\Rightarrow \quad b\sin(x+a)$

$\because \quad -1 \le \sin(x+\alpha) \le 1$

$\therefore \quad c - b \le b\sin(x+\alpha) + c \le b + c$

$\therefore \quad b\sin(x+\alpha) + c \in [c-b, c+b]$

● **Ex. 23.** *If* $\dfrac{\sin^3\theta - \cos^3\theta}{\sin\theta - \cos\theta} - \dfrac{\cos\theta}{\sqrt{(1+\cot^2\theta)}}$

$-2\tan\theta\cot\theta = -1, \theta \in [0, 2\pi]$, *then*

(a) $\theta \in \left(0, \frac{\pi}{2}\right) - \left\{\frac{\pi}{4}\right\}$ (b) $\theta \in \left(\frac{\pi}{2}, \pi\right) - \left\{\frac{3\pi}{4}\right\}$

(c) $\theta \in \left(\pi, \frac{3\pi}{2}\right) - \left\{\frac{5\pi}{4}\right\}$ (d) $\theta \in (0, \pi) - \left\{\frac{\pi}{4}, \frac{\pi}{2}\right\}$

Sol. (d) $\dfrac{\sin^3\theta - \cos^3\theta}{\sin\theta - \cos\theta} = \sin^2\theta + \cos^2\theta + \sin\theta\cos\theta$,

$\theta \ne \frac{\pi}{4}, \frac{5\pi}{4}$

$= 1 + \sin\theta\cos\theta$

and $\quad \dfrac{\cos\theta}{\sqrt{(1+\cot^2\theta)}} = \dfrac{\cos\theta}{|\operatorname{cosec}\theta|} = \sin\theta\cos\theta\ \forall\ \theta \in (0, \pi)$

and $\quad -2\tan\theta\cot\theta = -2, \theta \ne \frac{\pi}{2}$

Hence, $\quad$ LHS = RHS

But $\quad \theta \ne \frac{\pi}{4}, \frac{5\pi}{4}, \frac{\pi}{2}$

Hence, $\quad \theta \in (0, \pi) \sim \left\{\frac{\pi}{4}, \frac{\pi}{2}\right\}$

● **Ex. 24.** *If* $\cos x + \sin x = a\left(-\frac{\pi}{2} < x < -\frac{\pi}{4}\right)$, *then* $\cos 2x$ *is equal to*

(a) a^2 (b) $a\sqrt{(2-a)}$

(b) $a\sqrt{(2+a)}$ (d) $a\sqrt{(2-a^2)}$

Sol. (d) $\because -\frac{\pi}{2} < x < -\frac{\pi}{4}\left(-\pi < 2x < -\frac{\pi}{2}, \text{ i.e, in III quadrant}\right)$

$\Rightarrow \quad \cos x + \sin x = a$

Squaring both sides $\cos^2 x + \sin^2 x + 2\cos x\sin x = a^2$

$\Rightarrow \quad \sin 2x = (a^2 - 1)$

$\cos 2x = \sqrt{1 - (a^2-1)^2}$

$= \sqrt{a^2(2-a^2)}$

$= a\sqrt{(2-a^2)}$

● **Ex. 25.** *If* $S = \cos^2\frac{\pi}{2} + \cos^2\frac{2\pi}{n} + \ldots + \cos^2\frac{(n-1)\pi}{n}$, *then* S *equals*

(a) $\dfrac{n}{2(n+1)}$ (b) $\dfrac{1}{2(n-1)}$

(c) $\dfrac{1}{2(n-2)}$ (d) $\dfrac{n}{2}$

Sol. (c) $S = \cos^2\frac{\pi}{n} + \cos^2\frac{2\pi}{n} + \ldots + \cos^2(n-1)\frac{\pi}{n}$

$$\frac{1}{2}\left[\begin{matrix}1 + \cos\frac{2\pi}{n} + 1 + \cos\frac{4\pi}{n} + 1 + \cos\frac{6\pi}{n} \\ + \ldots + 1 + \cos 2(n-1)\frac{\pi}{n}\end{matrix}\right]$$

$$= \frac{1}{2}\left[n - 1 + \sum_{k=1}^{n-1}\cos\frac{2k\pi}{n}\right]$$

$$= \frac{1}{2}[n-1-1] = \frac{1}{2}(n-2)$$

● **Ex. 26.** *If* $\cos 5\theta = a\cos\theta + b\cos^3\theta + c\cos^5\theta + d$, *then*

(a) $a = 20$ (b) $b = -30$

(c) $a + b + c = 2$ (d) $a + b + c + d = 1$

Sol. (d) Put $\theta = \frac{\pi}{2}$ in the given inequality, we get $d = 0$

Put $\theta = 0$ in the given inequality, we get

$$a + b + c + d = 1 \qquad \ldots\text{(i)}$$

So, (d) is correct and (c) is not correct.

Now differentiate both sides with respect to θ, we get

$$-5\sin\theta = -a\sin\theta - 3b\cos^2\theta\sin\theta - 5c\cos^4\theta\sin\theta \quad \text{...(ii)}$$

Put $\theta = \frac{\pi}{2}$, then $a = 5$ ...(iii)

Again putting $\theta = \frac{\pi}{4}$ in the given expression or in (2), we get

$$4a + 2b + c = -4 \quad \text{...(iv)}$$

From (i), (iii) and (iv) we have $b = -20$ and $c = 16$

[**Note** We have found correct answer at the second step only however the complete solution is desired for better understanding of the solution.]

Alternates Solution

$\cos 5\theta = \cos(2\theta + 3\theta) = \cos 2\theta \cos 3\theta - \sin 2\theta \sin 3\theta$

$= (2\cos^2\theta - 1)(4\cos^3\theta - 3\cos\theta) - (2\sin\theta\cos\theta)(3\sin\theta - 4\sin^3\theta)$

$= 8\cos^5\theta - 10\cos^3\theta + 3\cos\theta - 2(1-\cos^2\theta)\cos\theta\{3 - 4(1-\cos^2\theta)\}$

$= 8\cos^5\theta - 10\cos^3\theta + 3\cos\theta - 2(\cos\theta - \cos^3\theta)(4\cos^2\theta - 1) = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.$

Ex. 27. *If A and B are acute positive angles satisfying the equations* $3\sin^2 A + 2\sin^2 B = 1$ *and* $3\sin 2A - 2\sin 3B = 0$, *then* $A + 2B$ *is equal to*

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$
(c) $\frac{3\pi}{4}$ (d) $\frac{2\pi}{3}$

Sol. (*b*) From the given relations, we have

$\sin 2B = \left(\frac{3}{2}\right)\sin 2A$ and $3\sin^2 A = 1 - 2\sin^2 B = \cos 2B$

so that

$\cos(A + 2B) = \cos A \cos 2B - \sin A \sin 2B$

$= \cos A \cdot 3\sin^2 A - \left(\frac{3}{2}\right)\sin A \sin 2A$

$= 3\cos A\sin^2 A - 3\sin^2 A\cos A = 0$

$A + 2B = \frac{\pi}{2}$

Ex. 28. *If* $A = \begin{bmatrix} \cos^2\alpha & \cos\alpha\sin\alpha \\ \cos\alpha\sin\alpha & \sin^2\alpha \end{bmatrix}$ *and* $B = \begin{bmatrix} \cos^2\beta & \cos\beta\sin\beta \\ \cos\beta\sin\beta & \sin^2\beta \end{bmatrix}$

are two matrices such that AB is the null matrix, then

(a) $\alpha = \beta$ (b) $\cos(\alpha - \beta) = 0$
(c) $\sin(\alpha - \beta) = 0$ (d) None of these

Sol. (*b*) $AB = 0$

$\Rightarrow \begin{bmatrix} \cos^2\alpha & \cos\alpha\sin\alpha \\ \cos\alpha\sin\alpha & \sin^2\alpha \end{bmatrix}\begin{bmatrix} \cos^2\beta & \cos\beta\sin\beta \\ \cos\beta\sin\beta & \sin^2\beta \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} \cos\alpha\cos\beta\cos(\alpha-\beta) & \cos\alpha\sin\beta\cos(\alpha-\beta) \\ \cos\beta\sin\alpha\cos(\alpha-\beta) & \sin\alpha\sin\beta\cos(\alpha-\beta) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \cos(\alpha - \beta) = 0$

Ex. 29. *If* $k_1 = \tan 27\theta - \tan\theta$ *and* $k_2 = \frac{\sin\theta}{\cos 3\theta} + \frac{\sin 3\theta}{\cos 9\theta} + \frac{\sin 9\theta}{\cos 27\theta}$ *then,*

(a) $k_1 = k_2$
(b) $k_1 = 2k_2$
(c) $k_1 + k_2 = 2$
(d) $k_2 = 2k_1$

Sol. (*b*) We can write

$k_1 = \tan 27\theta - \tan 9\theta + \tan 9\theta - \tan 3\theta + \tan 3\theta - \tan\theta$

But $\tan 3\theta - \tan\theta = \frac{\sin 3\theta\cos\theta - \cos 3\theta\sin\theta}{\cos 3\theta\cos\theta}$

$= \frac{\sin 2\theta}{\cos 3\theta\cos\theta}$

$= \frac{2\sin\theta}{\cos 3\theta}$

$\therefore \quad k_1 = 2\left[\frac{\sin 9\theta}{\cos 27\theta} + \frac{\sin 3\theta}{\cos 9\theta} + \frac{\sin\theta}{\cos 3\theta}\right] = 2k_2$

Ex. 30. *If* $a^2 - 2a\cos x + 1 = 674$ *and* $\tan\left(\frac{x}{2}\right) = 7$ *then the integral value of a is*

(a) 25 (b) 49
(c) 67 (d) 74

Sol. (*a*) $674 = a^2 - 2a\,\dfrac{1 - \tan^2\left(\frac{x}{2}\right)}{1 + \tan^2\left(\frac{x}{2}\right)} + 1$

$= a^2 - 2a \times \frac{1-49}{1+49} + 1$

$= a^2 + 2a \times \frac{48}{50} + 1$

$\Rightarrow \quad 25a^2 + 48a - 673 \times 25 = 0$

$\Rightarrow \quad (a - 25)(25a + 673) = 0$

$\Rightarrow \quad a = 25$ (taking the integral value of a).

Ex. 31. *The maximum value of* $(\cos\alpha_1)(\cos\alpha_2)\ldots(\cos\alpha_n)$ *under the restriction* $0 \le \alpha_1, \alpha_2, \ldots, \alpha_n \le \frac{\pi}{2}$ *and* $(\cot\alpha_1)(\cot\alpha_2)\ldots(\cot\alpha_n) = 1$ *is*

(a) $\frac{1}{2^{\frac{n}{2}}}$ (b) $\frac{1}{2^n}$

(c) $\frac{1}{2n}$ (d) 1

Sol. (*a*) From the given relations we have

$$\prod_{i=1}^{n}(\cos\alpha_i) = \prod_{i=1}^{n}(\sin\alpha_i)$$

$$\Rightarrow \prod_{i=1}^{n}(\cos^2\alpha_i) = \prod_{i=1}^{n}(\cos\alpha_i\sin\alpha_i) = \prod_{i=1}^{n}\left(\frac{\sin 2\alpha_i}{2}\right)$$

Since, $0 \le \alpha_i \le \frac{\pi}{2} \Rightarrow 0 \le 2\alpha_i \le \pi$

$\therefore \prod_{i=1}^{n}(\cos^2\alpha_i) \le \frac{1}{2^n}$ as max. value of $\sin 2\alpha_i$ is 1 for all i.

$$\Rightarrow \prod_{i=1}^{n}(\cos\alpha_i) \le \frac{1}{2^{\frac{n}{2}}}$$

So, the maximum value of the given expression is $\frac{1}{2^{\frac{n}{2}}}$.

Ex. 32. *The value of expression* $\frac{\sin^3 x}{1+\cos x} + \frac{\cos^3 x}{1-\sin x}$ *is/are*

(a) $\sqrt{2}\cos\left[\frac{\pi}{4} - x\right]$ (b) $\sqrt{2}\cos\left[\frac{\pi}{4} + x\right]$

(c) $\sqrt{2}\sin\left[\frac{\pi}{4} - x\right]$ (d) None of these

Sol. (*a*) Let $\frac{\sin^3 x}{1+\cos x} + \frac{\cos^3 x}{1-\sin x} = A$, then

$$A = \frac{(\sin^3 x + \cos^3 x) + (\cos^4 x - \sin^4 x)}{(1+\cos x)(1-\sin x)}$$

or $$A = \frac{\{(\sin^3 x + \cos^3 x)\} + \begin{Bmatrix}(\cos x + \sin x)(\cos x - \sin x) \\ (\cos^2 x + \sin^2 x)\end{Bmatrix}}{(1+\cos x)(1-\sin x)}$$

or $$A = \frac{(\sin x + \cos x)\{(1 - \sin x\cos x) + (\cos x - \sin x)\}}{1 + \cos x - \sin x - \sin x\cos x}$$

or $A = \sin x + \cos x$

or $$A = \sqrt{2}\left[\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right] \quad \text{...(i)}$$

or $$A = \sqrt{2}\left[\cos\frac{\pi}{4}\sin x + \sin\frac{\pi}{4}\cos x\right]$$

$$= \sqrt{2}\sin\left[\frac{\pi}{4} + x\right]$$

Again, by Eq. (i)

$$A = \sqrt{2}\left[\sin\frac{\pi}{4}\sin x + \cos\frac{\pi}{4}\cos x\right]$$

$$= \sqrt{2}\cos\left[\frac{\pi}{4} - x\right]$$

Ex. 33. *Let* $0 \le \theta \le \frac{\pi}{2}$ *and* $x = X\cos\theta + Y\sin\theta$, $y = X\sin\theta - Y\cos\theta$ *such that* $x^2 + 2xy + y^2 = aX^2 + bY^2$, *where a and b are constant, then*

(a) $a = -1, b = -3$ (b) $\theta = \frac{\pi}{2}$

(c) $a = 3, b = -1$ (d) $\theta = \frac{\pi}{3}$

Sol. (*c*) $x^2 + y^2 = X^2 + Y^2$,

$$xy = (X^2 - Y^2)\sin\theta\cdot\cos\theta - XY(\cos^2\theta - \sin^2\theta)$$

$$x^2 + 4xy + y^2 = X^2 + Y^2 + 2(X^2 - Y^2)\sin 2\theta - 2XY\cos 2\theta$$

$$= (1 + 2\sin 2\theta)X^2 + (1 - 2\sin 2\theta)Y^2 - 2\cos 2\theta\cdot XY$$

From the question,

$$a = 1 + 2\sin 2\theta,\ b = 1 - 2\sin 2\theta,\ \cos 2\theta = 0$$

$$\cos 2\theta = 0 \Rightarrow \theta = \frac{\pi}{4}, \text{ then}$$

$$a = 1 + 2\sin\frac{\pi}{2},\ b = 1 - 2\sin\frac{\pi}{2}$$

$\therefore \quad a = 3, b = -1$

Ex. 34. *If* $0 < x < \frac{\pi}{2}$ *and* $\sin^n x + \cos^n x \ge 1$, *then*

(a) $n \in [2, \infty)$ (b) $n \in (-\infty, 2]$

(c) $n \in [-1, 1]$ (d) None of these

Sol. (*b*) Since, $0 < x < \frac{\pi}{2}$

$\therefore \quad 0 < \sin x < 1$ and $0 < \cos x < 1$

when $x = 2$, $\sin^n x + \cos^n x = 1$

when $n > 2$, both $\sin^n x$ and $\cos^n x$ will decreases and hence $\sin^n x + \cos^n x < 1$.

when $n > 2$, both $\sin^n x$ and $\cos^n x$ will increase and hence $\sin^n x + \cos^n x > 1$.

Thus, $\sin^n x + \cos^n x \ge 1$ for $n \le 2$.

● ***Ex. 35.*** *If* $a = \sin\frac{\pi}{18}\sin\frac{5\pi}{18}\sin\frac{7\pi}{18}$, *and* x *is the solution the equation* $y = 2[x] + 2$ *and* $y = 3[x-2]$, *where* $[x]$ *denotes the integral part of* x, *then* a *is equal to*

(a) $[x]$ (b) $\frac{1}{[x]}$

(c) $2[x]$ (d) $[x]^2$

Sol. (*b*) $a = \sin\frac{\pi}{18}\sin\frac{5\pi}{18}\sin\frac{7\pi}{18}$

$$= \sin 10^\circ \sin 50^\circ \sin 70^\circ$$
$$= \frac{1}{2}[2\sin 70^\circ \sin 10^\circ]\sin 50^\circ$$
$$= \frac{1}{2}[\cos 60^\circ - \cos 80^\circ]\sin 50^\circ$$
$$= \frac{1}{4}\sin 50^\circ - \frac{1}{4}(2\cos 80^\circ \sin 50^\circ)$$
$$= \frac{1}{4}\sin 50^\circ - \frac{1}{4}(\sin 130^\circ - \sin 30^\circ)$$
$$= \frac{1}{4}\sin 50^\circ - \frac{1}{4}\sin 50^\circ + \frac{1}{4}\cdot\frac{1}{2} = \frac{1}{8}$$

$$y = 2[x] + 2 \text{ and } y = 3[x-2]$$
$$\Rightarrow \quad 2[x] + 2 = 3[x-2]$$
$$= 3[x] + 3[-2] \;\Rightarrow\; [x] = 8$$
$$\therefore \quad a = \frac{1}{[x]}$$

● ***Ex. 36.*** *If the mapping* $f(x) = ax + b$, $a < 0$ *and maps* $[-1, 1]$ *onto* $[0, 2]$, *then for all values of* θ, $A = \cos^2\theta + \sin^4\theta$ *is such that*

(a) $f\left(\frac{1}{4}\right) \le A \le f(0)$ (b) $f(0) \le A \le f(-2)$

(c) $f\left(\frac{1}{3}\right) \le A \le f(0)$ (d) $f(-1) < A \le f(-2)$

Sol. (*a*) Given, $f(x) = ax + b$

$\therefore \quad f'(x) = a$

Since, $a < 0$, $f(x)$ is a decreasing function

$\therefore \quad f(-1) = 2$ and $f(1) = 0$

$\Rightarrow \quad -a + b = 2$ and $a + b = 0$

$\therefore \quad a = -1$ and $b = 1$

Thus, $\quad f(x) = -x + 1$

Clearly, $\quad f(0) = 1,\ f\left(\frac{1}{4}\right) = \frac{3}{4},\ f(-2) = 3,$

$$f\left(\frac{1}{3}\right) = \frac{2}{3},\ f(-1) = 2$$

Also, $\quad A = \frac{1 + \cos 2\theta}{2} + \left(\frac{1 - \cos 2\theta}{2}\right)^2$

$$= \frac{1}{2} + \frac{1}{2}\cos 2\theta + \frac{1}{4} - \frac{1}{2}\cos 2\theta + \frac{1}{4}\cos^2 2\theta$$
$$= \frac{3}{4} + \frac{1}{4}\left(\frac{1 + \cos 4\theta}{2}\right) = \frac{7}{8} + \frac{1}{8}\cos 4\theta$$
$$\therefore \quad \frac{3}{4} \le A \le 1 \Rightarrow f\left(\frac{1}{4}\right) \le A \le f(0)$$

● ***Ex. 37.*** *The value of* $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}$ *is equal to*

(a) 1 (b) -1

(b) $\frac{1}{2}$ (c) $-\frac{1}{2}$

Sol. (*d*) $\cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{6\pi}{7}\right)$

$$= \operatorname{Re}\left\{e^{\frac{2\pi i}{7}} + e^{\frac{4\pi i}{7}} + e^{\frac{6\pi i}{7}}\right\}$$
$$= \frac{e^{\frac{2\pi i}{7}} + e^{\frac{4\pi i}{7}} + e^{\frac{6\pi i}{7}} + e^{\frac{-4\pi i}{7}} + e^{\frac{-4\pi i}{7}} + e^{\frac{-6\pi i}{7}}}{2}$$
$$= \frac{-1 + \left(1 + e^{\frac{2\pi i}{7}} + e^{\frac{4\pi i}{7}} + e^{\frac{6\pi i}{7}} + e^{\frac{-2\pi i}{7}} + e^{\frac{-4\pi i}{7}} + e^{\frac{-6\pi i}{7}}\right)}{2}$$
$$= \frac{-1 + (\text{Sum of seven roots of unity})}{2}$$
$$= \frac{-1 + 0}{2} = -\frac{1}{2}$$

● ***Ex. 38.*** *The number of integral value of* k *for which the equation* $7\cos x + 5\sin x = 2k + 1$ *has a solution is*

(a) 4 (b) 8

(c) 10 (d) 12

Sol. (*b*) Since, $-\sqrt{a^2 + b^2} \le a\sin x + b\cos x \le \sqrt{a^2 + b^2}$

$\therefore \quad -\sqrt{74} \le 7\cos x + 5\sin x \le \sqrt{74}$

So, $\quad -\sqrt{74} < 2k + 1 < \sqrt{74}$

Therefore, $\quad 2k + 1 = \pm 8, \pm 7, \pm 6, \ldots, \pm 1, 0$

So, $\quad k = -4, \pm 3, \pm 2, \pm 1, 0,$ so, 8 values of k.

● ***Ex. 39.*** *If* $y = \frac{\sin^4 x - \cos^4 x + \sin^2 x\cos^2 x}{\sin^4 x + \cos^4 x + \sin^2 x\cos^2 x}$, $x \in \left(0, \frac{\pi}{2}\right)$, *then*

(a) $-\frac{3}{2} \le y \le \frac{1}{2}$ (b) $1 \le y \le \frac{1}{2}$

(c) $-\frac{5}{3} \le y \le 1$ (d) None of these

Sol. (*d*) $y = \frac{\sin^4 x - \cos^4 x + \sin^2 x\cos^2 x}{\sin^4 x + \cos^4 x + \sin^2 x\cos^2 x}$

$$= \frac{(\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x) + \sin^2 x\cos^2 x}{(\sin^2 x + \cos^2 x)^2 - \sin^2 x\cos^2 x}$$

$$= \frac{-\cos 2x + \frac{1}{4}\sin^2 2x}{1 - \frac{1}{4}\sin^2 2x}$$

$$= \frac{-4\cos 2x + 1 - \cos^2 2x}{4 - 1 + \cos^2 2x}$$

$$= \frac{1 - 4\cos 2x - \cos^2 2x}{3 + \cos^2 2x}$$

$\Rightarrow \quad (1 + y)\cos^2 2x + 4\cos 2x + 3y - 1 = 0$

Since $\cos 2x$ is real, we have

$$16 - 4(3y - 1)(1 + y) > 0$$

or $\quad 3y^2 + 2y - 5 \le 0$

or $\quad (3y + 5)(y - 1) \le 0 \Rightarrow -\frac{5}{3} \le y \le 1$

But $y = 1$ implies $\cos 2x = -1$ i.e. $x = \frac{\pi}{2}$ which is not permissible.

- **Ex. 40.** *The distance between the two parallel lines is 1 unit. A pont 'A' is chosen to lie between the lines at a distance 'd' from one of them. Triangle ABC is equilateral with B on one line and C on the other parallel line. The length of the side of the equilateral triangle is*

(a) $\frac{2}{3}\sqrt{d^2 + d + 1}$ (b) $2\sqrt{\frac{d^2 - d + 1}{3}}$

(c) $2\sqrt{d^2 - d + 1}$ (d) $\sqrt{d^2 - d + 1}$

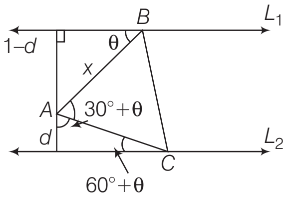
Sol. (*b*) From, figure

$$x\cos(\theta + 30^\circ) = d \quad \text{...(i)}$$

and $\quad x\sin\theta = 1 - d \quad$ ...(ii)

Dividing $\sqrt{3}\cot\theta = \frac{1 + d}{1 - d}$, squaring Eq. (ii) and putting the value of $\cot\theta$, we get

$$x^2 = \frac{1}{3}(4d^2 - 4d + 4)$$



- **Ex. 41.** *If $a\sin x + b\cos(x + \theta) + b\cos(x - \theta) = d$, then the minimum value of $|\cos\theta|$ is equal to*

(a) $\frac{1}{2|b|}\sqrt{d^2 - a^2}$ (b) $\frac{1}{2|a|}\sqrt{d^2 - a^2}$

(c) $\frac{1}{2|d|}\sqrt{d^2 - a^2}$ (d) None of these

Sol. (*a*) $a\sin x + b\cos(x + \theta) + b\cos(x - \theta) = d$

$\Rightarrow \quad a\sin x + 2b\cdot\cos x\cdot\cos\theta = d$

$\Rightarrow \quad |d| \le \sqrt{a^2 + 4b^2\cdot\cos^2\theta}$

$\Rightarrow \quad \frac{d^2 - a^2}{4b^2} \le \cos^2\theta \Rightarrow |\cos\theta| \ge \frac{\sqrt{d^2 - a^2}}{2|b|}$

- **Ex. 42.** *The set of values of $\lambda \in R$ such that $\tan^2\theta + \sec\theta = \lambda$ holds for some θ is*

(a) $(-\infty, 1]$ (b) $(-\infty, -1]$

(c) ϕ (d) $[1, \infty)$

Sol. (*d*) $\because \tan^2\theta + \sec\theta = \lambda$

$\Rightarrow \quad \sec^2\theta + \sec\theta - 1 - \lambda = 0$

$\therefore \quad \sec\theta = \frac{-1 \pm \sqrt{(4\lambda + 5)}}{2}$

for real $\sec\theta$, $\quad 4\lambda + 5 \ge 0$ i.e, $\lambda \ge -\frac{5}{4}$

But we know that

$$\sec\theta \le -1 \text{ and } \sec\theta \ge 1$$

$\therefore \quad \frac{-1 \pm \sqrt{(4\lambda + 5)}}{2} \le -1$ and $\frac{-1 \pm \sqrt{(4\lambda + 5)}}{2} \ge 1$

$\Rightarrow \quad -1 - \sqrt{(4\lambda + 5)} \le -2$ and $-1 + \sqrt{(4\lambda + 5)} \ge 2$

$\Rightarrow \quad \sqrt{(4\lambda + 5)} \ge 1$ and $\sqrt{(4\lambda + 5)} \ge 3$

$\Rightarrow \quad \sqrt{4\lambda + 5} \ge 3$

or $\quad 4\lambda + 5 \ge 9$ or $\lambda \ge 1$

$\therefore \lambda \in [1, \infty)$

- **Ex. 43.** *For $0 < \phi, \frac{\pi}{2}$, if $x = \sum_{n=0}^{\infty}\cos^{2n}\phi$, $y = \sum_{n=0}^{\infty}\sin^{2n}\phi$ and $z = \sum_{n=0}^{\infty}\cos^{2n}\phi\sin^{2n}\phi$, then*

(a) $xyz = xz + y$ (b) $xyz = xy + y$

(c) $xyz = x + y + z$ (d) $xyz = yz + x$

Sol. (*c*) We have,

$$x = \sum_{n=0}^{\infty}\cos^{2n}\phi = 1 + \cos^2\phi + \cos^4\phi + \ldots$$

$$= \frac{1}{1 - \cos^2\phi} = \frac{1}{\sin^2\phi}$$

Similarly, $\quad y = \frac{1}{1 - \sin^2\phi} = \frac{1}{\cos^2\phi}$

and $\quad z = \frac{1}{1 - \sin^2\phi\cos^2 x}$

$\therefore \quad z = \frac{1}{1 - \frac{1}{x}\cdot\frac{1}{y}} = \frac{xy}{xy - 1}$

$\Rightarrow \quad xyz = xy + z$

Ex. 44. *If* $\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha=1$, $\frac{x}{a}\cos\beta+\frac{y}{b}\sin\beta=1$ *and* $\frac{\cos\alpha\cos\beta}{a^2}+\frac{\sin\alpha\sin\beta}{b^2}=0$, *then*

(a) $\tan\alpha\,\tan\beta=\frac{b^2(x^2+a^2)}{a^2(y^2+b^2)}$

(b) $x^2+y^2=a^2+b^2$

(c) $\tan\alpha\,\tan\beta=\frac{a^2}{b^2}$

(c) None of the above

Sol. (*b*) $\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha=1$ ...(i)

$$\frac{x}{a}\cos\beta+\frac{y}{b}\sin\beta=1 \quad \text{...(ii)}$$

$$\frac{\cos\alpha\cdot\cos\beta}{a^2}+\frac{\sin\alpha\cdot\sin\beta}{b^2}=0 \quad \text{...(iii)}$$

From Eqs. (i) and (ii), we get

$$x=a\frac{\sin\alpha-\sin\beta}{\sin(\alpha-\beta)}$$

and $$y=b\frac{\cos\beta-\cos\alpha}{\sin(\alpha-\beta)}$$

Now, $$x^2+y^2=\frac{a^2(\sin\alpha-\sin\beta)^2+b^2(\cos\beta-\cos\alpha)^2}{\sin^2(\alpha-\beta)}$$

$$=\frac{a^2(\sin^2\alpha\sin^2\beta)+b^2(\cos^2\alpha+\cos^2\beta)-2(a^2\sin\alpha\sin\beta+b^2\cos\alpha\cos\beta)}{\sin^2(\alpha-\beta)}$$

$$=\frac{a^2(\sin^2\alpha+\sin^2\beta)+b^2(\cos^2\alpha+\cos^2\beta)}{\sin^2(\alpha-\beta)}$$

[from Eq. (iii)]

$$=a^2+b^2+\frac{a^2\{\sin^2\alpha+\sin^2\beta-\sin^2(\alpha-\beta)\}+b^2\{\cos^2\alpha+\cos^2\beta-\sin^2(\alpha-\beta)\}}{\sin^2(\alpha-\beta)}$$

$$=a^2+b^2+\frac{1}{\sin^2(\alpha-\beta)}\left[a^2\begin{pmatrix}\sin^2\alpha+\sin^2\beta-\sin^2\alpha\cos^2\beta\\ -\cos^2\alpha\sin^2\beta+2\sin\alpha\sin\beta\\ \cos\alpha\cos\beta\end{pmatrix}+b^2\begin{Bmatrix}\cos^2\alpha(1-\sin^2\beta)\\ +\cos^2\beta(1-\sin^2\alpha)\\ +2\sin\alpha\sin\beta\sin\alpha\\ \sin\beta\cos\alpha\cos\beta\end{Bmatrix}\right]$$

$$=a^2+b^2+\frac{2a^2\sin\alpha\sin\beta\cos(\alpha-\beta)+2b^2\cos\alpha\cos\beta\cos(\alpha-\beta)}{\sin^2(\alpha-\beta)}$$

$$=a^2+b^2+\frac{2a^2b^2\cos(\alpha-\beta)}{\sin^2(\alpha-\beta)}\left[\frac{\sin\alpha\sin\beta}{b^2}+\frac{\cos\alpha\cos\beta}{a^2}\right] \quad \text{[from Eq. (iii)]}$$

$$=a^2+b^2+0=a^2+b^2$$

Thus, $x^2+y^2=a^2+b^2$

Now, $x^2+y^2=a^2+b^2$

$\Rightarrow$ $x^2-a^2=-(y^2-b^2)$

$\Rightarrow$ $\frac{x^2-a^2}{y^2-b^2}=-1$

$\Rightarrow$ $\frac{b^2(x^2-a^2)}{a^2(y^2-b^2)}=-\frac{b^2}{a^2}$

$=\tan\alpha\,\tan\beta$ [from Eq. (iii)]

Ex. 45. *If* α,β,γ *are acute angles and* $\cos\theta=\sin\beta/\sin\alpha$, $\cos\phi=\sin\gamma/\sin\alpha$ *and* $\cos(\theta-\phi)=\sin\beta\sin\gamma$, *then the value of* $\tan^2\alpha-\tan^2\beta-\tan^2\gamma$ *is equal to*

(a) -1 (b) 0

(c) 1 (d) 2

Sol. (*b*) From the third relation we get

$$\cos\theta\cos\phi+\sin\theta\sin\phi=\sin\beta\sin\gamma$$

$\Rightarrow$ $\sin^2\theta\sin^2\phi=(\cos\theta\cos\phi-\sin\beta\sin\gamma)^2$

$\Rightarrow$ $\left(1-\frac{\sin^2\beta}{\sin^2\alpha}\right)\left(1-\frac{\sin^2\gamma}{\sin^2\alpha}\right)=\left(\frac{\sin\beta\sin\gamma}{\sin^2\alpha}-\sin\beta\sin\gamma\right)^2$

$\Rightarrow$ $(\sin^2\alpha-\sin^2\beta)(\sin^2\alpha-\sin^2\gamma)=\sin^2\beta\sin^2\gamma(1-\sin^2\alpha)^2$

$\Rightarrow$ $\sin^4\alpha(1-\sin^2\beta\sin^2\gamma)-\sin^2\alpha(\sin^2\beta+\sin^2\gamma-2\sin^2\beta\sin^2\gamma)=0$

$\therefore$ $\sin^2\alpha=\frac{\sin^2\beta+\sin^2\gamma-2\sin^2\beta\sin^2\gamma}{1-\sin^2\beta\sin^2\gamma}$

and $\cos^2\alpha=\frac{1-\sin^2\beta-\sin^2\gamma+\sin^2\beta\sin^2\gamma}{1-\sin^2\beta\sin^2\gamma}$

$\Rightarrow$ $\tan^2\alpha=\frac{\sin^2\beta-\sin^2\beta\sin^2\gamma+\sin^2\gamma-\sin^2\beta\sin^2\gamma}{\cos^2\beta-\sin^2\gamma(1-\sin^2\beta)}$

$$=\frac{\sin^2\beta\cos^2\gamma+\cos^2\beta\sin^2\gamma}{\cos^2\beta\cos^2\gamma}$$

$$=\tan^2\beta+\tan^2\gamma$$

$\Rightarrow$ $\tan^2\alpha-\tan^2\beta-\tan^2\gamma=0$

Ex. 46. *If* $\sqrt{2}\cos A = \cos B + \cos^3 B$, *and* $\sqrt{2}\sin A = \sin B - \sin^3 B$ *then* $\sin(A-B) =$

(a) ± 1 (b) $\pm\frac{1}{2}$

(c) $\pm\frac{1}{3}$ (d) $\pm\frac{1}{4}$

Sol. (*c*) $\sqrt{2}\cos A = \cos B + \cos^3 B$ ...(i)

and $\sqrt{2}\sin A = \sin B - \sin^3 B$ ...(ii)

$$\Rightarrow \sqrt{2}\sin A\cos B - \sqrt{2}\cos A\sin B$$
$$= (\sin B - \sin^3 B)\cos B - (\cos B + \cos^3 B)\sin B$$
$$= -\sin B\cos B$$
$$\Rightarrow \sin(A-B) = \frac{-1}{2\sqrt{2}}\sin 2B$$

Now squaring and adding Eqs. (i) and (ii), we get

$$2 = \cos^2 B + \sin^2 B + \cos^6 B + \sin^6 B + 2(\cos^4 B - \sin^4 B)$$
$$\Rightarrow 1 = (\cos^2 A + \sin^2 A)^3 - 3\cos^2 A\sin^2 A(\cos^2 A + \sin^2 A) + 2\cos 2B$$
$$\Rightarrow 1 = 1 - \left(\frac{3}{4}\right)\sin^2 2B + 2\cos 2B$$
$$\Rightarrow -3\sin^2 2B + 8\cos 2B = 0$$
$$\Rightarrow 3\cos^2 2B + 8\cos 2B - 3 = 0$$
$$\Rightarrow \cos 2B = \frac{1}{3}$$
$$\Rightarrow \sin 2B = \pm\frac{2\sqrt{2}}{3}$$
$$\therefore \sin(A-B) = \pm\frac{1}{3}$$

Ex. 47. *If* x_1 *and* x_2 *are two distinct roots of the equation* $a\cos x + b\sin x = c$, *then* $\tan\frac{x_1+x_2}{2}$ *is equal to*

(a) $\frac{a}{b}$ (b) $\frac{b}{a}$

(c) $\frac{c}{a}$ (d) $\frac{a}{c}$

Sol. (*b*) $a\cos x + b\sin x = c$

$$\Rightarrow \frac{a\left(1-\tan^2\frac{x}{2}\right)}{1+\tan^2\frac{x}{2}} + \frac{2b\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = c$$
$$\Rightarrow (c+a)\tan^2\frac{x}{2} - 2b\tan\frac{x}{2} + c - a = 0$$
$$\Rightarrow \tan\frac{x_1}{2} + \tan\frac{x_2}{2} = \frac{2b}{c+a},$$
$$\tan\frac{x_1}{2}\cdot\tan\frac{x_2}{2} = \frac{c-a}{c+a}$$

Thus, $$\tan\left(\frac{x_1+x_2}{2}\right) = \frac{\tan\frac{x_1}{2} + \tan\frac{x_2}{2}}{1-\tan\frac{x_1}{2}\tan\frac{x_2}{2}}$$
$$= \frac{\frac{2b}{c+a}}{1-\left(\frac{c-a}{c+a}\right)} = \frac{b}{a}$$

Ex. 48. *The minimum value of the function*
$f(x) = \frac{\sin x}{\sqrt{1-\cos^2 x}} + \frac{\cos x}{\sqrt{1-\sec^2 x}} + \frac{\tan x}{\sqrt{\sec^2 x - 1}} + \frac{\cot x}{\sqrt{\operatorname{cosec}^2 x - 1}}$ *whenever it is defined is*

(a) 4 (b) -2

(c) 0 (d) 2

Sol. (*b*) $$f(x) = \frac{\sin x}{\sqrt{1-\cos^2 x}} + \frac{\cos x}{\sqrt{1-\sin^2 x}} + \frac{\tan x}{\sqrt{\sec^2 x - 1}} + \frac{\cot x}{\sqrt{\operatorname{cosec}^2 x - 1}}$$
$$= \frac{\sin x}{|\sin x|} + \frac{\cos x}{|\cos x|} + \frac{\tan x}{|\tan x|} + \frac{\cot x}{|\cot x|}$$
$$= \begin{cases} 4, & x \in \text{1st quadrant} \\ -2, & x \in \text{2nd quadrant} \\ 0, & x \in \text{3rd quadrant} \\ -2, & x \in \text{4th quadrant} \end{cases}$$
$$f(x)_{\text{min.}} = -2$$

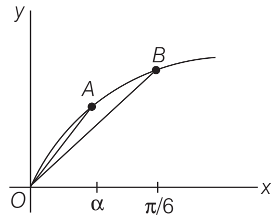
Ex. 49. *If* $0 < \alpha < \frac{\pi}{6}$, *then* $\alpha(\operatorname{cosec}\alpha)$ *is*

(a) less than $\frac{\pi}{6}$ (b) greater than $\frac{\pi}{6}$

(c) less than $\frac{\pi}{3}$ (d) greater than $\frac{\pi}{3}$

Sol. (*c*) In the graph of $y = \sin x$. Let

$$A \equiv (\alpha, \sin\alpha),\ B = \left(\frac{\pi}{6}, \sin\frac{\pi}{6}\right)$$



Clearly, slope of $OA >$ slope of OB, so

$$\frac{\sin\alpha}{\alpha} > \frac{\sin\frac{\pi}{6}}{\frac{\pi}{6}} = \frac{3}{\pi} \Rightarrow \frac{\alpha}{\sin\alpha} < \frac{\pi}{3}.$$

- **Ex. 50.** *In which one of the following intervals the inequality* $\sin x < \cos x < \tan x < \cot x$ *can hold good?*

(a) $\left(\frac{7\pi}{4}, 2\pi\right)$ (b) $\left(\frac{3\pi}{4}, \pi\right)$

(c) $\left(\frac{5\pi}{4}, \frac{3\pi}{2}\right)$ (d) $\left(0, \frac{\pi}{4}\right)$

Sol. (*d*) In the second quadrant, $\sin x < \cos x$ is false, as $\sin x$ is positive and $\cos x$ is negative.

In the fourth quadrant, $\cos x < \tan x$ is false, as $\cos x$ is positive and $\tan x$ is negative.

In the third quadrant, i.e. $\left(\frac{5\pi}{4}, \frac{3\pi}{2}\right)$ if $\tan x < \cot x$ then $\tan^2 x < 1$, which is false.

Now, $\sin x < \cos x$ is true in $\left(0, \frac{\pi}{4}\right)$ and $\tan x < \cot x$ is also true.

Further, $\cos x < \tan x$, as $\tan x = \frac{(\sin x)}{(\cos x)}$ and $\cos x < 1$.

JEE Type Solved Examples : More than One Correct Option Type Questions

- **Ex. 51.** *If* $x \in (0, \pi)$ *and* $\cos x + \sin x = \frac{1}{2}$, *then* $\tan x$ *is equal to*

(a) $\frac{4-\sqrt{7}}{3}$ (b) $\frac{4+\sqrt{7}}{3}$

(c) $\frac{-(4+\sqrt{7})}{3}$ (d) $\frac{-4+\sqrt{7}}{3}$

Sol. (*c,d*) Given, $\cos x + \sin x = \frac{1}{2}$

$$\Rightarrow \quad 1 + \sin 2x = \frac{1}{4}$$

$$\sin 2x = -\frac{3}{4} \quad \Rightarrow \quad 2x \in (\pi, 2\pi)$$

$$\Rightarrow \quad x \in \left(\frac{\pi}{2}, \pi\right) \quad \Rightarrow \quad \tan x < 0$$

$$\frac{2t}{1+t^2} = -\frac{3}{4} \quad \Rightarrow \quad 8t = -3 - 3t^2$$

$\Rightarrow 3t^2 + 8t + 3 = 0$, where $t = \tan x$

$$t = \frac{-8 \pm \sqrt{64-36}}{2\cdot 3};$$

$$t = \frac{-8 \pm \sqrt{28}}{2\cdot 3};$$

$$t = \frac{-(4+\sqrt{7})}{3}$$

or $$= \frac{-4+\sqrt{7}}{3}$$

- **Ex. 52.** *The value of the expression* $\tan\frac{\pi}{7} + 2\tan\frac{2\pi}{7} + 4\tan\frac{4\pi}{7} + 8\cot\frac{8\pi}{7}$ *is equal to*

(a) $\operatorname{cosec}\frac{2\pi}{7} + \cot\frac{2\pi}{7}$ (b) $\tan\frac{\pi}{14} - \cot\frac{\pi}{14}$

(c) $\dfrac{\sin\frac{2\pi}{7}}{1-\cos\frac{2\pi}{7}}$ (d) $\dfrac{1+\cos\frac{\pi}{7}+\cos\frac{2\pi}{7}}{\sin\frac{\pi}{7}+\sin\frac{2\pi}{7}}$

Sol. (*a,c,d*) $\tan\frac{\pi}{7} + 2\tan\frac{2\pi}{7} + 4\tan\frac{4\pi}{7} + 8\cot\frac{8\pi}{7} = \cot\frac{\pi}{7}$

$$\left[\tan\theta + 2\tan 2\theta + 4\tan 4\theta + 8\cot 8\theta = \cot\theta \ \text{ when } \theta = \frac{\pi}{7}\right]$$

(a) $\operatorname{cosec} 2\theta + \cot 2\theta = \dfrac{1+\cot 2\theta}{\sin 2\theta} = \cot\theta = \cot\frac{\pi}{7}$

(where, $\theta = \frac{\pi}{7}$)

(b) $\tan\frac{\pi}{14} - \cot\frac{\pi}{14} = -2\cot\frac{\pi}{7}$

(c) $\dfrac{\sin\frac{2\pi}{7}}{1-\cos\frac{2\pi}{7}} = \dfrac{2\sin\frac{\pi}{7}\cos\frac{\pi}{7}}{2\sin^2\frac{\pi}{7}} = \cot\frac{\pi}{7}$

(d) $\dfrac{\left(1+\cos\frac{2\pi}{7}\right)+\cos\frac{\pi}{7}}{2\sin\frac{\pi}{7}\cos\frac{\pi}{7}+\sin\frac{\pi}{7}} = \dfrac{2\cos^2\frac{\pi}{7}+\cos\frac{\pi}{7}}{2\sin\frac{\pi}{7}\left(\cos\frac{\pi}{7}+1\right)}$

$$= \cot\frac{\pi}{7}$$

Ex. 53. *Two parallel chords are drawn on the same side of the centre of a circle of radius R. It is found that they subtend and angle of θ and 2θ at the centre of the circle. The perpendicular distance between the chords is*

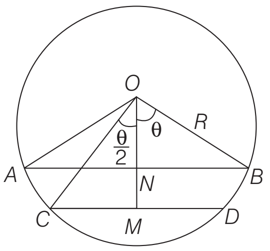
(a) $2R\sin\dfrac{3\theta}{2}\sin\dfrac{\theta}{2}$

(b) $\left(1-\cos\dfrac{\theta}{2}\right)\left(1+2\cos\dfrac{\theta}{2}\right)R$

(c) $\left(1+\cos\dfrac{\theta}{2}\right)\left(1-2\cos\dfrac{\theta}{2}\right)R$

(d) $2R\sin\dfrac{3\theta}{4}\sin\dfrac{\theta}{4}$

Sol. (b,d) $OM = p_1 = R\cos\dfrac{\theta}{2}$



$ON = p_2 = R\cos\theta$

$MN = p_1 - p_2 = R\left(\cos\dfrac{\theta}{2} - \cos\theta\right)$

$= R\,2\sin\dfrac{3\theta}{4}\sin\dfrac{\theta}{4}$ (d)

Again convert $\cos\theta = 2\cos^2\dfrac{\theta}{2} - 1$ and factorise, we get

$= R(1-\cos\theta/2)^2\,(1+2\cos\theta/2)$

Ex. 54. *If $2x$ and $2y$ are complementary angles and $\tan(x+2y) = 2$, then which of the following is(are) correct ?*

(a) $\sin(x+y) = \dfrac{1}{2}$ (b) $\tan(x-y) = \dfrac{1}{7}$

(c) $\cot x + \cot y = 5$ (d) $\tan x\tan y = 6$

Sol. (b,c) We have, $2x + 2y = \dfrac{\pi}{2}$

$\Rightarrow\quad x + y = \dfrac{\pi}{4} \Rightarrow \sin(x+y) = \dfrac{1}{\sqrt{2}}$

Also, $y = \left(\dfrac{\pi}{4} - x\right)$,

So, $\tan(x+2y) = \tan\left(x + \dfrac{\pi}{2} - 2x\right)$

$= \tan\left(\dfrac{\pi}{2} - x\right) = \cot x$

$\therefore\quad 2 = \cot x \Rightarrow \tan x = \dfrac{1}{2}$

Similarly, $x = \left(\dfrac{\pi}{4} - y\right)$

So, $\tan(x+2y) = \tan\left(\dfrac{\pi}{4} + y\right) = \dfrac{1+\tan y}{1-\tan y}$

$\Rightarrow\quad 2 = \dfrac{1+\tan y}{1-\tan y} \Rightarrow \tan y = \dfrac{1}{3}$

$\cot y = 3$

Also, $\tan(x-y) - \dfrac{\tan x - \tan y}{1+\tan x\tan y} = \dfrac{\frac{1}{2}-\frac{1}{3}}{1+\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)}$

$= \left(\dfrac{1}{6}\right)\left(\dfrac{6}{7}\right) = \dfrac{1}{7}$

Now, verify alternatives.

Ex. 55. *If $2\cos\theta + 2\sqrt{2} = 3\sec\theta$, where $\theta \in (0, 2\pi)$, then which of the following can be correct?*

(a) $\cos\theta = \dfrac{1}{\sqrt{2}}$ (b) $\tan\theta = 1$

(c) $\sin\theta = -\dfrac{1}{\sqrt{2}}$ (d) $\cot\theta = -1$

Sol. (a,b,c,d) $2\cos\theta + 2\sqrt{2} = 3\sec\theta$

$\therefore\quad 2\cos^2\theta + 2\sqrt{2}\cos\theta - 3 = 0$

$\cos\theta = \dfrac{-2\sqrt{2} \pm \sqrt{32}}{4} = \dfrac{-2\sqrt{2} \pm 4\sqrt{2}}{4}$

$\therefore\quad \cos\theta = \dfrac{1}{\sqrt{2}}$ or $\cos\theta = -\dfrac{3}{\sqrt{2}}$ (rejected)

$\therefore\quad \theta = \dfrac{\pi}{4}$ or $\dfrac{7\pi}{4} \Rightarrow \sin\theta = -\dfrac{1}{\sqrt{2}}$;

$\cot\theta = -1$; $\tan\theta = 1$

Ex. 56. *The value of x in $(0, \pi/2)$ satisfying the equation,* $\dfrac{\sqrt{3}-1}{\sin x} + \dfrac{\sqrt{3}+1}{\cos x} = 4\sqrt{2}$ *is*

(a) $\dfrac{\pi}{12}$ (b) $\dfrac{5\pi}{12}$

(c) $\dfrac{7\pi}{24}$ (d) $\dfrac{11\pi}{36}$

Sol. (a,d) $\dfrac{\sqrt{3}-1}{2\sqrt{2}\sin x} + \dfrac{\sqrt{3}+1}{2\sqrt{2}\cos x} = 2$

$\sin\dfrac{\pi}{12}\cos x + \cos\dfrac{\pi}{12}\sin x = \sin 2x$

$\sin 2x = \sin\left(x + \dfrac{\pi}{12}\right)$

$\therefore\quad 2x = x + \dfrac{\pi}{12}$

or $\quad 2x = \pi - x - \dfrac{\pi}{12}$

$x = \dfrac{\pi}{12}$

or $\quad 3x = \dfrac{11\pi}{12}$

$\therefore\quad x = \dfrac{\pi}{12}$ or $\dfrac{11\pi}{36}$

Ex. 57. *Which of the following statements are always correct? (where, Q denotes the set of rationals)*

(a) $\cos 2\theta \in Q$ and $\sin 2\theta \in Q \Rightarrow \tan\theta \in Q$ (if defined)
(b) $\tan\theta \in Q \Rightarrow \sin 2\theta$, $\cos 2\theta$ and $\tan 2\theta \in Q$ (if defined)
(c) If $\sin\theta \in Q$ and $\cos\theta \in Q \Rightarrow \tan 3\theta \in Q$ (if defined)
(d) If $\sin\theta \in Q \Rightarrow \cos 3\theta \in Q$

Sol. (*a,b,c*)

(a) $\tan\theta = \dfrac{1-\cos 2\theta}{\sin 2\theta} \Rightarrow$ (a) is correct

(b) $\sin 2\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$; $\cos 2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$;

$\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta} \Rightarrow$ (b) is correct

(c) $\tan 3\theta = \dfrac{\sin 3\theta}{\cos 3\theta} \Rightarrow$ (c) is correct

(d) $\sin\theta = \dfrac{1}{3}$ which is rational but

$\cos 3\theta = \cos\theta(4\cos^2\theta - 3)$ which is irrational $\Rightarrow$ (d) is incorrect.

Ex. 58. *In* ΔABC, $\tan B + \tan C = 5$ *and* $\tan A \tan C = 3$, *then*

(a) ΔABC is an acute angled triangle
(b) ΔABC is an obtuse angled triangle
(c) sum of all possible values of $\tan A$ is 10
(d) sum of all possible values of $\tan A$ is 9

Sol. (*a,c*) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

$$\Rightarrow \tan A + 5 = 3\tan B$$
$$\Rightarrow 5 + \tan A = 3(5 - \tan C)$$
$$\Rightarrow 5 + \tan A = 15 - \frac{9}{\tan A}$$
$$\Rightarrow \tan^2 A - 10\tan A + 9 = 0$$
$$\Rightarrow \tan A = 1 \text{ or } \tan A = 9$$

$\Rightarrow \tan B$ and $\tan C$ are 2, 3 or $\dfrac{14}{3}, \dfrac{1}{3}$, respectively

$\Rightarrow \Delta ABC$ is always on acute angled triangle and sum of all possible values of $\tan A$ is 10.

Ex. 59. $(m+2)\sin\theta + (2m-1)\cos\theta = 2m+1$, *if*

(a) $\tan\theta = \dfrac{3}{4}$ (b) $\tan\theta = \dfrac{4}{3}$

(c) $\tan\theta = \dfrac{2m}{(m^2-1)}$ (d) $\tan\theta = \dfrac{2m}{(m^2+1)}$

Sol. (*b, c*) The given relation can be written as

$(m+2)\tan\theta + (2m-1) = (2m+1)\sec\theta$

$$\Rightarrow (m+2)^2\tan^2\theta + 2(m+2)(2m-1)\tan\theta + (2m-1)^2 = (2m+1)^2(1+\tan^2\theta)$$

$$\Rightarrow [(m+2^2) - (2m+1)^2]\tan^2\theta + 2(m+2)\tan\theta + (2m-1)^2 - (2m+1)^2 = 0$$

$$\Rightarrow 3(1-m^2)\tan^2\theta + (4m^2+6m-4)\tan\theta - 8m = 0$$

$$\Rightarrow (3\tan\theta - 4)[(1-m)^2\tan\theta + 2m] = 0$$

which is true if $\tan\theta = \dfrac{4}{3}$ or $\tan\theta = \dfrac{2m}{(m^2-1)}$.

Ex. 60. *If* $x\cos\alpha + y\sin\alpha = x\cos\beta + y\sin\beta = 2a\left(0 < \alpha, \beta < \dfrac{\pi}{2}\right)$, *then*

(a) $\cos\alpha + \cos\beta = \dfrac{4ax}{x^2+y^2}$

(b) $\cos\alpha\cos\beta = \dfrac{4a^2-y^2}{x^2+y^2}$

(c) $\sin\alpha + \sin\beta = \dfrac{4ay}{x^2+y^2}$

(d) $\sin\alpha\sin\beta = \dfrac{4a^2-x^2}{x^2+y^2}$

Sol. (*a, b, c, d*) We find out the given relations that α and β are the roots of the equation

$x\cos\theta + y\sin\theta = 2a$

$$\Rightarrow (x\cos\theta - 2a)^2 = (-y\sin\theta)^2$$

$$\Rightarrow x^2\cos^2\theta - 4ax\cos\theta + 4a^2 = y^2\sin^2\theta = y^2(1-\cos^2\theta)$$

$$\Rightarrow (x^2+y^2)\cos^2\theta - 4ax\cos\theta + 4a^2 - y^2 = 0$$

which, being quadratic in $\cos\theta$, has two roots $\cos\alpha$ and $\cos\beta$, such that

$$\cos\alpha + \cos\beta = \frac{4ax}{x^2+y^2}$$

and $$\cos\alpha\cos\beta = \frac{4a^2-y^2}{x^2+y^2}$$

Similarly, we can write (1) as a quadratic in $\sin\theta$, giving two values $\sin\alpha$ and $\sin\beta$, such that

$$\sin\alpha + \sin\beta = \frac{4ay}{x^2+y^2}$$

and $$\sin\alpha\sin\beta = \frac{4a^2-x^2}{x^2+y^2}.$$

Ex. 61. *Let* $y = \sin^2 x + \cos^4 x$. *Then, for all real x*

(a) the maximum value of y is 2
(b) the minimum value of y is $\dfrac{3}{4}$
(c) $y \le 1$
(d) $y \ge \dfrac{1}{4}$

Sol. (*b, c*) $y = \cos^4 x - \cos^2 x + 1$

$$= \left(\cos^2 x - \frac{1}{2}\right)^2 + \frac{3}{4}$$

$\therefore y_{\min} = \frac{3}{4}$ and y is maximum when $\left(\cos^2 x - \frac{1}{2}\right)^2$ is then maximum

$$\therefore \quad y_{\max} = \frac{1}{4} + \frac{3}{4} = 1$$

Ex. 62. *If in ΔABC, $\tan A + \tan B + \tan C = 6$ and $\tan A \tan B = 2$, then $\sin^2 A : \sin^2 B : \sin^2 C$ is*

(a) 8 : 9 : 5 (b) 8 : 5 : 9
(c) 5 : 9 : 5 (d) 5 : 8 : 5

Sol. (b, c) $\tan A + \tan B + \tan C = 6$ ...(i)

$\Rightarrow \tan A \tan B \tan C = 6$

$$2 \tan C = 6$$

$$\therefore \quad \tan C = 3$$

$$\therefore \quad \sin^2 C = \frac{\tan^2 C}{1 + \tan^2 C} = \frac{9}{1+9} = \frac{9}{10}$$

From Eq. (i), $\tan A + \tan B = 3$ and $\tan A \tan B = 2$

$\tan A - \tan B$

$$= \pm \sqrt{\{(\tan A + \tan B)^2 - 4 \tan A \tan B)\}}$$

$$= \pm 1$$

we get, $\tan A = 2, 1$ and $\tan B = 1, 2$

$$\therefore \quad \sin^2 A = \frac{4}{1+4}, \frac{1}{1+1} \text{ and } \sin^2 B = \frac{1}{1+1}, \frac{4}{1+4}$$

$$\Rightarrow \quad \sin^2 A = \frac{8}{10}, \frac{5}{10} \text{ and } \sin^2 B = \frac{5}{10}, \frac{8}{10}$$

$$\therefore \quad \sin^2 A : \sin^2 B : \sin^2 C = 8:5:9 \text{ or } 5:8:9$$

Ex. 63. *If $0 \le x, y \le 180°$ and $\sin(x-y) = \cos(x+y) = \frac{1}{2}$, then the values of x and y are given by*

(a) $x = 45°, y = 15°$ (b) $x = 45°, y = 135°$
(c) $x = 165°, y = 15°$ (d) $x = 165°, y = 135°$

Sol. (a, d) $\sin(x - y) = \frac{1}{2} \Rightarrow x - y = 30°$ or $150°$ (1)

and $\cos(x + 1) = \frac{1}{2} \Rightarrow x + y = 60°$ or $300°$ (2)

Since x and y lie between 0° and 180°, (1) and (2) are simultaneously true when $x = 45°$, $y = 15°$, or $x = 165°$, $y = 135°$. But, for the values given by (b) or (c), (1) and (2) do not hold simultaneously.

Ex. 64. *If $\sin\alpha + \sin\beta = l$, $\cos\alpha \cos\beta = m$ and $\tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right) = n (\ne 1)$, then*

(a) $\cos(\alpha - \beta) = \frac{l^2 + m^2 - 2}{2}$

(b) $\cos(\alpha + \beta) = \frac{m^2 - l^2}{m^2 + l^2}$

(c) $\frac{1+n}{1-n} = \frac{l^2 + m^2}{2n}$

(d) $\alpha + \beta = \frac{\pi}{2}$ if $l = m$

Sol. (a, b, c, d) Now, $l^2 = \sin^2\alpha + \sin^2\beta + 2\sin\alpha\sin\beta$ and $m^2 = \cos^2\alpha + \cos^2\beta + 2\cos\alpha\cos\beta$

$$2\cos(\alpha - \beta) = l^2 + m^2 - 2 \quad \text{(by adding)}$$

$$\Rightarrow \quad 2\cos 2\alpha + \cos 2\beta) = m^2 - l^2 \quad \text{(by subtracting)}$$

$$\Rightarrow \quad 2\cos(\alpha+\beta)\cos(\alpha-\beta) + 2\cos(\alpha+\beta) = m^2 - l^2$$

$$\Rightarrow \quad \cos(\alpha+\beta) = \frac{m^2 - l^2}{m^2 + l^2}.$$

Ex. 65. *Let $f(x) = ab \sin x + b\sqrt{1-a^2} \cos x + c$, where $|a| < 1, b > 0$ then*

(a) maximum value of $f(x)$ if b is $c = 0$
(b) difference of maximum and minimum values of $f(x)$ is $2b$
(c) $f(x) = c$ if $x = -\cos^{-1} a$
(d) $f(x) = c$ if $x = \cos^{-1} a$

Sol. (a, b, c) $f(x) = ab \sin x + b\sqrt{1-a^2} \cos x + c$, where $|a| < 1, b < 0$

$$f(x) = \sqrt{a^2b^2 + b^2 - b^2a^2} \sin(x + \alpha) + c$$

$$= b\sin(x+\alpha) + c, \text{ where } \tan\alpha = \frac{b\sqrt{1-a^2}}{ab} = \frac{\sqrt{1-a^2}}{a}$$

$$= b\cos(x - \alpha) + c, \text{ where } \tan\alpha = \frac{ab}{b\sqrt{1-a^2}}$$

$$= \frac{a}{\sqrt{1-a^2}}$$

$$f(x)_{\max} - f(x)_{\min} = c + b - (c - b) = 2b$$

$$f(x) = c \text{ if } x + \alpha = 0$$

or $\quad x = -\alpha \quad$ or $\quad x = -\cos^{-1} a$

Ex. 66. *If $(x - a)\cos\theta + y\sin\theta = (x-a)\cos\phi + y\sin\phi = a$ and $\tan\left(\frac{\theta}{2}\right) - \tan\left(\frac{\phi}{2}\right) = 2b$, then*

(a) $y^2 = 2ax - (1 - b^2)x^2$ (b) $\tan\frac{\theta}{2} = \frac{1}{x}(y + bx)$

(c) $y^2 = 2bx - (1 - a^2)x^2$ (d) $\tan\frac{\phi}{2} = \frac{1}{x}(y - bx)$

Sol. (a, b) Let, $\tan\left(\frac{\theta}{2}\right) = \alpha$ and $\tan\left(\frac{\phi}{2}\right) = \beta$, so that $\alpha - \beta = 2b$.

$$\text{Also,} \quad \cos\theta = \frac{1 - \tan^2\left(\frac{\theta}{2}\right)}{1 + \tan^2\left(\frac{\theta}{2}\right)} = \frac{1-\alpha^2}{1+\alpha^2}$$

And $\quad \sin\theta = \dfrac{2\tan\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)} = \dfrac{2\alpha}{1+\alpha^2}$

Similarly, $\cos\phi = \dfrac{1-\beta^2}{1+\beta^2}$ and $\sin\phi = \dfrac{2\beta}{1+\beta^2}$

Therefore, we have from the given relations

$$(x-a)\frac{1-\alpha^2}{1+\alpha^2} + y\left(\frac{2\alpha}{1+\alpha^2}\right) = a$$

$\Rightarrow \quad x\alpha^2 - 2y\alpha + 2a - x = 0$

Similarly $\quad x\beta^2 - 2y\beta + 2a = 0$

We see that α and β are roots of the equation

$$xz^2 - 2yz + 2a - x = 0,$$

So that $\quad \alpha+\beta = \dfrac{2y}{x}$ and $\alpha\beta = \dfrac{(2a-x)}{x}$.

Now, from $(\alpha+\beta)^2 = (\alpha-\beta)^2 + 4\alpha\beta$, we get

$\Rightarrow \quad \left(\dfrac{2y}{x}\right)^2 = (2b)^2 + \dfrac{4(2a-x)}{x}$

$\Rightarrow \quad y^2 = 2ax - (1-b^2)x^2$

Also, from $\alpha+\beta = \dfrac{2y}{x}$ and $\alpha-\beta = 2b$, we get

$$\alpha = \frac{y}{x} + b \text{ and } \beta = \frac{y}{x} - b$$

$\Rightarrow \quad \tan\dfrac{\theta}{2} = \dfrac{1}{x}(y+bx)$

and $\quad \tan\dfrac{\phi}{2} = \dfrac{1}{x}(y-bx)$

- **Ex. 67.** *If* $\cos(\beta-\gamma)+\cos(\gamma-\alpha)+\cos(\alpha-\beta) = -\dfrac{3}{2}$ *then*

(a) $\Sigma\cos\alpha = 0$ (b) $\Sigma\sin\alpha = 0$
(c) $\Sigma\cos\alpha\sin\alpha = 0$ (d) $\Sigma(\cos\alpha+\sin\alpha) = 0$

Sol. (*a, b, d*) The given expression can be written as

$$2[\cos\beta\cos\gamma + \cos\gamma\cos\alpha + \cos\alpha\cos\beta] + 2[\sin\beta\sin\gamma + \sin\gamma\sin\alpha + \sin\alpha\sin\beta] + (\sin^2\alpha+\cos^2\alpha) + (\sin^2\beta+\cos^2\beta) + (\sin^2\gamma+\cos^2\gamma) = 0$$

$\Rightarrow (\cos\alpha+\cos\beta+\cos\gamma)^2 + (\sin\alpha+\sin\beta+\sin\gamma)^2 = 0$

$\Rightarrow \quad \Sigma\cos\alpha = 0$ and $\Sigma\sin\alpha = 0$

$\Rightarrow \quad \Sigma(\cos\alpha+\sin\alpha) = 0$

JEE Type Solved Examples : Statement I and II Type Questions

- **Ex. 68.** **Statement I** $\tan 5\theta - \tan 3\theta - \tan 2\theta = \tan 5\theta \tan 3\theta \tan 2\theta$.

Statement II $x = y + z$
$\Rightarrow \tan x - \tan y - \tan z = \tan x \tan y \tan z$

(a) A (b) B
(c) C (d) D

Sol. (*a*) $\because 5\theta = 3\theta + 2\theta$

$\Rightarrow \quad \tan 5\theta = \tan(3\theta+2\theta) = \dfrac{\tan 3\theta + \tan 2\theta}{1-\tan 3\theta\tan 2\theta}$

$\Rightarrow \tan 5\theta - \tan 5\theta\tan 3\theta\tan 2\theta = \tan 3\theta + \tan 2\theta$

$\Rightarrow \quad \tan 5\theta - \tan 3\theta - \tan 2\theta = \tan 5\theta\tan 3\theta\tan 2\theta$

- **Ex. 69.** **Statement I** *The maximum value of* $\sin\theta+\cos\theta$ *is 2.*

Statement II *The maximum value of* $\sin\theta$ *is 1 and that of* $\cos\theta$ *is also 1.*

(a) A (b) B
(c) C (d) D

Sol. (*d*) $\because -\sqrt{2} \le \sin\theta + \cos\theta \le \sqrt{2}$

$\therefore$ Maximum value of $\sin\theta+\cos\theta$ is $\sqrt{2}$

But maximum value of $\sin\theta$ is 1 and that of $\cos\theta$ is also 1 which is always true.

- **Ex. 70.** **Statement I** *If* $a, b, c \in R$ *and not all equal, then* $\sec\theta = \dfrac{(bc+ca+ab)}{(a^2+b^2+c^2)}$,

Statement II $\sec\theta \le -1$ *and* $\sec\theta \ge 1$

(a) A (b) B
(c) C (d) D

Sol. (*d*) $\because a^2+b^2+c^2-ab-bc-ca$

$$= \frac{1}{2}\{(a-b)^2+(b-c)^2+(c-a)^2\} > 0$$

$\Rightarrow \quad a^2+b^2+c^2 > ab+bc+ca$

or $\quad \dfrac{ab+bc+ca}{a^2+b^2+c^2} < 1$

$\Rightarrow \quad \sec\theta < 1$, which is false.

- **Ex. 71.** **Statement I** $\prod_{r=1}^{n}(1+\sec 2^r\theta) = \tan 2^n\theta\cot\theta$

Statement II $\prod_{r=1}^{n}\cos(2^{r-1}\theta) = \dfrac{\sin(2^n\theta)}{2^n\sin\theta}$

(a) A (b) B
(c) C (d) D

Sol. (*a*) $\because \prod_{r=1}^{n}(1+\sec 2^r\theta)=\dfrac{\prod_{r=1}^{n}(1+\cos 2^r\theta)}{\prod_{r=1}^{n}\cos 2^r\theta}$

$$=\frac{\prod_{r=1}^{n}2\cos^2(2^{r-1}\theta)}{\prod_{r=1}^{n}\cos(2^r\theta)}$$

$$=\frac{2^n\cdot\prod_{r=1}^{n}\cos(2^{r-1}\theta)\prod_{r=1}^{n}\cos(2^{r-1}\theta)}{\dfrac{\cos(2^n\theta)}{\cos\theta}\prod_{r=1}^{n}\cos(2^{r-1}\theta)}$$

$$=\frac{\dfrac{2^n\cdot\sin(2^n\theta)}{2^n\sin\theta}\cdot\cos\theta}{\cos(2^n\theta)}$$

$$=\tan(2^n\theta)\cdot\cot\theta$$

● ***Ex. 72.*** **Statement I** $\cos 36^\circ>\sin 36^\circ$

Statement II $\cos 36^\circ>\tan 36^\circ$

(a) A (b) B

(c) C (d) D

Sol. (*b*) Since, $\cos\theta>\sin\theta$ for $0\le\theta\le\dfrac{\pi}{4}$

So, Statement I is true.

Now, $\cos 36^\circ>\tan 36^\circ$

$\Rightarrow$ $\cos 36^\circ>\dfrac{\sin 36^\circ}{\cos 36^\circ}$

$\Rightarrow$ $\cos^2 36^\circ>\sin 36^\circ$

$\Rightarrow$ $1+\cos 72^\circ>2\sin 36^\circ=2\sin(30^\circ+6^\circ)$

$\Rightarrow$ $1+2\sin 9^\circ\cos 9^\circ>\cos 6^\circ+2\cos 30^\circ\sin 6^\circ$

which is true

● ***Ex. 73.*** **Statement I** $\cos^3\alpha+\cos^3\left(\alpha+\dfrac{2\pi}{3}\right)+\cos^3\left(\alpha+\dfrac{4\pi}{3}\right)=3\cos\alpha\cos\left(\alpha+\dfrac{2\pi}{3}\right)\cos\left(\alpha+\dfrac{4\pi}{3}\right)$

Statement II *If* $a+b+c=0\Leftrightarrow a^3+b^3+c^3=3abc$

(a) A (b) B

(c) C (d) D

Sol. (*a*) $\because$ $\cos\alpha+\cos\left(\alpha+\dfrac{2\pi}{3}\right)+\cos\left(\alpha+\dfrac{4\pi}{3}\right)$

$$=\cos\alpha+2\cos(\alpha+\pi)\cos\frac{\pi}{3}$$

$$=\cos\alpha+(-2\cos\alpha)\left(\frac{1}{2}\right)=0$$

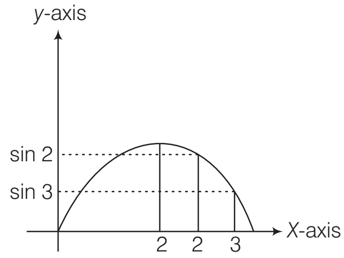
$\therefore \cos^3\alpha+\cos^3\left(\alpha+\dfrac{2\pi}{3}\right)+\cos^3\left(\alpha+\dfrac{4\pi}{3}\right)$

$$=3\cos\alpha\cos\left(\alpha+\frac{2\pi}{3}\right)\cos\left(\alpha+\frac{4\pi}{3}\right)$$

● ***Ex. 74.*** **Statement I** $\sin 2>\sin 3$

Statement II *If* $x, y\in\left(\dfrac{\pi}{2},\pi\right)$, $x<y$, *then* $\sin x>\sin y$

Sol. (*a*)



● ***Ex. 75.*** *Let* $\alpha,\beta,\gamma>0$ *and* $\alpha+\beta+\gamma=\dfrac{\pi}{2}$

Statement I $\left|\tan\alpha\tan\beta-\dfrac{a!}{6}\right|+\left|\tan\beta\tan\gamma-\dfrac{b!}{2}\right|+\left|\tan\gamma\tan\alpha-\dfrac{c!}{3}\right|\le 0$, *where* $n!=1.2\ldots.n$, *then* $\tan\alpha\tan\beta$, $\tan\beta\tan\gamma$, $\tan\gamma\tan\alpha$ *are in AP.*

Statement II $\tan\alpha\tan\beta+\tan\beta\tan\gamma+\tan\gamma\tan\alpha=1$

Sol. (*d*) Statement II $\alpha+\beta=\dfrac{\pi}{2}-\gamma$

$$\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\frac{1}{\tan\gamma}$$

$\Rightarrow$ $\Sigma\tan\alpha\tan\beta=1$

$\therefore$ Statement II is true.

Statement I $\tan\alpha\tan\beta=\dfrac{a!}{6}$,

$\tan\beta\tan\gamma=\dfrac{b!}{2}$

and $\tan\alpha\tan\beta=\dfrac{c!}{3}$

$$\frac{a!}{6}+\frac{b!}{2}+\frac{c!}{3}=1$$

$\Rightarrow$ $a!=1\ b!=1\ c!=1$

$\Rightarrow\tan\alpha\tan\beta$, $\tan\gamma\tan\alpha$ and $\tan\beta\tan\gamma$ are not in AP.

$\therefore$ Statement I is false.

Hence, (d) is the correct answer.

● ***Ex. 76.*** **Statement I** *The triangle so obtained is an equilateral triangle.*

Statement II *If roots of the equation be* $\tan A$, $\tan B$ *and* $\tan C$, *then* $\tan A+\tan B+\tan C=3\sqrt{3}$

Sol. (*b*) $\tan A + \tan B + \tan C = 3\sqrt{3}$

and $\tan A \tan B \tan C = 3$

$\therefore$ $\tan A + \tan B + \tan C \neq \tan A \tan B \tan C$

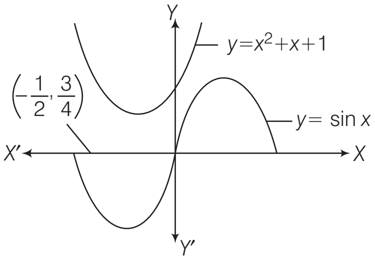
$\Rightarrow$ triangle does not exists.

● **Ex. 77.** *Let us define the function* $f(x) = x^2 + x + 1$

Statement I *The equation* $\sin x = f(x)$ *has no solution.*

Statement II *The curve* $y = \sin x$ *and* $y = f(x)$ *do not intersect each other when graph is observed.*

Sol. (*a*) Let $y = \sin x$ and $y = x^2 + x + 1$



Since, $-1 \leq \sin x \leq 1$ and $y = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$

It is clear from the graph that no two curves intersect each other.

JEE Type Solved Examples : Passage Based Questions

Passage I

(Ex. Nos. 78 to 80)

Consider, $f(x) = (x + 2a)(x + a - 4)\ (a \in R)$,

$g(x) = k(x^2 + x) + 3k + x\ (k \in R)$ and

$h(x) = (1 - \sin\theta)x^2 + 2(1 - \sin\theta)x - 3\sin\theta$

$\left(\theta \in R - (4n + 1)\frac{\pi}{2}, n \in I\right)$

● **Ex. 78.** *If* $f(x) < 0$ *for* $-1 \leq x \leq 1$, *then 'a' satisfies*

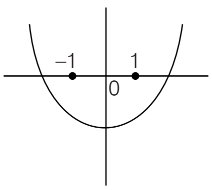
(a) $\frac{1}{2} < a < 3$ (b) $-\frac{1}{2} < a < \frac{1}{2}$

(c) $-3 < a < -\frac{1}{2}$ (d) $-3 < a < \frac{1}{2}$

Sol. (*a*) Given, $f(x) = (x + 2a)(x + a - 4)$

$= x^2 + (3a - 4)x + 2a(a - 4)$.

$\left.\begin{array}{r} f(-1) < 0 \\ f(1) < 0 \end{array}\right\} n$



On solving, we get $\frac{1}{2} < a < 3$

● **Ex. 79.** *If* $g(x) > -3$ *for all real x, then the values of k are given by*

(a) $-1 < k < \frac{1}{11}$ (b) $-1 < k < 0$

(c) $0 < k < \frac{1}{11}$ (d) $k < \frac{1}{11}$

Sol. (*d*) $g(x) = k(x^2 + x) + 3k + x > -3\ \forall\ x$

$\Rightarrow$ $k(x^2 + x) + 3k + x + 3 > 0\ \forall\ x$

or $kx^2 + (k + 1)x + (3k + 3) > 0\ \forall\ x$

$\left.\begin{array}{r} k > 0 \\ D < 0 \end{array}\right\} n$

Here, $D = (k + 1)^2 - 4k.3(k + 1) < 0$

$\Rightarrow$ $k^2 + 2k + 1 - 12k^2 - 12 < 0$

$\Rightarrow$ $11k^2 + 10k - 1 > 0$

$\Rightarrow$ $(k + 1)(11k - 1) > 0$

$\Rightarrow$ $k < -1$ or $k > \frac{1}{11}$

$\Rightarrow$ $k > \frac{1}{11}$ $(\because k > 0)$

● **Ex. 80.** *If the quadratic equation* $h(x) = 0$ *has both roots complex, then* θ *belongs to*

(a) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (b) $\left(0, \frac{3\pi}{2}\right)$

(c) $\left(\frac{\pi}{6}, \frac{7\pi}{6}\right)$ (d) $\left(\frac{7\pi}{6}, \frac{11\pi}{6}\right)$

Sol. (*d*) Given, $(1 - \sin\theta)x^2 + 2(1 - \sin\theta)x - 3\sin\theta = 0$ has both roots complex, then $D < 0$

$(1 - \sin\theta)(1 + 2\sin\theta) < 0$

$\underbrace{(\sin\theta - 1)}_{(-)\text{ ve number}}(2\sin\theta + 1) > 0$

$\Rightarrow$ $2\sin\theta + 1 < 0$

$\sin\theta < -\frac{1}{2}$

$\Rightarrow$ $\theta \in \left(\frac{7\pi}{6}, \frac{11\pi}{6}\right)$

Passage II

(Ex. Nos. 81 to 83)

Let $f(\theta) = \sin\theta - \cos^2\theta - 1$, where $\theta \in R$ and $m \le f(\theta) \le M$.

- **Ex. 81.** *Let N denotes the number of solution of the equation $f(\theta) = 0$ in $[0, 4\pi]$ then the value of*

$\log_{\sqrt{m^2}}(N) + \log_{\sqrt{m^2}}\left(\frac{1}{N+1}\right)$ *is equal to*

(a) $\frac{1}{2}$ (b) 1

(c) $\frac{-1}{2}$ (d) -1

Sol. (*c*) $f(\theta) = \sin\theta - (1 - \sin^2\theta) - 1$

$$= \sin^2\theta + \sin\theta - 2$$

$$= \left(\sin\theta + \frac{1}{2}\right)^2 - 2 - \frac{1}{4}$$

$$= \left(\sin\theta + \frac{1}{2}\right)^2 - \frac{9}{4}$$

$\therefore \quad f(\theta)_{\min} = 0 - \frac{9}{4} = \frac{-9}{4}$

$\Rightarrow \quad m = \frac{-9}{4}$

$$f(\theta)_{\max} = \left(1 + \frac{1}{2}\right)^2 - \frac{9}{4} = \frac{9}{4} - \frac{9}{4} = 0$$

$\Rightarrow \quad M = 0$

Hence, $m = \frac{-9}{4}$, $M = 0$

Now, $f(\theta) = 0$

$\Rightarrow \quad (\sin\theta + 2)(\sin\theta - 1) = 0$

$\Rightarrow \quad \sin\theta = 1$

$\Rightarrow \quad \theta = \frac{\pi}{2}$ and $\frac{5\pi}{2}$

Hence, $N = 2$, i.e. number of solution s of $\sin\theta = 1$ in $[0, 4\pi]$.

$\therefore \quad \log_{\sqrt{m^2}}(N) + \log_{\sqrt{m^2}}\left(\frac{1}{N+1}\right)$

$= \log_{|m|}\left(\frac{N}{N+1}\right)$v $= \log_{|m|}\left(\frac{2}{3}\right)$

$= \log_{\frac{9}{4}}\left(\frac{2}{3}\right) = \frac{-1}{2}$.

- **Ex. 82.** *The value of $(4m + 13)$ is equal to*

(a) 0 (b) 4

(c) 5 (d) 6

Sol. (*b*) As $m = \frac{-9}{4}$, so $(4m + 13) = 4$

- **Ex. 83.** *Sum of all values of x satisfying the equation*

$$x = \sqrt{\frac{1}{|m|} + \sqrt{\frac{1}{|m|} + \sqrt{\frac{1}{|m|} + \ldots\ldots\ldots\ldots \infty}}}, \text{ is}$$

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$

(c) $\frac{3}{3}$ (d) $\frac{4}{3}$

Sol. (*d*) $x = \sqrt{\frac{4}{9} + \sqrt{\frac{4}{9} + \sqrt{\frac{4}{9} + \ldots\ldots\ldots\ldots \infty}}}$

$\Rightarrow \quad x = \sqrt{\frac{4}{9} + x}$

$\Rightarrow \quad x^2 = \frac{4}{9} + x$

$9x^2 = 4 + 9x$

$\Rightarrow \quad 9x^2 - 9x - 4 = 0$

$\Rightarrow \quad 9x^2 - 12x + 3x - 4 = 0$

$\Rightarrow \quad (3x - 4)(3x + 1) = 0$

$\therefore \quad x = \frac{4}{3}$ and $x = \frac{-1}{3}$ (rejected)

Passage III

(Ex. Nos. 84 to 88)

The method of eliminating 'θ' from two given equations involving trigonometrical functions of 'θ'. By using given equations involving 'θ' and trigonometrical identities, we shall obtain an equation not involving 'θ'.

On the basis of above information answer the following questions.

- **Ex. 84.** *If $x\sin^3\theta + y\cos^3\theta = \sin\theta\cos\theta$ and $x\sin\theta - y\cos\theta = 0$ then (x, y) lie one*

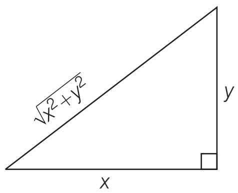
(a) a circle (b) a parabola

(c) an ellipse (d) a hyperbola

Sol. (*a*) We have, $x\sin^3\theta + y\cos^3\theta = \sin\theta\cos\theta$ ...(i)

and $\quad x\sin\theta - y\cos\theta = 0$ ...(ii)

From Eq. (ii), $\tan\theta = \frac{y}{x}$



$\therefore \quad \sin\theta = \frac{y}{\sqrt{(x^2 + y^2)}}$ and $\cos\theta = \frac{x}{\sqrt{(x^2 + y^2)}}$

From Eq., (i) $x \times \dfrac{y^3}{(x^2+y^2)^{\frac{3}{2}}} + y \times \dfrac{x^3}{(x^2+y^2)^{\frac{3}{2}}}$

$$= \frac{xy}{(x^2+y^2)}$$

or $\quad \dfrac{(x^2+y^2)}{(x^2+y^2)^{\frac{3}{2}}} = \dfrac{1}{(x^2+y^2)} \Rightarrow (x^2+y^2)^{\frac{1}{2}} = 1$

or $\quad x^2 + y^2 = 1$ which is a circle

Ex. 85. *If* $\dfrac{x}{a\cos\theta} = \dfrac{y}{b\sin\theta}$ ...(i)

and $\dfrac{ax}{\cos\theta} - \dfrac{by}{\sin\theta} = a^2 - b^2$*, then* (x, y) *lie on*

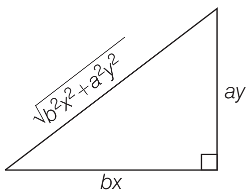
(a) a circle (b) a parabola

(c) an ellipse (d) a hyperbola

Sol. (*c*) $\because \quad \dfrac{x}{a\cos\theta} = \dfrac{y}{b\sin\theta}$ ...(i)

and $\quad \dfrac{ax}{\cos\theta} - \dfrac{by}{\sin\theta} = a^2 - b^2$ ...(ii)

From Eq. (i), $\tan\theta = \dfrac{ay}{bx}$



From Eq. (ii),

$$\frac{ax}{\dfrac{bx}{\sqrt{(b^2x^2+a^2y^2)}}} - \frac{by}{\dfrac{ay}{\sqrt{(b^2x^2+a^2y^2)}}} = (a^2-b^2)$$

$\Rightarrow \quad (a^2-b^2)\sqrt{(b^2x^2+a^2y^2)} = ab(a^2-b^2)$

$\Rightarrow \quad b^2x^2 + a^2y^2 = a^2b^2$

$\therefore \quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ which is an ellipse.

Ex. 86. *If* $\tan\theta + \sin\theta = m$ *and* $\tan\theta - \sin\theta = n$*, then* $(m^2-n^2)^2$ *is*

(a) $4\sqrt{mn}$ (b) $4mn$

(c) $16\sqrt{mn}$ (d) $16mn$

Sol. (*d*) $\because m + n = 2\tan\theta,\ m - n = 2\sin\theta$ ...(i)

and $\quad mn = \tan^2\theta - \sin^2\theta = \sin^2\theta(\sec^2\theta - 1)$

$$= \sin^2\theta\tan^2\theta$$

$$= \left(\frac{m-n}{2}\right)^2\left(\frac{m+n}{2}\right)^2 \quad \text{[from Eq. (i)]}$$

$\therefore \quad (m^2-n^2)^2 = 16\,mn$

Ex. 87. *If* $\sin\theta + \cos\theta = a$ *and* $\sin^3\theta + \cos^2\theta = b$*, then we get* $\lambda a^3 + \mu b + va = 0$ *when* λ, μ, v *are independent of* θ*, then the value of* $\lambda^3 + \mu^3 + v^3$ *is*

(a) -6 (b) -18 (c) -36 (d) -98

Sol. (*b*) $\quad \sin\theta + \cos\theta = a$ ...(i)

$\sin^3\theta + \cos^3\theta = b$ ...(ii)

From Eq. (i),

$$\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta = a^2$$

or $\quad \sin\theta\cos\theta = \dfrac{a^2-1}{2}$ ...(iii)

From Eq. (ii),

$$(\sin\theta + \cos\theta)^3 + 3\sin\theta\cos\theta(\sin\theta+\cos\theta) = b$$

$\Rightarrow \quad a^3 - \dfrac{3(a^2-1)}{2}a = b \quad$ [from Eqs. (i) and (iii)]

$\Rightarrow \quad 2a^3 - 3a^3 + 3a = 2b \Rightarrow a^3 + 2b - 3a = 0$

On comparing, we get

$$\lambda = 1, \mu = 2, v = -3$$

$\therefore \quad \lambda + \mu + v = 0$

$\therefore \quad \lambda^3 + \mu^3 + v^3 = 3\lambda\mu v = 3(1)(2)(-3) = -18$

Ex. 88. *After eliminating 'θ' from equations* $\dfrac{x\cos\theta}{a} + \dfrac{y\sin\theta}{b} = 1$ *and* $x\sin\theta - y\cos\theta = \sqrt{(a^2\sin^2\theta + b^2\cos^2\theta}$*, we get*

(a) $x^2 + y^2 = a^2 + b^2$ (b) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

(c) $\dfrac{x^2}{a(a+b)} + \dfrac{y^2}{b(a+b)} = 1$ (d) $x^2 + y^2 = (a+b)^2$

Sol. (*c*) $\because \dfrac{x\cos\theta}{a} + \dfrac{y\sin\theta}{b} = 1$ ...(i)

and $\quad x\sin\theta - y\cos\theta = \sqrt{(a^2\sin^2\theta + b^2\cos^2\theta}$ ...(ii)

Squaring Eq. (i), we get

$$\frac{x^2}{a^2}\cos^2\theta + \frac{y^2}{b^2}\sin^2\theta + \frac{2xy}{ab}\sin\theta\cos\theta$$

$$= 1 = \sin^2\theta + \cos^2\theta$$

or $\quad \left(\dfrac{x^2}{a^2} - 1\right)\cos^2\theta + \left(\dfrac{y^2}{b^2} - 1\right)\sin^2\theta$

$$+ \frac{2xy}{ab}\sin\theta\cos\theta = 0 \quad \text{...(iii)}$$

and squaring Eq. (ii), we get

$$x^2\sin^2\theta + y^2\cos^2\theta - 2xy\sin\theta\cos\theta$$

$$= a^2\sin^2\theta + b^2\cos^2\theta$$

$$(x^2-a^2)\sin^2\theta + (y^2-b^2)\cos^2\theta - 2xy\sin\theta\cos\theta = 0$$

$\Rightarrow \quad \left(\dfrac{x^2-a^2}{ab}\right)\sin^2\theta + \left(\dfrac{y^2-b^2}{ab}\right)\cos^2\theta$

$$-\frac{2xy}{ab}\sin\theta\cos\theta = 0 \quad \text{...(iv)}$$

Adding Eqs. (iii) and (iv), we get

$$\left(\frac{x^2-a^2}{a}\right)\left(\frac{\sin^2\theta}{b}+\frac{\cos^2\theta}{a}\right)+\left(\frac{y^2-b^2}{b}\right)\left(\frac{\sin^2\theta}{b}+\frac{\cos^2\theta}{a}\right)=0$$

or $$\frac{x^2-a^2}{a}+\frac{y^2-b^2}{b}=0 \quad \text{or} \quad \frac{x^2}{a}+\frac{y^2}{b}=(a+b)$$

JEE Type Solved Examples : Matching Type Questions

● **Ex. 89.** *Match the statement of Column I with values of Column II.*

Column-I	Column-II
(A) The number of real roots of the equation $\cos^7 x + \sin^4 x = 1$ in $(-\pi, \pi)$ is	(p) 1
(B) The value of $\sqrt{3}$ cosec $20° -$ sec $20°$ is	(q) 4
(C) $4\cos 36° - 4\cos 72° + 4\sin 18°\cos 36°$ equals	(r) 3
(D) The number of values of x where $x \in [-2\pi, 2\pi]$, which satisfy cosec $x = 1 + \cot x$	(s) 2

Sol. (A) → (r), (B) → (q), (C) → (r), (D) → (s)

(A) $\cos^7 x + \sin^4 x = 1$

$\cos^7 x = (1+\sin^2 x)\cos^2 x$

$\Rightarrow \quad \cos x = 0$ or $\cos^5 x = 1 + \sin^2 x$

$\cos x = 0 \Rightarrow x - \frac{\pi}{2}, \frac{\pi}{2}$; $\cos^5 x = 1 + \sin^2 x$

$\Rightarrow \quad x = 0 \quad [\because \text{LHS} \le 1 \text{ and RHS} \ge 1]$

$\therefore \quad x = -\frac{\pi}{2}, 0, \frac{\pi}{2}.$

(B) $\sqrt{3}$ cosec $20° -$ sec $20°$

$$= \frac{\sqrt{3}}{\sin 20°} - \frac{1}{\cos 20°} = \frac{\sqrt{3}\cos 20° - \sin 20°}{\sin 20°\cos 20°}$$

$$= \frac{2\left(\frac{\sqrt{3}}{2}\cos 20° - \frac{1}{2}\sin 20°\right)}{\sin 20°\cos 20°} = \frac{4\sin 40°}{\sin 40°} = 4$$

(C) $4\cos 36° - 4\cos 72° + 4\sin 18° \cdot \cos 36°$

$$= 4\left(\frac{\sqrt{5}+1}{4}\right) - 4\left(\frac{\sqrt{5}-1}{4}\right) + 4\left(\frac{\sqrt{5}-1}{4}\right)\left(\frac{\sqrt{5}+1}{4}\right)$$

$$= \sqrt{5}+1-\sqrt{5}+1+1 = 3$$

(D) cosec $x = 1 + \cot x$; $\frac{1}{\sin x} = \frac{\sin x + \cos x}{\sin x} \Rightarrow$

$\sin x + \cos x = 1$ and $\sin x \ne 0$

$$\cos\left(x - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$\Rightarrow \quad x - \frac{\pi}{4} = -2\pi + \frac{\pi}{4}, \frac{\pi}{4}$

$$\left(\because x - \frac{\pi}{4} \in \left[-2\pi - \frac{\pi}{4}, 2\pi - \frac{\pi}{4}\right]\right)$$

$\Rightarrow \quad x = -\frac{3\pi}{2}, \frac{\pi}{2}$

● **Ex. 90.** *Match the statement of Column I with values of Column II.*

Column-I	Column-II
(A) The number of solutions of the equation $\lvert \cot x \rvert = \cot x + \frac{1}{\sin x}$ $(0 < x < \pi)$	(p) No solution
(B) If $\sin\theta + \sin\phi = \frac{1}{2}$ and $\cos\theta + \cos\phi = 2$, then $\cot\left(\frac{\theta+\phi}{2}\right)$	(q) $\frac{1}{3}$
(C) $\sin^2\alpha + \sin\left(\frac{\pi}{3}-\alpha\right)\sin\left(\frac{\pi}{3}-\alpha\right)$	(r) 1
(D) If $\tan\theta = 3\tan\phi$, then maximum value of $\tan^2(\theta-\phi)$ is	(s) 4

Sol. (A) → (r), (B) → (s), (C) → (p), (D) → (q)

(A) $\lvert \cot x \rvert = \cot x + \frac{1}{\sin x}$

If $0 < x < \frac{\pi}{2} \Rightarrow \cot x > 0$

So, $\cot x = \cot x + \frac{1}{\sin x} \Rightarrow \frac{1}{\sin x} = 0$ no solution

If $\frac{\pi}{2} < \cot x < \pi, -\cot x = \cot x + \frac{1}{\sin x}$

$$\frac{2\cos x}{\sin x} + \frac{1}{\sin x} = 0$$

$1 + 2\cos x = 0$ and $x \ne 0 \Rightarrow x = \frac{2\pi}{3}$

(B) Since, $\sin\phi + \sin\theta = \frac{1}{2}$ and $\cos\theta + \cos\phi = 2$ has no solution.

(C) $\sin^2\alpha + \sin\left(\frac{\pi}{3}-\alpha\right)\cdot\sin\left(\frac{\pi}{3}+\alpha\right)$

$$= \sin^2\alpha + \sin^2\frac{\pi}{3} - \sin^2\alpha = \frac{3}{4}$$

(D) $\tan\theta = 3\tan\phi$

$$\tan(\theta-\phi) = \frac{\tan\theta - \tan\phi}{1+\tan\theta\tan\phi} = \frac{2\tan\phi}{1+3\tan^2\phi}$$

$$= \frac{2}{\cot\phi + 3\tan\phi} \cdot \text{Max of } \tan\phi > 0$$

$$\frac{\cot\phi + 3\tan\phi}{2} \geq \sqrt{3} \qquad \text{(using AM} \geq \text{GM)}$$

$$\Rightarrow \quad (\cot\phi + 3\tan\phi)^2 \geq 12 \;\Rightarrow\; \tan^2(\theta-\phi) \leq \frac{1}{3}$$

Ex. 91. *Match the statement of Column I with values of Column II.*

	Column-I		Column-II
(A)	The tangents of two acute angles are 3 and 2. The sine of twice their difference is	(p)	1
(B)	If $n = \frac{\pi}{4\alpha}$, then $\tan\alpha\tan 2\alpha\tan 3\alpha \ldots \tan(2n-1)\alpha$ is equal to	(q)	0
(C)	If $x = y\cos\frac{2\pi}{3} = z\cos\frac{4\pi}{3}$, then $xy + yz + zx =$	(r)	$\frac{1}{2}$
(D)	The ratio of the greatest value of $2-\cos x + \sin^2 x$ to its least value is	(s)	$\frac{7}{25}$
		(t)	$\frac{13}{4}$

Sol. (A) $\to$ (s), (B) $\to$ (p), (C) $\to$ (q), (D) $\to$ (t)

(A) Given, $\tan\alpha = 3$ and $\tan\beta = 2$

$$\therefore \quad \tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1-\tan\alpha\tan\beta} = \frac{3-2}{1+3\times 2} = \frac{1}{7}$$

$$\Rightarrow \quad \sin(\alpha-\beta) = \frac{1}{\sqrt{50}} \text{ and } \cos(\alpha-\beta) = \frac{7}{\sqrt{50}}$$

$$\therefore \quad \sin 2(\alpha-\beta) = 2\sin(\alpha-\beta)\cos(\alpha-\beta)$$

$$= 2\times\frac{1}{\sqrt{50}}\times\frac{7}{\sqrt{50}} = \frac{7}{25}$$

(B) We have, $\tan\alpha\cdot\tan(2n-1)\alpha = \tan\alpha\cdot\tan\left(\frac{\pi}{2\alpha}-1\right)\alpha$

$$= \tan\alpha\cdot\tan\left(\frac{\pi}{2}-\alpha\right) = \tan\alpha\cot\alpha = 1$$

$\therefore$ The given expression = 1.

(C) We have, $x = y\cos\frac{2\pi}{2} = z\cos\frac{4\pi}{3} = k$ (say)

$$\Rightarrow \quad \frac{1}{x} = \frac{1}{k}, \frac{1}{y} = \frac{\cos\frac{2\pi}{3}}{k}, \frac{1}{z} = \frac{\cos\frac{4\pi}{3}}{k}$$

$$\therefore \quad \frac{1}{x}+\frac{1}{y}+\frac{1}{z} = \frac{1}{k}\left(1+\cos\frac{2\pi}{3}+\cos\frac{4\pi}{3}\right)$$

$$= \frac{1}{k}\left(1-\frac{1}{2}-\frac{1}{2}\right) = 0$$

$$\Rightarrow \quad xy + xz + yz = 0$$

(D) We have, $2-\cos x + \sin^2 x = 2 - \cos x + 1 - \cos^2 x$

$$= -(\cos^2 x + \cos x) + 3 = -\left[\left(\cos x + \frac{1}{2}\right)^2 - \frac{1}{4}\right] + 3$$

$$= \frac{13}{4} - \left(\cos x + \frac{1}{2}\right)^2$$

$\therefore$ Maximum value occurs at $\cos x = -\frac{1}{2}$ and it is 1 $\frac{13}{4}$ and minimum value occurs at $\cos x = 1$ and it is

$\therefore$ The required ratio is $\frac{13}{4}$.

Ex. 92. *Match the statement of Column I with values of Column II*

	Column-I		Column-II
(A)	If α, β, γ and δ are four solutions of the equation $\tan\left(\theta+\frac{\pi}{4}\right) = 3\tan 3\theta$, no two of which have equal tangents, then the value of $\tan\alpha + \tan\beta + \tan\gamma + \tan\delta$ is	(p)	$\sqrt{2}$
(B)	If $\frac{\cos(\theta_1-\theta_2)}{\cos(\theta_1+\theta_2)} + \frac{\cos(\theta_3+\theta_4)}{\cos(\theta_3-\theta_4)} = 0$ then $\tan\theta_1\tan\theta_2\tan\theta_3\tan\theta_4 =$	(q)	$\sqrt{3}$
(C)	If $\sec(\alpha-\beta), \sec\alpha$ and $\sec(\alpha+\beta)$ are in A.P. (with $\beta \neq 0$), then $\cos\alpha\sec\frac{\beta}{2} =$	(r)	-1
(D)	If $\cos\alpha = \frac{2\cos\beta - 1}{2-\cos\beta}$ $(0<\alpha<\beta<\pi)$, then $\frac{\tan\frac{\alpha}{2}}{\tan\frac{\beta}{2}}$ is equal to	(s)	0

Sol. (A) $\to$ (s), (B) $\to$ (r), (C) $\to$ (p), (D) $\to$ (q)

(A) Using $\tan\left(\theta+\frac{\pi}{4}\right) = \frac{1+\tan\theta}{1-\tan\theta}$

and $\quad 3\tan 3\theta = \frac{3(3\tan\theta - \tan^3\theta)}{1-3\tan^2\theta}$

the given equation becomes

$$3\tan^4\theta - 6\tan^2\theta + 8\tan\theta - 1 = 0$$

If $\tan\alpha, \tan\beta, \tan\gamma$ and $\tan\delta$ are the roots of this equation, then the sum of these roots, $\tan\alpha + \tan\beta + \tan\gamma + \tan\delta$ equals zero, since the coefficient of $\tan^3\theta$ is zero.

(B) The given equation can be written as

$$\Rightarrow \quad \frac{\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2}{\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2} + \frac{\cos\theta_3\cos\theta_4 - \sin\theta_3\sin\theta_4}{\cos\theta_3\cos\theta_4 + \sin\theta_3\sin\theta_4}$$

$$\Rightarrow \quad \frac{1+\tan\theta_1\tan\theta_2}{1-\tan\theta_1\tan\theta_2} + \frac{1-\tan\theta_3\tan\theta_4}{1+\tan\theta_3\tan\theta_4} = 0$$

$$\Rightarrow \quad \frac{2+2\tan\theta_1\tan\theta_2\tan\theta_3\tan\theta_4}{(1-\tan\theta_1\tan\theta_2)(1+\tan\theta_3\tan\theta_4)}=0$$

Showing that $\tan\theta_1\tan\theta_2\tan\theta_3\tan\theta_4=-1$.

(C) For the given A.P., we have

$2\sec\alpha=\sec(\alpha-\beta)+\sec(\alpha+\beta)$, which gives

$$\frac{2}{\cos\alpha}=\frac{1}{\cos(\alpha-\beta)}+\frac{1}{\cos(\alpha+\beta)}$$

$$=\frac{2\cos\alpha\cos\beta}{\cos^2\alpha-\sin^2\beta}$$

$$\Rightarrow \quad \cos^2\alpha-\sin^2\beta=\cos^2\alpha\cos\beta$$

$$\Rightarrow \quad \cos^2\alpha(1-\cos\beta)=\sin^2\beta$$

$$\Rightarrow \quad \cos^2\alpha\left(2\sin^2\frac{\beta}{2}\right)=4\sin^2\frac{\beta}{2}\cos^2\frac{\beta}{2}$$

$$\Rightarrow \quad \cos^2\alpha\sec^2\frac{\beta}{2}=2$$

(D) $1+\cos\alpha=1+\dfrac{2\cos\beta-1}{2-\cos\beta}$

$$=\frac{2-\cos\beta+2\cos\beta-1}{2-\cos\beta}=\frac{1+\cos\beta}{2-\cos\beta}$$

$$\Rightarrow \quad \cos^2\frac{\alpha}{2}=\frac{\cos^2\frac{\beta}{2}}{1+2\sin^2\frac{\beta}{2}} \quad \text{...(i)}$$

$$\Rightarrow \quad 1-\cos^2\frac{\alpha}{2}=1-\frac{\cos^2\frac{\beta}{2}}{1+2\sin^2\frac{\beta}{2}}$$

$$=\frac{1+2\sin^2\frac{\beta}{2}-\cos^2\frac{\beta}{2}}{1+2\sin^2\frac{\beta}{2}}$$

$$=\frac{1+2\sin^2\frac{\beta}{2}-\left[1-\sin^2\frac{\beta}{2}\right]}{1+2\sin^2\frac{\beta}{2}}$$

$$\Rightarrow \quad \sin^2\frac{\alpha}{2}=\frac{3\sin^2\frac{\beta}{2}}{1+2\sin^2\frac{\beta}{2}} \quad \text{...(ii)}$$

Dividing equation (ii) by (i), we get

$$\tan^2\frac{\alpha}{2}=3\tan^2\frac{\beta}{2}$$

$$\Rightarrow \quad \frac{\tan\frac{\alpha}{2}}{\tan\frac{\beta}{2}}=\sqrt{3}$$

JEE Type Solved Examples : Single Integer Answer Type Questions

● ***Ex. 93.*** $\tan 46^\circ\tan 14^\circ-\tan 74^\circ\tan 14^\circ+\tan 74^\circ\tan 46^\circ$ *is equal to*

Sol. (3) $\dfrac{\tan 46^\circ+\tan 14^\circ}{1-\tan 46^\circ\tan 14^\circ}=\tan(46^\circ+14^\circ)=\sqrt{3}$ ...(i)

$$\frac{\tan 74^\circ-\tan 14^\circ}{1+\tan 74^\circ\tan 14^\circ}=\tan(74^\circ-14^\circ)$$

$$=\sqrt{3} \quad \text{...(ii)}$$

$$\frac{\tan 74^\circ+\tan 46^\circ}{1-\tan 74^\circ\tan 46^\circ}=\tan(74^\circ+46^\circ)$$

$$=-\sqrt{3} \quad \text{...(iii)}$$

From Eqs. (i), (ii) and (iii)

$$\tan 46^\circ\tan 14^\circ=1-\frac{\tan 46^\circ+\tan 14^\circ}{\sqrt{3}}$$

$$\tan 74^\circ\tan 14^\circ=\frac{\tan 74^\circ-\tan 14^\circ}{\sqrt{3}}-1$$

$$\tan 74^\circ\tan 46^\circ=1-\frac{\tan 74^\circ+\tan 46^\circ}{-\sqrt{3}}$$

$\therefore$ $\tan 46^\circ\tan 14^\circ-\tan 74^\circ\tan 14^\circ+\tan 74^\circ\tan 46^\circ=3$

● ***Ex. 94.*** *Maximum value of the expression* $\log_3(9-2\cos^2\theta-4\sec^2\theta)$ *is equal to*

Sol. (1) For the expression $a\cos^2\theta+b\sec^2\theta$ if $b>a$, then minimum value attains at $\cos^2\theta=\sec^2\theta=1$

$\Rightarrow$ max of $\{9-(2\cos^2\theta+4\sec^2\theta)\}=3$

So, maximum of $\log_3(9-2\cos^2\theta+4\sec^2\theta))=1$

● ***Ex. 95.*** *Let* $x\in\left(0,\dfrac{\pi}{2}\right)$ *and* $\log_{24\sin x}(24\cos x)=\dfrac{3}{2}$, *then find the value of* $\operatorname{cosec}^2 x$.

Sol. (9) $(24\sin x)^{3/2}=24\cos x$

$$\Rightarrow \quad \sqrt{24}\,(\sin x)^{3/2}=\cos x$$

$$\Rightarrow \quad 24\sin^3 x=\cos^2 x=1-\sin^2 x$$

Put $\sin x=t$, we get

$$24t^3+t^2-1=0$$

$$\Rightarrow \quad (3t-1)\underbrace{(8t^2+3t+1)}_{>0}=0$$

$$\Rightarrow \qquad t = \frac{1}{3}$$

$$\therefore \qquad t = \frac{1}{3} \text{ i.e. } \sin x = \frac{1}{3}$$

$$\Rightarrow \qquad \operatorname{cosec} x = 3$$

$$\Rightarrow \qquad \operatorname{cosec}^2 x = 9$$

● **Ex. 96.** *If x and y are non-zero real numbers satisfying $xy(x^2 - y^2) = x^2 + y^2$, then find the minimum value of $x^2 + y^2$.*

Sol. (4) Put $x = r\cos\theta$ and $y = r\sin\theta$

Hence, we have to minimise r^2?

Now, $r^2 \cos\theta \sin\theta r^2(\cos^2\theta - \sin^2\theta) = r^2$

$$r^2 \sin 2\theta \cos 2\theta = 2$$

$$r^2 \frac{\sin 4\theta}{4} = 1$$

$$r^2 = \frac{4}{\sin 4\theta}$$

$$r^2 = 4\operatorname{cosec}^2 4\theta$$

$$\therefore \qquad r^2{}_{\min} = 4$$

● **Ex. 97.** *Using the identity $\sin^4 x = \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$ or otherwise, if the value of $\sin^4\left(\frac{\pi}{7}\right) + \sin^4\left(\frac{3\pi}{7}\right) + \sin^4\left(\frac{5\pi}{7}\right) = \frac{a}{b}$, where a and b are coprime, find the value of $(a - b)$.*

Sol. (5) $\sin^4\left(\frac{\pi}{7}\right) = \frac{3}{8} - \frac{1}{2}\cos\left(\frac{2\pi}{7}\right) + \frac{1}{8}\cos\left(\frac{4\pi}{7}\right)$ ...(i)

and $\sin^4\left(\frac{3\pi}{7}\right) = \frac{3}{8} - \frac{1}{2}\cos\left(\frac{6\pi}{7}\right) + \frac{1}{8}\cos\left(\frac{12\pi}{7}\right)$

or $\sin^4\left(\frac{3\pi}{7}\right) = \frac{3}{8} + \frac{1}{2}\cos\left(\frac{\pi}{7}\right) - \frac{1}{8}\cos\left(\frac{5\pi}{7}\right)$ ...(ii)

Similarly, $\sin^4\left(\frac{5\pi}{7}\right) = \frac{3}{8} - \frac{1}{2}\cos\left(\frac{10\pi}{7}\right) + \frac{1}{8}\cos\left(\frac{20\pi}{7}\right)$

or $\sin^4\left(\frac{5\pi}{7}\right) = \frac{3}{8} + \frac{1}{2}\cos\left(\frac{3\pi}{7}\right) - \frac{1}{8}\cos\left(\frac{\pi}{7}\right)$ ...(iii)

On adding Eqs. (i), (ii) and (iii), we get

$$\sin^4\left(\frac{\pi}{7}\right) + \sin^4\left(\frac{3\pi}{7}\right) + \sin^4\left(\frac{5\pi}{7}\right)$$

$$= \frac{9}{8} + \frac{1}{2}\cos\left(\frac{5\pi}{7}\right) - \frac{1}{8}\cos\left(\frac{3\pi}{7}\right) + \frac{1}{2}\cos\left(\frac{\pi}{7}\right) - \frac{1}{8}\cos\left(\frac{5\pi}{7}\right) + \frac{1}{2}\cos\left(\frac{3\pi}{7}\right) - \frac{1}{8}\cos\left(\frac{\pi}{7}\right)$$

$$= \frac{9}{8} + \frac{3}{8}\underbrace{\left[\cos\left(\frac{\pi}{7}\right) + \cos\left(\frac{3\pi}{7}\right) + \cos\left(\frac{5\pi}{7}\right)\right]}_{\frac{1}{2}\left(\text{use } S = \frac{\sin\frac{n\alpha}{2}}{\sin\frac{\alpha}{2}}\cos\left(\frac{\theta_1 + \theta_2}{2}\right)\right)}$$

$$= \frac{9}{8} + \frac{3}{16} = \frac{21}{16}$$

$$\therefore \qquad a - b = 21 - 16 = 5.$$

● **Ex. 98.** *In any triangle, if $(\sin A + \sin B + \sin C)(\sin A + \sin B - \sin C) = 3\sin A \sin B$, then the angle $\frac{C}{10}$ (in degree).*

Sol. (6) We have, $(\sin A + \sin B)^2 - \sin^2 C = 3\sin A \sin B$

$$\sin^2 A - \sin^2 C + \sin^2 B = \sin A \sin B$$

$$\sin(A + C)\sin(A - C) + \sin^2 B = \sin A \sin B$$

$$\sin B[\sin(A - C) + \sin(A + C)] = \sin A \sin B$$

[using, $\sin(A + C) = \sin B$]

$$2\sin A \cos C = \sin A \qquad (\sin B \neq 0)$$

$$\cos C = \frac{1}{2} \Rightarrow \frac{C}{10} = \frac{60^\circ}{10} = 6$$

● **Ex. 99.** *Find the exact value of the expression*

$$T = \frac{\sin 40^\circ}{\sin 80^\circ} + \frac{\sin 80^\circ}{\sin 20^\circ} - \frac{\sin 20^\circ}{\sin 40^\circ}$$

Sol. (3) We have, $\frac{1}{2\cos 40^\circ} + 4\cos 40^\circ \cdot \cos 20^\circ - \frac{1}{2\cos 20^\circ}$

$$= \frac{1}{2}\left[\frac{1}{\cos 40^\circ} - \frac{1}{\cos 20^\circ}\right] + 2(\cos 60^\circ + \cos 20^\circ)$$

$$= \frac{1}{2}\left[\frac{\cos 20^\circ - \cos 40^\circ}{\cos 40^\circ \cos 20^\circ}\right] + 1 + 2\cos 20^\circ$$

$$= \frac{2\sin 30^\circ \sin 10^\circ}{2\cos 40^\circ \cos 20^\circ} + 1 + 2\cos 20^\circ$$

$$= \frac{2\sin 10^\circ \sin 20^\circ}{\sin 80^\circ} + 1 + 2\cos 20^\circ$$

$$= \frac{2\sin 10^\circ \, 2\sin 10^\circ \cos 10^\circ}{\cos 10^\circ} + 2\cos 20^\circ + 1$$

$$= 2(1 - \cos 20^\circ) + 2\cos 20^\circ + 1 = 3$$

Alternatively $T_1 = \frac{2\sin 20^\circ \cos 20^\circ}{\cos 10^\circ} = 2 \cdot 2 \cdot \sin 10^\circ \cdot \cos 20^\circ$

$$= 2(\sin 30^\circ - \sin 10^\circ)$$

$$T_1 = 1 - 2\sin 10^\circ$$

$$T_2 = \frac{\sin 80^\circ}{\sin 20^\circ} = \frac{2\sin 40^\circ \cos 40^\circ}{\sin 20^\circ}$$

$$= 4\cos 20^\circ \cdot \cos 40^\circ$$

$$T_2 = 2[\cos 60^\circ + \cos 20^\circ] = 1 + 2\cos 20^\circ$$

$$T_2 = \frac{\sin 20^\circ}{\sin 40^\circ} = \frac{1}{2\cos 20^\circ}$$

$$\therefore\ T = T_1 + T_2 + T_3$$

$$= (1 - 2\sin 10^\circ) + (1 + 2\cos 20^\circ) - \frac{1}{2\cos 20^\circ}$$

$$= 2 + 2(\cos 20^\circ - \sin 10^\circ) - \frac{1}{2\cos 20^\circ}$$

$$= 2 + 2(\cos 20^\circ - \cos 80^\circ) - \frac{1}{2\cos 20^\circ}$$

$$= 2 + 2\cdot 2\sin 50^\circ \cdot \sin 30^\circ - \frac{1}{2\cos 20^\circ}$$

$$= 2 + 2\sin 50^\circ - \frac{1}{2\cos 20^\circ}$$

$$T = 2 + \frac{4\sin 50^\circ \cos 20^\circ - 1}{2\cos 20^\circ}$$

$$= 2 + \frac{2[\sin 70^\circ + \sin 30^\circ] - 1}{2\cos 20^\circ}$$

$$= 2 + \frac{2\sin 70^\circ}{2\cos 20^\circ} = 2 + 1 = 3$$

Ex. 100. *If* $\cot(\theta - \alpha), 3\cot\theta, \cot(\theta + \alpha)$ *are in* AP *(where,* $\theta \neq \frac{n\pi}{2}, \alpha \neq k\pi, n, k \in I$*), then* $\frac{2\sin^2\theta}{\sin^2\alpha}$ *is equal to*

Sol. (3) We have, $6\cot\theta = \cot(\theta - \alpha) + \cot(\theta + \alpha)$

$$\Rightarrow \quad \frac{6\cos\theta}{\sin\theta} = \frac{\sin 2\theta}{\sin(\theta - \alpha)\sin(\theta + \alpha)}$$

$$\Rightarrow \quad \frac{6\cos\theta}{\sin\theta} = \frac{2\sin\theta\cos\theta}{\sin^2\theta - \sin^2\alpha}$$

$$\Rightarrow \quad 3(\sin^2\theta - \sin^2\alpha) = \sin^2\theta$$

$$\text{or} \quad 2\sin^2\theta = 3\sin^2\alpha$$

$$\Rightarrow \quad \frac{2\sin^2\theta}{\sin^2\alpha} = 3$$

Ex. 101. *If* $4\sin^2 x + \operatorname{cosec}^2 x, a, \sin^2 y + 4\operatorname{cosec}^2 y$ *are in* AP, *then minimum value of* $(2a)$ *is*

Sol. (9) $2a = 4\sin^2 x + \operatorname{cosec}^2 x + \sin^2 y + 4\operatorname{cosec}^2 y$

$$= (2\sin x - \operatorname{cosec} x)^2 + 4 + (\sin y - \operatorname{cosec} y)^2 + 3\operatorname{cosec}^2 y + 2$$

$$= 6 + (2\sin x - \operatorname{cosec} x)^2 + (\sin y - \operatorname{cosec} y)^2 + 3\operatorname{cosec}^2 y$$

$\therefore$ Minimum value of

$$2a = 6 + 3 = 9,$$

when $\quad 2\sin x = \operatorname{cosec} x$

and $\quad \sin y = \operatorname{cosec} y$

Ex. 102. *If* $\sin\alpha, \sin\beta, \sin\gamma$ *are in* AP *and* $\cos\alpha, \cos\beta, \cos\gamma$ *are in* GP, *then the value of* $\frac{\cos^2\alpha + \cos^2\gamma + 4\cos\alpha\cos\gamma - 2\sin\alpha\sin\gamma - 2}{1 - 2\sin^2\beta}$, *where* $\beta \neq \frac{\pi}{4}$, *is equal to*

Sol. (4) Now, $\sin\alpha + \sin\gamma = 2\sin\beta$ and $\cos^2\beta = \cos\alpha\,.\,\cos\gamma$

$$= \frac{\cos^2\alpha + \cos^2\gamma + 4\cos\alpha\cos\gamma - 2 - 2\sin\alpha\sin\gamma}{1 - 2\sin^2\beta}$$

$$= \frac{-\sin^2\alpha - \sin^2\gamma - 2\sin\alpha\sin\gamma + 4\cos\alpha\cos\gamma}{1 - 2\sin^2\beta}$$

$$= \frac{-(\sin\alpha + \sin\gamma)^2 + 4\cos\alpha\cos\gamma}{1 - 2\sin^2\beta}$$

$$= \frac{-4\sin^2\beta + 4\cos^2\beta}{\cos 2\beta} = 4$$

Ex. 103. *Let*

$$\prod_{r=1}^{51} \tan\left(\frac{\pi}{3}\left(1 + \frac{3^r}{3^{50} - 1}\right)\right) = k\prod_{r=1}^{51}\cot\left[\frac{\pi}{3}\left(1 - \frac{3^r}{3^{50} - 1}\right)\right]$$

On solving equation, we get, $1 - 3\tan^2\left(\frac{\pi}{3^{50} - 1}\right) = \frac{a}{bk - 1}$, *(*$a, b \in I$*), then value of* $(a - b)$ *is equal to*

Sol. (5) We have,

$$\prod_{r=1}^{51} \tan\left(\frac{\pi}{3}\left(1 + \frac{3^r}{3^{50} - 1}\right)\right) = \prod_{r=1}^{51}\cot\left[\frac{\pi}{3}\left(1 - \frac{3^r}{3^{50} - 1}\right)\right]$$

$$\text{Let } \frac{3^{r-1}\pi}{3^{50} - 1} = \theta_r$$

$$\prod_{r-1}^{51} \tan\left(\frac{\pi}{3} + \theta_r\right)\tan\left(\frac{\pi}{3} + \theta_r\right) = k$$

$$\prod_{r-1}^{51} \frac{\tan 3\theta_r}{\tan\theta^r} = k$$

$$k = \frac{\tan\theta_2}{\tan\theta_1} \times \frac{\tan\theta_3}{\tan\theta_2} \times \dots\dots \frac{\tan\theta_{52}}{\tan\theta_{51}}$$

$$= \frac{\tan\theta_{52}}{\tan\theta} = \frac{\tan\left(\frac{3^{51}\pi}{3^{50} - 1}\right)}{\tan\left(\frac{\pi}{3^{50} - 1}\right)}$$

$$= \frac{\tan\left(3\pi + \frac{3\pi}{3^{50} - 1}\right)}{\tan\left(\frac{\pi}{3^{50} - 1}\right)}$$

$$= \frac{\tan\left(\frac{3\pi}{3^{50}-1}\right)}{\tan\left(\frac{\pi}{3^{50}-1}\right)}$$

Let, $\alpha = \frac{\pi}{3^{50}-1}$;

$k = \frac{\tan 3\alpha}{\tan\alpha}$;

$1 - 3\tan^2\alpha = \frac{8}{3k-1}$

So, $a = 8, b = 3$

$a - b = 5$

Ex. 104. *If* $\sec A \tan B + \tan A \sec B = 91$, *then the value of* $(\sec A \sec B + \tan A \tan B)^2$ *is equal to*

Sol. (8282) $(\sec A \sec B + \tan A \tan B)^2$

$$- (\sec A \tan B + \tan A \sec B)^2$$

$$= \left[\frac{1 + \sin A \sin B}{\cos A \cos B}\right]^2 - \left[\frac{\sin B + \sin A}{\cos A \cos B}\right]^2$$

$$= \frac{1 + \sin^2 A \sin^2 B - \sin^2 B - \sin^2 A}{\cos^2 A \cos^2 B}$$

$$= \frac{1 - \sin^2 B \cos^2 A - \sin^2 A}{\cos^2 A \cos^2 B}$$

$$= \frac{\cos^2 A \cos^2 B}{\cos^2 A \cos^2 B} = 1$$

$\Rightarrow (\sec A \sec B + \tan A \tan B)^2 = (91)^2 + 1 = 8282.$

Ex. 105. *If* $(25)^2 + a^2 + 50a\cos\theta = (31)^2 + b^2 + 62\,b\cos\theta = 1$ *and* $775 + ab + (31a + 25b)\cos\theta = 0$, *then the value of* $\text{cosec}^2\,\theta$ *is*

Sol. (1586) We can write $(a + 25\cos\theta)^2 + (25)^2 - (25\cos\theta)^2 = 1$ and

$\Rightarrow \quad (a + 25\cos\theta)^2 = 1 - (25\sin\theta)^2$

Similarly $\quad (b + 31\cos\theta)^2 = 1 - (31\sin\theta)^2$

Multiplying we get

$$[(a + 25\cos\theta)(b + 31\cos\theta)]^2 = [1 - (25\sin\theta)^2][1 - (31\sin\theta)^2]$$

$\Rightarrow \quad [ab + (31a + 25b)\cos\theta + 775\cos^2\theta]^2$

$$= 1 - (625 + 961)\sin^2\theta + (775\sin^2\theta)^2$$

$\Rightarrow \quad (-775 + 775\cos^2\theta)^2 = 1 - 1586\sin^2\theta + (775\sin^2\theta)^2$

$\Rightarrow \quad \text{cosec}^2\,\theta = 1586$

Ex. 106. *If* $\sin x_1 + \sin x_2 + \sin x_3 + \ldots + \sin x_{2008} = 2008$, *then find the value of* $\sin^{2008} x_1 + \sin^{2008} x_2 + \sin^{2008} x_3 + \ldots + \sin^{2008} x_{2008}$.

Sol. (2008) We know that, $\sin x_i \le 1 \;\forall\; i$

$\Rightarrow \sin x_1 + \sin x_2 + \sin x_3 + \ldots + \sin x_{2008} \le 2008$

Thus, equality holds only when each of the terms is 1

i.e. $\quad \sin x_i = 1 \;\forall\; i = 1, 2, 3, \ldots, 2008.$

and consequently

$$\cos x_i = 0, \forall i = 1, 2, 3, \ldots, 2008$$

Now, $\sin^{2008} x_1 + \sin^{2008} x_2 + \sin^{2008} x_3 + \ldots \sin^{2008} x_{2008}$

$$= 1 + 1 + 1 + \ldots + 1 = 2008$$

Ex. 107. *If* $4\sin 27° = \sqrt{\alpha} + \sqrt{\beta}$, *then the value of* $(\alpha + \beta - \alpha\beta + 2)^4$ *must be*

Sol. (400), We know $(\cos 27° + \sin 27°)^2$

$= 1 + \sin 54 = 1 + \cos 36°$

$\Rightarrow \quad \cos 27° + \sin 27° = \sqrt{(1 + \cos 36°)} \qquad [\because \text{LHS} > 0]$

Also, $\quad \cos 27° - \sin 27° = \sqrt{(1 - \cos 36°)}$

$[\because \cos 27° > \sin 27°]$

$\therefore \quad 2\sin 27° = \sqrt{(1 + \cos 36°)} - \sqrt{(1 - \cos 36°)}$

$$= \sqrt{\left(1 + \left(\frac{\sqrt{5}+1}{4}\right)\right)} - \sqrt{\left(1 - \left(\frac{\sqrt{5}+1}{4}\right)\right)}$$

$\therefore \quad 4\sin 27° = \sqrt{(5 + \sqrt{5})} - \sqrt{(3 - \sqrt{5})}$

On comparing, we get

$$\alpha = 5 + \sqrt{5}, \beta = 3 - \sqrt{5}$$

$\therefore \quad \alpha + \beta = 8, \alpha\beta = 10 - 2\sqrt{5}$

$\alpha + \beta - \alpha\beta + 2 = 2\sqrt{5}$

$\therefore (\alpha + \beta - \alpha\beta + 2)^4 = 400$

Ex. 108. *If* $0 < A < \frac{\pi}{2}$ *and* $\sin A + \cos A + \tan A + \cot A + \sec A + \text{cosec}\,A = 7$ *and* $\sin A$ *and* $\cos A$ *are the roots of the equation* $4x^2 - 3x + a = 0$, *then the value of* $25a$ *must be*

Sol. (28) $\sin A$ and $\cos A$ are the roots of the equation $4x^2 - 3x + a = 0$, then

$$\sin A + \cos A = \frac{3}{4}, \sin A\cos A = \frac{a}{4} \qquad \ldots\text{(i)}$$

Also, $\sin A + \cos A + \tan A + \cot A + \sec A + \text{cosec}\,A = 7$

$$\Rightarrow \quad (\sin A + \cos A) + \left(\frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}\right) + \left(\frac{1}{\cos A} + \frac{1}{\sin A}\right) = 7$$

$$\Rightarrow \quad (\sin A + \cos A) + \frac{1}{\sin A \cos A} + \frac{(\sin A + \cos A)}{\sin A \cos A} = 7$$

$$\Rightarrow \quad \frac{3}{4} + \frac{4}{a} + \frac{3}{a} = 7$$

$$\Rightarrow \quad \frac{3}{4} + \frac{7}{a} = 7$$

$$\Rightarrow \quad \frac{7}{a} = 7 - \frac{3}{4} = \frac{25}{4}$$

$$\therefore \quad 25a = 28$$

● **Ex. 109.** *Given that,* $f(n\theta) = \dfrac{2\sin\theta}{\cos 2\theta - \cos 4n\theta}$, *and* $f(\theta) + f(2\theta) + f(3\theta) + \ldots + f(n\theta) = \dfrac{\sin \lambda\theta}{\sin\theta \sin\mu\theta}$, *then the value of* $\mu - \lambda$ *is*

Sol. (1) $f(n\theta) = \dfrac{2\sin 2\theta}{\cos 2\theta - \cos 4n\theta}$

$$= \frac{2\sin 2\theta}{2\sin(2n+1)\theta \sin(2n-1)\theta}$$

$$= \frac{\sin((2n+1)\theta - (2n-1)\theta)}{\sin(2n+1)\theta \sin(2n-1)\theta}$$

$$= \frac{\sin(2n+1)\theta \cos(2n-1)\theta - \cos(2n+1)\theta \sin(2n-1)\theta}{\sin(2n+1)\theta \sin(2n-1)\theta}$$

$$= \cot(2n-1)\theta - \cot(2n+1)\theta$$

$\therefore f(\theta) + f(2\theta) + f(3\theta) + \ldots + f(n\theta)$

$$= \cot\theta - \cot(2n+1)\theta$$

$$= \frac{\sin(2n+1)\theta\cos\theta - \cos(2n+1)\theta\sin\theta}{\sin(2n+1)\theta\sin\theta}$$

$$= \frac{\sin 2n\theta}{\sin(2n+1)\theta\sin\theta}$$

$\therefore \quad \pi = 2n$ and $\mu = 2n+1$

Hence, $\mu - \lambda = 2n + 1 - 2n = 1$

● **Ex. 110.** *If* $\dfrac{1}{\cos 290^\circ} + \dfrac{1}{\sqrt{3}\sin 250^\circ} = \lambda$, *then the value of* $9\lambda^4 + 81\lambda^2 + 97$ *must be*

Sol. (785) Here, $\cos 290^\circ = \cos(270^\circ + 20^\circ) = \sin 20^\circ$ and $\sin 250^\circ = \sin(270^\circ - 20^\circ) = -\cos 20^\circ$

$\therefore$ The given expression $= \dfrac{1}{\sin 20^\circ} - \dfrac{1}{\sqrt{3}\cos 120^\circ} = \lambda$

$$\Rightarrow \quad \frac{1}{\sin 20^\circ} - \frac{\cos 60^\circ}{\sin 60^\circ \cos 20^\circ} = \lambda$$

$$\Rightarrow \quad \frac{\sin 60^\circ \cos 20^\circ - \cos 60^\circ \sin 20^\circ}{\sin 20^\circ \cos 20^\circ \sin 60^\circ} = \lambda$$

$$\Rightarrow \quad \frac{\sin(60^\circ - 20^\circ)}{\dfrac{\sin 40^\circ}{2} \times \dfrac{\sqrt{3}}{2}} = \lambda$$

$$\therefore \quad \lambda = \frac{4}{\sqrt{3}}$$

$$\Rightarrow \quad \lambda^2 = \frac{16}{3}$$

Then, $9\lambda^4 + 81\lambda^2 + 97 = 9 \times \dfrac{256}{9} + 81 \times \dfrac{16}{3} + 97$

$$= 256 + 432 + 97 = 785$$

● **Ex. 111.** *If* $\log_{10}\sin x + \log_{10}\cos x = -1$ *and* $\log_{10}(\sin x + \cos x) = \dfrac{(\log_{10} n) - 1}{2}$, *then the value of 'n/3' is ....*

Sol. (4) Given, $\log_{10}\left(\dfrac{\sin 2x}{2}\right) = -1$

or $\quad \dfrac{\sin 2x}{2} = \dfrac{1}{10}$

or $\quad \sin 2x = \dfrac{1}{5}$

Also $\quad \log_{10}(\sin x + \cos x) = \dfrac{\log_{10}\left(\dfrac{n}{10}\right)}{2}$

or $\quad \log_{10}(\sin x + \cos x)^2 = \log_{10}\left(\dfrac{n}{10}\right)$

or $\quad 1 + \sin 2x = \dfrac{n}{10}$

or $\quad 1 + \dfrac{1}{5} = \dfrac{n}{10}$ or $\dfrac{6}{5} = \dfrac{n}{10}$

or $\quad \dfrac{n}{3} = 4$

● **Ex. 112.** *If* $498[16\cos x + 12\sin x] = 2k + 60$, *then the maximum value of* k *is*

Sol. (4950) $16\cos x + 12\sin x = \sqrt{16^2 + 12^2}\cos(x - \alpha)$, $\alpha = \tan^{-1}\left(\dfrac{3}{4}\right)$.

$$\Rightarrow \quad |2k + 60| \le 498 \times 20 \text{ as } |\cos(x-\alpha)| \le 1$$

$$\Rightarrow \quad k \le 4950$$

Ex. 113. *If* $a\tan\alpha + \sqrt{(a^2-1)}\tan\beta + \sqrt{(a^2+1)}\tan\gamma = 2a$, *where a is constant and* α, β, γ *are variable angles. Then the least value of 2727* $(\tan^2\alpha + \tan^2\beta + \tan^2\gamma)$ *must be*

Sol. (3636) We have,

$$(a\tan\beta - \sqrt{(a^2-1)}\tan\alpha)^2 + (\sqrt{(a^2-1)}\tan\gamma - \sqrt{(a^2+1)}\tan\beta)^2 + (\sqrt{(a^2+1)}\tan\alpha - a\tan\gamma)^2 \geq 0$$

$$\Rightarrow (a^2 + a^2 - 1 + a^2 + 1)(\tan^2\alpha + \tan^2\beta + \tan^2\gamma) - \{a\tan\alpha + \sqrt{(a^2-1)}\tan\beta + \sqrt{(a^2+1)}\tan\gamma\}^2 \geq 0$$

(using Lagrange's identity)

$$\Rightarrow 3a^2(\tan^2\alpha + \tan^2\beta + \tan^2\gamma) - (2a)^2 \geq 0$$

$$\therefore \quad 3(\tan^2\alpha + \tan^2\beta + \tan^2\gamma) \geq 4$$

Hence, $2727 \quad (\tan^2\alpha + \tan^2\beta + \tan^2\gamma) \geq 3636$

$\therefore$ Least value is 3636.

Ex. 114. *If* $\frac{\tan x}{2} = \frac{\tan y}{3} = \frac{\tan z}{5}$, $x + y + z = \pi$ *and* $\tan^2 x + \tan^2 y + \tan^2 z = \frac{38}{K}$ *then* $K = ...$

Sol. (3) $\tan x = 2t$, $\tan y = 3t$, $\tan z = 5t$

Also $\quad x + y + z = \pi$

$\therefore \tan x + \tan y + \tan z = \tan x \tan y \tan z$

$$\Rightarrow \quad t^2 = \frac{1}{3}$$

$$\Rightarrow \quad \tan^2 x + \tan^2 y + \tan^2 z = t^2(4 + 9 + 25) = 38t^2,$$

$$\therefore \quad K = 3$$

Ex. 115. *If* $\tan\left(\frac{3\pi}{11}\right) + 4\sin\left(\frac{2\pi}{11}\right) = \lambda$, *then the value of* $1 + \lambda^2 + \lambda^4 + \lambda^6$ *must be.*

Sol. (1464) Let $\lambda = \tan\left(\frac{3\pi}{11}\right) + 4\sin\left(\frac{2\pi}{11}\right)$

$$= \frac{1}{\cos\left(\frac{3\pi}{11}\right)}\left\{\sin\left(\frac{3\pi}{11}\right) + 4\sin\left(\frac{2\pi}{11}\right)\cos\left(\frac{3\pi}{11}\right)\right\}$$

$$\lambda\cos\left(\frac{3\pi}{11}\right) = \sin\left(\frac{3\pi}{11}\right) + 4\sin\left(\frac{2\pi}{11}\right)\cos\left(\frac{3\pi}{11}\right)$$

$$\lambda^2\cos^2\left(\frac{3\pi}{11}\right) = \sin^2\left(\frac{3\pi}{11}\right) + 16\sin^2\left(\frac{2\pi}{11}\right)\cos^2\left(\frac{3\pi}{11}\right) + 8\sin\left(\frac{2\pi}{11}\right)\sin\left(\frac{3\pi}{11}\right)\cos\left(\frac{3\pi}{11}\right)$$

$$\lambda^2\left(2\cos^2\left(\frac{3\pi}{11}\right)\right) = 2\sin^2\left(\frac{3\pi}{11}\right) + 32\sin^2\left(\frac{2\pi}{11}\right)\cos^2\left(\frac{3\pi}{11}\right) + 8\sin\left(\frac{2\pi}{11}\right)\sin\left(\frac{6\pi}{11}\right)$$

$$= \left(1 - \cos\left(\frac{6\pi}{11}\right)\right) + 8\left(1 - \cos\frac{4\pi}{11}\right)\left(1 + \cos\frac{6\pi}{11}\right) + 4\left(\cos\frac{4\pi}{11} - \cos\frac{8\pi}{11}\right)$$

$$= 9 + 7\cos\left(\frac{6\pi}{11}\right) - 4\cos\left(\frac{4\pi}{11}\right) - 8\cos\left(\frac{4\pi}{11}\right)\cos\left(\frac{6\pi}{11}\right) - 4\cos\left(\frac{8\pi}{11}\right)$$

$$= 9 + 7\cos\left(\frac{6\pi}{11}\right) - 4\cos\left(\frac{4\pi}{11}\right) - 4\left(\cos\left(\frac{10x}{11}\right) + \cos\left(\frac{2\pi}{11}\right)\right) - 4\cos\left(\frac{8\pi}{11}\right)$$

$$= 9 + 11\cos\left(\frac{6\pi}{11}\right) - 4\left\{\cos\left(\frac{2\pi}{11}\right) + \cos\left(\frac{4\pi}{11}\right) + \cos\left(\frac{6\pi}{11}\right) + \cos\left(\frac{8\pi}{11}\right) + \cos\left(\frac{10\pi}{11}\right)\right\}$$

$$= 9 + 11\cos\left(\frac{6\pi}{11}\right) - \frac{4\cdot\cos\left(\frac{6\pi}{11}\right)\sin\left(\frac{5\pi}{11}\right)}{\sin\left(\frac{\pi}{11}\right)}$$

$$= 9 + 11\cos\left(\frac{6\pi}{11}\right) - \frac{2\left\{\sin\pi - \sin\left(\frac{\pi}{11}\right)\right\}}{\sin\left(\frac{\pi}{11}\right)}$$

$$= 9 + 11\cos\left(\frac{6\pi}{11}\right) + 2$$

$$= 11\left(1 + \cos\left(\frac{6\pi}{11}\right)\right) = 11\left(2\cos^2\left(\frac{3\pi}{11}\right)\right)$$

$$\therefore \quad \lambda^2 = 11$$

Then, $1 + \lambda^2 + \lambda^4 + \lambda^6 = 1 + 11 + 121 + 1331 = 1464$

Subjective Type Examples

Ex. 116. *For all* θ *in* $[0, \pi/2]$, *show that* $\cos(\sin\theta) > \sin(\cos\theta)$.

Sol. We know,

$$\cos\theta + \sin\theta = \sqrt{2}\left[\frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta\right]$$

$$= \sqrt{2}\left[\sin\frac{\pi}{4}\cos\theta + \cos\frac{\pi}{4}\sin\theta\right]$$

$$= \sqrt{2}\sin\left(\frac{\pi}{4} + \theta\right)$$

$$\Rightarrow \quad \cos\theta + \sin\theta \le \sqrt{2} < \frac{\pi}{2} \qquad \{\text{as } \sqrt{2} = 1.414\}$$

$$\Rightarrow \quad \cos\theta + \sin\theta < \frac{\pi}{2} \qquad [\pi/2 = 1.57 \text{ approx}]$$

$$\Rightarrow \quad \cos\theta < \frac{\pi}{2} - \sin\theta$$

On taking sine both sides;

$$\sin(\cos\theta) < \sin\left(\frac{\pi}{2} - \sin\theta\right)$$

$$\Rightarrow \quad \sin(\cos\theta) < \cos(\sin\theta)$$

$$\therefore \quad \cos(\sin\theta) > \sin(\cos\theta)$$

Alternate Method

For $0 \le x \le \dfrac{\pi}{2}$

$$x \ge \sin x \qquad \text{...(i)}$$

Replace x by $\cos\theta$, we get

$$\cos\theta \ge \sin(\cos\theta) \qquad \text{...(ii)}$$

Also, we know $\cos\theta$ is decreasing for $0 \le \theta \le \dfrac{\pi}{2}$.

As $\theta_1 < \theta_2 \Rightarrow \cos\theta_1 > \cos\theta_2$ when $\theta_1, \theta_2 \in [0, \pi/2]$

$\therefore$ Taking cos on both side of Eq. (i) and putting θ for x, we get

$$\cos\theta \le \cos(\sin\theta) \qquad \text{...(iii)}$$

Using Eqs. (ii) and (iii),

$$\cos(\sin\theta) \ge \cos\theta \ge \sin(\cos\theta)$$

$$\Rightarrow \quad \cos(\sin\theta) > \sin(\cos\theta)$$

Ex. 117. *Show that* $2^{\sin x} + 2^{\cos x} \ge 2^{1 - \frac{1}{\sqrt{2}}}$ *for all real* x.

Sol. Clearly, $2^{\sin x}$ and $2^{\cos x}$ are positive, so their AM $\ge$ GM

$$\therefore \quad \frac{2^{\sin x} + 2^{\cos x}}{2} \ge \sqrt{2^{\sin x} . 2^{\cos x}} = \sqrt{2^{\sin x + \cos x}} \qquad \text{...(i)}$$

As we know,

$$\sin x + \cos x \ge -\sqrt{2}$$

$$\left[\text{using } -\sqrt{a^2 + b^2} \le a\sin x + b\cos x \le \sqrt{a^2 + b^2}\right]$$

$$2^{\sin x + \cos x} \ge 2^{-\sqrt{2}}$$

$$\text{or} \quad \sqrt{2^{\sin x + \cos x}} \ge 2^{-\sqrt{2}/2}$$

$$\text{or} \quad \sqrt{2^{\sin x + \cos x}} \ge 2^{-1/\sqrt{2}} \qquad \text{...(ii)}$$

From Eqs. (i) and (ii),

$$\frac{2^{\sin x} + 2^{\cos x}}{2} \ge \sqrt{2^{\sin x + \cos x}} \ge 2^{-1/\sqrt{2}}$$

$$\therefore \quad 2^{\sin x} + 2^{\cos x} \ge 2 \cdot 2^{-1/\sqrt{2}}$$

$$\text{or} \quad 2^{\sin x} + 2^{\cos x} \ge 2^{1 - 1/\sqrt{2}} \text{ for all values of } x.$$

Ex. 118. *Eliminate* θ *and* ϕ *if*

$$a\sin^2\theta + b\cos^2\theta = m$$

$$b\sin^2\phi + a\cos^2\phi = n$$

and $\quad a\tan\theta = b\tan\phi$

Sol. Dividing $a\sin^2\theta + b\cos^2\theta = m$ by $\cos^2\theta$, we get

$$a\tan^2\theta + b = m\sec^2\theta$$

$$\text{or} \quad (a - m)\tan^2\theta = (m - b) \qquad \text{...(i)}$$

Dividing $b\sin^2\phi + a\cos^2\phi = n$ by $\cos^2\phi$, we get

$$b\tan^2\phi + a = n\sec^2\phi$$

$$\text{or} \quad (b - n)\tan^2\phi = (n - a) \qquad \text{...(ii)}$$

On dividing Eq. (i) by Eq. (ii), we get

$$\frac{(a - m)}{(b - n)} \cdot \left(\frac{\tan\theta}{\tan\phi}\right)^2 = \frac{m - b}{n - a}$$

$$\Rightarrow \quad \left(\frac{a - m}{b - n}\right) \cdot \frac{b^2}{a^2} = \frac{m - b}{n - a} \qquad [\text{given, } a\tan\theta = b\tan\phi]$$

$$\text{or} \quad b^2(a - m)(n - a) = a^2(b - n)(m - b)$$

$$\text{or} \quad b^2\{(m + n)a - a^2 - mn\} = a^2\{(m + n)b - b^2 - mn\}$$

$$\text{or} \quad (m + n)(ab^2 - a^2 b) + mn(a^2 - b^2) = 0$$

$$\text{or} \quad (m + n)ab(b - a) + mn(a - b)(a + b) = 0$$

$$\text{or} \quad (m + n)(ab) = mn(a + b)[a - b \ne 0]$$

Ex. 119. *Let* $\cos A + \cos B + \cos C = \dfrac{3}{2}$ *in a* ΔABC, *show that the triangle is equilateral.*

Sol. In a triangle, $A + B + C = \pi$

$$\Rightarrow \quad \cos A + \cos B + \cos C = 2\cos\left(\frac{A + B}{2}\right) \cdot \cos\left(\frac{A - B}{2}\right) + \cos C = \frac{3}{2}$$

$$\Rightarrow \quad 2\cos\left(\frac{\pi}{2} - \frac{C}{2}\right) \cdot \cos\left(\frac{A - B}{2}\right) + 1 - 2\sin^2\frac{C}{2} = \frac{3}{2}$$

$$\Rightarrow \quad 2\sin^2\frac{C}{2} - 2\sin\left(\frac{C}{2}\right)\cos\left(\frac{A-B}{2}\right) - 1 + \frac{3}{2} = 0$$

$$\Rightarrow \quad 4\sin^2\frac{C}{2} - 4\sin\left(\frac{C}{2}\right)\cos\left(\frac{A-B}{2}\right) + 1 = 0 \qquad \text{...(i)}$$

Now, Eq. (i) is quadratic in $(\sin C/2)$ and is real.

$$\therefore \quad D \geq 0$$

$$\Rightarrow \quad 16\cos^2\left(\frac{A-B}{2}\right) - 16 \geq 0 \Rightarrow \cos^2\left(\frac{A-B}{2}\right) - 1 \geq 0$$

$$\Rightarrow \quad \cos^2\left(\frac{A-B}{2}\right) \geq 1$$

which is only possible if $\cos^2\left(\frac{A-B}{2}\right) = 1$

$$\Rightarrow \quad \cos^2\left(\frac{A-B}{2}\right) = 1$$

$$\frac{A-B}{2} = 0$$

$$\Rightarrow \quad A = B \qquad \text{...(ii)}$$

Similarly, we can show $B = C, C = A$. Hence, the triangle is equilateral.

● **Ex. 120.** *If $\frac{\tan 3A}{\tan A} = k$, show that $\frac{\sin 3A}{\sin A} = \frac{2k}{k-1}$ and hence or otherwise prove that either $k > 3$ or $k < \frac{1}{3}$.*

Sol. $$\frac{\tan 3A}{\tan A} = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}\cdot\frac{1}{\tan A} = k$$

$$\Rightarrow \quad \frac{3 - \tan^2 A}{1 - 3\tan^2 A} = k$$

$$\Rightarrow \quad (3 - \tan^2 A) = k(1 - 3\tan^2 A)$$

$$\Rightarrow \quad (3k-1)\tan^2 A = k - 3$$

$$\Rightarrow \quad \tan^2 A = \left(\frac{k-3}{3k-1}\right) \qquad \text{...(i)}$$

Now, $$\frac{\sin 3A}{\sin A} = \frac{3\sin A - 4\sin^3 A}{\sin A} = 3 - 4\sin^2 A$$

$$\Rightarrow \quad 3 - 4\sin^2 A = 3 - \frac{4}{\operatorname{cosec}^2 A} = 3 - \frac{4}{1 + \cot^2 A}$$

$$= 3 - \frac{4}{1 + \frac{1}{\tan^2 A}}$$

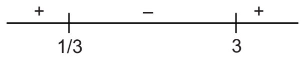
$$\Rightarrow \quad 3 - \frac{4}{1 + \left(\frac{3k-1}{k-3}\right)} \qquad \text{[using Eq. (i)]}$$

$$\Rightarrow \quad 3 - \frac{4(k-3)}{4(k-1)} = \frac{3k - 3 - k + 3}{k-1} = \frac{2k}{k-1} \qquad \text{...(ii)}$$

Again from Eq. (i),

$$\tan^2 A = \frac{k-3}{3k-1} \qquad [\tan A \neq 0 \text{ and } \tan^2 A > 0]$$

$$\Rightarrow \quad \frac{k-3}{3k-1} > 0, \text{ using number line rule.}$$



which shows $k < 1/3$ or $k > 3$

● **Ex. 121.** *Let A, B, C be three angles such that $A = \pi/4$ and $\tan B \tan C = p$. Find all possible values of p such that A, B, C are the angles of triangles.*

Sol. Let us assume ΔABC.

$$\therefore \quad A + B + C = \pi$$

$$\Rightarrow \quad B + C = \pi - \frac{\pi}{4} = \frac{3\pi}{4} \qquad \text{...(i) } [\because A = \pi/4, \text{ given}]$$

Also, $\quad 0 < B, C < 3\pi/4$

$$\Rightarrow \quad \tan B \tan C = p$$

$$\Rightarrow \quad \frac{\sin B\cdot\sin C}{\cos B\cdot\cos C} = \frac{p}{1}$$

$$\Rightarrow \quad \frac{\sin B\cdot\sin C + \cos B\cdot\cos C}{\cos B\cdot\cos C - \sin B\cdot\sin C} = \frac{1+p}{1-p}$$

$$\Rightarrow \quad \frac{\cos(B-C)}{\cos(B+C)} = \frac{1+p}{1-p}$$

$$\Rightarrow \quad \cos(B-C) = \left(\frac{1+p}{1-p}\right)\cos\left(\frac{3\pi}{4}\right)$$

[using Eq. (i), $B + C = 3\pi/4$]

$$\Rightarrow \quad \cos(B-C) = \frac{1+p}{\sqrt{2}(p-1)} \qquad \text{...(ii)}$$

Since, B or C can vary from 0 to $3\pi/4$.

$$\therefore \quad 0 \leq (B - C) < 3\pi/4$$

$$\Rightarrow \quad -\frac{1}{\sqrt{2}} < \cos(B-C) \leq 1 \qquad \text{...(iii)}$$

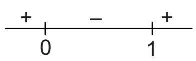
From Eqs. (ii) and (iii), we get

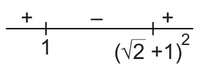
$$-\frac{1}{\sqrt{2}} < \frac{1+p}{\sqrt{2}(p-1)} \leq 1$$

$$\therefore \quad -\frac{1}{\sqrt{2}} < \frac{1+p}{\sqrt{2}(p-1)} \text{ and } \frac{1+p}{\sqrt{2}(p-1)} \leq 1$$

$$\Rightarrow \quad 0 < 1 + \frac{p+1}{p-1} \text{ and } \frac{(p+1) - \sqrt{2}(p-1)}{\sqrt{2}(p-1)} \leq 0$$

$$\Rightarrow \quad \frac{2p}{(p-1)} > 0 \text{ and } \frac{[p - (\sqrt{2}+1)^2]}{(p-1)} \geq 0$$





$$\Rightarrow \quad p < 0 \text{ or } p > 1 \text{ and } p < 1 \text{ or } p \geq (\sqrt{2}+1)^2$$

The combining above expressions;

$$p < 0 \text{ or } p \geq (\sqrt{2}+1)^2$$

i.e. $$p \in (-\infty, 0) \cup [(\sqrt{2}+1)^2, \infty).$$

Ex. 122. *If ABC is a triangle and* $\tan\frac{A}{2}, \tan\frac{B}{2}, \tan\frac{C}{2}$ *are in HP, then find minimum value of* $\cot\frac{B}{2}$.

Sol. $A+B+C=\pi$

$$\Rightarrow \frac{A}{2}+\frac{B}{2}=\frac{\pi}{2}-\frac{C}{2}$$

$$\Rightarrow \cot\left(\frac{A}{2}+\frac{B}{2}\right)=\cot\left(\frac{\pi}{2}-\frac{C}{2}\right)$$

$$\Rightarrow \frac{\cot\frac{A}{2}\cdot\cot\frac{B}{2}-1}{\cot\frac{A}{2}+\cot\frac{B}{2}}=\tan\left(\frac{C}{2}\right)=\frac{1}{\cot\left(\frac{C}{2}\right)}$$

$$\Rightarrow \cot\frac{A}{2}\cdot\cot\frac{B}{2}\cdot\cot\frac{C}{2}=\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2} \quad \text{...(i)}$$

But $\tan\frac{A}{2}, \tan\frac{B}{2}, \tan\frac{C}{2}$ are in HP

$\Rightarrow$ $\cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2}$ are in AP

$$\therefore \cot\frac{A}{2}+\cot\frac{C}{2}=2\cot\frac{B}{2} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$\cot\frac{A}{2}\cdot\cot\frac{B}{2}\cot\frac{C}{2}=3\cot\frac{B}{2}$$

$$\Rightarrow \cot\frac{A}{2}\cdot\cot\frac{C}{2}=3 \quad \text{...(iii)}$$

As we know, AM $\geq$ GM

$$\Rightarrow \frac{\cot\frac{A}{2}+\cot\frac{C}{2}}{2}\geq\sqrt{\cot\frac{A}{2}\cdot\cot\frac{C}{2}}$$

$$\Rightarrow \frac{2\cot\frac{B}{2}}{2}\geq\sqrt{3} \quad \text{[\{using Eq. (iii)]}$$

$$\Rightarrow \cot\frac{B}{2}\geq\sqrt{3}$$

$\therefore$ Minimum value of $\cot\frac{B}{2}$ is $\sqrt{3}$.

Ex 123. *(i) If* $\tan A-\tan B=x$ *and* $\cot B-\cot A=y$.

Prove that $\cot(A-B)=\frac{1}{x}+\frac{1}{y}$.

(ii) If $2\cos A=x+\frac{1}{x}$, $2\cos B=y+\frac{1}{y}$, *then show that* $2\cos(A-B)=\frac{x}{y}+\frac{y}{x}$.

Sol. (i) If $\cot B-\cot A=y \Rightarrow \frac{\tan A-\tan B}{\tan A\tan B}=y$

$$\therefore \frac{x}{y}=\tan A\tan B$$

Now, $$\cot(A-B)=\frac{1}{\tan(A-B)}=\frac{1+\tan A\tan B}{\tan A-\tan B}=\frac{1+\frac{x}{y}}{x}=\frac{1}{x}+\frac{1}{y}=\text{RHS}$$

(ii) $2\cos A=x+\frac{1}{x}$, since $4\sin^2 A=4$

$$-4\cos^2 A=4-\left(x+\frac{1}{x}\right)^2$$

$$\therefore 4\sin^2 A=-\left[\left(x+\frac{1}{x}\right)^2-4\right]$$

$$\text{or} \quad 4\sin^2 A=i^2\left[\left(x-\frac{1}{x}\right)^2\right]$$

$$\Rightarrow 2\sin A=i\left(x-\frac{1}{x}\right) \quad \text{...(i)}$$

Similarly, $$2\cos B=y+\frac{1}{y}$$

$$\Rightarrow 2\sin B=i\left(y-\frac{1}{y}\right) \quad \text{...(ii)}$$

Now, $2\cos(A-B)=2\,[\cos A\cos B+\sin A\sin B]$

$$=\frac{2}{4}\left[\left(x+\frac{1}{x}\right)\left(y+\frac{1}{y}\right)+i^2\left(x-\frac{1}{x}\right)\left(y-\frac{1}{y}\right)\right]$$

$$=\frac{1}{2}\left[\left\{xy+\frac{1}{xy}+\frac{y}{x}+\frac{x}{y}\right\}-\left\{xy+\frac{1}{xy}-\frac{y}{x}-\frac{x}{y}\right\}\right]$$

$$=\frac{1}{2}\left[2\frac{y}{x}+2\frac{x}{y}\right]=\frac{x}{y}+\frac{y}{x}=\text{RHS}$$

Ex. 124. *If* $\tan\theta\tan\phi=\sqrt{\frac{a-b}{a+b}}$, *then prove that* $(a-b\cos 2\theta)(a-b\cos 2\phi)$ *is independent of* θ *and* ϕ.

Sol. Let us put,

$$\tan\theta=t_1 \text{ and } \tan\phi=t_2$$

$$\therefore t_1^2.t_2^2=\frac{a-b}{a+b} \quad \text{...(i)}$$

$$\left\{\text{given}, \tan\theta\tan\phi=\sqrt{\frac{a-b}{a+b}}\right\}$$

Also, $$\cos 2\theta=\frac{1-\tan^2\theta}{1+\tan^2\theta}=\frac{1-t_1^2}{1+t_1^2} \quad \text{...(ii)}$$

$$\cos 2\phi=\frac{1-\tan^2\phi}{1+\tan^2\phi}=\frac{1-t_2^2}{1+t_2^2} \quad \text{...(iii)}$$

Now, $(a-b\cos 2\theta).(a-b\cos 2\phi)$

$$= \left\{a - b\left(\frac{1-t_1^2}{1+t_1^2}\right)\right\}\cdot\left\{a - b\left(\frac{1-t_2^2}{1+t_2^2}\right)\right\} \text{[using Eqs. (ii) and (iii)]}$$

$$= \left\{\frac{(a-b)+(a+b)\,t_1^2}{1+t_1^2}\right\}\cdot\left\{\frac{(a-b)+(a+b)\,t_2^2}{1+t_2^2}\right\}$$

$$= \frac{(a+b)}{(1+t_1^2)}\left[\frac{a-b}{a+b}+t_1^2\right]\cdot\frac{(a+b)}{(1+t_2^2)}\left[\frac{a-b}{a+b}+t_2^2\right]$$

$$= \frac{(a+b)}{(1+t_1^2)}[t_1^2\,t_2^2+t_1^2]\cdot\frac{(a+b)}{(1+t_2^2)}[t_1^2t_2^2+t_2^2] \quad \text{[using Eq. (i)]}$$

$$= (a+b)^2\cdot(t_1^2.t_2^2) = (a+b)^2\cdot\frac{(a-b)}{(a+b)} = (a^2-b^2)$$

So, $(a - b\cos 2\theta)(a - b\cos 2\phi) = a^2 - b^2$, which is independent of θ and ϕ.

Ex. 125. *Find all possible real values of x and y satisfying.*

$$\sin^2 x + 4\sin^2 y - \sin x - 2\sin y - 2\sin x\,\sin y + 1 = 0,$$
$$\forall\; x, y \in [0, \pi/2]$$

Sol. Given, equation can be rewritten as,

$$\sin^2 x - \sin x(1 + 2\sin y) + (4\sin^2 y - 2\sin y + 1) = 0$$

$$\Rightarrow \sin x = \frac{(1+2\sin y) \pm \sqrt{(1+2\sin y)^2 - 4(4\sin^2 y - 2\sin y + 1)}}{2}$$

$$= \frac{(1+2\sin y) \pm \sqrt{-3 - 12\sin^2 y + 12\sin y}}{2}$$

$$= \frac{(1+2\sin y) \pm \sqrt{-3(2\sin y - 1)^2}}{2} \quad \text{...(i)}$$

Since, $\sin x$ is real.

$\therefore$ From equation (i) is real only if,

$$2\sin y - 1 = 0 \text{ or } \sin y = \frac{1}{2} \quad \text{and} \quad \sin x = \frac{1+1}{2} = 1.$$

$$\Rightarrow \quad y = \frac{\pi}{6} \text{ and } x = \frac{\pi}{2} \text{ as } x, y \in [0, \pi/2].$$

Ex. 126. *Find the roots of the following cubic equations* $2x^3 - 3x^2\cos(A-B) - 2x\cos^2(A+B) + \sin 2A\,\sin 2B\cos(A-B) = 0.$

Sol. We know,

$$\sin 2A \sin 2B = \frac{1}{2}[\cos(2A-2B) - \cos(2A-2B)]$$

$$= \frac{1}{2}[2\cos^2(A-B) - 1 - 2\cos^2(A+B) + 1]$$

$$= \cos^2(A-B) - \cos^2(A+B)$$

$$\therefore \quad \sin 2A.\sin 2B = \cos^2(A-B) - \cos^2(A+B) \quad \text{...(i)}$$

Now, $2x^3 - 3x^2\cos(A-B) - 2x\cos^2(A+B) + \sin 2A.\sin 2B.\cos(A-B) = 0$

$$\Rightarrow 2x^3 - 3x^2\cos(A-B) - 2x\cos^2(A+B) + \cos^3(A-B) - \cos^2(A+B).\cos(A-B) = 0$$

By inspection, we find that $x = -\frac{1}{2}\cos(A-B)$ because

$$\left(-\frac{1}{4} - \frac{3}{4} + 1\right)\cos^3(A-B) + \cos^2(A+B)\cos(A-B) - \cos^2(A+B)\cos(A-B) = 0$$

Hence, $2x + \cos(A-B)$ is factor of the given equation which when divided by it, given the other factor as,

$$x^2 - 2x\cos(A-B) + \cos^2(A-B) - \cos^2(A+B) = 0$$

$$\text{So, } x = \frac{2\cos(A-B) \pm \sqrt{4\cos^2(A-B) - 4\cos^2(A-B) + 4\cos^2(A+B)}}{2}$$

$$x = \frac{2\cos(A-B) \pm 2\cos(A+B)}{2}$$

$x = \cos(A-B) + \cos(A+B)$ or $\cos(A-B) - \cos(A+B)$

Hence, the roots are,

$$2\cos A\cos B,\ 2\sin A\sin B \text{ and } -\frac{1}{2}\cos(A-B).$$

Ex. 127. *If* $m^2 + m'^2 + 2mm'\cos\theta = 1.$

$$n^2 + n'^2 + 2nn'\cos\theta = 1$$

and $mn + m'n' + (mn' + m'n)\cos\theta = 0$, *then prove that* $m^2 + n^2 = \operatorname{cosec}^2\theta.$

Sol. $m^2 + m'^2 + 2mm'\cos\theta = 1$

or $\quad (m^2\cos^2\theta + m^2\sin^2\theta) + m'^2 + 2mm'\cos\theta = 1$

or $\quad m^2\cos^2\theta + 2mm'\cos\theta + m'^2 = 1 - m^2\sin^2\theta$

or $\quad (m\cos\theta + m')^2 = 1 - m^2\sin^2\theta \quad$ ...(i)

Similarly, $\quad n^2 + n'^2 + 2nn'\cos\theta = 1$

$\Rightarrow \quad (n\cos\theta + n')^2 = 1 - n^2\sin^2\theta \quad$ ...(ii)

Finally, $mn + m'n' + (mn' + m'n)\cos\theta = 0$

$$\Rightarrow (mn\cos^2\theta + mn\sin^2\theta) + m'n' + mn'\cos\theta + m'n\cos\theta = 0$$

$$\Rightarrow mn\cos^2\theta + m'n\cos\theta + m'n' + mn'\cos\theta = -mn\sin^2\theta$$

$$\Rightarrow n\cos\theta(m\cos\theta + m') + n'(m' + m\cos\theta) = -mn\sin^2\theta$$

$$\Rightarrow (m\cos\theta + m')(n\cos\theta + n') = -mn\sin^2\theta$$

$$\text{or} \quad (m\cos\theta + m')^2(n\cos\theta + n')^2 = m^2n^2\sin^4\theta$$

$$\Rightarrow (1 - m^2\sin^2\theta)(1 - n^2\sin^2\theta) = m^2n^2\sin^4\theta.$$

[using Eqs. (i) and (ii)]

$$\Rightarrow 1 - (m^2 + n^2)\sin^2\theta + m^2n^2\sin^4\theta = m^2n^2\sin^4\theta$$

$$\Rightarrow (m^2 + n^2)\sin^2\theta = 1$$

$$\Rightarrow m^2 + n^2 = \operatorname{cosec}^2\theta$$

● **Ex 128.** *Prove that from the equality*

$$\frac{\sin^4\alpha}{a}+\frac{\cos^4\alpha}{b}=\frac{1}{a+b}$$ *follows the relation,*

$$\frac{\sin^8\alpha}{a^3}+\frac{\cos^8\alpha}{b^3}=\frac{1}{(a+b)^3} \text{ and}$$

$$\frac{\sin^{4n}x}{a^{2n-1}}+\frac{\cos^{4n}x}{b^{2n-1}}=\frac{1}{(a+b)^{2n-1}}.$$

Sol. Given condition can be rewritten as,

$$b(\sin^2\alpha)^2+a\cos^4\alpha=\frac{ab}{a+b}$$

$$\Rightarrow \quad b(1-\cos^2\alpha)^2+a\cos^4\alpha=\frac{ab}{a+b}$$

$$\Rightarrow \quad b(\cos^4\alpha-2\cos^2\alpha+1)+a\cos^4\alpha=\frac{ab}{a+b}$$

$$\Rightarrow \quad (a+b)^2\cos^4\alpha-2b(a+b)\cos^2\alpha+b(a+b)=ab$$

$$(a+b)^2\cos^4\alpha-2b(a+b)\cos^2\alpha+b^2=0$$

$$\Rightarrow \quad [(a+b)\cos^2\alpha-b]^2=0$$

$$\Rightarrow \quad \cos^2\alpha=\frac{b}{a+b} \Rightarrow \sin^2\alpha=\frac{a}{a+b} \qquad \text{...(i)}$$

$$\therefore \quad \frac{\sin^8\alpha}{a^3}+\frac{\cos^8\alpha}{b^3}=\frac{a^4}{a^3(a+b)^4}+\frac{b^4}{b^3(a+b)^4}$$

$$=\frac{a}{(a+b)^4}+\frac{b}{(a+b)^4}$$

$$=\frac{a+b}{(a+b)^4}=\frac{1}{(a+b)^3}$$

$$\therefore \quad \frac{\sin^8\alpha}{a^3}+\frac{\cos^8\alpha}{b^3}=\frac{1}{(a+b)^3}$$

Now, $$\frac{\sin^{4n}x}{a^{2n-1}}+\frac{\cos^{4n}x}{b^{2n-1}}=\frac{a^{2n}}{a^{2n-1}(a+b)^{2n}}+\frac{b^{2n}}{b^{2n-1}(a+b)^{2n}}$$

$$=\frac{a+b}{(a+b)^{2n}}=\frac{1}{(a+b)^{2n-1}}.$$

● **Ex 129.** *If* $a_{r+1}=\sqrt{\frac{1}{2}(1+a_r)}$, *the prove that*

$$\cos\left(\frac{\sqrt{1-a_0^2}}{a_1\cdot a_2\cdot a_3\ldots\text{to}\infty}\right)=a_0.$$

Sol. Let $a_0=\cos\theta$, then $a_{r+1}=\sqrt{\frac{1}{2}(1+a_r)}$ gives

$$a_1=\sqrt{\frac{1}{2}(1+a_0)}=\sqrt{\frac{1}{2}(1+\cos\theta)}=\cos\frac{\theta}{2}$$

$$a_2=\sqrt{\frac{1}{2}(1+a_1)}=\sqrt{\frac{1}{2}\left(1+\cos\frac{\theta}{2}\right)}=\cos\frac{\theta}{2^2}$$

$$a_3=\sqrt{\frac{1}{2}(1+a_2)}=\sqrt{\frac{1}{2}\left(1+\cos\frac{\theta}{2^2}\right)}=\cos\frac{\theta}{2^3}, \text{... etc.}$$

$$\therefore \quad a_1\cdot a_2\cdot a_3\ldots a_n$$

$$\Rightarrow \quad \cos\frac{\theta}{2}\cdot\cos\frac{\theta}{2^2}\cdot\cos\frac{\theta}{2^3}\ldots\cos\frac{\theta}{2^n}$$

$$\Rightarrow \quad \frac{\cos\frac{\theta}{2}\cdot\cos\frac{\theta}{2^2}\cdot\cos\frac{\theta}{2^3}\ldots\cos\frac{\theta}{2^n}\cdot 2\sin\frac{\theta}{2^n}}{2\sin\frac{\theta}{2^n}}$$

$$\Rightarrow \quad \frac{\cos\frac{\theta}{2}\cdot\cos\frac{\theta}{2^2}\ldots\cos\frac{\theta}{2^{n-1}}\cdot\sin\frac{\theta}{2^{n-1}}}{2\sin\frac{\theta}{2^n}}$$

$$[\because 2\sin\alpha.\cos\alpha=\sin 2\alpha]$$

$$\Rightarrow \quad \frac{\cos\frac{\theta}{2}\cdot\cos\frac{\theta}{2^2}\ldots\cos\frac{\theta}{2^{n-2}}\cdot 2\sin\frac{\theta}{2^{n-2}}}{2^2\sin\frac{\theta}{2^n}}$$

$$=\ldots\ldots\ldots\ldots=\frac{\cos\frac{\theta}{2^{n-(n-1)}}\cdot\sin\frac{\theta}{2^{n-(n-1)}}}{2^{n-1}\cdot\sin\frac{\theta}{2^n}}$$

$$=\frac{\cos\frac{\theta}{2}\cdot\sin\frac{\theta}{2}}{2^{n-1}\cdot\sin\frac{\theta}{2^n}}=\frac{\sin\theta}{2^n.\sin\frac{\theta}{2^n}}$$

$$\therefore \quad a_1.a_2\ldots\text{ to }\infty=\lim_{n\to\infty}\frac{\sin\theta}{2^n.\sin\frac{\theta}{2^n}}=\lim_{n\to\infty}\frac{\sin\theta}{\theta\cdot\frac{\sin(\theta/2^n)}{(\theta/2^n)}}=\frac{\sin\theta}{\theta}$$

$$\therefore \quad \frac{\sqrt{1-a_0^2}}{a_1a_2a_3\ldots\text{to}\infty}=\frac{\sqrt{1-\cos^2\theta}}{\frac{\sin\theta}{\theta}}=\theta$$

$$\therefore \quad \cos\left(\frac{\frac{\sqrt{1-a_0^2}}{a_1}}{a_1a_2a_3\ldots\text{to}\infty}\right)=\cos\theta=a_0.$$

● **Ex. 130.** *Evaluate* $\sum_{r=2}^{n}\sin r\alpha$, *where* $(n+2)\alpha=2\pi$ *(without using formula.)*

Sol. Let $S=\sum_{r=2}^{n}\sin r\alpha=\sin 2\alpha+\sin 3\alpha+\sin 4\alpha+\ldots+\sin n\alpha$

$$\therefore \quad 2\sin\frac{\alpha}{2}\cdot S=2\sin\alpha/2\sin 2\alpha+2\sin\frac{\alpha}{2}\cdot\sin 3\alpha+2\sin\frac{\alpha}{2}\cdot\sin 4\alpha+\ldots+2\sin\alpha/2.\sin n\alpha$$

$$=\left\{\cos\frac{3\alpha}{2}-\cos\frac{5\alpha}{2}\right\}+\left\{\cos\frac{5\alpha}{2}-\cos\frac{7\alpha}{2}\right\}+\ldots+\left\{\cos\left(n-\frac{1}{2}\right)\alpha-\cos\left(n+\frac{1}{2}\right)\alpha\right\}$$

$$=\cos\frac{3\alpha}{2}-\cos\left(n+\frac{1}{2}\right)\alpha$$

$$= 2\sin\left(\frac{\frac{3\alpha}{2}+\left(n+\frac{1}{2}\right)\alpha}{2}\right)\sin\left(\frac{\left(n+\frac{1}{2}\right)\alpha-\frac{3\alpha}{2}}{2}\right)$$

$$2\sin\frac{\alpha}{2}\cdot S = 2\sin\frac{(n+2)\alpha}{2}\cdot\sin\frac{(n-1)\alpha}{2}$$

$$\Rightarrow \quad S = \frac{\sin\frac{(n-1)\alpha}{2}}{\sin\frac{\alpha}{2}}\cdot\sin\frac{(n+2)\alpha}{2}$$

$$= \frac{\sin\frac{(n-1)\alpha}{2}}{\sin\frac{\alpha}{2}}\cdot\sin\frac{2\pi}{2} = 0$$

● **Ex. 131.** *Sum the series* $\sqrt{1+\cos\alpha}+\sqrt{1+\cos 2\alpha}+\sqrt{1+\cos 3\alpha}+\ldots$ *to n terms.*

Sol. We have,

$$\sqrt{1+\cos\alpha}+\sqrt{1+\cos 2\alpha}+\sqrt{1+\cos 3\alpha}+\ldots+\sqrt{1+\cos n\alpha}$$

$$= \sqrt{2\cos^2\alpha/2}+\sqrt{2\cos^2\alpha}+\sqrt{2\cos^2\frac{3\alpha}{2}}+\ldots \text{ to } n \text{ terms}$$

$$= \sqrt{2}\left\{\cos\frac{\alpha}{2}+\cos\frac{2\alpha}{2}+\cos\frac{3\alpha}{2}+\ldots+\text{ to } n \text{ terms}\right\}$$

$$= \sqrt{2}\cdot\frac{\sin\frac{n\alpha}{4}}{\sin\frac{\alpha}{4}}\cdot\cos\left\{\frac{\alpha}{2}+(n-1)\frac{\alpha}{4}\right\} \qquad \text{\{using formula\}}$$

$$= \sqrt{2}\frac{\sin\frac{n\alpha}{4}}{\sin\frac{\alpha}{4}}.\cos\left\{(n+1)\frac{\alpha}{4}\right\}$$

● **Ex. 132.** *If* $A+B+C=\pi$, *show that*

$\cot A+\cot B+\cot C-\operatorname{cosec} A\operatorname{cosec} B\cdot\operatorname{cosec} C = \cot A\cdot\cot B\cdot\cot C$

Sol. $\text{LHS} = \dfrac{\cos A}{\sin A}+\dfrac{\cos A}{\sin B}+\dfrac{\cos C}{\sin C}-\dfrac{1}{\sin A\cdot\sin B\cdot\sin C}$

$$= \frac{\cos A\cdot\sin B\sin C+\cos B\sin A\sin C+\cos C\sin A\sin B-1}{\sin A\sin B\sin C}$$

$$= \frac{\sin C(\cos A\sin B+\cos B\sin A)+\cos C\sin A\sin B-1}{\sin A\sin B\sin C}$$

$$= \frac{\sin C\sin(A+B)+\cos C\sin A\sin B-1}{\sin A\sin B\sin C}$$

$$= \frac{\sin^2 C+\cos C\sin A\sin B-1}{\sin A\sin B\sin C}$$

[using $\sin(A+B)=\sin(\pi-C)=\sin C$]

$$= \frac{\cos C.\sin A\sin B-\cos^2 C}{\sin A\sin B\sin C}$$

$$= \frac{\cos C\{\sin A\sin B-\cos C\}}{\sin A\sin B\sin C}$$

[$\because \cos C=\cos(\pi-(A+B))=-\cos(A+B)$]

$$= \frac{\cos C\{\sin A\sin B+\cos(A+B)\}}{\sin A\sin B\sin C}$$

$$= \frac{\cos C\{\sin A\sin B+\cos A\cos B-\sin A\sin B\}}{\sin A\sin B\sin C}$$

$$= \frac{\cos A\cos B\cos C}{\sin A\sin B\sin C} = \cot A\cdot\cot B\cdot\cot C = \text{RHS}$$

● **Ex. 133.** *In* ΔABC, *if* $\cot\theta=\cot A+\cot B+\cot C$, *prove that* $\sin^3\theta=\sin(A-\theta)\sin(B-\theta)\sin(C-\theta)$.

Sol. We have, $\cot\theta=\cot A+\cot B+\cot C$

$$\Rightarrow \quad \cot(\theta)-\cot(A)=\cot B+\cot C$$

$$\Rightarrow \quad \frac{\cos\theta}{\sin\theta}-\frac{\cos A}{\sin A}=\frac{\cos B}{\sin B}+\frac{\cos C}{\sin C}$$

$$\Rightarrow \quad \frac{\cos\theta\sin A-\cos A\sin\theta}{\sin\theta\sin A}=\frac{\cos B\sin C+\sin B\cos C}{\sin B\sin C}$$

$$\Rightarrow \quad \frac{\sin(A-\theta)}{\sin A\sin\theta}=\frac{\sin(B+C)}{\sin B\sin C}$$

$$\Rightarrow \quad \sin(A-\theta)=\frac{\sin^2 A\sin\theta}{\sin B\sin C} \qquad \text{...(i)}$$

Similarly, $$\sin(B-\theta)=\frac{\sin^2 B\sin\theta}{\sin A\sin C} \qquad \text{...(ii)}$$

and $$\sin(C-\theta)=\frac{\sin^2 C\sin\theta}{\sin A\sin B} \qquad \text{...(iii)}$$

Multiplying Eqs. (i), (ii) and (iii), we get

$$\sin(A-\theta)\sin(B-\theta)\sin(C-\theta)=\sin^3\theta.$$

● **Ex. 134.** *If A, B, C and D are angles of a quadrilateral and* $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{D}{2}=\frac{1}{4}$, *then prove that* $A=B=C=D=\pi/2$.

Sol. Now, $\left(2\sin\frac{A}{2}\cdot\sin\frac{B}{2}\right)\cdot\left(2\sin\frac{C}{2}\cdot\sin\frac{D}{2}\right)=1$

$$\Rightarrow \left\{\cos\left(\frac{A-B}{2}\right)-\cos\left(\frac{A+B}{2}\right)\right\}\left\{\cos\left(\frac{C-D}{2}\right)-\cos\left(\frac{C+D}{2}\right)\right\}=1$$

Since, $A+B=2\pi-(C+D)$, the above equation becomes,

$$\left(\cos\frac{A-B}{2}-\cos\frac{A+B}{2}\right)\left(\cos\frac{C-D}{2}+\cos\frac{A+B}{2}\right)=1$$

$$\Rightarrow \cos^2\left(\frac{A+B}{2}\right)-\cos\left(\frac{A+B}{2}\right)\left\{\cos\left(\frac{A-B}{2}\right)-\cos\left(\frac{C-D}{2}\right)\right\}+1$$

$$-\cos\left(\frac{A-B}{2}\right)\cos\left(\frac{C-D}{2}\right)=0.$$

This is a quadratic equation in $\cos\left(\frac{A+B}{2}\right)$ which has real roots.

$$\Rightarrow \left\{\cos\left(\frac{A-B}{2}\right)-\cos\left(\frac{C-D}{2}\right)\right\}^2 - 4\left\{1-\cos\left(\frac{A-B}{2}\right)\cdot\cos\left(\frac{C-D}{2}\right)\right\} \ge 0$$

$$\Rightarrow \left(\cos\frac{A-B}{2}+\cos\frac{C-D}{2}\right)^2 \ge 4$$

$$\Rightarrow \cos\frac{A-B}{2}+\cos\frac{C-D}{2} \ge 2$$

Now, both $\cos\frac{A-B}{2}$ and $\cos\frac{C-D}{2} \le 1$

$$\Rightarrow \cos\frac{A-B}{2} = 1 = \cos\frac{C-D}{2}$$

$$\Rightarrow \frac{A-B}{2} = 0 = \frac{C-D}{2}$$

$$\Rightarrow A = B, C = D$$

Similarly, $A = C, B = D$

$$\Rightarrow A = B = C = D = \pi/2$$

● ***Ex. 135.*** *If α, β are two different values of θ which satisfy is $bc\cos\theta\cos\phi + ac\sin\theta\sin\phi = ab$, then prove that $(b^2+c^2-a^2)\cos\alpha\cos\beta + ac\sin\alpha\sin\beta = a^2+b^2-c^2$.*

Sol. We have, $bc\cos\theta\cos\phi = ab - ac\sin\theta\sin\phi$.

$$\Rightarrow b^2c^2\cos^2\theta\cos^2\phi = a^2b^2 + a^2c^2\sin^2\theta\sin^2\phi - 2a^2bc\sin\theta\sin\phi$$

$$\Rightarrow (a^2c^2\sin^2\phi + b^2c^2\cos^2\phi)\sin^2\theta - 2a^2bc\sin\theta\sin\phi + a^2b^2 - b^2c^2\cos^2\phi = 0$$

$$\Rightarrow \sin\alpha\sin\beta = \frac{a^2b^2 - b^2c^2\cos^2\phi}{a^2c^2\sin^2\phi + b^2c^2\cos^2\phi} \quad \text{...(i)}$$

Similarly, $ac\sin\theta\sin\phi = ab - bc\cos\phi\cos\theta$

$$\Rightarrow a^2c^2\sin^2\theta\sin^2\phi = a^2b^2 + b^2c^2\cos^2\theta\cos^2\phi - 2ab^2c\cos\theta\cos\phi$$

$$\therefore \cos\alpha\cos\beta = \frac{a^2b^2 - a^2c^2\sin^2\phi}{a^2c^2\sin^2\phi + b^2c^2\cos^2\phi} \quad \text{...(ii)}$$

On substituting the value from Eqs. (i) and (ii) in $(b^2+c^2-a^2)\cos\alpha\cos\beta + ac\sin\alpha\sin\beta$, we get

$$\Rightarrow \frac{(b^2+c^2-a^2)(a^2b^2-a^2c^2\sin^2\phi) + ac(a^2b^2-b^2c^2\cos^2\phi)}{a^2c^2\sin^2\phi + b^2c^2\cos^2\phi}$$

$$\Rightarrow (a^2+b^2-c^2) = \text{RHS}$$

● ***Ex. 136.*** *Find all number pairs x, y that satisfy the equaion;*

$$\tan^4 x + \tan^4 y + 2\cot^2 x\cot^2 y = 3 + \sin^2(x+y);\ \forall\ x, y \in \left[0, \frac{\pi}{2}\right]$$

Sol. We know, $a^4 + b^4 \ge 2a^2b^2$ {AM $\ge$ GM}

$$\therefore \tan^4 x + \tan^4 y \ge 2\tan^2 x\tan^2 y \quad \text{...(i)}$$

Equality occurring only when $\tan^2 x = \tan^2 y = 1$.

Also, $\tan^2 x\tan^2 y + \cot^2 x\cot^2 y \ge 2$ ...(ii)

Since, $a + \frac{1}{a} \ge 2$ and equality occurring only when

$a = 1$, i.e. $\tan^2 x\tan^2 y = 1$

From Eqs. (i) and (ii);

$$\tan^4 x + \tan^4 y + 2\cot^2 x\cot^2 y \ge 4 \quad \text{...(iii)}$$

Also, $\text{RHS} = 3 + \sin^2(x+y) \le 4$ ...(iv)

From Eqs. (iii) and (iv),

$$\text{LHS} = \text{RHS} = 4$$

$$\Rightarrow \tan^2 x = \tan^2 y = \tan^2 x\tan^2 y = 1$$

$$\Rightarrow \tan x = \tan y = \pm 1$$

$$\Rightarrow \tan x = \tan y = 1 \quad \{\text{as } x, y \in [0, \pi/2]\}$$

$$\therefore x = y = \pi/4$$

Only one solution i.e. $(x = \pi/4, y = \pi/4)$.

● ***Ex. 137.*** *Prove that $\tan\frac{3\pi}{11} + 4\sin\frac{2\pi}{11} = \sqrt{11}$.*

Sol. Let $y = \tan\frac{3\pi}{11} + 4\sin\frac{2\pi}{11} = \frac{1}{\cos\frac{3\pi}{11}}\left(\sin\frac{3\pi}{11} + 4\sin\frac{2\pi}{11}\cos\frac{3\pi}{11}\right)$

$$y^2.\cos^2\frac{3\pi}{11} = \sin^2\frac{3\pi}{11} + 16\sin^2\frac{2\pi}{11}\cdot\cos^2\frac{3\pi}{11} + 8\sin\frac{2\pi}{11}\cdot\cos\frac{3\pi}{11}\cdot\sin\frac{3\pi}{11}$$

$$\Rightarrow 2\cos^2\frac{3\pi}{11}y^2 = 2\sin^2\frac{3\pi}{11} + 32\sin^2\frac{2\pi}{11}\cdot\cos^2\frac{3\pi}{11} + 8\sin\frac{2\pi}{11}\cdot\sin\frac{6\pi}{11}$$

$$= \left(1-\cos\frac{6\pi}{11}\right) + 8\left(1-\cos\frac{4\pi}{11}\right)\cdot\left(1+\cos\frac{6\pi}{11}\right) + 4\left(\cos\frac{4\pi}{11}-\cos\frac{8\pi}{11}\right)$$

$$= 9 + 7\cos\frac{6\pi}{11} - 4\cos\frac{4\pi}{11} - 8\cos\frac{4\pi}{11}\cdot\cos\frac{6\pi}{11} - 4\cos\frac{8\pi}{11}$$

$$= 9 + 7\cos\frac{6\pi}{11} - 4\cos\frac{4\pi}{11} - 4\left(\cos\frac{10\pi}{11}+\cos\frac{2\pi}{11}\right) - 4\cos\frac{8\pi}{11}$$

$$= 9 + 11\cos\frac{6\pi}{11} - 4\left(\cos\frac{2\pi}{11}+\cos\frac{4\pi}{11}+\cos\frac{6\pi}{11}+\cos\frac{8\pi}{11}+\cos\frac{10\pi}{11}\right)$$

$$= 9 + 11\cos\frac{6\pi}{11} - 4\left(\frac{\cos\left(\frac{2\pi}{11}+2\cdot\frac{2\pi}{11}\right)\cdot\sin\left(\frac{5\pi}{11}\right)}{\sin\pi/11}\right)$$

$$= 9 + 11\cos\frac{6\pi}{11} - \frac{4\cos\frac{6\pi}{11}\cdot\sin\frac{5\pi}{11}}{\sin\frac{\pi}{11}}$$

$$= 9 + 11\cos\frac{6\pi}{11} - \frac{2\sin\frac{12\pi}{11}}{\sin\frac{\pi}{11}}$$

$$= 9 + 11\cos\frac{6\pi}{11} + 2 = 11\left(1 + \cos\frac{6\pi}{11}\right)$$

$$2y^2 \cdot \left(\cos^2\frac{3\pi}{11}\right) = 22\cos^2\frac{3\pi}{11} \Rightarrow y^2 = 11$$

$$\Rightarrow \quad y = \sqrt{11} \qquad [\text{as } y > 0]$$

● **Ex. 138.** *Prove that* $\sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7} = \frac{\sqrt{7}}{2}$.

Sol. Put, $7\theta = 2n\pi$, where n is any integer, then

$$4\theta = 2n\pi - 3\theta$$

$$\Rightarrow \quad \sin(4\theta) = \sin(2n\pi - 3\theta) = -\sin 3\theta \qquad \text{...(i)}$$

This means $\sin\theta$ takes the values; $0, \pm\sin\frac{2\pi}{7}, \pm\sin\frac{4\pi}{7}$ and $\pm\sin\frac{8\pi}{7}$.

Since, $\sin\frac{6\pi}{7} = -\sin\left(\frac{8\pi}{7}\right)$

From Eq. (i), we now get $2\sin 2\theta . \cos 2\theta = 4\sin^3\theta - 3\sin\theta$

$$\Rightarrow \quad 4\sin\theta\cos\theta(1 - 2\sin^2\theta) = \sin\theta(4\sin^2\theta - 3)$$

$$\Rightarrow \quad 4\cos\theta(1 - 2\sin^2\theta) = 4\sin^2\theta - 3$$

$$\Rightarrow \quad 16\cos^2\theta(1 - 2\sin^2\theta)^2 = (4\sin^2\theta - 3)^2$$

$$\Rightarrow 16(1 - \sin^2\theta)(1 - 4\sin^2\theta + 4\sin^4\theta) = 16\sin^4\theta - 24\sin^2\theta + 9$$

$$\Rightarrow \quad 64\sin^6\theta - 112\sin^4\theta + 56\sin^2\theta - 7 = 0$$

This is a cubic in $\sin^2\theta$ with the roots,

$$\sin^2\left(\frac{2\pi}{7}\right), \sin^2\left(\frac{4\pi}{7}\right), \sin^2\left(\frac{8\pi}{7}\right)$$

$\therefore$ Sum of the roots is

$$\sin^2\left(\frac{2\pi}{7}\right) + \sin^2\left(\frac{4\pi}{7}\right) + \sin^2\left(\frac{8\pi}{7}\right) = \frac{112}{64} = \frac{7}{4}$$

We already proved

$$\sin\frac{2\pi}{7}\cdot\sin\frac{4\pi}{7} + \sin\frac{4\pi}{7}\cdot\sin\frac{8\pi}{7} + \sin\frac{8\pi}{7}\sin\frac{\pi}{7} = 0$$

So, $$\left(\sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7}\right)^2 = \frac{7}{4}$$

$$\Rightarrow \quad \sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7} = \frac{\sqrt{7}}{2}$$

Alternate Method

$$x^7 - 1 = 0$$

[assuming x as the seventh root of unity]

$$x^7 = 1 + 0.i = \cos(2k\pi) + i\sin(2k\pi)$$

$$x = \left(\cos\frac{2k\pi}{7} + i\sin\frac{2k\pi}{7}\right)$$

$$\Rightarrow \quad x = e^{i2k\pi/7} \qquad [\text{where, } k = 0, 1, 2, 3, 4, 5, 6]$$

$$\Rightarrow \quad \sum_{k=0}^{6} e^{i2k\pi/7} = 0$$

$$\Rightarrow \quad 1 + \sum_{k=1}^{6} e^{i2k\pi/7} = 0$$

$$\Rightarrow \quad 1 + \sum_{k=1}^{3} (e^{i2k\pi/7} + e^{-2k\pi/7}) = 0$$

$$\Rightarrow \quad 1 + \sum_{k=1}^{3} 2.\cos\frac{2k\pi}{7} = 0$$

$$\Rightarrow \quad 1 + 2\sum_{k=1}^{3}\left(1 - 2\sin^2\frac{k\pi}{7}\right) = 0$$

$$\Rightarrow \quad 1 + 2\left[3 - 2\left(\sin^2\frac{\pi}{7} + \sin^2\frac{2\pi}{7} + \sin^2\frac{3\pi}{7}\right)\right] = 0$$

$$\Rightarrow \quad \sin^2\frac{\pi}{7} + \sin^2\frac{2\pi}{7} + \sin^2\frac{3\pi}{7} = \frac{7}{4}$$

$$\Rightarrow \quad \sin^2\frac{8\pi}{7} + \sin^2\frac{2\pi}{7} + \sin^2\frac{4\pi}{7} = \frac{7}{4}$$

$$\Rightarrow \quad \sin^2\frac{2\pi}{7} + \sin^2\frac{4\pi}{7} + \sin^2\frac{8\pi}{7} = \frac{7}{4} \qquad \text{...(i)}$$

and $$\sin\frac{2\pi}{7}\cdot\sin\frac{4\pi}{7} + \sin\frac{4\pi}{7}\cdot\sin\frac{8\pi}{7} + \sin\frac{8\pi}{7}\cdot\sin\frac{2\pi}{7}$$

$$= \frac{1}{2}\left[\cos\frac{2\pi}{7} - \cos\frac{6\pi}{7} + \cos\frac{4\pi}{7} - \cos\frac{12\pi}{7} + \cos\frac{6\pi}{7} - \cos\frac{10\pi}{7}\right]$$

$$= \frac{1}{2}\left[\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} - \cos\left(2\pi - \frac{2\pi}{7}\right) - \cos\left(2\pi - \frac{4\pi}{7}\right)\right]$$

$$= \frac{1}{2}\left[\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} - \cos\frac{2\pi}{7} - \cos\frac{4\pi}{7}\right] = 0 \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$\left(\sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7}\right)^2 = \sin^2\frac{2\pi}{7} + \sin^2\frac{4\pi}{7} + \sin^2\frac{8\pi}{7} = \frac{7}{4}$$

$$\Rightarrow \quad \sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7} = \frac{\sqrt{7}}{2}$$

● **Ex. 139.** *In a* ΔABC, $\tan A + \tan B + \tan C = k$, *then find the interval in which k should lie so that*

(A) there exists exactly one isosceles triangle ABC

(B) there exists exactly two isosceles triangle ABC

(C) can there exist three non-similar isosceles triangles for any real value of k.

Sol. Let $A = B$, then $2A + C = 180^\circ$

and $\quad 2\tan A + \tan C = k \qquad \ldots(\text{i})$

Now, $\quad 2A + C = 180^\circ$

$\Rightarrow \quad \tan 2A = -\tan C$

Also, $\quad 2\tan A + \tan C = k$

$\Rightarrow \quad 2\tan A + \tan(180 - 2A) = k$

$\Rightarrow \quad 2\tan A - \tan 2A = k$

$$\Rightarrow \quad 2\tan A - \frac{2\tan A}{1 - \tan^2 A} = k$$

$\Rightarrow \quad 2\tan A(1 - \tan^2 A - 1) = k - k\tan^2 A$

$\Rightarrow \quad 2\tan^3 A - k\tan^2 A + k = 0$

Let, $\quad \tan A = x,\ x > 0$ (as $A < 90^\circ$)

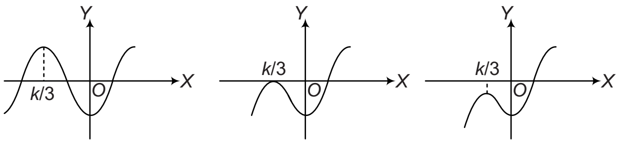
Then let, $\quad f(x) = 2x^3 - kx^2 + k \qquad \ldots(\text{ii})$

$\quad f'(x) = 6x^2 - 2kx = 0$

$\Rightarrow \quad x = k/3, 0$

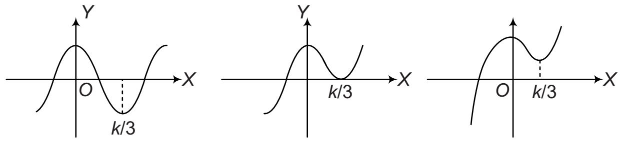
Following cases arises

(i) $k < 0$, three graphs of cubic equation (ii) are possible. Clearly, in all these case, only one triangle is possible and the condition for that triangle to be possible is $f(0) < 0 \Rightarrow k < 0$ so for $k < 0$ only one isosceles triangle is possible.



(ii) $k > 0$, three graphs of the cubic equation (ii) are possible. In fig. (i), two such triangle are possible. The condition is $f(k/3) < 0$.

$$\Rightarrow \quad k\left(1 - \frac{k^2}{27}\right) < 0 \Rightarrow k > 3\sqrt{3}$$



In figure (ii), one such triangle is possible. The condition is $f(k/3) = 0$

$\Rightarrow \quad k = 3\sqrt{3}$.

In figure (iii), no such triangle is possible. The condition is $f(k/3) > 0$

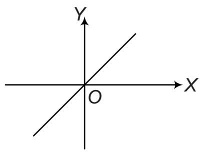
$$\Rightarrow \quad k\left(1 - \frac{k^2}{27}\right) > 0$$

$\Rightarrow \quad k < 3\sqrt{3}$.

(iii) $k = 0$, graph will be shown as, so no such triangle is possible. Hence, the solution for mentioned conditions;

$\therefore$ (A) either $k < 0$ or $k = 3\sqrt{3}$

(B) $k > 3\sqrt{3}$



(C) Clearly, there will never exists three or more than three non-similar isosceles triangle for any value of k.

Trigonometric Functions and Identities Exercise 1 : Single Option Correct Type Questions

1. The value of $\sum_{n=1}^{10}\left(\sin\frac{2n\pi}{11}-\cos\frac{2n\pi}{11}\right)$ is equal to

(a) 2 (b) 1
(c) 0 (d) -1

2. Given, $a^2+2a+\text{cosec}^2\left(\frac{\pi}{2}(a+x)\right)=0$, then which of the following holds good?

(a) $a=1;\ \frac{x}{2}\in 1$
(b) $a=-1;\ \frac{x}{2}\in I$
(c) $a\in R;\ x\in\phi$
(d) a, x are finite but not possible to find

3. The minimum value of the function $f(x)=(3\sin x-4\cos x-10)(3\sin x+4\cos x-10)$, is

(a) 49 (b) $\frac{195-60\sqrt{2}}{2}$
(c) 84 (d) 48

4. The value of expression $\sum_{\theta=0}^{8}\frac{1}{1+\tan^3(10\theta)^\circ}$ equal is to

(a) 5 (b) $\frac{21}{4}$
(c) $\frac{14}{3}$ (d) $\frac{9}{2}$

5. The value of $\sqrt{1-\sin^2 110^\circ}\cdot\sec\ 110^\circ$ is equal to

(a) 2 (b) -1
(c) -2 (d) 1

6. If $\tan\alpha$, $\tan\beta$ are the roots of the equation $x^2+px+q=0$, then the value of $\sin^2(\alpha+\beta)+p\sin(\alpha+\beta)\cos(\alpha+\beta)+q\cos^2(\alpha+\beta)$ is

(a) independent of p but dependent on q
(b) independent of q but dependent on p
(c) independent of both p and q
(d) dependent on both p and q

7. The value of the product

$$\sin\left(\frac{\pi}{2^{2009}}\right)\cos\left(\frac{\pi}{2^{2009}}\right)\cos\left(\frac{\pi}{2^{2008}}\right)\cos\left(\frac{\pi}{2^{2007}}\right)\cos\left(\frac{\pi}{2^{2006}}\right)\dots\cos\left(\frac{\pi}{2^3}\right)\cos\left(\frac{\pi}{2^2}\right),$$ is

(a) $\frac{1}{2^{2007}}$ (b) $\frac{1}{2^{2008}}$
(c) $\frac{1}{2^{2009}}$ (d) $\frac{1}{2^{2010}}$

8. If $\tan B=\frac{n\sin A\cos A}{1-n\cos^2 A}$, then $\tan(A+B)$ equals to

(a) $\frac{\sin A}{(1-n)\cos A}$ (b) $\frac{(n-1)\cos A}{\sin A}$
(c) $\frac{\sin A}{(n-1)\cos A}$ (d) $\frac{\sin A}{(n+1)\cos A}$

9. If $P=(\tan(3^{n+1}\theta)-\tan\theta)$ and $Q=\sum_{r=0}^{n}\frac{\sin(3^r\theta)}{\cos(3^{r+1}\theta)}$, then

(a) $P=2Q$ (b) $P=3Q$
(c) $2P=Q$ (d) $3P=Q$

10. The value of
$(\cos^4 1^\circ+\cos^4 2^\circ+\cos^4 3^\circ+\dots+\cos^4 179^\circ)-$
$(\sin^4 1^\circ+\sin^4 2^\circ+\sin^4 3^\circ+\dots+\sin^4 179^\circ)$ equals to

(a) $2\cos 1^\circ$ (b) -1
(c) $2\sin 1^\circ$ (d) 0

11. Suppose that 'a' is a non-zero real number for which $\sin x+\sin y=a$ and $\cos x+\cos y=2a$. The value of $\cos(x-y)$, is

(a) $\frac{3a^2-2}{2}$ (b) $\frac{7a^2-2}{2}$
(c) $\frac{9a^2-2}{2}$ (d) $\frac{5a^2-2}{2}$

12. Let $P(x)=$
$\sqrt{(\cos x+\cos 2x+\cos 3x)^2+(\sin x+\sin 2x+\sin 3x)^2}$,
then $P(x)$ is equal to

(a) $1+2\cos x$ (b) $1+\sin 2x$
(c) $1-2\cos x$ (d) None of these

13. If the maximum value of the expression $\frac{1}{5\sec^2\theta-\tan^2\theta+4\,\text{cosec}^2\theta}$ is equal to $\frac{p}{q}$ (where, p and q are coprime), then the value of $(p+q)$ is

(a) 14 (b) 15
(c) 16 (d) 18

14. Let $f_n(a)=\frac{\sin\alpha+\sin 3\alpha+\sin 5\alpha+\dots+\sin(2n-1)\alpha}{\cos\alpha+\cos 3\alpha+\cos 5\alpha+\dots+\cos(2n-1)\alpha}$.

Then, the value of $f_4\left(\frac{\pi}{32}\right)$ is equal to

(a) $\sqrt{2}+1$ (b) $\sqrt{2}-1$
(c) $2+\sqrt{3}$ (d) $2-\sqrt{3}$

15. The minimum value of $\left|\sin x+\cos x+\frac{\cos x+\sin x}{\cos^4 x-\sin^4 x}\right|$ is

(a) 2 (b) $\frac{3}{2}$ (c) $\sqrt{2}$ (d) 1

16. If $a = \cos(2012\,\pi)$, $b = \sec(2013\,\pi)$ and $c = \tan(2014\,\pi)$, then

(a) $a < b < c$ (b) $b < c < a$
(c) $c < b < a$ (d) $a = b < c$

17. In a ΔABC, the minimum value of $\sec^2\dfrac{A}{2} + \sec^2\dfrac{B}{2} + \sec^2\dfrac{C}{2}$ is equal to

(a) 3 (b) 4
(c) 5 (d) 6

18. The number of ordered pairs (x, y) of real numbers satisfying $4x^2 - 4x + 2 = \sin^2 y$ and $x^2 + y^2 \le 3$, is equal to

(a) 0 (b) 2
(c) 4 (d) 8

19. In a ΔABC, $3\sin A + 4\cos B = 6$ and $3\cos A + 4\sin B = 1$, then $\angle C$ can be

(a) $30°$ (b) $60°$
(c) $90°$ (d) $150°$

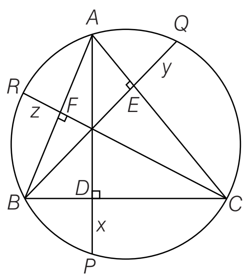
20. An equilateral triangle has side length 8. The area of the region containing all points outside the triangle but not more than 3 units from a point on the triangle is :

(a) $9(8 + \pi)$
(b) $8(9 + \pi)$
(c) $9\left(8 + \dfrac{\pi}{2}\right)$
(d) $8\left(9 + \dfrac{\pi}{2}\right)$

21. If $a\cos^3\alpha + 3a\cos\alpha\sin^2\alpha = m$ and $a\sin^3\alpha + 3a\cos^2\alpha\sin\alpha = n$. Then, $(m+n)^{2/3} + (m-n)^{2/3}$ is equal to

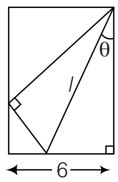
(a) $2a^2$ (b) $2a^{1/3}$
(c) $2a^{2/3}$ (d) $2a^3$

22. As shown in the figure, AD is the altitude on BC and AD produced meets the circumcircle of ΔABC at P where $DP = x$. Similarly, $EQ = y$ and $FR = z$. If a, b, c respectively denotes the sides BC, CA and AB, then $\dfrac{a}{2x} + \dfrac{b}{2y} + \dfrac{c}{2z}$ has the value equal to



(a) $\tan A + \tan B + \tan C$
(b) $\cot A + \cos B + \cot C$
(c) $\cos A + \cos B + \cos C$
(d) $\operatorname{cosec} A + \operatorname{cosec} B + \operatorname{cosec} C$

23. One side of a rectangular piece of paper is 6 cm, the adjacent sides being longer than 6 cm. One corner of the paper is folded so that is sets on the opposite longer side. If the length of the crease is l cm and it makes an angle θ with the long side as shown, then l is



(a) $\dfrac{3}{\sin\theta\cos^2\theta}$ (b) $\dfrac{6}{\sin^2\theta\cos\theta}$
(c) $\dfrac{3}{\sin\theta\cos\theta}$ (d) $\dfrac{3}{\sin^2\theta}$

24. The average of the numbers $n\sin n°$ for $n = 2, 4, 6, \dots 180$

(a) 1 (b) $\cot 1°$
(c) $\tan 1°$ (d) $\dfrac{1}{2}$

25. A circle is inscribed inside a regular pentagon and another circle is circumscribed about this pentagon. Similarly, a circle is inscribed in a regular heptagon and another circumscribed about the heptagon. The area of the regions between the two circles in two cases are A_1 and A_2, respectively. If each polygon has a side length of 2 units, then which one of the following is true ?

(a) $A_1 = \dfrac{5}{7}A_2$ (b) $A_1 = \dfrac{25}{49}A_2$
(c) $A_1 = \dfrac{49}{25}A_2$ (d) $A_1 = A_2$

26. The value of $\sum_{r=1}^{18}\cos^2(5r)°$, where $x°$ denotes the x degrees, is equal to

(a) 0 (b) $\dfrac{7}{2}$
(c) $\dfrac{17}{2}$ (d) $\dfrac{25}{2}$

27. Minimum value of $4x^2 - 4x|\sin x| - \cos^2\theta$ is equal to

(a) -2 (b) -1
(c) $-\dfrac{1}{2}$ (d) 0

28. If in a triangle ABC, $\cos 3A + \cos 3B + \cos 3C = 1$, then one angle must be exactly equal to

(a) $\dfrac{\pi}{3}$ (b) $\dfrac{2\pi}{3}$ (c) π (d) $\dfrac{4\pi}{3}$

29. If $|\tan A| < 1$ and $|A|$ is acute, then $\dfrac{\sqrt{(1+\sin 2A)} + \sqrt{(1-\sin 2A)}}{\sqrt{(1+\sin 2A)} - \sqrt{(1-\sin 2A)}}$ is equal to

(a) $\tan A$ (b) $-\tan A$
(c) $\cot A$ (d) $-\cot A$

30. For any real θ, the maximum value of $\cos^2(\cos\theta) + \sin^2(\sin\theta)$ is

(a) 1 (b) $1 + \sin^2 1$
(c) $1 + \cos^2 1$ (d) does not exist

31. Minimum value of $27^{\cos 2x} \cdot 81^{\sin 2x}$ is

(a) -5 (b) $\frac{1}{5}$
(c) $\frac{1}{243}$ (d) $\frac{1}{27}$

32. $ABCD$ is a trapezium, such that AB and CD are parallel $BC \perp CD$. If $\angle ADB = \theta$, $BC = p$ and $CD = q$, then AB is equal to

(a) $\frac{(p^2+q^2)\sin\theta}{p\cos\theta + q\sin\theta}$ (b) $\frac{p^2+q^2\cos\theta}{p\cos\theta + q\sin\theta}$
(c) $\frac{p^2+q^2}{p^2\cos\theta + q^2\sin\theta}$ (d) $\frac{(p^2+q^2)\sin\theta}{(p\cos\theta + q\sin\theta)^2}$

33. If $4n\alpha = \pi$, then $\cot\alpha \cot 2\alpha \cot 3\alpha \ldots \cot(2n-1)\alpha$ is equal to

(a) 0 (b) 1
(c) n (d) None of these

34. If in a triangle $ABC(\sin A + \sin B + \sin C)$ $(\sin A + \sin B - \sin C) = 3\sin A \sin B$, then angle C is equal to

(a) 30° (b) 45°
(c) 60° (d) 75°

35. If α, β, γ are acute angles and $\cos\theta = \frac{\sin\beta}{\sin\alpha}$, $\cos\phi = \frac{\sin\gamma}{\sin\alpha}$ and $\cos(\theta - \phi) = \sin\beta\sin\gamma$, then $\tan^2\alpha - \tan^2\beta - \tan^2\gamma$ is equal to

(a) -1 (b) 0
(c) 1 (d) None of these

36. If $\tan\beta = \frac{n\sin\alpha\cos\alpha}{1 - n\sin^2\alpha}$, then $\tan(\alpha - \beta)$ is equal to

(a) $n\tan\alpha$ (b) $(1-n)\tan\alpha$
(c) $(1+n)\tan\alpha$ (d) None of these

37. If $\frac{\cos\theta}{a} = \frac{\sin\theta}{b}$, then $\frac{a}{\sec 2\theta} + \frac{b}{\operatorname{cosec} 2\theta}$ is equal to

(a) a (b) b
(c) $\frac{a}{b}$ (d) $a + b$

38. The graph of the function $\cos x\cos(x+2) - \cos^2(x+1)$ is

(a) a straight line passing through $(0, -\sin^2\theta)$ with slope 2
(b) a straight line passing through $(0, 0)$
(c) a parabola with vertex $(1, -\sin^2 1)$
(d) a straight line passing through the point $\left(\frac{\pi}{2}, -\sin^2 1\right)$ and parallel to the X-axis

39. $f(\theta) = |\sin\theta| + |\cos\theta|, \theta \in R$, then

(a) $f(\theta) \in [0, 2]$ (b) $f(\theta) \in [0, \sqrt{2}]$
(c) $f(\theta) \in [0, 1]$ (d) $f(\theta) \in [1, \sqrt{2}]$

40. If $A = \cos(\cos x) + \sin(\cos x)$ the least and greatest value of A are

(a) 0 and 2 (b) -1 and 1
(c) $-\sqrt{2}$ and $\sqrt{2}$ (d) 0 and $\sqrt{2}$

41. If $U_n = \sin n\theta \sec^n \theta$, $V_n = \cos n\theta \sec^n \theta \neq 1$, then $\frac{V_n - V_{n-1}}{U_{n-1}} + \frac{1}{n}\frac{U_n}{V_n}$ is equal to

(a) 0 (b) $\tan\theta$
(c) $-\tan\theta + \frac{\tan n\theta}{n}$ (d) $\tan\theta + \frac{\tan n\theta}{n}$

42. If $0 \le x \le \frac{\pi}{3}$ then range of $f(x) = \sec\left(\frac{\pi}{6} - x\right) + \sec\left(\frac{\pi}{6} + x\right)$ is

(a) $\left(\frac{4}{\sqrt{3}}, \infty\right)$ (b) $\left[\frac{4}{\sqrt{3}}, \infty\right)$
(c) $\left(0, \frac{4}{\sqrt{3}}\right]$ (d) $\left(0, \frac{4}{\sqrt{3}}\right)$

43. If $A = \sin^8\theta + \cos^{14}\theta$, then for all values of θ,

(a) $A \ge 1$ (b) $0 < A \le 1$
(c) $1 < 2a \le 3$ (d) None of these

44. The expression $3\left\{\sin^4\left(\frac{3\pi}{2} - \alpha\right) + \sin^4(3\pi + \alpha)\right\} - 2\left\{\sin^6\left(\frac{\pi}{2} + \alpha\right) + \sin^6(5\pi - \alpha)\right\}$ is equal to

(a) 0 (b) -1
(c) 1 (d) 3

45. The maximum value of $\sin\left(x + \frac{\pi}{6}\right) + \cos\left(x + \frac{\pi}{6}\right)$ in the interval $\left(0, \frac{\pi}{2}\right)$ is attained at

(a) $\frac{\pi}{12}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

46. If $\cot^2 x = \cot(x-y)\cdot\cot(x-z)$, then $\cot 2x$ is equal to $\left(x \neq \pm\frac{\pi}{4}\right)$

(a) $\frac{1}{2}(\tan y + \tan z)$ (b) $\frac{1}{2}(\cot y + \cot z)$
(c) $\frac{1}{2}(\sin y + \sin z)$ (d) None of these

47. The minimum value of the expression $\sin\alpha + \sin\beta + \sin\gamma$, where α, β, γ are real numbers satisfying $\alpha + \beta + \gamma = \pi$, is

(a) positive (b) zero
(c) negative (d) None of these

48. If $\cos x - \sin\alpha \cot\beta \sin x = \cos\alpha$, then $\tan\frac{x}{2}$ is equal to

(a) $\cot\frac{\alpha}{2}\tan\frac{\beta}{2}$ (b) $\cot\frac{\beta}{2}\tan\frac{\alpha}{2}$

(c) $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}$ (d) None of these

49. If $\cos^4\theta\sec^2\alpha, \frac{1}{2}$ and $\sin^4\theta\operatorname{cosec}^2\alpha$ are in AP, then $\cos^8\theta\sec^6\alpha, \frac{1}{2}$ and $\sin^8\theta\cdot\operatorname{cosec}^6\alpha$ are in

(a) AP (b) GP

(c) HP (d) None of these

50. The maximum value of $\cos\alpha_1\cdot\cos\alpha_2\cdot\cos\alpha_3\cdot\ldots\cdot\cos\alpha_n$ under the restriction $0\le\alpha_1,\alpha_2,\ldots,\alpha_n\le\frac{\pi}{2}$ and $\cot\alpha_1\cdot\cot\alpha_2\cdot\ldots\cdot\cot\alpha_n=1$ is

(a) $\frac{1}{2^{\frac{n}{2}}}$ (b) $\frac{1}{2^n}$

(c) $\frac{-1}{2^n}$ (d) 1

51. If $x\in(0,\pi)$ and $\sin x\cdot\cos^3 x>\cos x\cdot\sin^3 x$, then complete set of values of x is

(a) $x\in\left(0,\frac{\pi}{4}\right)\cup\left(\frac{\pi}{2},\frac{3\pi}{4}\right)$

(b) $x\in\left(\frac{\pi}{4},\frac{\pi}{2}\right)\cup\left(\frac{3\pi}{4},\pi\right)$

(c) $x\in\left(\frac{\pi}{4},\frac{\pi}{2}\right)$

(d) None of the above

52. If $u=\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}+\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}$ then the difference between the maximum and minimum values of u^2 is given by

(a) $2(a^2+b^2)$ (b) $2\sqrt{a^2+b^2}$

(c) $(a+b)^2$ (d) $(a-b)^2$

53. For a positive integer n, let $f_n(\theta)=\frac{\tan\theta}{2}(1+\sec\theta)(1+\sec 2\theta)\ldots(1+\sec 2^n\theta)$, then

(a) $f_2\left(\frac{\pi}{16}\right)=0$ (b) $f_3\left(\frac{\pi}{32}\right)=-1$

(c) $f_4\left(\frac{\pi}{64}\right)=-1$ (d) $f_5\left(\frac{\pi}{128}\right)=1$

Trigonometric Functions and Identities Exercise 2 :
More than One Option Correct Type Questions

54. Suppose $\cos x=0$ and $\cos(x+z)=\frac{1}{2}$. Then, the possible value(s) of z is (are).

(a) $\frac{\pi}{6}$ (b) $\frac{5\pi}{6}$

(c) $\frac{7\pi}{6}$ (d) $\frac{11\pi}{6}$

55. Let $f_n(\theta)=2\sin\frac{\theta}{2}\sin\frac{3\theta}{2}+2\sin\frac{\theta}{2}\sin\frac{5\theta}{2}+2\sin\frac{\theta}{2}\sin\frac{7\theta}{2}+\ldots+2\sin\frac{\theta}{2}\sin(2n+1)\frac{\theta}{2}, n\in N$, then which of the following is/are correct?

(a) $f_9\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$ (b) $f_n\left(\frac{2\pi}{n}\right)=0, n\in N$

(c) $f_5\left(\frac{2\pi}{7}\right)=0$ (d) $f_9\left(\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}}$

56. Let $P=\sin 25^\circ\sin 35^\circ\sin 60^\circ\sin 85^\circ$ and $Q=\sin 20^\circ\sin 40^\circ\sin 75^\circ\sin 80^\circ$. Which of the following relation(s) is (are) correct ?

(a) $P+Q=0$ (b) $P-Q=0$

(c) $P^2+Q^2=1$ (d) $P^2-Q^2=0$

57. For $0<\theta<\frac{\pi}{2}$, if $x=\sum_{n=0}^{\infty}\cos^{2n}\theta, y=\sum_{n=0}^{\infty}\sin^{2n}\theta$,

$$z=\sum_{n=0}^{\infty}\cos^{2n}\theta\sin^{2n}\theta, \text{ then}$$

(a) $xyz=xz+y$

(b) $xyz=xy+z$

(c) $xyz=x+y+z$

(d) $xyz=yz+x$

58. Let $P(x)=\cot^2 x\left(\frac{1+\tan x+\tan^2 x}{1+\cot x+\cot^2 x}\right)+\left(\frac{\cos x-\cos 3x+\sin 3x-\sin x}{2(\sin 2x+\cos 2x)}\right)^2$. Then, which of the following is (are) correct?

(a) The value of $P(18^\circ)+P(72^\circ)$ is 2.

(b) The value of $P(18^\circ)+P(72^\circ)$ is 3.

(c) The value of $P\left(\frac{4\pi}{3}\right)+P\left(\frac{7\pi}{6}\right)$ is 3.

(d) The value of $P\left(\frac{4\pi}{3}\right)+P\left(\frac{7\pi}{6}\right)$ is 2.

59. It is known that $\sin\beta = \frac{4}{5}$ and $0 < \beta < \pi$, then thes value of $\dfrac{\sqrt{3}\sin(\alpha+\beta) - \dfrac{2}{\cos\frac{11\pi}{6}}\cos(\alpha+\beta)}{\sin\alpha}$ is

(a) independent of α for all β in $(0, \pi)$

(b) $\frac{5}{\sqrt{3}}$ for $\tan\beta > 0$

(c) $\frac{\sqrt{3}(7 + 24\cot\alpha)}{15}$ for $\tan\beta < 0$

(d) zero for $\tan\beta > 0$

60. In cyclic quadrilateral $ABCD$, if $\cot A = \frac{3}{4}$ and $\tan B = \frac{-12}{5}$, then which of the following is (are) correct?

(a) $\sin D = \frac{12}{13}$ (b) $\sin(A + B) = \frac{16}{65}$

(c) $\cos D = \frac{-15}{13}$ (d) $\sin(C + D) = \frac{-16}{65}$

61. If the equation $2\cos^2 x + \cos x - a = 0$ has solutions, then a can be

(a) $\frac{-1}{4}$ (b) $\frac{-1}{8}$

(c) 2 (d) 5

62. If $A = \sin 44^\circ + \cos 44^\circ$, $B = \sin 45^\circ + \cos 45^\circ$ and $C = \sin 46^\circ + \cos 46^\circ$. Then, correct option(s) is/are

(a) $A < B < C$ (b) $C < B < A$

(c) $B > A$ (d) $A = C$

63. If $\tan(2\alpha+\beta) = x$ & $\tan(\alpha + 2\beta) = y$, then $[\tan 3(\alpha+\beta)]$. $[\tan(\alpha-\beta)]$ is equal to (wherever defined)

(a) $\frac{x^2+y^2}{1-x^2y^2}$ (b) $\frac{x^2-y^2}{1+x^2y^2}$

(c) $\frac{x^2+y^2}{1+x^2y^2}$ (d) $\frac{x^2-y^2}{1-x^2y^2}$

64. If $x = \sec\phi - \tan\phi$ and $y = \operatorname{cosec}\phi + \cot\phi$, then

(a) $x = \frac{y+1}{y-1}$ (b) $x = \frac{y-1}{y+1}$

(c) $y = \frac{1+x}{1-x}$ (d) $xy + x - y + 1 = 0$

65. If $\tan\left(\frac{x}{2}\right) = \operatorname{cosec} x - \sin x$, then $\tan^2\left(\frac{x}{2}\right)$ is equal to

(a) $2 - \sqrt{5}$ (b) $\sqrt{5} - 2$

(c) $(9 - 4\sqrt{5})(2 + \sqrt{5})$ (d) $(9 + 4\sqrt{5})(2 - \sqrt{5})$

66 If $y = \dfrac{\sqrt{1-\sin 4A}+1}{\sqrt{1+\sin 4A}-1}$, then one of the value of y is

(a) $-\tan A$ (b) $\cot A$

(c) $\tan\left(\frac{\pi}{4}+A\right)$ (d) $-\cot\left(\frac{\pi}{4}+A\right)$

67. If $3\sin\beta = \sin(2\alpha+\beta)$, then

(a) $[\cot\alpha + \cot(\alpha+\beta)][\cot\beta - 3\cot(2\alpha+\beta)] = 6$

(b) $\sin\beta = \cos(\alpha+\beta)\sin\alpha$

(c) $2\sin\beta = \sin(\alpha+\beta)\cos\alpha$

(d) $\tan(\alpha+\beta) = 2\tan\alpha$

68. Let $P_n(u)$ be a polynomial is u of degree n. Then, for every positive integer n, $\sin 2nx$ is expressible is

(a) $P_{2n}(\sin x)$ (b) $P_{2n}(\cos x)$

(c) $\cos x\, P_{2n-1}(\sin x)$ (d) $\sin x\, P_{2n-1}(\cos x)$

69. If $\tan\theta = \dfrac{\sin\alpha - \cos\alpha}{\sin\alpha + \cos\alpha}$, then

(a) $\sin\alpha - \cos\alpha = \pm\sqrt{2}\sin\theta$

(b) $\sin\alpha + \cos\alpha = \pm\sqrt{2}\cos\theta$

(c) $\cos 2\theta = \sin 2\alpha$

(d) $\sin 2\theta + \cos 2\alpha = 0$

70. If $\cos 5\theta = a\cos\theta + b\cos^3\theta + c\cos^5\theta + d$, then

(a) $a = 20$

(b) $b = -20$

(c) $c = 16$

(d) $d = 5$

71. $x = \sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha} = \sqrt{a^2\sin^2\alpha + b^2\cos^2\alpha}$ then $x^2 = a^2 + b^2 + 2\sqrt{p(a^2+b^2) - p^2}$, where p is equal to

(a) $a^2\cos^2\alpha + b^2\sin^2\alpha$

(b) $a^2\sin^2\alpha + b^2\cos^2\alpha$

(c) $\frac{1}{2}[a^2 + b^2 + (a^2 - b^2)\cos 2\alpha]$

(d) $\frac{1}{2}[a^2 + b^2 - (a^2 - b^2)\cos 2\alpha]$

72. $\left(\dfrac{\cos A + \cos B}{\sin A - \sin B}\right)^n + \left(\dfrac{\sin A + \sin B}{\cos A - \cos B}\right)^n$ (n, even or odd) is equal to

(a) $2\tan^n\left(\frac{A-B}{2}\right)$ (b) $2\cot^n\left(\frac{A-B}{2}\right)$

(c) 0 (d) None of these

73. Let $P(k) = \left(1 + \cos\frac{\pi}{4k}\right)\left(1 + \cos\frac{(2k-1)\pi}{4k}\right)\left(1 + \cos\frac{(2k+1)\pi}{4k}\right)\left(1 + \cos\frac{(4k-1)\pi}{4k}\right)$. Then

(a) $P(3) = \frac{1}{16}$ (b) $P(4) = \frac{2-\sqrt{2}}{16}$

(c) $P(5) = \frac{3-\sqrt{5}}{32}$ (d) $P(6) = \frac{2-\sqrt{3}}{16}$

74. If $x = a\cos^3\theta\sin^2\theta$, $y = a\sin^3\theta\cos^2\theta$ and $\dfrac{(x^2+y^2)^p}{(xy)^q}$ $(p, q \in N)$ is independent of θ, then

(a) $p = 4$ (b) $p = 5$

(c) $q = 4$ (d) $q = 5$

Trigonometric Functions and Identities Exercise 3: Statement I and II Type Questions

- This section contains 11 questions. Each question contains **Statement I** (Assertion) and **Statement II** (Reason). Each question has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are
  - (a) Both Statement I and Statement II are individually true and R is the correct explanation of Statement I.
  - (b) Both Statement I and Statement II are individually true but Statement II is not the correct explanation of Statement I.
  - (c) Statement I is true but Statement II is false.
  - (d) Statement I is false but Statement II is true.

75. **Statement I** $\tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 8\tan 8\alpha + 16\cot 16\alpha = \cot\alpha$

Statement II $\cot\alpha - \tan\alpha = 2\cot 2\alpha$

76. **Statement I** If $xy + yz + zx = 1$, then $\Sigma \dfrac{x}{(1+x^2)} = \dfrac{2}{\sqrt{\Pi(1+x^2)}}$.

Statement II In a ΔABC $\sin 2A + \sin 2B - \sin 2C = 4\cos A\cos B\sin C$

77. **Statement I** If α and β are two distinct solutions of the equation $a\cos x + b\sin x = c$, then $\tan\left(\dfrac{\alpha+\beta}{2}\right)$ is independent of c.

Statement II Solution of $a\cos x + b\sin x = c$ is possible, if $-\sqrt{(a^2+b^2)} \le c \le \sqrt{(a^2+b^2)}$

78. **Statement I** If A is obtuse angle in ΔABC, then $\tan B\tan C > 1$.

Statement II In ΔABC, $\tan A = \dfrac{\tan B + \tan C}{\tan B\tan C - 1}$

79. **Statement I** $\sin\left(\dfrac{2\pi}{7}\right) + \sin\left(\dfrac{4\pi}{7}\right) + \sin\left(\dfrac{8\pi}{7}\right) = -\dfrac{1}{2}$.

Statement II $\cos\dfrac{2\pi}{7} + i\sin\dfrac{2\pi}{7}$ is complex 7th root of unity.

80. **Statement I** The curve $y = 81^{\sin^2 x} + 81^{\cos^2 x} - 30$ intersects X-axis at eight points in the region $-\pi \le x \le \pi$.

Statement II The curve $y = \sin x$ or $y = \cos x$ intersects the X-axis at infinitely many points.

81. **Statement I** The numbers $\sin 18^\circ$ and $-\sin 54^\circ$ are the roots of a quadratic equation with integer coefficients.

Statement II If $x = 18^\circ$, $\cos 3x = \sin 2x$ and if $y = -54^\circ$ $\sin 2y = \cos 3y$.

82. **Statement I** The minimum value of the expression $\sin\alpha + \sin\beta + \sin\gamma$ where α, β, γ are real numbers such that $\alpha + \beta + \gamma = \pi$ is negative.

Statement II If $\alpha + \beta + \gamma = \pi$, then α, β, γ are the angles of a triangle.

83. **Statement I** If $2\sin\left(\dfrac{\theta}{2}\right) = \sqrt{1+\sin\theta} + \sqrt{1-\sin\theta}$ then $\dfrac{\theta}{2}$ lies between $2n\pi + \dfrac{\pi}{4}$ and $2n\pi + \dfrac{3\pi}{4}$.

Statement II If $\dfrac{\pi}{4} \le 0 \le \dfrac{3\pi}{4}$ then $\sin\dfrac{\theta}{2} > 0$.

84. **Statement I** If $2\cos\theta + \sin\theta = 1\left(\theta \ne \dfrac{\pi}{2}\right)$ then the value of $7\cos\theta + 6\sin\theta$ is 2.

Statement II If $\cos 2\theta - \sin\theta = \dfrac{1}{2}, 0 < \theta < \dfrac{\pi}{2}$, then $\sin\theta + \cos 6\theta = 0$.

85. **Statement I** If $A > 0$, $B > 0$ and $A + B = \dfrac{\pi}{3}$, then the maximum value of $\tan A\tan B$ is $\dfrac{1}{3}$.

Statement II If $a_1 + a_2 + a_3 + \ldots + a_n = k$ (constant), then the value $a_1 a_2 a_3 \ldots a_n$ is greatest when $a_1 = a_2 = a_3 = \ldots = a_n$

Trigonometric Functions and Identities Exercise 4 : Passage Based Questions

Passage I

(Q. Nos. 86 and 87)

If a, b, c are the sides of $\Delta\ ABC$ such that $3^{2a^2} - 2\cdot 3^{a^2+b^2+c^2} + 3^{2b^2+2c^2} = 0$, then

86. Triangle ABC is

(a) equilateral (b) right angled
(c) isosceles right angled (d) obtuse angled

87. If sides of ΔPQR are a, $b\sec C$, $c\operatorname{cosec} C$. Then, area of ΔPQR is

(a) $\frac{\sqrt{3}}{4}a^2$ (b) $\frac{\sqrt{3}}{4}b^2$ (c) $\frac{\sqrt{3}}{4}c^2$ (d) $\frac{1}{2}abc$

Passage II

(Q. Nos. 88 to 90)

For $0 < x < \frac{\pi}{2}$, let $P_{mn}(x) = m\log_{\cos x}(\sin x) + n\log_{\cos x}(\cot x)$; where $m, n \in \{1, 2, \ldots, 9\}$

[For example :

$P_{29}(x) = 2\log_{\cos x}(\sin x) + 9\log_{\cos x}(\cot x)$ and
$P_{77}(x) = 7\log_{\cos x}(\sin x) + 7\log_{\cos x}(\cot x)$]

On the basis of above information , answer the following questions :

88. Which of the following is always correct?

(a) $P_{mn}(x) \geq m\ \forall\ m \geq n$ (b) $P_{mn}(x) \geq n\ \forall\ m \geq n$
(c) $2P_{mn}(x) \leq n\ \forall\ m \leq n$ (d) $2P_{mn}(x) \leq m\ \forall\ m \leq n$

89. The mean proportional of numbers $P_{49}\left(\frac{\pi}{4}\right)$ and $P_{94}\left(\frac{\pi}{4}\right)$ is equal to

(a) 4 (b) 6
(c) 9 (d) 10

90. If $P_{34}(x) = P_{22}(x)$, then the value of $\sin x$ is expressed as $\left(\frac{\sqrt{q}-1}{p}\right)$, then $(p+q)$ equals

(a) 3 (b) 4
(c) 7 (d) 9

Note Mean proportional of a and b $(a > 0, b > 0)$ is $\sqrt{ab}$]

Passage III

(Q. Nos. 91 to 93)

If $7\theta = (2n+1)\pi$, where $n = 0, 1, 2, 3, 4, 5, 6$, then answer the following questions.

91. The equations whose roots are $\cos\frac{\pi}{7}, \cos\frac{3\pi}{7}, \cos\frac{5\pi}{7}$ is

(a) $8x^3 + 4x^2 + 4x + 1 = 0$
(b) $8x^3 - 4x^2 - 4x - 1 = 0$
(c) $8x^3 - 4x^2 - 4x - 1 = 0$
(d) $8x^3 + 4x^2 + 4x - 1 = 0$

92. The value of $\sec\frac{\pi}{7} + \sec\frac{3\pi}{7} + \sec\frac{5\pi}{7}$ is

(a) 4 (b) -4
(c) 3 (d) -3

93. The value of $\sec^2\frac{\pi}{7} + \sec^2\frac{3\pi}{7} + \sec^2\frac{5\pi}{7}$ is

(a) -24 (b) 80 (c) 24 (d) -80

Passage IV

(Q. Nos. 94 to 96)

If $1 + 2\sin x + 3\sin^2 x + 4\sin^3 x + \ldots$ upto infinite terms = 4 and number of solutions of the equation in $\left[\frac{-3\pi}{2}, 4\pi\right]$ is k.

94. The value of k is equal to

(a) 4 (b) 5 (c) 6 (d) 7

95. The value of $\left|\frac{\cos 2x - 1}{\sin 2x}\right|$ is equal to

(a) 1 (b) $\sqrt{3}$
(c) $2-\sqrt{3}$ (d) $\frac{1}{\sqrt{3}}$

96. Sum of all internal angles of a k-sided regular polygon is

(a) 5π (b) 4π
(c) 3π (d) 2π

Passage V

(Q. Nos. 97 to 98)

Let α is a root of the equation $(2\sin x - \cos x)(1+\cos x) = \sin^2 x$, β is a root of the equation $3\cos^2 x - 10\cos x + 3 = 0$ and γ is a root of the equation $1 - \sin 2x = \cos x - \sin x$, $0 \leq \alpha, \beta, \gamma \leq \frac{\pi}{2}$.

97. $\cos\alpha + \cos\beta + \cos\gamma$ can be equal to

(a) $\frac{3\sqrt{6}+2\sqrt{2}+6}{6\sqrt{2}}$ (b) $\frac{3\sqrt{3}+8}{6}$
(c) $\frac{3\sqrt{3}+2}{6}$ (d) None of these

98. $\sin(\alpha - \beta)$ is equal to

(a) 1 (b) 0
(c) $\frac{1-2\sqrt{6}}{6}$ (d) $\frac{\sqrt{3}-2\sqrt{2}}{6}$

Trigonometric Functions and Identities Exercise 5: Matching Type Questions

99. Match the statement of Column I with values of Column II.

Column I		Column II
(A) If $\theta + \phi = \frac{\pi}{2}$, where θ and ϕ are positive, then $(\sin\theta + \sin\phi)\sin\left(\frac{\pi}{4}\right)$ is always less than	(p)	1
(B) If $\sin\theta - \sin\phi = a$ and $\cos\theta + \cos\phi = b$, then $a^2 + b^2$ cannot exceed	(q)	2
(C) If $3\sin\theta + 5\cos\theta = 5$, $(\theta \neq 0)$ then the value of $5\sin\theta - 3\cos\theta$ is	(r)	3
	(s)	4
	(t)	5

100. Match the statement of Column I with values of Column II.

Column I		Column II
(A) If maximum and minimum values of $\frac{7 + 6\tan\theta - \tan^2\theta}{(1 + \tan^2\theta)}$ for all real values of $\theta \sim \frac{\pi}{2}$ are λ and μ respectively, then	(p)	$\lambda + \mu = 2$
(B) If maximum and minimum values of $5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) + 3$ for all real values of θ are λ and μ respectively, then	(q)	$\lambda - \mu = 6$
(C) If maximum and minimum values of $1 + \sin\left(\frac{\pi}{4} + \theta\right) + 2\cos\left(\frac{\pi}{4} - \theta\right)$ for all real values of θ and λ and μ respectively, then	(r)	$\lambda + \mu = 6$
	(s)	$\lambda - \mu = 10$
	(t)	$\lambda - \mu = 14$

101. Match the statement of Column I with values of Column II.

Column I		Column II
(A) The number of solutions of the equation $\lvert\cot x\rvert = \cot x + \frac{1}{\sin x}$ $(0 < x < \pi)$ is	(p)	no solution
(B) If $\sin\theta + \sin\phi = \frac{1}{2}$ and $\cos\theta + \cos\phi = 2$, then value of $\cot\left(\frac{\theta + \phi}{2}\right)$ is	(q)	$\frac{1}{3}$
(C) The value of $\sin^2\alpha + \sin\left(\frac{\pi}{3} - \alpha\right)\sin\left(\frac{\pi}{3} + \alpha\right)$ is	(r)	1
(D) If $\tan\theta = 3\tan\phi$, then maximum value of $\tan^2(\theta - \phi)$ is	(s)	2
	(t)	4

102. Match the statement of Column I with values of Column II.

Column I		Column II
(A) In a ΔABC, $\sin\left(\frac{A}{2}\right) + \sin\left(\frac{B}{2}\right) + \sin\left(\frac{C}{2}\right) =$	(p)	$-1 + 4\sin\left(\frac{\pi + A}{4}\right)\sin\left(\frac{\pi + B}{4}\right)\cos\left(\frac{\pi + C}{4}\right)$
(B) In a ΔABC, $\sin\left(\frac{A}{2}\right) + \sin\left(\frac{B}{2}\right) - \sin\left(\frac{C}{2}\right) =$	(q)	$4\cos\left(\frac{\pi + A}{4}\right)\cos\left(\frac{\pi + B}{4}\right)\cos\left(\frac{\pi - C}{4}\right)$
(C) In a ΔABC, $\cos\left(\frac{A}{2}\right) + \cos\left(\frac{B}{2}\right) - \cos\left(\frac{C}{2}\right)$	(r)	$1 + 4\sin\left(\frac{\pi - A}{4}\right)\sin\left(\frac{\pi - B}{4}\right)\sin\left(\frac{\pi - C}{4}\right)$
	(s)	$-1 + 4\cos\left(\frac{\pi - A}{4}\right)\cos\left(\frac{\pi - B}{4}\right)\sin\left(\frac{\pi - C}{4}\right)$
	(t)	$1 + 4\cos\left(\frac{\pi + A}{4}\right)\cos\left(\frac{\pi + B}{4}\right)\sin\left(\frac{\pi - C}{4}\right)$

Trigonometric Functions and Identities Exercise 6: Single Integer Answer Type Questions

103. In a ΔABC, $\dfrac{1}{1+\tan^2\frac{A}{2}}+\dfrac{1}{1+\tan^2\frac{B}{2}}+\dfrac{1}{1+\tan^2\frac{C}{2}}=k\left[1+\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right]$, then the value of k is

104. If $\dfrac{\sin\alpha}{\sin\beta}=\dfrac{\cos\gamma}{\cos\delta}$, then $\dfrac{\sin\left(\frac{\alpha-\beta}{2}\right)\cdot\cos\left(\frac{\alpha+\beta}{2}\right)\cdot\cos\delta}{\sin\left(\frac{\delta-\gamma}{2}\right)\cdot\sin\left(\frac{\delta+\gamma}{2}\right)\cdot\sin\beta}$ is equal to

105. Find the exact value of the expression $\tan\dfrac{\pi}{20}-\tan\dfrac{3\pi}{20}+\tan\dfrac{5\pi}{20}-\tan\dfrac{7\pi}{20}+\tan\dfrac{9\pi}{20}$.

106. Let $x=\dfrac{\sum_{n=1}^{44}\cos n^\circ}{\sum_{n=1}^{44}\sin n^\circ}$, find the greatest integer that does not exceed.

107. Find θ (in degree) satisfying the equation, $\tan 15^\circ\cdot\tan 25^\circ\cdot\tan 35^\circ=\tan\theta$, where $\theta\in(0, 45^\circ)$

108. Find the exact value of $\operatorname{cosec}10^\circ+\operatorname{cosec}50^\circ-\operatorname{cosec}70^\circ$.

109. If $\cos 5\alpha=\cos^5\alpha$, where $\alpha\in\left(0,\frac{\pi}{2}\right)$, then find the possible values of $(\sec^2\alpha+\operatorname{cosec}^2\alpha+\cot^2\alpha)$.

110. Compute the value of the expression $\tan^2\dfrac{\pi}{16}+\tan^2\dfrac{2\pi}{16}+\tan^2\dfrac{3\pi}{16}+\ldots+\tan^2\dfrac{7\pi}{16}$.

111. Compute the square of the value of the expression $\dfrac{4+\sec 20^\circ}{\operatorname{cosec}20^\circ}$

112. In ΔABC, if $\dfrac{\sin A}{3}=\dfrac{\cos B}{3}=\dfrac{\tan C}{2}$, then the value of $\left(\dfrac{\sin A}{\cot 2A}+\dfrac{\cos B}{\cot 2B}+\dfrac{\tan C}{\cot 2C}\right)$ is

113. Let f and g be function defined by $f(\theta)=\cos^2\theta$ and $g(\theta)=\tan^2\theta$, suppose α and β satisfy $2f(\alpha)-g(\beta)=1$, then value of $2f(\beta)-g(\alpha)$ is

114. If sum of the series $1+x\log_{\left|\frac{1-\sin x}{\cos x}\right|}\left(\dfrac{1+\sin x}{\cos x}\right)^{1/2}+x^2\log_{\left|\frac{1-\sin x}{\cos x}\right|}\left(\dfrac{1+\sin x}{\cos x}\right)^{1/4}+\ldots\infty$ (wherever defined) is equal to $\dfrac{k(1-x)}{(2-x)}$, then k is equal to

115. If $\dfrac{9x}{\cos\theta}+\dfrac{5y}{\sin\theta}=56$ and $\dfrac{9x\sin\theta}{\cos^2\theta}-\dfrac{5y\cos\theta}{\sin^2\theta}=0$ then the value of $\left[(9x)^{\frac{2}{3}}+(5y)^{\frac{2}{3}}\right]^3$ is

116. The angle A of the ΔABC is obtuse. $x=2635-\tan B\tan C$, if $[x]$ denotes the greatest integer function, the value of $[x]$ is

117. If $4\cos 36^\circ+\cot\left(7\frac{1}{2}^\circ\right)=\sqrt{n_1}+\sqrt{n_2}+\sqrt{n_3}+\sqrt{n_4}+\sqrt{n_5}+\sqrt{n_6}$, then the value of $\sum_{i=1}^{6}n_i^2$ must be

118. If $\sin^2 A=x$ and $\prod_{r=1}^{4}\sin(rA)=ax^2+bx^3+cx^4+dx^5$, then the value of $10a-7b+15c-5d$ must be

119. If $x, y\in R$ satisfies $(x+5)^2+(y-12)^2=(14)^2$, then the minimum value of $\sqrt{x^2+y^2}$ is ....

120. The least degree of a polynomial with integer coefficient whose one of the roots may be $\cos 12^\circ$ is

121. If $A+B+C=180^\circ$, $\dfrac{\sin 2A+\sin 2B+\sin 2C}{\sin A+\sin B+\sin C}=k\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$ then the value of $3k^3+2k^2+k+1$ is equal to

122. The value of $f(x)=x^4+4x^3+2x^2-4x+7$, when $x=\cot\dfrac{11\pi}{8}$ is ....

123. In any ΔABC, then minimum value of $2020\sum\dfrac{\sqrt{(\sin A)}}{(\sqrt{(\sin B)}+\sqrt{(\sin C)}-\sqrt{(\sin A)})}$ must be

124. If $\sin\theta+\sin^2\theta+\sin^3\theta=1$, then the value of $\cos^6\theta-4\cos^4\theta+8\cos^2\theta$ must be

125. $16\left(\cos\theta-\cos\frac{\pi}{8}\right)\left(\cos\theta-\cos\frac{3\pi}{8}\right)\left(\cos\theta-\cos\frac{5\pi}{8}\right)\left(\cos\theta-\cos\frac{7\pi}{8}\right)=\lambda\cos 4\theta$, then the value of λ is ....

126. If $\frac{1}{\sin 20^\circ}+\frac{1}{\sqrt{3}\cos 20^\circ}=2k\cos 40^\circ$, then $18k^4+162k^2+369$ is equal to

Trigonometric Functions and Identities Exercise 7 :
Subjective Type Questions

127. Prove that $\tan 82\frac{1}{2}^\circ=(\sqrt{3}+\sqrt{2})(\sqrt{2}+1)$

or $\cot 7\frac{1}{2}^\circ=\sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{6}$.

128. If $m\sin(\alpha+\beta)=\cos(\alpha-\beta)$, prove that

$$\frac{1}{1-m\sin 2\alpha}+\frac{1}{1-m\sin 2\beta}=\frac{2}{1-m^2}.$$

129. If $\alpha+\beta+\gamma=\pi$ and

$\tan\frac{1}{4}(\beta+\gamma-\alpha)\tan\frac{1}{4}(\gamma+\alpha-\beta)\tan\frac{1}{4}(\alpha+\beta-\gamma)=1$,

then prove that $1+\cos\alpha+\cos\beta+\cos\gamma=0$.

130. Find the value of a for which the equation $\sin^4 x+\cos^4 x=a$ has real solutions.

131. If a and b are positive quantities and $a\geq b$, then find the minimum positive values of $a\sec\theta-b\tan\theta$.

132. If a, b, c and k are constant quantities and α,β,γ are variable subjects to the relation $a\tan\alpha+b\tan\beta+c\tan\gamma=k$, then find the minimum value of $\tan^2\alpha+\tan^2\beta+\tan^2\gamma$.

133. If $\frac{x}{\tan(\theta+\alpha)}=\frac{y}{\tan(\theta+\beta)}=\frac{z}{\tan(\theta+\gamma)}$, prove that :

$$\Sigma\frac{x+y}{x-y}\sin^2(\alpha-\beta)=0.$$

134. Let $a_1,a_2,\dots,a_n$ be real constants, x be a real variable and $f(x)=\cos(a_1+x)+\frac{1}{2}\cos(a_2+x)+\frac{1}{4}\cos(a_3+x)+\dots+\frac{1}{2^{n-1}}\cos(a_n+x)$

Given that $f(x_1)=f(x_2)=0$, prove that $x_2-x_1=m\pi$ for some integer m.

135. Eliminate θ from the equations
$\tan(n\theta+\alpha)-\tan(n\theta+\beta)=x$ and
$\cot(n\theta+\alpha)-\cot(n\theta+\beta)=y$.

136. If $\{\sin(\alpha-\beta)+\cos(\alpha+2\beta)\sin\beta\}^2=4\cos\alpha\sin\beta\sin(\alpha+\beta)$. Then, prove that $\tan\alpha+\tan\beta=\frac{\tan\beta}{(\sqrt{2}\cos\beta-1)^2}$; $\alpha,\beta\in(0,\pi/4)$.

137. If A, B, C are the angle of a triangle and

$$\begin{vmatrix}\sin A & \sin B & \sin C\\ \cos A & \cos B & \cos C\\ \cos^3 A & \cos^3 B & \cos^3 C\end{vmatrix}=0,$$ then show that ΔABC is an isosceles.

138. In any ΔABC, prove that

$$\Sigma\frac{\sqrt{\sin A}}{\sqrt{\sin B}+\sqrt{\sin C}-\sqrt{\sin A}}\geq 3$$

and the equality holds if and only if triangle is equilateral.

139. If $2(\cos(\alpha-\beta)+\cos(\beta-\gamma)+\cos(\gamma-\alpha)+3=0$, prove that

$$\frac{d\alpha}{\sin(\beta+\theta)\sin(\gamma+\theta)}+\frac{d\beta}{\sin(\alpha+\theta)\sin(\gamma+\theta)}+\frac{d\gamma}{\sin(\alpha+\theta)\sin(\beta+\theta)}=0,$$

where, 'θ' is any real angle such that $\alpha+\theta,\beta+\theta,\gamma+\theta$ are not the multiple of π.

140. If the quadratic equation $4^{\sec^2\alpha}x^2+2x+\left(\beta^2-\beta+\frac{1}{2}\right)=0$ have real roots, then find all the possible values of $\cos\alpha+\cos^{-1}\beta$.

141. Four real constants a, b, A, B are given and $f(\theta)=1-a\cos\theta-b\sin\theta-A\cos 2\theta-B\sin 2\theta$. Prove that if $f(\theta)\geq 0$, $\forall\,\theta\in R$, then $a^2+b^2\leq 2$ and $A^2+B^2\leq 1$.

142. If $\frac{\cos\theta_1}{\cos\theta_2}+\frac{\sin\theta_1}{\sin\theta_2}=\frac{\cos\theta_0}{\cos\theta_2}+\frac{\sin\theta_0}{\sin\theta_2}=1$, where θ_1 and θ_0 do not differ by an even multiple of π, prove that

$$\frac{\cos\theta_1\cdot\cos\theta_0}{\cos^2\theta_2}+\frac{\sin\theta_1\cdot\sin\theta_0}{\sin^2\theta_2}=-1$$

143. Prove that

$$\sum_{k=1}^{n-1} {}^nC_k\,[\cos kx\,.\cos(n+k)x+\sin(n-k)x\,.\sin(2n-k)x]=(2^n-2)\cos nx.$$

144. Determine all the values of x in the interval $x \in [0, 2\pi]$ which satisfy the inequality $2\cos x \le |\sqrt{1+\sin 2x}-\sqrt{1-\sin 2x}| \le \sqrt{2}$.

145. Find all the solutions of this equation $x^2 - 3\left[\sin\left(x-\frac{\pi}{6}\right)\right] = 3$, where [.] represents the greatest integer function.

146. In a ΔABC, prove that

$$\sum_{r=0}^{n} {}^nC_r a^r b^{n-r} \cos(rB-(n-r)A) = c^n.$$

147. Resolve z^5+1 into linear and quadratic factors with real coefficients. Hence, or otherwise deduce that, $4\sin\left(\frac{\pi}{10}\right)\cdot\cos\left(\frac{\pi}{5}\right)=1$.

148. Prove that the roots of the equation $8x^3-4x^2-4x+1=0$ are $\cos\frac{\pi}{7}$, $\cos\frac{3\pi}{7}$, $\cos\frac{5\pi}{7}$ and hence, show that $\sec\frac{\pi}{7}+\sec\frac{3\pi}{7}+\sec\frac{5\pi}{7}=4$ and deduce the equation whose roots are $\tan^2\frac{\pi}{7}$, $\tan^2\frac{3\pi}{7}$, $\tan^2\frac{5\pi}{7}$.

Trigonometric Functions and Identities Exercise 8 :
Questions Asked in Previous 10 Years Exam

149. Let α and β be non-zero real numbers such that $2(\cos\beta-\cos\alpha)+\cos\alpha\cos\beta=1$. Then which of the following is/are true?

[More than one correct option 2017 Adv.]

(a) $\sqrt{3}\tan\left(\frac{\alpha}{2}\right)-\tan\left(\frac{\beta}{2}\right)=0$

(b) $\tan\left(\frac{\alpha}{2}\right)-\sqrt{3}\tan\left(\frac{\beta}{2}\right)=0$

(c) $\tan\left(\frac{\alpha}{2}\right)+\sqrt{3}\tan\left(\frac{\beta}{2}\right)=0$

(d) $\sqrt{3}\tan\left(\frac{\alpha}{2}\right)+\tan\left(\frac{\beta}{2}\right)=0$

150. Let $-\frac{\pi}{6}<\theta<-\frac{\pi}{12}$. Suppose α_1 and β_1 are the roots of the equation $x^2-2x\sec\theta+1=0$, and α_2 and β_2 are the roots of the equation $x^2+2x\tan\theta-1=0$. If $\alpha_1>\beta_1$ and $\alpha_2>\beta_2$, then $\alpha_1+\beta_2$ equals to

[Single correct option 2016 Adv.]

(a) $2(\sec\theta-\tan\theta)$ (b) $2\sec\theta$
(c) $-2\tan\theta$ (d) 0

151. The value of $\sum_{k=1}^{13}\frac{1}{\sin\left(\frac{\pi}{4}+\frac{(k-1)\pi}{6}\right)\sin\left(\frac{\pi}{4}+\frac{k\pi}{6}\right)}$ is equal to

[Single correct option 2016 Adv.]

(a) $3-\sqrt{3}$ (b) $2(3-\sqrt{3})$ (c) $2(\sqrt{3}-1)$ (d) $2(2+\sqrt{3})$

152. Let $f:(-1,1)\to R$ be such that $f(\cos 4\theta)=\frac{2}{2-\sec^2\theta}$ for $\theta\in\left(0,\frac{\pi}{4}\right)\cup\left(\frac{\pi}{4},\frac{\pi}{2}\right)$. Then, the value(s) of $f\left(\frac{1}{3}\right)$ is/are

[More than one correct option 2012]

(a) $1-\sqrt{\frac{3}{2}}$ (b) $1+\sqrt{\frac{3}{2}}$ (c) $1-\sqrt{\frac{2}{3}}$ (d) $1+\sqrt{\frac{2}{3}}$

153. The number of all possible values of θ, where $0<\theta<\pi$, for which the system of equations
$(y+z)\cos 3\theta=(xyz)\sin 3\theta$
$x\sin 3\theta=\frac{2\cos 3\theta}{y}+\frac{2\sin 3\theta}{z}$,
and $(xyz)\sin 3\theta=(y+2z)\cos 3\theta+y\sin 3\theta$ have a solution (x_0, y_0, z_0) with $y_0z_0\ne 0$, is ......

[Integer Answer Type 2010]

154. For $0<\theta<\frac{\pi}{2}$, the solution(s) of

$$\sum_{m=1}^{6}\operatorname{cosec}\left(\theta+\frac{(m-1)\pi}{4}\right)\operatorname{cosec}\left(\theta+\frac{m\pi}{4}\right)=4\sqrt{2}\text{ is/are}$$

[More than one correct option 2009]

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{12}$ (d) $\frac{5\pi}{12}$

155. If $\frac{\sin^4 x}{2}+\frac{\cos^4 x}{3}=\frac{1}{5}$, then

[Single correct option 2009]

(a) $\tan^2 x=\frac{2}{3}$ (b) $\frac{\sin^8 x}{8}+\frac{\cos^8 x}{27}=\frac{1}{125}$
(c) $\tan^2 x=\frac{1}{3}$ (d) $\frac{\sin^8 x}{8}+\frac{\cos^8 x}{27}=\frac{2}{125}$

156. Let $\theta\in\left(0,\frac{\pi}{4}\right)$ and $t_1=(\tan\theta)^{\tan\theta}$, $t_2=(\tan\theta)^{\cot\theta}$, $t_3=(\cot\theta)^{\tan\theta}$ and $t_4=(\cot\theta)^{\cot\theta}$, then

[Single correct option 2006]

(a) $t_1>t_2>t_3>t_4$ (b) $t_4>t_3>t_1>t_2$
(c) $t_3>t_1>t_2>t_4$ (d) $t_2>t_3>t_1>t_4$

157. The number of ordered pairs (α, β), where $\alpha, \beta \in (-\pi, \pi)$ satisfying $\cos(\alpha - \beta) = 1$ and $\cos(\alpha + \beta) = \frac{1}{e}$ is **[Single correct option 2005]**

(a) 0 (b) 1
(c) 2 (d) 4

II. JEE Mains and AIEEE

158. $5(\tan^2 x - \cos^2 x) = 2\cos 2x + 9$, then the value of $\cos 4x$ is **[2017 JEE Main]**

(a) $-\frac{3}{5}$ (b) $\frac{1}{3}$ (c) $\frac{2}{9}$ (d) $-\frac{7}{9}$

159. If $f_k(x) = \frac{1}{k}(\sin^k x + \cos^k x)$, where $x \in R$, $k \geq 1$, then $f_4(x) - f_6(x)$ is equal to **[2014 JEE Main]**

(a) $\frac{1}{6}$ (b) $\frac{1}{3}$ (c) $\frac{1}{4}$ (d) $\frac{1}{12}$

160. The expression $\frac{\tan A}{1 - \cot A} + \frac{\cot A}{1 - \tan A}$ can be written as **[2013 JEE Main]**

(a) $\sin A \cos A + 1$ (b) $\sec A \operatorname{cosec} A + 1$
(c) $\tan A + \cot A$ (d) $\sec A + \operatorname{cosec} A$

161. In a ΔPQR, if $3\sin P + 4\cos Q = 6$ and $4\sin Q + 3\cos P = 1$, then the angle R is equal to **[2012 AIEEE]**

(a) $\frac{5\pi}{6}$ (b) $\frac{\pi}{6}$
(c) $\frac{\pi}{4}$ (d) $\frac{3\pi}{4}$

162. If $A = \sin^2 x + \cos^4 x$, then for all real x **[2011 AIEEE]**

(a) $\frac{13}{16} \leq A \leq 1$ (b) $1 \leq A \leq 2$
(c) $\frac{3}{4} \leq A \leq \frac{13}{16}$ (d) $\frac{3}{4} \leq A \leq 1$

163. Let $\cos(\alpha + \beta) = \frac{4}{5}$ and $\sin(\alpha - \beta) = \frac{5}{13}$, where $0 \leq \alpha, \beta \leq \frac{\pi}{4}$. Then, $\tan 2\alpha$ is equal to **[2010 AIEEE]**

(a) $\frac{25}{16}$ (b) $\frac{56}{33}$ (c) $\frac{19}{12}$ (d) $\frac{20}{7}$

164. Let A and B denote the statements

$A : \cos\alpha + \cos\beta + \cos\gamma = 0$

$B : \sin\alpha + \sin\beta + \sin\gamma = 0$

If $\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta) = -\frac{3}{2}$, then **[2009 AIEEE]**

(a) A is true and B is false
(b) A is false and B is true
(c) Both A and B are true
(d) Both A and B are false

165. A triangular park is enclosed on two sides by a fence and on the third side by a straight river bank. The two sides having fence are of same length x. The maximum area enclosed by the park is **[2006 AIEEE]**

(a) $\sqrt{\frac{x^3}{8}}$ (b) $\frac{1}{2}x^2$ (c) πx^2 (d) $\frac{3}{2}x^2$

166. If $0 < x < \pi$ and $\cos x + \sin x = \frac{1}{2}$, then $\tan x$ is **[2006 AIEEE]**

(a) $\frac{(4 - \sqrt{7})}{3}$ (b) $-\frac{(4 + \sqrt{7})}{3}$
(c) $\frac{(1 + \sqrt{7})}{4}$ (d) $\frac{(1 - \sqrt{7})}{4}$

167. In a ΔPQR, $\angle R = \frac{\pi}{2}$. If $\tan\left(\frac{P}{2}\right)$ and $\tan\left(\frac{Q}{2}\right)$ are the roots of $ax^2 + bx + c = 0$, $a \neq 0$, then **[2005 AIEEE]**

(a) $b = a + c$ (b) $b = c$
(c) $c = a + b$ (d) $a = b + c$

Answers

Exercise for Session 1

1. 72°, 18° 2. $\frac{\pi}{2}$ cm 3. 70 m 4. 45° 5. 8π
6. $\left(\frac{11}{90}\right)^c$ 7. 252 cm 8. 880 cm/s 9. 1.7 cm 10. 7

Exercise for Session 2

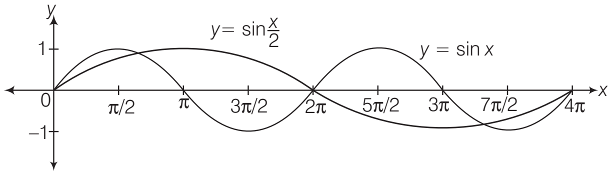
3. -3 4. $\pm\sqrt{a^2 + b^2 - c^2}$ 5. 13 6. 2 7. $\frac{1}{12}$
10. 4

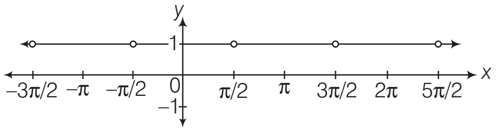
Exercise for Session 3

1. $\frac{2k}{k^2 + 1}$ 2. 1 4. $x = 1$, $y = 0$ 6. 0
7. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$ 9. 10 10. $\frac{7}{5}$

Exercise for Session 4

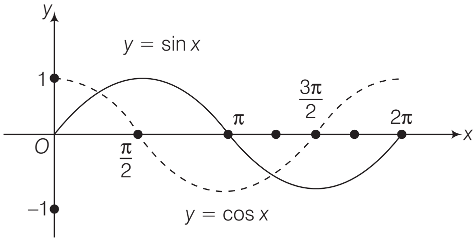
1. 28 3. -2
4.

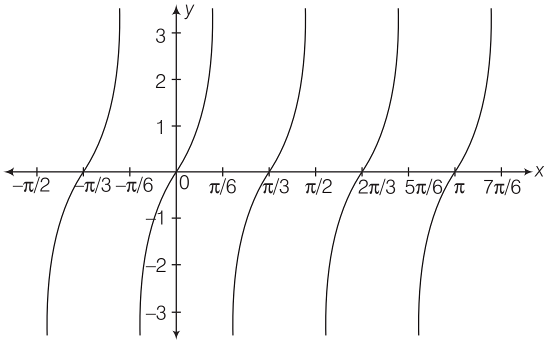


5. 

From the graph, the period of the function is π.

7. $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$

8. 

9. 

10. $\frac{1}{4}$

Exercise for Session 5

1. $\sqrt{3}$ **2.** Negative **3.** -1 **4.** 1 **5.** 1 **6.** 1
7. 2 **8.** does not exists any real solutions **9.** 2
10. 0

Exercise for Session 6

1. I and IV **3.** $\frac{\pi}{4}$ **4.** $\frac{65}{33}$ **5.** $\tan\beta + 2\tan\gamma$
6. $a^2 + b^2$ **8.** 27

Exercise for Session 7

4. $\frac{4}{5}$ **5.** 0 **6.** $2\cot^n\left(\frac{A-B}{2}\right)$ **7.** $\frac{1}{8}$ **8.** $\frac{2\pi}{8}$

Exercise for Session 8

1. False **2.** $\tan\theta$ **3.** $\frac{2}{\sqrt{5}}$ **5.** $\alpha = -1, \beta = 3$ **6.** $\frac{24}{7}$
7. -1 **8.** 1 **9.** 15 **10.** 1

Exercise for Session 9

1. $-2 \pm \sqrt{5}$ **2.** $\frac{3+2\sqrt{2}}{8}$ **3.** $\frac{4}{\sqrt{3}}$ **4.** $2\cos n\theta$, **10.** $\cot\frac{\pi}{5}$

Exercise for Session 10

4. $\frac{1}{2}$ **5.** 2 **7.** -1 **8.** 1

Exercise for Session 11

2. 0 **4.** 2 **5.** positive **6.** $\frac{3}{4} \le A \le 1$ **7.** 4.
8. $-\sqrt{2}$ and $\sqrt{2}$ **10.** 13 : 4 or 1 : 4

Chapter Exercises

1. (b) **2.** (b) **3.** (a) **4.** (a) **5.** (b) **6.** (a)
7. (b) **8.** (a) **9.** (a) **10.** (b) **11.** (d) **12.** (d)
13. (d) **14.** (b) **15.** (a) **16.** (b) **17.** (b) **18.** (b)
19. (a) **20.** (a) **21.** (c) **22.** (a) **23.** (a) **24.** (b)
25. (d) **26.** (c) **27.** (b) **28.** (b) **29.** (c) **30.** (b)
31. (c) **32.** (a) **33.** (b) **34.** (c) **35.** (b) **36.** (b)
37. (a) **38.** (d) **39.** (d) **40.** (c) **41.** (c)
42. (b) **43.** (b) **44.** (c) **45.** (a) **46.** (b) **47.** (c)
48. (b) **49.** (a) **50.** (a) **51.** (a) **52.** (d) **53.** (a)
54. (a,b,c,d) **55.** (a,b,c) **56.** (b,d) **57.** (b,c) **58.** (b,c)
59. (b,c) **60.** (a,b,d) **61.** (b,c) **62.** (c,d) **63.** (d) **64.** (b, c, d)
65. (b, c) **66.** (a, b, c, d) **67.** (a, b, c, d) **68.** (c, d)
69. (a, b, c, d) **70.** (b, c) **71.** (a, b, c, d) **72.** (b, c)
73. (a, b, c, d) **74.** (b, c) **75.** (a) **76.** (b) **77.** (b)
78. (d) **79.** (d) **80.** (a) **81.** (a) **82.** (c) **83.** (b)
84. (b) **85.** (b) **86.** (b) **87.** (a) **88.** (b) **89.** (b)
90. (c) **91.** (b) **92.** (a) **93.** (c) **94.** (b) **95.** (d)
96. (c) **97.** (b) **98.** (c)
99. A—(p, q, r, s, t); B—(s, t); C—(r)
100. A—(r, s); B—(r, t); C—(p, q)
101. A—(r); B—(p); C—(p); D—(q)
102. A—(r, t); B—(p, s); C—q
103. (2) **104.** (1) **105.** (5) **106.** (2) **107.** (5) **108.** (6)
109. (5) **110.** (35) **111.** (3) **112.** (2) **113.** (1) **114.** (2)
115. (3136) **116.** (2634) **117.** (91) **118.** (3448)
119. (1) **120.** (4) **121.** (1673) **122.** (6) **123.** (6060)
124. (4) **125.** (2) **126.** (1745)
130. $\frac{1}{2} \le a \le 1$ **131.** Minimum value is $\sqrt{a^2 - b^2}$
132. Minimum value is $\left(\frac{k^2}{a^2 + b^2 + c^2}\right)$
135. $\cot(\alpha+\beta) = \frac{1}{x} - \frac{1}{y}$
140. $\cos\alpha - \cos^{-1}\beta = \begin{cases} \frac{\pi}{3} - 1, \text{ when } n \text{ is an even integer} \\ \frac{\pi}{3} + 1, \text{ when } n \text{ is an even integer} \end{cases}$
144. $x \in \left[\frac{\pi}{4}, \frac{7\pi}{4}\right]$
145. Only two solutions, $x = 0, \sqrt{3}$
148. Required equation is, $z^3 - 21z^2 + 35z - 7 = 0$ whose roots are $\tan^2\frac{\pi}{7}, \tan^2\frac{3\pi}{7}, \tan^2\frac{5\pi}{7}$
149. (b, c) **150.** (c) **151.** (c) **152.** (a,b) **153.** (3)
154. (c,d) **155.** (b) **156.** (b) **157.** (d) **158.** (d) **159.** (d)
160. (b) **161.** (b) **162.** (d) **163.** (b) **164.** (c) **165.** (b)
166. (b) **167.** (c)

Solutions

1. Given series

$$= \left(\sin\frac{2\pi}{11} - \cos\frac{2\pi}{11}\right) + \left(\sin\frac{4\pi}{11} - \cos\frac{4\pi}{11}\right) + \left(\sin\frac{6\pi}{11} - \cos\frac{6\pi}{11}\right) + \ldots + \left(\sin\frac{20\pi}{11} - \cos\frac{20\pi}{11}\right)$$

$$= \left(\sin\frac{2\pi}{11} + \sin\frac{4\pi}{11} + \ldots + \sin\frac{20\pi}{11}\right) - \left(\cos\frac{2\pi}{11} + \cos\frac{4\pi}{11} + \ldots + \cos\frac{20\pi}{11}\right)$$

$$= \frac{\sin\pi \cdot \sin\frac{10\pi}{11}}{\sin\frac{\pi}{11}} - \frac{\cos\pi \cdot \sin\frac{10\pi}{11}}{\sin\frac{\pi}{11}}$$

$$= 0 + \frac{\sin\left(\pi - \frac{\pi}{11}\right)}{\sin\frac{\pi}{11}} = 1$$

2. $(a+1)^2 + \operatorname{cosec}^2\left(\frac{\pi a}{2} + \frac{\pi x}{2}\right) - 1 = 0$

or $\quad (a+1)^2 + \cot^2\left(\frac{\pi a}{2} + \frac{\pi x}{2}\right) = 0$

From option [b], if $a = -1$ and $\cot^2\left(\frac{-\pi}{2} + \frac{\pi x}{2}\right) = 0$

$$\Rightarrow \quad \tan^2\left(\frac{\pi x}{2}\right) = 0$$

$$\Rightarrow \quad \frac{\pi}{2} = 1$$

3. $f(x) = 9\sin^2 x - 16\cos^2 x - 10(3\sin x - 4\cos x)$

$$-10(3\sin x + 4\cos x) + 100$$

$$= 25\sin^2 x - 60\sin x + 84$$

$$= (5\sin x - 6)^2 + 48$$

$\therefore$ $f(x)_{\min}$ occurs when $\sin x = 1$

Minimum value = 49

4. $S = \frac{1}{1+\tan^3 0°} + \frac{1}{1+\tan^3 10°} + \ldots + \frac{1}{1+\tan^3 80°}$

Now, $\frac{1}{1+\tan^3\theta} + \frac{1}{1+\tan^3(90-\theta)}$

$$= \frac{1}{1+\tan^3\theta} + \frac{1}{1+\cot^3\theta}$$

$$= \frac{1}{1+\tan^3\theta} + \frac{\tan^3\theta}{1+\tan^3\theta}$$

$$= \frac{1+\tan^3\theta}{1+\tan^3\theta} = 1$$

Hence, $S = 1 + (1+1+1+1) = 5$

5. Clearly, $\sqrt{1-\sin^2 110°}\cdot\sec 110°$

$$= |\cos 110°|\sec 110°$$

$$= -\cos 110°\sec 110° = -1$$

6. $\tan\alpha + \tan\beta = -p$

$\tan\alpha\ \tan\beta = q$

$\tan(\alpha+\beta) = \frac{-p}{1-q} = \frac{p}{q-1}$

$$\frac{1}{1+\tan^2(\alpha+\beta)}[\tan^2(\alpha+\beta) + p\tan(\alpha+\beta) + q]$$

$$= \frac{1}{1+\frac{p^2}{(q-1)^2}}\left[\frac{p^2}{(q-1)^2} + \frac{p^2}{(q-1)} + q\right]$$

$$= \frac{1}{(q-1)^2+p^2}[p^2 + p^2(q-1) + q(q-1)^2]$$

$$= \frac{1}{p^2+(q-1)^2}[p^2 q + q(q-1)^2]$$

$$= q\left[\frac{p^2+(q-1)^2}{p^2+(q-1)^2}\right] = q$$

7. Let A be the expression. Multiplying A by 2^{2008} and using $2\sin\theta\cos\theta = \sin 2\theta$,

we have $\quad 2^{2008}A = \sin\frac{\pi}{2} = 1.\ A = \frac{1}{2^{2008}}$

Alternatively $\sin\left(\frac{\pi}{2^{2009}}\right)\cos\left(\frac{\pi}{2^{2009}}\right) = \frac{1}{2}\sin\left(\frac{\pi}{2^{2008}}\right)$

$$= \frac{1}{2^2}\cdot 2\sin\left(\frac{\pi}{2^{2008}}\right)\cos\left(\frac{\pi}{2^{2008}}\right)$$

$$= \frac{1}{2^2}\sin\left(\frac{\pi}{2^{2007}}\right)$$

Similarly, continued product upto,

$$\cos\left(\frac{\pi}{2^2}\right) = \frac{1}{2^{2008}}\sin\left(\frac{\pi}{2}\right) = \frac{1}{2^{2008}}$$

8. $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\ \tan B}$

$$= \frac{\tan A + \frac{n\sin A\cos A}{1-n\cos^2 A}}{1 - \tan A\cdot\frac{n\sin A\cos A}{1-n\cos^2 A}}$$

$$= \frac{\sin A(1-n\cos^2 A) + n\sin A\cos^2 A}{\cos A(1-n\cos^2 A) - n\sin^2 A\cos A}$$

$$= \frac{\sin A - 0}{\cos A(1-n\cos^2 A - n\sin^2 A)}$$

$$= \frac{\sin A}{(1-n)\cos A}$$

9. We have, $\quad Q = \sum_{r=0}^{n}\frac{\sin(3^r\theta)}{\cos(3^{r+1}\theta)}$

$$= \frac{\sin\theta}{\cos 3\theta} + \frac{\sin 3\theta}{\cos 9\theta} + \frac{\sin 9\theta}{\cos 27\theta} + \ldots\ldots + \frac{\sin(3^n\theta)}{\cos(3^{n+1}\theta)}$$

As, $\dfrac{\sin\theta}{\cos 3\theta} = \dfrac{2\sin\theta\cos\theta}{2\cos\theta\cos 3\theta} = \dfrac{\sin 2\theta}{2\cos\theta\cos 3\theta}$

$= \dfrac{1}{2}\left[\dfrac{\sin(3\theta - \theta)}{\cos\theta\cos 3\theta}\right]$

$= \dfrac{1}{2}(\tan 3\theta - \tan\theta)$

$\therefore \quad Q = \dfrac{1}{2}[(\tan 3\theta - \tan\theta) + (\tan 9\theta$

$- \tan 3\theta) + \ldots + \tan 3^{n+1}\theta - \tan 3^n\theta)]$

$\Rightarrow \quad Q = \dfrac{P}{2} \Rightarrow P = 2Q$

10. Expression

$(\cos^4 1° + \cos^4 2° + \cos^4 3° + \ldots + \cos^4 179°)$

$- (\sin^4 1° + \sin^4 2° + \sin^4 3° + \ldots + \sin^4 179°)$

$= \cos 2° + \cos 4° + \cos 6° + \ldots + \cos(358°)$

$= \cos\dfrac{\left(\dfrac{2° + 358°}{2}\right) \cdot \sin(179 \times 1°)}{\sin 1°}$

$= \cos(180°) = -1.$

11. $\sin x + \sin y = a$ ...(i)

$\cos x + \cos y = 2a$ ...(ii)

On squaring and adding Eqs. (i) and (ii), we get

$2 + 2\cos(x - y) = 5a^2$

$\cos(x - y) = \dfrac{5a^2 - 2}{2}$

12. $P(x) = \sqrt{3 + 2(\cos x + \cos x + \cos 2x)}$

$= \sqrt{3 + 2(2\cos x + 2\cos x^2 x - 1}$

$= \sqrt{4\cos^2 x + 4\cos^2 x + 1}$

$= |2\cos x + 1|$

13. Consider $y = 5\,\text{secc}^2\theta - \tan^2\theta + 4\,\text{cosec}^2\theta$

$\therefore \quad y = 5 + 5\tan^2\theta - \tan^2\theta + 4 + \cot^2\theta$

$y = 9 + 4(\tan^2 + \cot^2)$

$= 9 + 4[(\tan\theta - \cot\theta)^2 + 2]$

$\therefore \quad y_{\min} = 9 + 8 = 17$

$\Rightarrow$ Maximum value of the expression is $\dfrac{1}{17} = \dfrac{p}{q}$

$\Rightarrow \quad p + q = 1 + 17 = 18$

14. $f_n(\alpha) = \tan n\alpha$ and $f_n\left(\dfrac{\pi}{32}\right) = \tan\dfrac{\pi}{8} = \sqrt{2} - 1$

15. Let $\sin x + \cos x = t$

$\therefore \quad y = \left|t + \dfrac{1}{t}\right|$

Hence, minimum value of y is 2.

16. $a = \cos(2012\,\pi) = 1$

$b = \sec(2013\,\pi) = -1$

$c = \tan(2014\,\pi) = 0$

$\therefore b < c < a$

17. InΔABC, $\sum \tan\dfrac{A}{2} \cdot \tan\dfrac{B}{2} = 1$

$\therefore \quad \sum \tan^2\dfrac{A}{2} \geq \sum \tan\dfrac{A}{2}\tan\dfrac{B}{2} = 1$

$[\because a^2 + b^2 + c^2 - ab - bc - ca \geq 0, \forall\, a, b, c \in R]$

$\therefore \quad 3 + \sum \tan^2\dfrac{A}{2} \geq 4$

$\Rightarrow \quad 3 + \tan^2\dfrac{A}{2} + \tan^2\dfrac{B}{2} + \tan^2\dfrac{C}{2} \geq 1 + 3$

$\Rightarrow \quad \sec^2\dfrac{A}{2} + \sec^2\dfrac{B}{2} + \sec^2\dfrac{C}{2} \geq 4$

18. The given equation can be rewritten as

$(2x - 1)^2 + 1 = \sin^2 y$, which is possible only when $x = \dfrac{1}{2}$,

$\sin^2 y = 1$

$\Rightarrow y = \dfrac{-\pi}{2}, \dfrac{\pi}{2}$ [as $x^2 + y^2 \leq 3$]

Thus, there are only two pairs (x, y) satisfying the given equation. They are $\left(\dfrac{1}{2}, \dfrac{-\pi}{2}\right)$ and $\left(\dfrac{1}{2}, \dfrac{\pi}{2}\right)$.]

19. Given,

$3\sin A + 4\cos B = 6$ ...(i)

$3\cos A + 4\sin B = 1$ ...(ii)

On squaring and adding Eqs. (i) and (ii), we get

$9 + 16 + 24\sin(A + B) = 37$

$24\sin(A + B) = 12$

$\sin(A + B) = \dfrac{1}{2}$

$\Rightarrow \quad \sin C = \dfrac{1}{2}$

$C = 30°$ or $150°$

If $C = 150°$, then even of $B = 0$ and $A = 30°$.

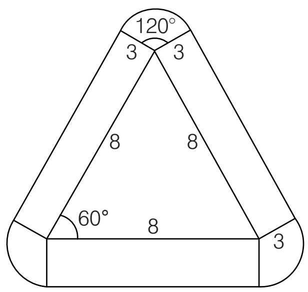
The quantity $3\sin A + 4\cos B$

$3 \cdot \dfrac{1}{2} + 4 = 5\dfrac{1}{2} < 6$

Hence, $C = 150°$ is not possible

$\Rightarrow \quad \angle C = 30°$ only

20. Area $= 3 \cdot (8 \cdot 3) + 3 \cdot \dfrac{1}{2} r^2\theta$



$= 72 + \dfrac{3}{2} \cdot 9 \cdot \dfrac{2\pi}{3}$

$= 72 + 9\pi$

$= 9(8 + \pi)$

21. $m+n=a\{(\cos^3\alpha+\sin^3\alpha)+3\cos\alpha\sin\alpha(\cos\alpha+\sin\alpha)\}$

$m+n=a\{\cos\alpha+\sin\alpha\}^3$

Similarly, $\quad m-n=a\,\{\cos\alpha-\sin\alpha\}^3$

$$(m+n)^{2/3}=a^{2/3}(\cos\alpha+\sin\alpha)^2 \quad \text{...(i)}$$

Similarly, $\quad (m-n)^{2/3}=a^{2/3}(\cos\alpha-\sin\alpha)^2 \quad$ ...(ii)

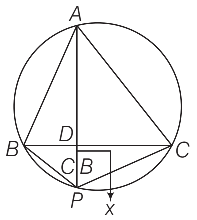
On adding Eqs. (i) and (ii), we get

$(m+n)^{2/3}+(m-n)^{2/3}=a^{2/3}\,(2)$

$\Rightarrow \qquad\qquad =2a^{2/3}$

22. $BD=x\tan C$ in ΔPDB

and $DC=x\tan B$ for ΔPDC



$\therefore \quad BD+DC=a=x(\tan B+\tan C)$

$$\frac{a}{x}=\tan B+\tan C$$

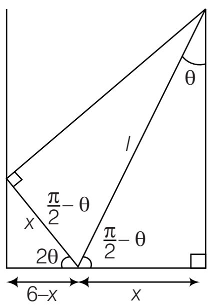
Similarly, $\quad \dfrac{b}{y}=\tan A+\tan C$

$$\frac{c}{z}=\tan A+\tan B$$

$$\therefore \quad \frac{a}{2x}+\frac{b}{2y}+\frac{c}{2z}=\frac{1}{2}\left(\frac{a}{x}+\frac{b}{y}+\frac{c}{z}\right)\tan A+\tan B+\tan C$$

23. $\sin\theta=\dfrac{x}{l} \quad$ ...(i)

Also, $\qquad \cos 2\theta=\dfrac{6-x}{x}$



$$1+\cos 2\theta=\frac{6}{x};$$

$$2\cos^2\theta=\frac{6}{l\sin\theta}$$

[substituting $x=l\sin\theta$ from Eq. (i)]

$$l=\frac{3}{\sin\theta\cos^2\theta}$$

24. $S=2\sin 2^\circ+4\sin 4^\circ+\ldots+178\sin 178+180^\circ\sin 180^\circ$

$S=2[\sin 2^\circ+2\sin 4^\circ+3\sin 6^\circ+\ldots+89\sin 178^\circ] \quad$ ...(i)

$S=2[89\sin 178^\circ+88\sin 176^\circ+\ldots+1\cdot\sin 2^\circ] \quad$ ...(ii)

On adding Eqs. (i) and (ii), we get

[converting in reverse order]

$2S=2[90\,(\sin 2^\circ+\sin 4^\circ+\sin 6^\circ+\ldots+\sin 178^\circ)]$

$$S=90\cdot\frac{\sin\left(\dfrac{n\theta}{2}\right)}{\sin\dfrac{\theta}{2}}\sin\left(\frac{(n+1)\theta}{2}\right)$$

$$=\frac{90\sin(89^\circ)}{\sin 1^\circ}\cdot\sin 90^\circ \qquad [\theta=2^\circ]$$

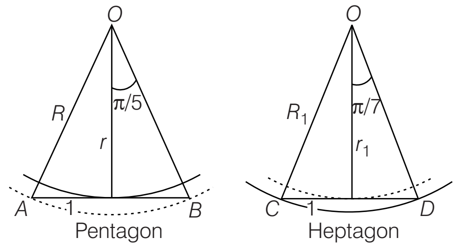
$S=90\cot 1^\circ$

Average value $=\dfrac{90\cot 1^\circ}{90}=\cot 1^\circ$

25. In 1st case, $r=\cot\dfrac{\pi}{5}$; $R=\operatorname{cosec}\dfrac{\pi}{5}$

2nd case, $\quad r_1=\cot\dfrac{\pi}{7}$; $R_1=\operatorname{cosec}\dfrac{\pi}{7}$

$$\therefore \qquad A_1=\pi(R^2-r^2)=\pi\left(\operatorname{cosec}^2\frac{\pi}{5}-\cot^2\frac{\pi}{5}\right)=\pi$$



M $\qquad A_2=\pi(R_1^2-r_1^2)$

$$=\pi\left(\operatorname{cosec}^2\frac{\pi}{7}-\cot^2\frac{\pi}{7}\right)=\pi$$

$\Rightarrow \qquad A_1=A_2$

26. $\displaystyle\sum_{r=1}^{18}\cos^2(5r)^\circ=\cos^2 5^\circ+\cos^2 10^\circ$

$+\cos^2 15^\circ+\ldots+\cos^2 85^\circ+\cos^2 90^\circ$

$=(\cos^2 5^\circ+\cos^2 85^\circ)+(\cos^2 10^\circ+\cos^2 80^\circ)$

$+(\cos^2 15^\circ+\cos^2 75^\circ)+\ldots+(\cos^2 40^\circ+\cos^2 50^\circ)+\cos^2 45^\circ$

$=(\cos^2 5^\circ+\sin^2 5^\circ)+(\cos^2 10^\circ+\sin^2 10^\circ)$

$+(\cos^2 15^\circ+\sin^2 15^\circ)+\ldots+(\cos^2 40^\circ+\sin^2 40^\circ)+\cos^2 45^\circ$

$$=1+1+1+\ldots+1+\frac{1}{2}=8+\frac{1}{2}=\frac{17}{2}$$

27. $4x^2-4x\,|\sin\theta|-(1-\sin^2\theta)$

$=-1+(2x-|\sin\theta|)^2$

$\therefore$ Minimum value $=-1$

28. $\because \cos 3A+\cos 3B+\cos 3C=1$

$\Rightarrow \qquad \cos 3A+\cos 3B+\cos 3C-1=0$

$\Rightarrow \qquad \cos 3A+\cos 3B+\cos 3C+\cos 3\pi=0$

$$\Rightarrow 2\cos\left(\frac{3A+3B}{2}\right)\cos\left(\frac{3A-3B}{2}\right)+2\cos\left(\frac{3\pi+3C}{2}\right)$$

$$\cos\left(\frac{3\pi-3C}{2}\right)=0$$

$$\Rightarrow 2\cos\left(\frac{3\pi-3C}{2}\right)\left\{\cos\left(\frac{3A-3B}{2}\right)+\cos\left(\frac{3\pi+3C}{2}\right)\right\}=0$$

$$\Rightarrow\ 2\cos\left(\frac{3\pi}{2}-\frac{3C}{2}\right)\cdot 2\cos\left(\frac{3\pi+3C+3A-3B}{4}\right)$$

$$\cdot\cos\left(\frac{3\pi+3C-3A+3B}{4}\right)=0$$

$$\Rightarrow\ 2\cos\left(\frac{3\pi}{2}-\frac{3C}{2}\right)\cdot 2\cos\left(\frac{3\pi}{2}-\frac{3B}{2}\right)\cdot\cos\left(\frac{3\pi}{2}-\frac{3A}{2}\right)=0$$

$$\Rightarrow\ -4\sin\left(\frac{3A}{2}\right)\sin\left(\frac{3B}{2}\right)\sin\left(\frac{3C}{2}\right)=0$$

$$\Rightarrow\ \sin\left(\frac{3A}{2}\right)\sin\left(\frac{3B}{2}\right)\sin\left(\frac{3C}{2}\right)=0$$

$$\therefore\ \frac{3A}{2}=\pi \text{ or } \frac{3B}{2}=\pi \text{ or } \frac{3C}{2}=\pi$$

$$\therefore\ A=\frac{2\pi}{3} \text{ or } B=\frac{2\pi}{3} \text{ or } C=\frac{2\pi}{3}$$

29. $\because |\tan A|<1$

$$\Rightarrow\ -1<\tan A<1 \text{ and } 0\le |A|<\frac{\pi}{2}$$

$$\Rightarrow\ -\frac{\pi}{2}<A<\frac{\pi}{2}$$

$$\because\ \sqrt{(1+\sin 2A)}=\sqrt{1+\frac{2\tan A}{1+\tan^2 A}}$$

$$=\frac{|1+\tan A|}{\sqrt{(1+\tan^2 A)}}=\frac{(1+\tan A)}{\sqrt{(1+\tan^2 A)}}$$

and $\sqrt{(1-\sin 2A)}=\sqrt{1-\left(\dfrac{2\tan A}{1+\tan^2 A}\right)}$

$$=\frac{|1-\tan A|}{\sqrt{(1+\tan^2 A)}}$$

$$=\frac{(1-\tan A)}{\sqrt{(1-\tan^2 A)}}$$

$$\therefore\ \frac{\sqrt{(1+\sin 2A)}+\sqrt{(1-\sin 2A)}}{\sqrt{(1+\sin 2A)}-\sqrt{(1-\sin 2A)}}=\frac{2}{2\tan A}=\cot A$$

30. Let $f(\theta)=\cos^2(\cos\theta)+\sin^2(\sin\theta)$

$\because\ -1\le\cos\theta\le 1$ and $-1\le\sin\theta\le 1$

$\therefore\ \cos 1\le\cos(\cos\theta)\le 1$ and $-\sin 1\le\sin(\sin\theta)\le\sin 1$

$\therefore\ \cos^2 1\le\cos^2(\cos\theta)\le 1$ and $0\le\sin^2(\sin\theta)\le\sin^2 1$

$\therefore$ Maximum value of $f(\theta)=1+\sin^2 1$

31. Let $f(x)=27^{\cos 2x}\,81^{\sin 2x}=3^{3\cos 2x+4\sin 2x}$

$$=3^{5\left(\frac{3}{5}\cos 2x+\frac{4}{5}\sin 2x\right)}$$

Let $\dfrac{3}{5}=\sin\phi$ and $\dfrac{4}{5}=\cos\phi$

Thus, $f(x)=3^{5(\sin\phi\cos 2x+\cos\phi\sin 2x)}=3^{5[(\sin(\phi+2x)]}$

For minimum value of given function, $\sin(\phi+2x)$ will be minimum.

i.e. $\sin(\phi+2x)=-1$

$$\therefore\ f(x)=3^{5(-1)}=\frac{1}{243}$$

Alternate Method

Let $f(x)=27^{\cos 2x}\,81^{\sin 2x}=3^{4\sin 2x}=3^{3\cos 2x+4\sin 2x}$

For minimum value of given function, $3\cos 2x+4\sin 2x$ will be minimum.

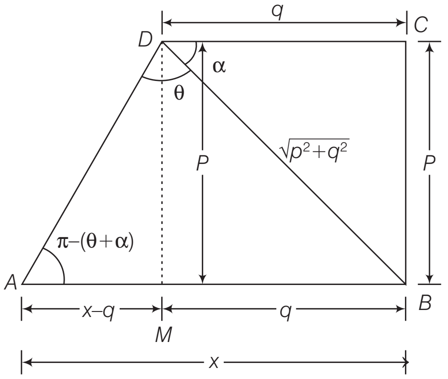
$$\therefore\ -\sqrt{3^2+4^2}\le 3\cos 2x+4\sin 2x\le\sqrt{3^2+4^2}$$

$$\Rightarrow\ -5\le 3\cos 2x+4\sin 2x\le 5$$

$\therefore$ Minimum of $3\cos 2x+4\sin 2x=-5$

So, $\min f(x)=3^{-5}=\dfrac{1}{243}$

32. Let $AB=x$ and $\angle BDC=\alpha$



In ΔDAM, $\tan(\pi-\theta-\alpha)=\dfrac{p}{x-q}$

$$\Rightarrow\ \tan(\theta+\alpha)=\frac{p}{q-x}\Rightarrow q-x=p\cos(\theta+\alpha)$$

$$\Rightarrow\ x=q-p\cot(\theta+\alpha)=q-p\left(\frac{\cot\theta\cot\alpha-1}{\cot\alpha+\cot\theta}\right)$$

$$=\frac{q(\cot\alpha+\cot\theta)+p(\cot\theta\cot\alpha)+p}{\cot\alpha+\cot\theta}$$

$$=\frac{q\left(\dfrac{q}{p}+\dfrac{\cos\theta}{\sin\theta}\right)-p\left(\dfrac{q}{p}\dfrac{\cos\theta}{\sin\theta}\right)+p}{\dfrac{q}{p}+\dfrac{\cos\theta}{\sin\theta}}\qquad\left[\because \cot\alpha=\frac{q}{p}\right]$$

$$=\frac{\dfrac{q^2}{p}+\dfrac{q\cos\theta}{\sin\theta}-q\dfrac{\cos\theta}{\sin\theta}+p}{\dfrac{q\sin\theta+p\cos\theta}{p\sin\theta}}=\frac{(q^2+p^2)\sin\theta}{q\sin\theta+p\sin\theta}$$

33. Given $4n\alpha=\pi\ \Rightarrow\ 2n\alpha=\dfrac{\pi}{2}$

Now $\cot\alpha\cdot\cot(2n-1)\alpha=\cot\alpha\cot\left(\dfrac{\pi}{2}-\alpha\right)$

$$=\cot\alpha\cdot\tan\alpha=1$$

Similarly, $\cot 2\alpha\ \cos(2n-2)\alpha=1$,

$\cot 3\alpha\cdot\cot(2n-3)\alpha=1,\ \dots,\ \cot(n-1)\alpha\ \cot(n+1)\alpha=1$

Thus $\cot\alpha\ \cot 2\alpha\ \cot 3\alpha\ \dots\ \cot(2n-1)\alpha$

$=\{\cot\alpha\ \cot(2n-1)\alpha\}\{\cot 2\alpha\ \cot(2n-2)\alpha\}$

$\dots\{\cot(n-1)\alpha\ \cot(n+1)\alpha\}\cdot\cot n\alpha$

$=1\cdot 1\cdot 1\dots 1\cdot 1\qquad\left[\because \cot n\alpha=\cot\dfrac{\pi}{4}=1\right]$

$=1$

34. We have $(\sin A+\sin B+\sin C)(\sin A+\sin B-\sin C)$

$$=3\sin A\sin B$$

$$\Rightarrow\ (\sin A+\sin B)^2-\sin^2 C=3\sin A\sin B$$

$$\Rightarrow\ \sin^2 A+\sin^2 B-\sin^2 C=\sin A\sin B$$

$$\Rightarrow \quad \sin^2 A + \sin(B+C)\sin(B-C) = \sin A \sin B$$

$$\Rightarrow \quad \sin A\,[\sin(B+C) + \sin(B-C)] = \sin A \sin B$$

$$[\because A + B + C = \pi]$$

$$\Rightarrow \quad \sin A(2\sin B\cos C) = \sin A \sin B$$

$$\therefore \quad \cos C = \frac{1}{2} \Rightarrow C = 60^\circ$$

35. From the third relation we get

$$\cos\theta\cos\phi + \sin\theta\sin\phi = \sin\beta\sin\gamma$$

$$\Rightarrow \sin^2\theta\sin^2\phi = (\cos\theta\cos\phi - \sin\beta\sin\gamma)^2$$

$$\Rightarrow \left(1 - \frac{\sin^2\beta}{\sin^2\alpha}\right)\left(1 - \frac{\sin^2\gamma}{\sin^2\alpha}\right) = \left(\frac{\sin\beta\sin\gamma}{\sin^2\alpha} - \sin\beta\sin\gamma\right)^2$$

[from the first and second relations]

$$\Rightarrow (\sin^2\alpha - \sin^2\beta)(\sin^2\alpha - \sin^2\gamma)$$

$$= \sin^2\beta\sin^2\gamma(1 - \sin^2\alpha)^2$$

$$\Rightarrow \sin^4\alpha(1 - \sin^2\beta\sin^2\gamma)$$

$$- \sin^2\alpha(\sin^2\beta + \sin^2\gamma - 2\sin^2\beta\sin^2\gamma) = 0$$

$$\therefore \quad \sin^2\alpha = \frac{\sin^2\beta - \sin^2\gamma - 2\sin^2\beta\sin^2\gamma}{1 - \sin^2\beta\sin^2\gamma} \quad [\because \sin\alpha \neq 0]$$

and $\cos^2\alpha = \dfrac{1 - \sin^2\beta - \sin^2\gamma + \sin^2\beta\sin^2\gamma}{1 - \sin^2\beta\sin^2\gamma}$

$$\Rightarrow \quad \tan^2\alpha = \frac{\sin^2\beta - \sin^2\beta - \sin^2\gamma + \sin^2\gamma - \sin^2\beta\sin^2\gamma}{\cos^2\beta - \sin^2\gamma(1 - \sin^2\beta)}$$

$$= \frac{\sin^2\beta\cos^2\gamma + \cos^2\beta\sin^2\gamma}{\cos^2\beta\cos^2\gamma}$$

$$= \tan^2\beta + \tan^2\gamma$$

$$\Rightarrow \tan^2\alpha - \tan^2\beta - \tan^2\gamma = 0$$

36. $\tan\beta = \dfrac{n\tan\alpha}{1 + \tan^2\alpha - n\tan^2\alpha}$

$$= \frac{n\tan\alpha}{1 + (1-n)\tan^2\alpha}$$

$$\Rightarrow \tan(\alpha - \beta) = \frac{\tan\alpha - \dfrac{n\tan\alpha}{1 + (1-n)\tan^2\alpha}}{1 + \dfrac{\tan\alpha \cdot n\tan\alpha}{1 + (1-n)\tan^2\alpha}}$$

$$= \frac{\tan\alpha + (1-n)\tan^3\alpha - n\tan\alpha}{1 + (1-n)\tan^2\alpha + n\tan^2\alpha}$$

$$= \frac{(1-n)\tan\alpha(1 + \tan^2\alpha)}{1 + \tan^2\alpha} = (1-n)\tan\alpha$$

37. Let $\dfrac{(\cos\theta)}{a} = \dfrac{(\sin\theta)}{b} = k$, so that $\cos\theta = ak$ and $\sin\theta = bk$. Then

$$a\cos 2\theta + b\sin 2\theta = a(1 - 2\sin^2\theta) + 2b\sin\theta\cos\theta$$

$$= a - 2ab^2k^2 + 2b \cdot bk \cdot ak$$

$$= a - 2ab^2k^2 + 2ab^2k^2 = a$$

38. Let $y = \cos x\cos(x+2) - \cos^2(x+1)$

$$= \frac{1}{2}[\cos(2x+2) + \cos 2] - \cos^2(x+1)$$

$$= \frac{1}{2}[2\cos^2(x+1) - 1 + \cos 2] - \cos^2(x+1)$$

$$= -\frac{1}{2}(1 - \cos 2) = -\frac{1}{2}(2\sin^2 1) = -\sin^2 1$$

This shows that $y = -\sin^2 1$ is a straight line which is parallel to X-axis and clearly passes through the point $\left(\dfrac{\pi}{2}, -\sin^2 1\right)$.

39. $f(\theta) = |\sin\theta| + |\cos\theta|, \forall\, \theta \in R$ Clearly, $f(\theta) > 0$.

Also, $\quad f^2(\theta) = \sin^2\theta + \cos^2\theta + |2\sin\theta\cdot\cos\theta|$

$$= 1 + |\sin 2\theta|$$

$$0 \leq |\sin 2\theta| \leq 1$$

$$\Rightarrow \quad 1 \leq f^2(\theta) \leq 2 \Rightarrow 1 \leq f(\theta) \leq \sqrt{2}$$

40. $A = \cos(\cos x) + \sin(\cos x)$

$$= \sqrt{2}\left\{\cos(\cos x)\cos\frac{\pi}{4} + \sin(\cos x)\sin\frac{\pi}{4}\right\}$$

$$= \sqrt{2}\left\{\cos\left(\cos x - \frac{\pi}{4}\right)\right\}$$

$$\because \quad -1 \leq \cos\left(\cos x - \frac{\pi}{4}\right) \leq 1$$

$$\therefore \quad -\sqrt{2} \leq A \leq \sqrt{2}$$

41. We have, $\dfrac{U_n}{V_n} = \tan n\theta$

and $\dfrac{V_n - V_{n-1}}{U_{n-1}} = \dfrac{\cos n\theta\sec^n\theta - \cos(n-1)\theta\sec^{n-1}\theta}{\sin(n-1)\theta\sec^{n-1}\theta}$

$$= \frac{\cos n\theta\sec\theta - \cos(n-1)\theta}{\sin(n-1)\theta}$$

$$= \frac{\cos n\theta - \cos(n-1)\theta\cos\theta}{\cos\theta\sin(n-1)\theta}$$

$$= \frac{\cos(n-1)\theta\cos\theta - \sin(n-1)\theta\sin\theta - \cos(n-1)\theta\cos\theta}{\cos\theta\sin(n-1)\theta}$$

$$= -\tan\theta$$

So, that $\dfrac{V_n - V_{n-1}}{U_{n-1}} + \dfrac{1}{n}\dfrac{U_n}{V_n} = -\tan\theta + \dfrac{\tan n\theta}{n} \neq 0$

42. If $a, b > 0$

Using A.M. $\geq$ G.M., we get

$$\frac{1}{a} + \frac{1}{b} \geq \frac{2}{\sqrt{ab}}$$

$$\Rightarrow \quad f(x) \geq \frac{2}{\sqrt{\cos\left(\dfrac{\pi}{6} - x\right)\cos\left(\dfrac{\pi}{6} + x\right)}}$$

$$= \frac{2}{\sqrt{\cos^2\dfrac{\pi}{6} - \sin^2 x}} = \frac{2}{\sqrt{\dfrac{3}{4} - \dfrac{1 - \cos 2x}{2}}}$$

$$= \frac{2}{\sqrt{\dfrac{1}{4} + \dfrac{\cos 2x}{2}}}$$

Now for $0 \le x \le \frac{\pi}{3}$, $\frac{-1}{2} \le \cos 2x \le 1$

$$\Rightarrow \quad 0 \le \sqrt{\frac{1}{4} + \frac{\cos 2x}{2}} \le \frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad f(x) \ge \frac{4}{\sqrt{3}}$$

Since 'f' is continuous range of 'f' is $\left[\frac{4}{\sqrt{3}}, \infty\right)$.

43. $\because 0 \le \sin^2 \theta \le 1$ and $0 \le \cos^2 \theta \le 1$

$\Rightarrow \quad 0 \le \sin^8 \theta \le \sin^2 \theta$ and $0 \le \cos^{14} \theta \le \cos^2 \theta$

$\therefore \quad 0 < \sin^8 \theta + \cos^{14} \theta \le \sin^2 \theta + \cos^2 \theta$

Hence, $\quad 0 < A \le 1$

44. $\sin\left(\frac{3\pi}{2} - \alpha\right) = -\cos\alpha$, $\sin\left(\frac{\pi}{2} + \alpha\right) = \cos\alpha$

$\sin(3\pi + \alpha) = -\sin\alpha$

$\sin(5\pi - \alpha) = -\sin\alpha$

$$\therefore \quad 3\left\{\sin^4\left(\frac{3\pi}{2} - \alpha\right) + \sin^4(3\pi + \alpha)\right\} - 2\left[\sin^6\left(\frac{\pi}{2} + \alpha\right) + \sin^6(5\pi - \alpha)\right]$$

$= 3\{\cos^4 \alpha + \sin^4 \alpha\} - 2\{\cos^6 \alpha + \sin^6 \alpha\}$

$= 3\{1 - 2\sin^2 \alpha \cos^2 \alpha\} - 2\{1 - 3\sin^2 \alpha \cos^2 \alpha\} = 1$

45. $\sin\left(x + \frac{\pi}{6}\right) + \cos\left(x + \frac{\pi}{6}\right)$

$= \sqrt{2}\left[\sin\left(x + \frac{\pi}{6} + \frac{\pi}{4}\right)\right]$

$= \sqrt{2}\sin\left(x + \frac{5\pi}{12}\right) \le \sqrt{2}$

Equality holds when $x + \frac{5\pi}{12} = \frac{\pi}{2}$ ie, $x = \frac{\pi}{12}$

Therefore, maximum value of given expression is attained at

$$x = \frac{\pi}{12}$$

46. $\cot^2 x = \cot(x - y) \cdot \cot(x - z)$

$\Rightarrow \quad \cot^2 x = \left(\frac{\cot x \cot y + 1}{\cot y - \cot x}\right)\left(\frac{\cot x \cdot \cot z + 1}{\cot z - \cot x}\right)$

$\Rightarrow \cot^2 x \cdot \cot y \cdot \cot z - \cot^3 x \cdot \cot y - \cot^3 x \cot z + \cot^4 x$

$= \cot^2 x \cdot \cot y \cdot \cot z + \cot x \cdot \cot y + \cot x \cdot \cot z + 1$

$\Rightarrow \cot x \cot y(1 + \cot^2 x) + \cot x \cot z(1 + \cot^2 x) + 1 - \cot^4 x = 0$

$\Rightarrow \quad \cot x(\cot y + \cot z)(1 + \cot^2 x) + (1 - \cot^2 x)(1 + \cot^2 x) = 0$

$\Rightarrow \quad \cot x(\cot y + \cot z) + (1 - \cot^2 x) = 0$

$$\Rightarrow \quad \frac{\cot^2 x - 1}{2\cot x} = \frac{1}{2}(\cot y + \cot z)$$

$$\Rightarrow \quad \frac{1}{2}(\cot y + \cot z) = \cot 2x$$

47. Given that, $\alpha + \beta + \gamma = \pi$

Taking $\alpha = -\frac{\pi}{2}$; $\beta = -\frac{\pi}{2}$ and $\gamma = 2\pi$

$\therefore \quad \sin\alpha + \sin\beta + \sin\gamma = -1 - 1 + 0 = -2$

but $\sin\alpha + \sin\beta + \sin\gamma \ge -3$ for any α, β, γ

Hence, minimum value of $\sin\alpha + \sin\beta + \sin\gamma$ is negative.

48. $\cos x - \sin\alpha \cot\beta \sin x = \cos\alpha$

$\Rightarrow \quad \sin\beta\cos x - \sin\alpha\cos\beta\sin x = \cos\alpha\sin\beta$

$\Rightarrow \quad \sin\beta\left(1 - \tan^2\frac{x}{2}\right) - \sin\alpha\cos\beta \cdot 2\tan\frac{x}{2}$

$= \cos\alpha\sin\beta\left(1 + \tan^2\frac{x}{2}\right)$

$\Rightarrow \quad \tan^2\frac{x}{2}(-\sin\beta - \cos\alpha\sin\beta) - \sin\alpha\cos\beta \cdot 2\tan\frac{x}{2} + \sin\beta(1 - \cos\alpha) = 0$

$$\Rightarrow \quad \tan\frac{x}{2} = \frac{-2\sin\alpha\cos\beta \pm \sqrt{4[\sin^2\alpha\cos^2\beta + \sin^2\beta(1 + \cos\alpha)(1 - \cos\alpha)]}}{2\sin\beta(1 + \cos\alpha)}$$

$$= \frac{-\sin\alpha\cos\beta \pm \sqrt{\sin^2\alpha(\sin^2\beta + \cos^2\beta)}}{\sin\beta(1 + \cos\alpha)}$$

$$= \frac{-\sin\alpha\cos\beta \pm \sin\alpha}{\sin\beta(1 + \cos\alpha)} = \frac{\sin\alpha(1 - \cos\beta \pm 1)}{\sin\beta(1 + \cos\alpha)}$$

$$= \tan\frac{\beta}{2}\tan\frac{\alpha}{2} \text{ or } -\tan\frac{\alpha}{2}\cot\frac{\beta}{2}$$

49. $\because \cos^4\theta\sec^4\alpha$, $\frac{1}{2}$ and $\sin^4\theta\,\mathrm{cosec}^2\alpha$ are in AP

$1 = \cos^4\theta\sec^2\alpha + \sin^4\theta\,\mathrm{cosec}^2\alpha$

$$\Rightarrow \quad 1 = \frac{\cos^4\theta}{\cos^2\alpha} + \frac{\sin^4\theta}{\sin^2\alpha}$$

$$\Rightarrow \quad (\sin^2\theta + \cos^2\theta)^2 = \frac{\cos^4\theta}{\cos^2\alpha} + \frac{\sin^4\theta}{\sin^2\alpha}$$

$$\Rightarrow \quad \cos^4\theta\left(\frac{1}{\cos^2\alpha} - 1\right) + \sin^4\theta\left(\frac{1}{\sin^2\alpha} - 1\right) - 2\sin^2\theta\cos^2\theta = 0$$

$\Rightarrow \quad \sin^4\alpha\cos^4\theta + \sin^4\theta\cos^4\alpha - 2\sin^2\theta\cos^2\theta\sin^2\alpha\cos^2\alpha = 0$

$\Rightarrow \quad (\sin^2\alpha\cos^2\theta - \cos^2\alpha\sin^2\theta)^2 = 0$

$\Rightarrow \quad \tan^2\theta = \tan^2\alpha$

$\therefore \quad \theta = n\pi \pm \alpha, n \in I$

Now, $\quad \cos^8\theta\sec^6\alpha = \cos^8\alpha\sec^6\alpha = \cos^2\alpha$

and $\quad \sin^8\theta\,\mathrm{cosec}^6\alpha = \sin^8\alpha \cdot \mathrm{cosec}^6\alpha = \sin^2\alpha$

Hence, $\cos^8\theta\sec^6\alpha$, $\frac{1}{2}$, $\sin^8\theta\,\mathrm{cosec}^6\alpha$

ie, $\cos^2\alpha$, $\frac{1}{2}$, $\sin^2\alpha$ are in AP.

50. Given, $(\cot \alpha_1)\cdot(\cot \alpha_2)\ldots(\cot \alpha_n)=1$

$$\Rightarrow \quad \prod_{i=1}^{n} \cos \alpha_1 = \prod_{i=1}^{n} \sin \alpha_1$$

$$\Rightarrow \quad \prod_{i=1}^{n} \cos^2 \alpha_1 = \prod_{i=1}^{n} \sin \alpha_1 \cos \alpha_1 = \prod_{i=1}^{n} \frac{\sin 2\alpha_1}{2} \le \frac{1}{2^n}$$

$$\Rightarrow \quad \prod_{i=1}^{n} \cos \alpha_1 \le \frac{1}{2^{\frac{n}{2}}}$$

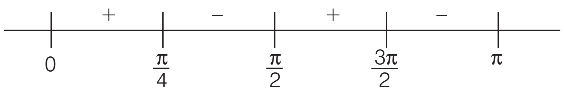
Hence, maximum value of $\prod_{i=1}^{n} \cos \alpha_1$ is $\frac{1}{2^{\frac{n}{2}}}$.

51. $\sin x \cdot \cos^3 x > \cos x \cdot \sin^3 x$

$$\Rightarrow \quad \sin x \cdot \cos x(\cos^2 x - \sin^2 x) > 0$$

$$\Rightarrow \quad \sin x \cdot \cos x \cdot \cos 2x > 0$$

$$\Rightarrow \quad \cos x \cdot \cos 2x > 0$$



$$\therefore \quad x \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$$

52. $u^2 = a^2 + b^2$

$$+ 2\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta} \times \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}$$

$$= a^2 + b^2 + 2\sqrt{\sin^2 \theta \cos^2 \theta(a^4 + b^4) + a^2b^2(\sin^4 \theta + \cos^4 \theta)}$$

$$= a^2 + b^2 + 2\sqrt{a^2b^2(1 - 2\sin^2\theta \cos^2\theta) + (a^4 + b^4)\sin^2\theta \cos^2\theta}$$

$$= (a^2 + b^2) + 2\sqrt{a^2b^2 + (a^2 - b^2)^2 \sin^2 \theta \cos^2 \theta}$$

$$= (a^2 + b^2) + 2\sqrt{a^2b^2 + \frac{(a^2 - b^2)^2}{4} \sin^2 2\theta}$$

Max. $u^2 = (a^2 + b^2) + 2\sqrt{a^2b^2 + \frac{(a^2 - b^2)^2}{4}}$

Min. $u^2 = (a^2 + b^2) + 2ab$

$$\Rightarrow \text{ Difference } 2\sqrt{a^2b^2 + \frac{(a^2 - b^2)^2}{4}} - 2ab$$

$$= \sqrt{4a^2b^2 + a^4 + b^4 - 2a^2b^2} - 2ab$$

$$= \sqrt{(a^2 + b^2)^2} - 2ab$$

$$= a^2 + b^2 - 2ab = (a - b)^2$$

53. $\because \left(\tan \frac{\theta}{2}\right)(1 + \sec \theta) = \tan\left(\frac{\theta}{2}\right)\left(\frac{1 + \cos \theta}{\cos \theta}\right)$

$$= \frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}} \cdot \frac{2\cos^2 \frac{\theta}{2}}{\cos \theta} = \frac{2\sin \frac{\theta}{2} \cdot \cos \frac{\theta}{2}}{\cos \theta}$$

$$= \frac{\sin \theta}{\cos \theta} = \tan \theta \qquad \text{...(i)}$$

$\therefore$ By repeated use of Eq. (i), we have

$$f_n(\theta) = \tan \theta\,(1 + \sec 2\theta)(1 + \sec 4\theta)\ldots(1 + \sec 2^n \theta)$$

$$= \tan 2\theta\,(1 + \sec 4\theta)\ldots(1 + \sec 2^n \theta)$$

$$= \tan 4\theta\,(1 + \sec 8\theta)\ldots(1 + \sec 2^n \theta)$$

$$= \ldots. = \tan 2^n \theta$$

Now, $f_2\left(\frac{\pi}{16}\right) = \tan\left(2^2 \frac{\pi}{16}\right) = \tan \frac{\pi}{4} = 1$

$$f_3\left(\frac{\pi}{32}\right) = \tan\left(2^3 \frac{\pi}{32}\right) = \tan \frac{\pi}{4} = 1$$

$$f_4\left(\frac{\pi}{64}\right) = \tan\left(2^4 \frac{\pi}{64}\right) = \tan \frac{\pi}{4} = 1$$

and $f_5\left(\frac{\pi}{128}\right) = \tan \frac{\pi}{4} = 1$

54. $\cos x = 0 \Rightarrow x = (2n + 1)\frac{\pi}{2}$

$$\Rightarrow \quad \cos\left((2n + 1)\frac{\pi}{2} + z\right) = \frac{1}{2}$$

$$\Rightarrow \quad \sin z = \frac{1}{2} \text{ or } \sin z = -\frac{1}{2}$$

$$\Rightarrow \quad z = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

55. $f_n(\theta) = \cos\theta - \cos 2\theta + \cos 2\theta - \cos 3\theta + \ldots + \cos(n)\theta - \cos(n + 1)\theta$

$$f_n(\theta) = \cos\theta - \cos(n + 1)\theta$$

Now, check options.

56. $P = \sin 25^\circ \sin 35^\circ \sin 60^\circ \sin 85^\circ$

$$= \sin 25^\circ \sin(60^\circ - 25^\circ)\ \sin 60^\circ \sin(60^\circ + 25^\circ)$$

$$= \sin 60^\circ \sin 25^\circ \sin(60^\circ - 25^\circ) \sin(60^\circ + 25^\circ)$$

$$\therefore \quad P = \sin 60^\circ \times \frac{1}{4} \sin 75^\circ \qquad \text{...(i)}$$

$$Q = \sin 20^\circ \sin 40^\circ \sin 75^\circ \sin 80^\circ$$

$$= \sin 20^\circ \sin(60^\circ - 20^\circ) \sin 75^\circ \sin(60^\circ + 20^\circ)$$

$$= \sin 75^\circ \sin 20^\circ \sin(60^\circ - 20^\circ) \sin(60^\circ + 20^\circ)$$

$$\therefore \quad Q = \sin 75^\circ \times \frac{1}{4} \times \sin 60^\circ \qquad \text{...(ii)}$$

Hence, $P = Q$

57. $x = \frac{1}{1 - \cos^2\theta} = \frac{1}{\sin^2\theta}, y = \frac{1}{\cos^2\theta},$

$$z = \frac{1}{1 - \sin^2\theta \cos^2\theta}$$

$$\Rightarrow \quad \frac{1}{x} + \frac{1}{y} = 1$$

$$\Rightarrow \quad xy = x + y \ \Rightarrow\ \frac{1}{z} = 1 - \frac{1}{xy}$$

$$\Rightarrow \quad xyz = xy + z = x + y + z$$

58. Given $P(x) = \cot^2 x\left(\frac{1 + \tan x + \tan^2 x}{1 + \cot x + \cot^2 x}\right) + \left(\frac{\cos x - \cos 3x + \sin 3x - \sin x}{2(\sin 2x + \cos 2x)}\right)^2$

$$= \frac{\cot^2 + \cot x + 1}{1 + \cot x \ \ \cot^2 x} + \left(\frac{2\sin x\,(\sin 2x + \cos 2x)}{2(\sin 2x + \cos 2x)}\right)^2$$

$$= 1 + \sin^2 x$$

$$\therefore \quad P(18°) = P(72°) = (1 + \sin^2 18°) + (1 + \sin^2 72°)$$

$$= 1 + 1 + (\sin^2 18° + \cos^2 18°) = 3$$

59. $E = \dfrac{3\sin(\alpha + \beta) - 4\cos(\alpha + \beta)}{\sqrt{3}\sin\alpha}$

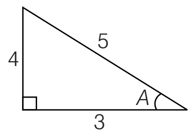
$$= \frac{3(\sin\alpha\cos\beta + \cos\alpha\sin\beta) - 4(\cos\alpha\cos\beta + \sin\alpha\sin\beta)}{\sqrt{3}\sin\alpha}$$

$$= \frac{5}{\sqrt{3}} \text{ for } 0 < \beta < \frac{\pi}{2}$$

and $E = \dfrac{\sqrt{3}\,(7 + 24\cot\alpha)}{15}$ for $\dfrac{\pi}{2} < \beta < \pi$.

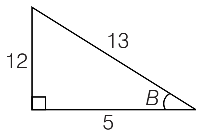
60. $\cot A = \dfrac{3}{4}$

$\Rightarrow \quad \cot C = \dfrac{-3}{4}$



$\Rightarrow$ C is obtuse angle.

$\therefore \quad \sin C = \dfrac{4}{5}, \cos C = -\dfrac{3}{5}$



$$\tan B = \frac{-12}{5}$$

$\Rightarrow \quad \tan D = \dfrac{12}{5}$

$\Rightarrow$ D is an acute angle

$\therefore \quad \sin D = \dfrac{12}{13} \quad \cos D = \dfrac{5}{13}$

Hence, $\sin(C + D) = \sin C \cdot \cos D + \cos C \cdot \sin D$

$$= \left(\frac{4}{5}\right)\left(\frac{5}{13}\right) + \left(\frac{-3}{5}\right)\left(\frac{12}{13}\right)$$

$$= \frac{20 - 36}{65} = \frac{-16}{65}$$

Also, $\sin(A + B) = \sin(2\pi - (C + D))$

$$= -\sin(C + D) = \frac{16}{65}.$$

61. $2\left(\cos^2 x + \dfrac{1}{2}\cos x\right) = a$

$$2\left(\cos x + \frac{1}{4}\right)^2 = a + \frac{1}{8}$$

$$\therefore \quad \left(\cos x + \frac{1}{4}\right)^2 = \frac{a}{2} + \frac{1}{16}$$

$$\left(\cos x + \frac{1}{4}\right)^2 \in \left[0, \frac{25}{16}\right]$$

$$\therefore \quad \frac{8a + 1}{16} \in \left[0, \frac{25}{16}\right]$$

$$\Rightarrow \quad 8a + 1 \in [0, 25]$$

$$\Rightarrow \quad a \in \left[\frac{-1}{8}, 3\right]$$

62. $A = \sin 44° + \cos 44°$

$$= \cos 46° + \sin 46° = C$$

$$B = \sin 45° + \cos 45° = \sqrt{2}\,[\sin 90°]$$

$$A = \sqrt{2}\left[\frac{1}{\sqrt{2}}\sin 44° + \frac{1}{\sqrt{2}}\cos 44°\right]$$

$$= \sqrt{2}\,[\sin 44° \cdot \cos 45° + \cos 44° \cdot \sin 45°]$$

$$= \sqrt{2}\sin 89°$$

$\Rightarrow \quad B > A$

63. $\tan(2\alpha + \beta) = x$

$\tan(\alpha + 2\beta) = y$

$\Rightarrow \quad \tan(3(\alpha + \beta)).\tan(\alpha - \beta)$

$$= \tan[(2\alpha + \beta) + (\alpha + 2\beta)]\,.$$

$$\tan[2(\alpha + \beta) - (\alpha + 2\beta)]$$

$$= \frac{\tan(2\alpha + \beta) + \tan(\alpha + 2\beta)}{1 - \tan(2\alpha + \beta).\tan(\alpha + 2\beta)} \cdot \frac{\tan(2\alpha + \beta) - \tan(\alpha + 2\beta)}{1 + \tan(2\alpha + \beta).\tan(\alpha + 2\beta)}$$

$$= \frac{x + y}{1 - xy} \cdot \frac{x - y}{1 + xy} = \frac{x^2 - y^2}{1 - x^2y^2}$$

64. We have $x = \dfrac{1 - \sin\phi}{\cos\phi}, y = \dfrac{1 + \cos\phi}{\sin\phi}$

Multiplying, we get $xy = \dfrac{(1 - \sin\phi)(1 + \cos\phi)}{\cos\phi\sin\phi}$

$$\Rightarrow \quad xy + 1 = \frac{1 - \sin\phi + \cos\phi - \sin\phi\cos\phi + \sin\phi\cos\phi}{\cos\phi\sin\phi}$$

$$= \frac{1 - \sin\phi + \cos\phi}{\cos\phi\sin\phi}$$

and $\quad x - y = \dfrac{(1 - \sin\phi)\sin\phi - \cos\phi(1 + \cos\phi)}{\cos\phi\sin\phi}$

$$= \frac{\sin\phi - \sin^2\phi - \cos\phi - \cos^2\phi}{\cos\phi\sin\phi}$$

$$= \frac{\sin\phi - \cos\phi - 1}{\cos\phi\sin\phi} = -(xy + 1)$$

Thus, $xy + x - y + 1 = 0$.

$$\Rightarrow \quad x = \frac{y - 1}{y + 1} \text{ and } y = \frac{1 + x}{1 - x}.$$

65. The given relation can be written as

$$\tan\left(\frac{x}{2}\right) = \frac{1 - \sin^2 x}{\sin x} = \frac{\cos^2 x}{\sin x}$$

$$\Rightarrow \quad 2\sin^2\left(\frac{x}{2}\right) = \left[\cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)\right]^2$$

$$\Rightarrow \quad 2\tan^2\left(\frac{x}{2}\right) = \left[1 - \tan^2\left(\frac{x}{2}\right)\right]^2 \Big/ \left[1 + \tan^2\left(\frac{x}{2}\right)\right]$$

$$\Rightarrow \quad 2y(1+y) = (1-y)^2 \qquad \left[\text{where } y = \tan^2\frac{x}{2}\right]$$

$$\Rightarrow \quad y^2 + 4y - 1 = 0$$

$$\Rightarrow \quad y = \frac{-4 \pm \sqrt{16+4}}{2} = -2 \pm \sqrt{5}$$

Since $y > 0$, we get

$$y = \sqrt{5} - 2 = \frac{(\sqrt{5}-2)^2}{\sqrt{5}+2} \cdot \frac{2+\sqrt{5}}{2+\sqrt{5}}$$

$$= (9 - 4\sqrt{5})(2 + \sqrt{5})$$

66. $y = \dfrac{\sqrt{(\cos 2A - \sin 2A)^2} + 1}{\sqrt{(\cos 2A + \sin 2A)^2} - 1}$

$$\Rightarrow \quad y = \frac{\pm(\cos 2A - \sin 2A) + 1}{\pm(\cos 2A + \sin 2A) - 1}$$

which gives us four values of y, say y_1, y_2, y_3 and y_4 We have,

$$y_1 = \frac{\cos 2A - \sin 2A + 1}{\cos 2A + \sin 2A - 1} = \frac{(1 + \cos 2A) - \sin 2A}{(\cos 2A - 1) + \sin 2A}$$

$$= \frac{2\cos^2 A - 2\sin A \cos A}{-2\sin^2 A + 2\sin A \cos A}$$

$$= \frac{\cos A(\cos A - \sin A)}{\sin A(\cos A - \sin A)} = \cot A$$

$$y_2 = \frac{-(\cos 2A - \sin 2A) + 1}{-(\cos 2A + \sin 2A) - 1}$$

$$= \frac{(1 - \cos 2A) + \sin 2A}{-(1 + \cos 2A) - \sin 2A}$$

$$= \frac{2\sin^2 A + 2\sin A \cos A}{-2\cos^2 A - 2\sin A \cos A} = -\tan A$$

$$y_3 = \frac{(\cos 2A - \sin 2A) + 1}{-(\cos 2A + \sin 2A) - 1}$$

$$= \frac{(1 + \cos 2A) - \sin 2A}{-(1 + \cos 2A) - \sin 2A}$$

$$= \frac{2\cos^2 A - 2\sin A \cos A}{-2\cos^2 A - 2\sin A \cos A}$$

$$= -\frac{\cos A - \sin A}{\cos A + \sin A}$$

$$= -\frac{1 - \tan A}{1 + \tan A} = -\tan\left(\frac{\pi}{4} - A\right) = -\cot\left(\frac{\pi}{4} + A\right)$$

$$y_4 = \frac{-(\cos 2A - \sin 2A) + 1}{(\cos 2A + \sin 2A) - 1}$$

$$= \frac{(1 - \cos 2A) + \sin 2A}{-(1 - \cos 2A) + \sin 2A}$$

$$= \frac{2\sin^2 A - 2\sin A \cos A}{-2\sin A + 2\sin A \cos A}$$

67. $\because \; 3\sin\beta = \sin(2\alpha + \beta)$

$$\Rightarrow \quad 2\sin\beta = \sin(2\alpha+\beta) - \sin\beta$$

$$= 2\cos(\alpha+\beta)\sin\alpha$$

$$\therefore \quad \sin\beta = \cos(\alpha+\beta)\sin\alpha \qquad \text{...(i)}$$

$\therefore$ **Alternate** (b) is correct

Also, $\quad \sin\beta = \sin(\alpha+\beta) - \alpha)$

$$= \sin(\alpha+\beta)\cos\alpha - \cos(\alpha+\beta)\sin\alpha \qquad \text{...(ii)}$$

From Eqs. (i) and (ii)

$$\sin\beta = \sin(\alpha+\beta)\cos\alpha - \sin\beta$$

$$\therefore \quad 2\sin\beta = \sin(\alpha+\beta)\cos\alpha$$

($\therefore$ Alternate (c) is correct)

Alternate (a)

LHS $= (\cot\alpha + \cos(\alpha+\beta))(\cot\beta - 3\cot(2\alpha+\beta))$

$$= \left(\frac{\sin(2\alpha+\beta)}{\sin\alpha \cdot \sin(\alpha+\beta)}\right)\left(\frac{\cos\beta}{\sin\beta} - \frac{3\cos(2\alpha+\beta)}{\sin(2\alpha+\beta)}\right)$$

$$= \left(\frac{3\sin\beta}{\sin\alpha \cdot \sin(\alpha+\beta)}\right)\left(\frac{\cos\beta}{\sin\beta} - \frac{3\cos(2\alpha+\beta)}{3\sin\beta}\right)$$

$(\because 3\sin\beta = \sin(2\alpha+\beta))$

$$= \left(\frac{3\sin\beta}{\sin\alpha \cdot \sin(\alpha+\beta)}\right)\left(\frac{\cos\beta - \cos(2\alpha+\beta)}{\sin\beta}\right)$$

$$= \left(\frac{3\sin\beta}{\sin\alpha \cdot \sin(\alpha+\beta)}\right)\left(\frac{2\sin(\alpha+\beta)\sin\alpha}{\sin\beta}\right)$$

$$= 6$$

Alternate (d)

$$\because \quad \tan(\alpha+\beta) = 2\tan\alpha$$

$$\Rightarrow \quad \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{2\sin\alpha}{\cos\alpha}$$

$$\Rightarrow \quad \sin(\alpha+\beta)\cos\alpha = 2\cos(\alpha+\beta)\sin\alpha$$

$$\Rightarrow \quad \sin(\alpha+\beta)\cos\alpha - \cos(\alpha+\beta)\sin\alpha = \cos(\alpha+\beta)\sin\alpha$$

$$\sin\beta = \cos(\alpha+\beta)\sin\alpha$$

[Alternate (b)]

68. $P_n(u)$ be a polynomial in u of degree n.

$$\therefore \quad \sin 2nx = 2\sin nx \cos nx$$

$$= \sin x\, P_{2n-1}(\cos x) \text{ or } \cos x\, P_{2n-1}(\sin x)$$

69. $\tan\theta = \dfrac{\sin\alpha - \cos\alpha}{\sin\alpha + \cos\alpha}$

$$= \frac{\tan\alpha - 1}{\tan\alpha + 1}$$

$$\tan\theta = \tan\left(\alpha - \frac{\pi}{4}\right)$$

$$\Rightarrow \quad \theta = n\pi + \alpha - \frac{\pi}{4}, n \in I$$

or $\quad 2\theta = 2n\pi + 2\alpha - \dfrac{\alpha}{2}$

$$\sin 2\theta = \sin\left(2\alpha - \frac{\pi}{2}\right) = -\cos 2\alpha$$

and $\quad \cos 2\theta = \cos\left(2\alpha - \dfrac{\pi}{2}\right) = \sin 2\alpha$

and $\sin\alpha - \cos\alpha = \sqrt{2}\sin\left(\alpha - \frac{\pi}{4}\right)$

$$=\sqrt{2}\sin\{\theta - n\pi\} = \pm\sqrt{2}\sin\theta$$

and $\sin\alpha + \cos\alpha = \sqrt{2}\sin\left(\alpha + \frac{\pi}{4}\right)$

$$= \sqrt{2}\sin\left\{\frac{\pi}{2} + \theta - n\pi\right\}$$

$$= \sqrt{2}\cos(\theta - n\pi) = \pm\sqrt{2}\cos\theta$$

70. $\cos 5\theta = \cos(4\theta + \theta) = \cos 4\theta\cos\theta - \sin 4\theta\sin\theta$

$$= (2\cos^2 2\theta - 1)\cos\theta - 2\sin 2\theta\cos 2\theta\sin\theta$$

$$= [2(2\cos^2\theta - 1)^2 - 1]\cos\theta - 2\cdot 2\cos\theta \sin^2\theta(2\cos^2\theta - 1)$$

$$= [2(4\cos^4\theta - 4\cos^2\theta + 1) - 1]\cos\theta - 4\cos\theta(2\cos^2\theta - 1)(1 - \cos^2\theta)$$

$$= \cos\theta(8\cos^4\theta - 8\cos^2\theta + 1) - 4\cos\theta(3\cos^2\theta - 2\cos^4\theta - 1)$$

$$= \cos\theta(16\cos^4\theta - 20\cos^2\theta + 5)$$

71. $x = \sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha} + \sqrt{a^2\sin^2\alpha + b^2\cos^2\alpha}$

$$\Rightarrow \quad x^2 = a^2 + b^2 + \sqrt{\begin{matrix}2(a^2\cos^2\alpha + b^2\sin^2\alpha)\\(a^2\sin^2\alpha + b^2\cos^2\alpha)\end{matrix}}$$

$a^2 + b^2 + 2k,$

where $k = \sqrt{\begin{matrix}[(a^2 + b^2) - (a^2\sin^2\alpha + b^2\cos^2\alpha)]\\ \times(a^2\sin^2\alpha + b^2\cos^2\alpha)\end{matrix}}$

$\therefore \quad x = a^2 + b^2 + 2\sqrt{(a^2 + b^2)p - p^2}$

where $p = a^2\sin^2\alpha + b^2\cos^2\alpha$

$$= \frac{a^2}{2}(1 - \cos 2\alpha) + \frac{b^2}{2}(1 + \cos 2\alpha)$$

72. $\left(\frac{\cos A + \cos B}{\sin A - \sin B}\right)^n + \left(\frac{\sin A + \sin B}{\cos A - \cos B}\right)^n$

$$= \cos^n\left(\frac{A - B}{2}\right) + \cot^n\left(\frac{B - A}{2}\right)$$

If n even, $2\cot^n\left(\frac{A-B}{2}\right)$, if n odd, O

73. $P(k) = \left(1 + \cos\frac{\pi}{4k}\right)\left(1 + \cos\left(\frac{\pi}{2} - \frac{\pi}{4k}\right)\right)\left(1 + \cos\left(\frac{\pi}{2} + \frac{\pi}{4k}\right)\right)\left(1 + \cos\left(\pi - \frac{\pi}{4k}\right)\right)$

$$= \left(1 + \cos\frac{\pi}{4k}\right)\left(1 + \sin\frac{\pi}{4k}\right)\left(1 - \sin\frac{\pi}{4k}\right)\left(1 - \cos\frac{\pi}{4k}\right)$$

$$= \left(1 - \cos^2\frac{\pi}{4k}\right)\left(1 - \sin^2\frac{\pi}{4k}\right)$$

$$= \frac{4\sin^2\frac{\pi}{4k}\cdot\cos^2\frac{\pi}{4k}}{4}$$

$$P(k) = \frac{1}{4}\sin^2\left(\frac{\pi}{2k}\right)$$

$$\Rightarrow \quad P(3) = \frac{1}{4}\cdot\frac{1}{4} = \frac{1}{16}$$

$$\Rightarrow \quad P(4) = \frac{\pi}{4}\sin^2\frac{\pi}{2k} = \frac{1}{4}\sin^2\frac{\pi}{8}$$

$$= \frac{1}{8}\left(1 - \cos\frac{\pi}{4}\right) = \frac{2 - \sqrt{2}}{16}$$

$$\Rightarrow \quad P(5) = \frac{1}{4}\sin^2\frac{\pi}{10} = \frac{1}{8}\left(2\sin^2\frac{\pi}{10}\right) = \frac{1}{8}(1 - \cos 36°)$$

$$= \frac{1}{8}\left(1 - \frac{\sqrt{5} + 1}{4}\right) = \frac{3 - \sqrt{5}}{32}$$

$$\Rightarrow \quad P(6) = \frac{1}{4}\sin^2\frac{\pi}{12} = \frac{1}{8}\left(2\sin^2\frac{\pi}{12}\right) = \frac{1}{8}\left(1 - \cos\frac{\pi}{6}\right)$$

$$= \frac{1}{8}\left(1 - \frac{\sqrt{3}}{2}\right) = \frac{2 - \sqrt{3}}{16}$$

74. $x^2 + y^2 = a^2\sin^4\theta\cos^4\theta$

$$xy = a^2\sin^5\theta\cos^5\theta$$

$$\therefore \quad \frac{(x^2 + y^2)^p}{(xy)^q} = \frac{a^{2p}(\sin\theta\cos\theta)^{4p}}{a^{2q}(\sin\theta\cos\theta)^{5q}}$$

which is independent of θ if $4p = 5q$

i.e. $\quad p = 5, q = 4.$

75. LHS

$$= \tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 8\tan 8\alpha + 16\cot 16\alpha$$

$$= \cot\alpha - (\cot\alpha - \tan\alpha) + 2\tan 2\alpha + 4\tan 4\alpha + 8\tan 8\alpha + 16\cot 16\alpha$$

$$= \cot\alpha - 2(\cot 2\alpha - \tan 2\alpha) + 4\tan 4\alpha + 8\tan 8\alpha + 16\cot 16\alpha$$

$$(\because \cot\alpha - \tan\alpha = 2\cot 2\alpha)$$

$$= \cot\alpha - 4(\cot 4\alpha - \tan 4\alpha) + 8\tan 8\alpha + 16\cot 16\alpha$$

$$(\because \cot 2\alpha - \tan 2\alpha = 2\cot 4\alpha)$$

$$= \cot\alpha - 8(\cot 8\alpha - \tan 8\alpha) + 16\cot 16\alpha$$

$$= \cot\alpha - 16\cot 16\alpha + 16\cot 16\alpha$$

$$(\because \cot 8\alpha - \tan 8\alpha = 2\cot 16\alpha)$$

$$= \cot\alpha = \text{RHS}$$

76. Let $x = \cot A, y = \cot B, z = \cot C$

$\therefore \quad \cot A\cot B + \cot B\cot C + \cot C\cot A = 1$

$\because \quad A + B + C = 180°$

$$\therefore \quad \Sigma\frac{x}{(1 + x^2)} = \Sigma\frac{\cot A}{(1 + \cot^2 A)}$$

$$= \frac{1}{2}\Sigma\frac{2\tan A}{(1 + \tan^2 A)} = \frac{1}{2}\Sigma\sin 2A$$

$$= \frac{1}{2}(\sin 2A + \sin 2B + \sin 2C)$$

$$= \frac{1}{2}(4\sin A\sin B\sin C) = 2\sin A\sin B\sin C$$

$$= \frac{2}{\sqrt{(1 + \cot^2 A)(1 + \cot^2 B)(1 + \cot^2 C)}}$$

$$= \frac{2}{\sqrt{(1 + x^2)(1 + y^2)(1 + z^2)}} = \frac{2}{\sqrt{\Pi(1 + x^2)}}$$

and $\sin 2A + \sin 2B - \sin 2C$

$= 2\sin(A+B)\cos(A-B) - 2\sin C\cos C$

$= 2\sin C\{\cos(A-B) - \cos C\}$

$= 2\sin C\{\cos(A-C) + \cos(A+B)\}$

$= 2\sin C(2\cos A\cos B)$

$= 4\cos A\cos B\sin C$

77. $\because a\cos x + b\sin x = c$

$$\Rightarrow \quad a\left\{\frac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}\right\} + b\left\{\frac{2\tan\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}\right\} = c$$

$$\Rightarrow \quad (a+c)\tan^2\left(\frac{x}{2}\right) - 2b\tan\left(\frac{x}{2}\right) + (c-a) = 0$$

$$\therefore \quad \tan\left(\frac{\alpha}{2}\right) + \tan\left(\frac{\beta}{2}\right) = \frac{2b}{(a+c)}$$

and $$\tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right) = \frac{c-a}{a+c}$$

Now, $$\tan\left(\frac{\alpha+\beta}{2}\right) = \frac{\tan\left(\frac{\alpha}{2}\right) + \tan\left(\frac{\beta}{2}\right)}{1-\tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right)}$$

$$= \frac{\frac{2b}{a+c}}{1-\left(\frac{c-a}{a+c}\right)} = \frac{b}{a} = \text{independent of } c$$

Also, $$-\sqrt{(a^2+b^2)} \le a\cos x + b\sin x \le \sqrt{(a^2+b^2)}$$

$$\therefore \quad -\sqrt{(a^2+b^2)} \le c \le \sqrt{(a^2+b^2)}$$

78. $\because A + B + C = 180°$

$$\Rightarrow \quad A = 180° - (B+C)$$

$$\therefore \quad \tan A = \tan(180° - (B+C))$$

$$= -\tan(B+C) = -\left\{\frac{\tan B + \tan C}{1 - \tan B\tan C}\right\}$$

$$= \left(\frac{\tan B + \tan C}{\tan B\tan C - 1}\right)$$

Now, $\because A$ is obtuse

$\therefore \quad \tan A < 0,$

then $\quad \tan B + \tan C > 0$

$\therefore \quad \tan B\tan C - 1 < 0$

$\Rightarrow \quad \tan B\tan C < 1$

79. Let $S = \sin\left(\frac{2\pi}{7}\right) + \sin\left(\frac{4\pi}{7}\right) + \sin\left(\frac{8\pi}{7}\right)$

and $\quad C = \cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{8\pi}{7}\right)$

$$\therefore \quad C + iS = \alpha + \alpha^2 + \alpha^4 \quad \text{...(i)}$$

Where $\alpha = \cos\left(\frac{2\pi}{7}\right) + i\sin\left(\frac{2\pi}{7}\right)$ is complex 7th root of unity.

Then, $$C - iS = \bar{\alpha} + \bar{\alpha}^2 + \bar{\alpha}^4$$

$$= \alpha^6 + \alpha^5 + \alpha^3 \quad \text{...(ii)}$$

Adding Eqs. (i) and (ii), then

$$2C = \alpha + \alpha^2 + \alpha^4 + \alpha^6 + \alpha^5 + \alpha^3 = -1$$

($\because$ sum of 7, 7th roots of unity is zero)

$$\therefore \quad C = -\frac{1}{2}$$

Also, multiplying Eqs. (i) and (ii), then $C^2 + S^2 = 2$

($\because \alpha^7 = 1$ and sum of 7, 7th roots of unity)

$$\Rightarrow \quad S^2 = 2 - \left(\frac{1}{2}\right)^2 = \frac{7}{4}$$

$$\therefore \quad S = \frac{\sqrt{7}}{2}$$

80. We observe that $y = 81^{\sin^2 x} + 81^{\cos^2 x} - 30 = 0$

$$\Rightarrow \quad 81^{\sin^2 x} + 81^{1-\sin^2 x} - 30 = 0$$

$$\Rightarrow \quad 81^{2\sin^2 x} - 30.81^{\sin^2 x} + 81 = 0$$

$$\Rightarrow \quad (81^{\sin^2 x} - 3)(81^{\sin^2 x} - 27) = 0$$

$$\Rightarrow \quad \sin^2 x = \frac{1}{4} \text{ or } \frac{3}{4}$$

$$\Rightarrow \quad \sin x = \pm\frac{1}{2} \text{ or } \sin x = \pm\frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad x = \pm\frac{\pi}{6}, \pm\frac{5\pi}{6}, \text{ or } x = \pm\frac{\pi}{3}, \pm\frac{2\pi}{3}$$

$\Rightarrow$ The graph $y = 81^{\sin^2 x} + 81^{\cos^2 x} - 30$

Intersects the X-axis at eight points in $(-\pi \le x \le \pi)$.

$\Rightarrow$ Statement-1 is true.

81. Statement-2 is correct, using it we have $\cos 3x = \sin 2x$

$$\Rightarrow \quad 4\cos^3 x - 3\cos x = 2\sin x\cos x$$

Similarly $4\cos^3 y - 3\cos y = 2\sin y\cos y$

So, $\quad 4(1-\sin^2 x) - 3 = 2\sin x$

$\Rightarrow \quad 4\sin^2 x + 2\sin x - 1 = 0$

and $\quad 4\sin^2 y + 2\sin y - 1 = 0$

Hence, $\sin x = \sin 18°$ and $\sin y = \sin(-54°) = -\sin 54°$ are the roots of a quadratic equation with integer coefficients.

82. The minimum value of the sum can be -3 provided $\sin\alpha = \sin\beta = \sin\gamma = -1$

$$\Rightarrow \quad \alpha = (4l-1)\frac{\pi}{2}, \ \beta = (4m-1)\frac{\pi}{2}, \ \gamma = (4n-1)\frac{\pi}{2}$$

Now $\quad \alpha + \beta + \gamma = \pi \Rightarrow [4(l+m+n) - 3]\frac{\pi}{2} = \pi$

$\Rightarrow 4(l+m+n) = 5$ which is not possible as l, m, n are integers.

1. minimum value can not be -3.

But for $\alpha = \frac{3\pi}{2}, \beta = \frac{3\pi}{2}, \gamma = -2\pi, \alpha + \beta + \gamma = \pi$

and $\quad \sin\alpha + \sin\beta + \sin\gamma = 2$

So, $\sin\alpha + \sin\beta + \sin\gamma$ can have negative values and thus the minimum value of the sum is negative proving that statement-1 is correct. But the statement-2 is false as $\alpha + \beta + \gamma = \pi$ for $\alpha = \beta = \frac{3\pi}{2}, \gamma = -2\pi$ which are not the angles of a triangle.

83. We have $2\sin\left(\frac{\theta}{2}\right)$

$$= \sqrt{\cos\left(\frac{\theta}{2}\right)+\sin\left(\frac{\theta}{2}\right)}^{2} + \sqrt{\left(\cos\left(\frac{\theta}{2}\right)-\sin\left(\frac{\theta}{2}\right)\right)^{2}}$$

$$= \left|\cos\left(\frac{\theta}{2}\right)+\sin\left(\frac{\theta}{2}\right)\right| + \left|\cos\left(\frac{\theta}{2}\right)-\sin\left(\frac{\theta}{2}\right)\right|$$

$$\Rightarrow \cos\left(\frac{\theta}{2}\right)+\sin\left(\frac{\theta}{2}\right) > 0 \text{ and } \cos\left(\frac{\theta}{2}\right)-\sin\left(\frac{\theta}{2}\right) < 0$$

$$\Rightarrow \sin\left(\frac{\theta}{2}+\frac{\pi}{4}\right) > 0 \text{ and } \cos\left(\frac{\theta}{2}+\frac{\pi}{4}\right) < 0$$

$$\Rightarrow 2n\pi + \frac{\pi}{2} < \frac{\theta}{2} + \frac{\pi}{4} > 2n\pi + \pi$$

$$\Rightarrow 2n\pi + \frac{\pi}{4} < \frac{\theta}{2} < 2n\pi + \frac{3\pi}{4}$$

So statement-1 is true but does not follow from statement-2 which is also true.

84. $2\cos\theta + \sin\theta = 1$

$$\Rightarrow \frac{2\left(1-\tan^2\left(\frac{\theta}{2}\right)\right)}{1+\tan^2\left(\frac{\theta}{2}\right)} + \frac{2\tan\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)} = 1$$

$$\Rightarrow 3\tan^2\left(\frac{\theta}{2}\right) - 2\tan\left(\frac{\theta}{2}\right) - 1 = 0$$

$$\Rightarrow \tan\left(\frac{\theta}{2}\right) = -\frac{1}{3} \text{ as } \theta \neq \frac{\pi}{2}$$

Now $7\cos\theta + 6\sin\theta$

$$= \frac{7\left(1-\tan^2\left(\frac{\theta}{2}\right)\right)}{1+\tan^2\left(\frac{\theta}{2}\right)} + \frac{6\times 2\tan\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)}$$

$$= \frac{7-7\tan^2\left(\frac{\theta}{2}\right)+12\tan\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)} = \frac{7-7\times\frac{1}{9}+12\left(-\frac{1}{3}\right)}{1+\frac{1}{9}} = 2$$

Showing that statement-1 is true.

In statement-2

$$\cos 2\theta - \sin\theta = \frac{1}{2}$$

$$\Rightarrow 2(1-2\sin^2\theta) - \sin\theta = 1$$

$$\Rightarrow 4\sin^2\theta + 2\sin\theta - 1 = 0$$

$$\Rightarrow \sin\theta = \frac{-2\pm\sqrt{4+16}}{8} = \frac{-1\pm\sqrt{5}}{4}$$

$$\Rightarrow \sin\theta = \frac{\sqrt{5}-1}{4} \Rightarrow \theta = 18°$$

$$\Rightarrow \cos 6\theta = \cos 108° = \cos(90° + 18°)$$

$$= -\sin 18°$$

$$\Rightarrow \sin\theta + \cos 6\theta = 0$$

So statement-2 is also true but does not lead to statement-1.

85. $\because A + B = \frac{\pi}{3}$

$$\therefore \tan(A+B) = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$$

$$\Rightarrow \frac{\tan A + \tan B}{1 - \tan A \tan B} = \sqrt{3}$$

$$\Rightarrow \tan A \tan B = 1 - \frac{1}{\sqrt{3}}(\tan A + \tan B)$$

$\therefore \tan A \tan B$ will be maximum if $\tan A + \tan B$ is minimum.

But the minimum value of $\tan A + \tan B$ is obtained when $\tan A = \tan B$

$$\Rightarrow A = B = \frac{\pi}{6}$$

Hence, the maximum value of $\tan A \tan B$

$$= \tan\frac{\pi}{6}\cdot\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}} = \frac{1}{3}$$

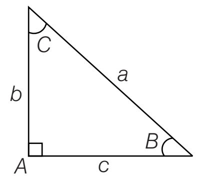
86. Let $3^{a^2} = A$ and $3^{b^2+c^2} = B$

$$\Rightarrow A^2 - 2A.B + B^2 = 0$$

$$\Rightarrow A = B$$

$$\Rightarrow a^2 = b^2 + c^2$$

87.



From figure, it is clear that $a = b\sec C = c\,\text{cosec}\,C$

$\Rightarrow$ Equilateral triangle $\Rightarrow$ Area $= \frac{\sqrt{3}}{4}a^2$

Sol. (Q. Nos. 88 to 90)

$$7\theta = (2n+1)\pi,\ n = 0 \text{ to } 6$$

$$4\theta = (2n+1)\pi - 3\theta$$

$$\cos 4\theta = -\cos 3\theta$$

$$\Rightarrow 2\cos^2 2\theta - 1 = -(4\cos^3\theta - 3\cos\theta)$$

$$\Rightarrow 2(2x^2-1) - 1 = -(4x^3 - 3x), \text{ where } x = \cos\theta$$

$$\Rightarrow 8x^4 + 4x^3 - 8x^2 - 3x + 1 = 0$$

$$(x+1)(8x^3 - 4x^2 - 4x - 1) = 0$$

88. $P_{mn} = m\log_{\cos x}(\sin x) + n\log_{\cos x}(\cot x)$

$$\geq n(\log_{\cos x}(\sin x) + \log_{\cos x}(\cot x))\ \forall\ m \geq n$$

$$= n(\log_{\cos x}(\sin x\cdot\cot x))$$

$$= n\log_{\cos x}\cos x = n$$

Thus, $P_{mn} \geq n\ \forall\ m \geq n$

89. Clearly, $P_{49}\left(\frac{\pi}{4}\right) = 4\log_{\frac{1}{\sqrt{2}}}\left(\frac{1}{\sqrt{2}}\right) + 9\log_{\frac{1}{\sqrt{2}}}(1) = 4$

Similarly $P_{94} = \left(\frac{\pi}{4}\right) = 9$

Mean proportional of $P_{49}\left(\frac{\pi}{4}\right)$ and $P_{94}\left(\frac{\pi}{4}\right)$ is $\sqrt{9\times 4} = 6$

90. $P_{34}(x) = P_{22}(x)$

$\Rightarrow\ 3\log_{\cos x}(\sin x) + 4\log_{\cos x}(\cot x)$
$= 2(\log_{\cos x}(\sin x) + \log_{\cos x}(\cot x))$

$\Rightarrow\ 3(\log_{\cos x}(\sin x) + \log_{\cos x}(\cot x)) + \log_{\cos x}(\cot x) = 2$

$\Rightarrow\ 3 + \log_{\cos x}(\cot x) = 2$

$\Rightarrow\ \log_{\cos x}(\cot x) = -1$

$\Rightarrow\ \cot x = (\cos x)^{-1} \Rightarrow \dfrac{\cos x}{\sin x} = \dfrac{1}{\cos x}$

$\Rightarrow\ \cos^2 x = \sin x$

$1 - \sin^2 x = \sin x \Rightarrow \sin^2 x + \sin x - 1 = 0$

$\Rightarrow\ \sin x = \dfrac{-1 \pm \sqrt{5}}{2}$

$\Rightarrow\ \sin x = \dfrac{\sqrt{5} - 1}{2}$ $\quad (\because \sin x \nless -1)$

Thus, $p + q = 7$

Sol. (Q. Nos. 91 to 93)

$7\theta = (2n+1)\pi,\ n = 0$ to 6

$4\theta = (2n+1)\pi - 3\theta$

$\cos 4\theta = -\cos 3\theta$

$\Rightarrow\ 2\cos^2 2\theta - 1 = -(4\cos^3\theta - 3\cos\theta)$

$\Rightarrow\ 2(2x^2 - 1) - 1 = -(4x^3 - 3x)$, where $x = \cos\theta$

$\Rightarrow\ 8x^4 + 4x^3 - 8x^2 - 3x + 1 = 0$

$(x+1)(8x^3 - 4x^2 - 4x - 1) = 0$

91. The roots are $\cos\dfrac{\pi}{7}, \cos\dfrac{2\pi}{7} \dots, \cos\dfrac{13\pi}{7}$

where $\cos\dfrac{\pi}{7} = \cos\dfrac{13\pi}{7}, \cos\dfrac{3\pi}{7} = \cos\dfrac{11\pi}{7}, \cos\dfrac{5\pi}{7} = \cos\dfrac{9\pi}{7}$

$\therefore$ The roots of $8x^3 - 4x^2 - 4x + 1 = 0$ are $\cos\dfrac{\pi}{7}, \cos\dfrac{3\pi}{7}, \cos\dfrac{5\pi}{7}$.

92. $\sec\dfrac{\pi}{7}, \sec\dfrac{3\pi}{7}, \sec\dfrac{5\pi}{7}$ are roots of $\dfrac{8}{x^3} - \dfrac{4}{x^2} - \dfrac{4}{x} + 1 = 0$

$\Rightarrow\ x^3 - 4x^2 - 4x + 8 = 0$

$\therefore\ \sec\dfrac{\pi}{7} + \sec\dfrac{3\pi}{7} + \sec\dfrac{5\pi}{7} = 4$

93. $\sec^2\dfrac{\pi}{7}, \sec^2\dfrac{3\pi}{7}, \sec^2\dfrac{5\pi}{7}$ are roots of $f(\sqrt{x}) = 0$

$\Rightarrow\ (\sqrt{x})^3 - 4(\sqrt{x})^2 - 4\sqrt{x} + 8 = 0$

$\Rightarrow\ x^3 - 24x^2 + 80x - 64 = 0$

$\therefore\ \sec^2\dfrac{\pi}{7} + \sec^2\dfrac{3\pi}{7} + \sec^2\dfrac{5\pi}{7} = 24$

Sol. (Q. Nos. 94 to 96)

Let $S = 1 + 2\sin x + 3\sin^2 x + 4\sin^3 x + \dots$

$\Rightarrow\ \sin x.\,S = \sin x + 2\sin^2 x + 3\sin^3 x + \dots$

$\therefore\ (1 - \sin x)\,S = 1\ \ \sin x + \sin^2 x + \dots$

$(1 - \sin x)\,S = \dfrac{1}{1 - \sin x}$

$\therefore\ S = \dfrac{1}{(1 - \sin x)^2}$

Given $S = 4 \Rightarrow \dfrac{1}{(1 - \sin x)^2} = 4$

$\Rightarrow\ \sin x = \dfrac{1}{2}$ or $\dfrac{3}{2}$ (rejected)

Number of solutions in $\left[\dfrac{-3\pi}{2}, 4\pi\right]$ is $k = 5$.

94. $k = 5$

95. $\left|\dfrac{\cos 2x - 1}{\sin 2x}\right| = \left|\dfrac{2\sin^2 x}{2\cos x \sin x}\right| = |\tan x| = \dfrac{1}{\sqrt{3}}$

96. Sum of interior angles $= (k-2)\pi = 3\pi$

97. Now, $(2\sin x - \cos x)(1 + \cos x) = \sin^2 x$

$\Rightarrow\ (1 + \cos x)[2\sin x - \cos x - 1 + \cos x] = 0$

$\Rightarrow\ (1 + \cos x)(2\sin x - 1) = 0$

$\Rightarrow\ \cos x = -1$ or $\sin x = \dfrac{1}{2}$

So, $\sin\alpha = \dfrac{1}{2}$ $\quad \left[\text{as } 0 \le \alpha \le \dfrac{\pi}{2}\right]$

$\therefore\ \cos\alpha = \dfrac{\sqrt{3}}{2}$

$3\cos^2 x - 10\cos x + 3 = 0$

$\Rightarrow\ \cos x = \dfrac{1}{3}, \cos x \ne 3$

$\Rightarrow\ \cos\beta = \dfrac{1}{3}, \sin\beta = \dfrac{2\sqrt{2}}{3}$

and $1 - \sin 2x = \cos x - \sin x$

$\Rightarrow\ \sin^2 x + \cos^2 x - 2\sin x \cos x = \cos x - \sin x$

$\Rightarrow (\cos x - \sin x)(\cos x - \sin x - 1) = 0$

$\Rightarrow\ \sin x = \cos x = \dfrac{1}{\sqrt{2}}$

or $\cos x - \sin x = 1$

$\Rightarrow\ \cos x = 1, \sin x = 0$

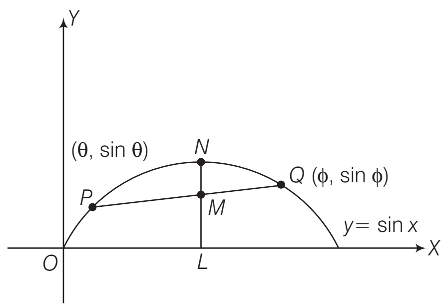
$\Rightarrow\ \cos\gamma = 1, \sin\gamma = 0$

$\therefore\ \cos\alpha + \cos\beta + \cos\gamma = \dfrac{\sqrt{3}}{2} + \dfrac{1}{3} + 1 = \dfrac{3\sqrt{3} + 8}{6}$

98. $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$= \dfrac{1}{2} \times \dfrac{1}{3} - \dfrac{\sqrt{3}}{2} \times \dfrac{2\sqrt{2}}{3} = \dfrac{1 - 2\sqrt{6}}{6}$

99. (A) If M is mid point of PQ, then $M = \left(\dfrac{\theta + \phi}{2}, \dfrac{\sin\theta + \sin\phi}{2}\right)$



Also, $N \equiv \left(\frac{\theta+\phi}{2}, \sin\left(\frac{\theta+\phi}{2}\right)\right)$

It is clear from the figure.

$$ML \le NL$$

$$\Rightarrow \quad \frac{\sin\theta + \sin\phi}{2} \le \sin\left(\frac{\theta+\phi}{2}\right)$$

$$\Rightarrow \quad \sin\theta + \sin\phi \le 2\sin\left(\frac{\theta+\phi}{2}\right) = 2\sin\left(\frac{\pi}{4}\right) = \sqrt{2}$$

$$\therefore \quad \sin\theta + \sin\phi \le \sqrt{2}$$

and $\quad (\sin\theta + \sin\phi)\sin\frac{\pi}{4} \le 1$ (p, q, r, s, t)

(B) $\because a^2 + b^2 = (\sin\theta - \sin\phi)^2 + (\cos\theta + \cos\phi)^2$

$$= 2 + 2\cos(\theta+\phi)$$

$$= 4\cos^2\left(\frac{\theta+\phi}{2}\right) \le 4 \text{ (s, t)}$$

(C) $\because 3\sin\theta + 5\cos\theta = 5$

$$\Rightarrow \quad 3\sin\theta = 5(1-\cos\theta)$$

Squaring both sides, then

$$9\sin^2\theta = 25(1-\cos\theta)^2$$

$$\Rightarrow \quad 9(1-\cos\theta)(1+\cos\theta) = 25(1-\cos\theta)^2$$

$$\Rightarrow \quad 9(1+\cos\theta) = 25(1-\cos\theta)\,(1-\cos\theta \ne 0)$$

$$\therefore \quad 34\cos\theta = 16$$

$$\cos\theta = \frac{8}{17}, \text{ then } \sin\theta = \frac{15}{17}$$

$$\therefore \quad 5\sin\theta - 3\cos\theta = \frac{75}{17} - \frac{24}{17} = 3 \text{ (r)}$$

Hence, $5\sin\theta - 3\cos\theta = 3$

100. (A) Let $y = \dfrac{7 + 6\tan\theta - \tan^2\theta}{(1+\tan^2\theta)}$

$$= 7\cos^2\theta + 6\sin\theta\cos\theta - \sin^2\theta$$

$$= 7\left(\frac{1+\cos\theta}{2}\right) + 3\sin 2\theta - \left(\frac{1-\cos 2\theta}{2}\right)$$

$$= 3\sin 2\theta + 4\cos 2\theta + 3$$

$$-\sqrt{(3^2+4^2)} + 3 \le 3\sin 2\theta + 4\cos 2\theta + 3 \le \sqrt{(3^2+4^2)} + 3$$

$$\therefore \quad -2 \le y \le 8 \Rightarrow \lambda = 8, \mu = -2$$

$$\Rightarrow \quad \lambda + \mu = 6, \lambda - \mu = 10 \text{ (R, S)}$$

(B) Let $y = 5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) + 3$

$$= 5\cos\theta + 3\left(\frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta\right) + 3$$

$$= \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta + 3$$

$$\therefore \quad 3 - \sqrt{\left(\frac{13}{2}\right)^2 + \left(\frac{-3\sqrt{3}}{2}\right)^2} \le \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta + 3 \le 3 + \sqrt{\left(\frac{13}{2}\right)^2 + \left(\frac{-3\sqrt{3}}{2}\right)^2}$$

$$\Rightarrow \quad 3 - 7 \le y \le 3 + 7$$

$$\Rightarrow \quad -4 \le y \le 10$$

$$\therefore \quad \lambda = 10, \mu = -4$$

$$\Rightarrow \quad \lambda + \mu = 6, \lambda - \mu = 14 \text{ (R, T)}$$

(C) Let $y = 1 + \sin\left(\frac{\pi}{4} + \theta\right) + 2\cos\left(\frac{\pi}{4} - \theta\right)$

$$= 1 + \cos\left(\frac{\pi}{2} - \left(\frac{\pi}{2} + \theta\right)\right) + 2\cos\left(\frac{\pi}{4} - \theta\right)$$

$$= 1 + \cos\left(\frac{\pi}{4} - \theta\right) + 2\cos\left(\frac{\pi}{4} - \theta\right)$$

$$= 1 + 3\cos\left(\frac{\pi}{4} - \theta\right)$$

$$\therefore \quad -1 \le \cos\left(\frac{\pi}{4} - \theta\right) \le 1$$

$$\Rightarrow \quad -3 \le 3\cos\left(\frac{\pi}{4} - \theta\right) \le 3$$

$$\Rightarrow \quad 1 - 3 \le 1 + 3\cos\left(\frac{\pi}{4} - \theta\right) \le 1 + 3$$

$$\therefore \quad -2 \le y \le 4$$

$$\Rightarrow \quad \lambda = 4, \mu = -2$$

$$\therefore \quad \lambda + \mu = 2, \lambda - \mu = 6 \text{ (P, Q)}$$

101. (A) $|\cot x| = \cot x + \dfrac{1}{\sin x}$

If $0 < x < \frac{\pi}{2} \Rightarrow \cot x > 0$

So $\cot x = \cot x + \dfrac{1}{\sin x} \Rightarrow \dfrac{1}{\sin x} = 0$, no solution

If $\frac{\pi}{2} < \cot x < \pi, -\cot x = \cot x + \dfrac{1}{\sin x}$

$$\Rightarrow \quad \frac{2\cos x}{\sin x} + \frac{1}{\sin x} = 0$$

$$\Rightarrow \quad 1 + 2\cos x = 0 \text{ and } \sin x \ne 0 \Rightarrow x = \frac{2\pi}{3}.$$

(B) since $\sin\phi + \sin\theta = \frac{1}{2}$ ...(i)

and $\cos\theta + \cos\phi = 2$ ...(ii)

(ii) is true only if $\theta = \phi = 0$ or 2π but $\theta = \phi = 0$ or 2π do not satisfy (i)

Hence given system of equation has no solution.

(C) $\sin^2\alpha + \sin\left(\frac{\pi}{3} - \alpha\right)\cdot\sin\left(\frac{\pi}{3} + \alpha\right)$

$$= \sin^2\alpha + \sin^2\frac{\pi}{3} - \sin^2\alpha = \frac{3}{4}.$$

(D) $\tan\theta = 3\tan\phi$

$$\tan(\theta - \phi) = \frac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi} = \frac{2\tan\phi}{1 + 3\tan\phi}$$

$$= \frac{2}{\cot\phi + 3\tan\phi} \cdot \text{Max if } \tan\phi > 0$$

$$\frac{\cos\phi + 3\tan\phi}{2} \ge \sqrt{3} \qquad \text{(Using AM} \ge \text{GM)}$$

$$\Rightarrow \quad (\cot\phi + 3\tan\phi)^2 \ge 12 \Rightarrow \left[\frac{2}{\tan(\theta-\phi)}\right]^2 \ge 12$$

$$\Rightarrow \quad \tan^2(\theta-\phi) \le \frac{1}{3}.$$

102. (A) $\sin\left(\frac{A}{2}\right) + \sin\left(\frac{B}{2}\right) + \sin\left(\frac{C}{2}\right)$

$$= 1 + \left(\sin\left(\frac{A}{2}\right) + \sin\left(\frac{B}{2}\right)\right) - \left(\sin\frac{\pi}{2} - \sin\frac{C}{2}\right)$$

$$= 1 + 2\sin\left(\frac{A+B}{4}\right)\cos\left(\frac{A-B}{4}\right) - 2\cos\left(\frac{\pi+C}{4}\right)\sin\left(\frac{\pi-C}{4}\right)$$

$$= 1 + 2\sin\left(\frac{\pi-C}{4}\right)\left\{\cos\left(\frac{A-B}{4}\right) - \cos\left(\frac{\pi+C}{4}\right)\right\}$$

$(\because A+B+C=\pi)$

$$= 1 + 2\sin\left(\frac{\pi-C}{4}\right)\left\{2\sin\left(\frac{\pi+C+A-B}{8}\right)\sin\left(\frac{\pi+C+B-A}{8}\right)\right\}$$

$$= 1 + 4\sin\left(\frac{\pi-C}{4}\right)\sin\left(\frac{\pi-B}{4}\right)\sin\left(\frac{\pi-A}{4}\right)$$

$$= 1 + 4\sin\left(\frac{\pi-A}{4}\right)\sin\left(\frac{\pi-B}{4}\right)\sin\left(\frac{\pi-C}{4}\right)$$

$$= 1 + 4\cos\left(\frac{\pi}{2} - \frac{\pi-A}{4}\right)\cos\left(\frac{\pi}{2} - \frac{\pi-B}{4}\right)\sin\left(\frac{\pi-C}{4}\right)$$

$$= 1 + 4\cos\left(\frac{\pi+A}{4}\right)\cos\left(\frac{\pi+B}{4}\right)\sin\left(\frac{\pi-C}{4}\right)$$

(B) $\sin\left(\frac{A}{2}\right) + \sin\left(\frac{B}{2}\right) - \sin\left(\frac{C}{2}\right)$

$$= -1 + \left(\sin\left(\frac{A}{2}\right) + \sin\left(\frac{B}{2}\right)\right) + \left(\sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{C}{2}\right)\right)$$

$$= -1 + 2\sin\left(\frac{A+B}{4}\right)\cos\left(\frac{A-B}{4}\right) + 2\cos\left(\frac{\pi+C}{4}\right)\sin\left(\frac{\pi-C}{4}\right)$$

$$= -1 + 2\sin\left(\frac{\pi-C}{4}\right)\left\{\cos\left(\frac{A-B}{4}\right) + \cos\left(\frac{\pi+C}{4}\right)\right\}$$

$(\because A+B+C=\pi)$

$$= -1 + 2\sin\left(\frac{\pi-C}{4}\right)\left\{2\cos\left(\frac{\pi+C+A-B}{8}\right)\cos\left(\frac{\pi+C+B-A}{8}\right)\right\}$$

$$= -1 + 4\sin\left(\frac{\pi-C}{4}\right)\cos\left(\frac{\pi-B}{4}\right)\cos\left(\frac{\pi-A}{4}\right)$$

$$= -1 + 4\cos\left(\frac{\pi-A}{4}\right)\cos\left(\frac{\pi-B}{4}\right)\sin\left(\frac{\pi-C}{4}\right)$$

$$= -1 + 4\sin\left(\frac{\pi}{2} - \frac{\pi-A}{4}\right)\sin\left(\frac{\pi}{2} - \frac{\pi-B}{4}\right)\cos\left(\frac{\pi}{2} - \frac{\pi-C}{4}\right)$$

$$= -1 + 4\sin\left(\frac{\pi+A}{4}\right)\sin\left(\frac{\pi+B}{4}\right)\cos\left(\frac{\pi+C}{4}\right)$$

(C) $\cos\left(\frac{A}{2}\right) + \cos\left(\frac{B}{2}\right) - \cos\left(\frac{C}{2}\right)$

$$= \left(\cos\left(\frac{A}{2}\right) + \cos\left(\frac{B}{2}\right)\right) - \left(\cos\left(\frac{C}{2}\right) + \cos\left(\frac{\pi}{2}\right)\right)$$

$$= 2\cos\left(\frac{A+B}{4}\right)\cos\left(\frac{A-B}{4}\right) - \cos\left(\frac{\pi+C}{4}\right)\cos\left(\frac{\pi-C}{4}\right)$$

$$= 2\cos\left(\frac{\pi-C}{4}\right)\left\{\cos\left(\frac{A-B}{4}\right) - \cos\left(\frac{\pi+C}{4}\right)\right\}$$

$(\because A+B+C=\pi)$

$$= 2\cos\left(\frac{\pi-C}{4}\right)\left\{2\sin\left(\frac{\pi+C+A-B}{8}\right)\sin\left(\frac{\pi+C+B-A}{8}\right)\right\}$$

$$= 4\cos\left(\frac{\pi-C}{4}\right)\sin\left(\frac{\pi-B}{4}\right)\sin\left(\frac{\pi-A}{4}\right)$$

$$= 4\cos\left(\frac{\pi-C}{4}\right)\cos\left(\frac{\pi}{2} - \frac{\pi-B}{4}\right)\cos\left(\frac{\pi}{2} - \frac{\pi-A}{4}\right)$$

$$= 4\cos\left(\frac{\pi+A}{4}\right)\cos\left(\frac{\pi+B}{4}\right)\cos\left(\frac{\pi-C}{4}\right)$$

103. $\frac{1}{1+\tan^2\frac{A}{2}} + \frac{1}{1+\tan^2\frac{B}{2}} + \frac{1}{1+\tan^2\frac{C}{2}}$

$$= k\left[1 + \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right]$$

$$\Rightarrow \cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2}$$

$$= 2\left[1 + \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right] \qquad \text{[by using identity]}$$

104. $\frac{\sin\alpha}{\sin\beta} = \frac{\cos\gamma}{\cos\delta}$

$$\Rightarrow \quad \frac{\sin\alpha - \sin\beta}{\sin\beta} = \frac{\cos\gamma - \cos\delta}{\cos\delta} \qquad \text{(using dividendo)}$$

$$\Rightarrow \quad \frac{2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)}{\sin\beta} = \frac{2\sin\left(\frac{\gamma+\delta}{2}\right)\sin\left(\frac{\delta-\gamma}{2}\right)}{\cos\delta}$$

105. Let $\frac{\pi}{20} = \theta \Rightarrow 10\theta = \frac{\pi}{2}$

$$\Rightarrow \quad 2\theta = 18^\circ \text{ or } \theta = 9^\circ$$

Now, $\tan\theta - \tan 3\theta + \tan 5\theta - \tan 7\theta + \tan 9\theta$

$\tan\theta - \tan 3\theta + \tan 5\theta - \cot 3\theta + \cot\theta$

$(\tan\theta + \cot\theta) - (\tan 3\theta + \cot 3\theta) + \tan 45^\circ$

[using $\tan 5\theta = \tan 45^\circ$]

$$E = \frac{2}{\sin 2\theta} - \frac{2}{\sin 6\theta} + 1$$

$$E = 2\left(\frac{1}{\sin 2\theta} - \frac{1}{\cos 4\theta}\right) + 1$$

$$E = 2\left(\frac{1}{\sin 18^\circ} - \frac{1}{\cos 36^\circ}\right) + 1$$

$$E = 2\left(\frac{4}{\sqrt{5}-1} - \frac{4}{\sqrt{5}+1}\right) + 1 = 5$$

Aliter $$E = 1 + 2\left(\frac{\sin 6\theta - \sin 2\theta}{\sin 2\theta \cdot \sin 6\theta}\right)$$

$$= 1 + 2\left(\frac{2\cos 4\theta \cdot \sin 2\theta}{\sin 2\theta \cdot \cos 4\theta}\right)$$

$$= 1 + 4 = 5$$

106. $$x = \frac{\cos 1^\circ + \cos 2^\circ + \ldots + \cos 44^\circ}{\sin 1^\circ + \sin 2^\circ + \ldots + \sin 44^\circ}$$

$$= \frac{\dfrac{\sin 22^\circ}{\sin(1/2)^\circ} \cdot \cos 22.5^\circ}{\dfrac{\sin 22^\circ}{\sin(1/2)^\circ} \cdot \sin 22.5^\circ} = \cot 22.5^\circ$$

[using the formula of sum of cos series]

$$S = \frac{\sin(n\theta/2)}{\sin(\theta/2)} \cos\frac{(n+1)\theta}{2},$$

for sine series, $S = \dfrac{\sin(n\theta/2)}{\sin(\theta/2)} \sin\dfrac{(n+1)\theta}{2}$

$$\cot\left(\frac{\pi}{8}\right) = \sqrt{2} + 1 = 2.414\ldots$$

$\therefore$ $x = 2.414\ldots$

Greatest integer $= 2$.

107. LHS $= \tan 15^\circ \cdot \tan(30^\circ - 5^\circ) \cdot \tan(30^\circ + 5^\circ)$

Let $t = \tan 30^\circ$ and $m = \tan 5^\circ$

$$\therefore \text{LHS} = \tan 15^\circ \cdot \frac{t-m}{1+tm} \cdot \frac{t+m}{1-tm} = \tan(3(5^\circ)) \cdot \frac{t^2 - m^2}{1 - t^2 m^2}$$

$$= \frac{3m - m^3}{1 - 3m^2} \; \frac{1 - 3m^2}{3 - m^2}$$

$$= \frac{m(3 - m^2)}{(1 - 3m^2)} \cdot \frac{(1 - 3m^2)}{3 - m^2} = m = \tan 5^\circ$$

Hence, $\tan\theta = \tan 5^\circ$

$\Rightarrow$ $\theta = 5^\circ$.

108. We have, $\dfrac{1}{\sin 10^\circ} + \dfrac{1}{\sin 50^\circ} - \dfrac{1}{\sin 70^\circ}$

$$= \frac{1}{\cos 80^\circ} + \frac{1}{\cos 40^\circ} - \frac{1}{\cos 20^\circ}$$

$$= \frac{\cos 40^\circ \cos 20^\circ + \cos 80^\circ \cos 20^\circ - \cos 40^\circ \cos 80^\circ}{\cos 20^\circ \,.\, \cos 40^\circ \,.\, \cos 80^\circ}$$

$$= 8\,[\cos 20^\circ (\cos 40^\circ + \cos 80^\circ) - \cos 40^\circ \cos 80^\circ]$$

$$= 8[2\cos 20^\circ \cos 60^\circ \cos 20^\circ - \cos 40^\circ \cos 80^\circ]$$

$$= 4[2\cos^2 20^\circ - 2\cos 40^\circ \cos 80^\circ]$$

$$= 4[1 + \cos 40^\circ - (\cos 120^\circ + \cos 40^\circ)]$$

$$= 4 \cdot \frac{3}{2} = 6$$

109. We have, $\cos 5\alpha = \cos^5 \alpha$

$\cos 5\alpha = \cos(3\alpha + 2\alpha) = \cos 3\alpha \cos 2\alpha - \sin 3\alpha \sin 2\alpha$

$$= (4\cos^3\alpha - 3\cos\alpha)(2\cos^2\alpha - 1) - (3\sin\alpha - 4\sin^3\alpha) 2\sin\alpha\cos\alpha$$

$$= (4\cos^3\alpha - 3\cos\alpha)(2\cos^2\alpha - 1) - (1 \cos^2\alpha)(3 - 4 + 4\cos^2\alpha) 2\cos\alpha$$

$$= (4\cos^3\alpha - 3\cos\alpha)(2\cos^2\alpha - 1) - (2\cos^2\alpha - 2\cos^3\alpha)(4\cos^2\alpha - 1)$$

$$= 8\cos^5\alpha - 4\cos^3\alpha - 6\cos^3\alpha + 3\cos\alpha - [8\cos^3\alpha - 2\cos\alpha - 8\cos^5\alpha + 2\cos^3\alpha]$$

$$= 16\cos^5\alpha - 20\cos^3\alpha + 5\cos\alpha$$

$\therefore$ $16\cos^5\alpha - 20\cos^3\alpha + 5\cos\alpha = \cos^5\alpha$

$$15\cos^5\alpha - 20\cos^3\alpha + 5\cos\alpha = 0$$

$$5\cos\alpha\,[3\cos^4\alpha - 4\cos^2\alpha + 1] = 0$$

Also $\cos\alpha = 0$

$$3\cos^4\alpha - 3\cos^2\alpha - \cos^2\alpha + 1 = 0$$

$$3\cos^2\alpha(\cos^2\alpha - 1) - (\cos^2\alpha - 1) = 0$$

$$(3\cos^2\alpha - 1)(1 - \cos^2\alpha) = 0$$

$$\cos^2\alpha = 1$$

$$\cos^2\alpha = \frac{1}{3}$$

$\Rightarrow$ $\sec^2\alpha = 3;\ \operatorname{cosec}^2\alpha = \dfrac{3}{2};\ \cot^2\alpha = \dfrac{1}{2}$

$\therefore$ $(\sec^2\alpha + \operatorname{cosec}^2\alpha + \cot^2\alpha) = 3 + \dfrac{3}{2} + \dfrac{1}{2} = 5$

110. We have,

$$\tan^2\frac{\pi}{16} + \tan^2\frac{2\pi}{16} + \tan^2\frac{3\pi}{16} + \ldots + \tan^2\frac{7\pi}{16}$$

$$= \left(\tan^2\frac{\pi}{16} + \cot^2\frac{\pi}{16} + 2\right) + \left(\tan^2\frac{2\pi}{16} + \cot^2\frac{2\pi}{16} + 2\right) + \left(\tan^2\frac{3\pi}{16} + \cot^2\frac{3\pi}{16} + 2\right) + \tan^2\frac{4\pi}{16} - 6$$

$$\left[\begin{array}{l} \text{If } A + B = \dfrac{\pi}{2}, \text{ then } \tan B = \tan\left(\dfrac{\pi}{2} - A\right) = \cot A \\ \text{So, } \tan^2\dfrac{7\pi}{16} = \tan^2\left(\dfrac{8\pi}{16} - \dfrac{\pi}{16}\right) = \cot^2\dfrac{\pi}{16} \text{ etc.} \end{array}\right]$$

$$= \left(\tan\frac{\pi}{16} + \cot\frac{\pi}{16}\right)^2 + \left(\tan\frac{2\pi}{16} + \cot\frac{2\pi}{16}\right)^2 + \left(\tan\frac{3\pi}{16} + \cot\frac{3\pi}{16}\right)^2 - 5$$

$$= \frac{4}{\sin^2\frac{\pi}{8}} + \frac{4}{\sin^2\frac{3\pi}{8}} + \left(\frac{4}{\sin^2\frac{\pi}{4}} - 5\right)$$

$$= \frac{4\left(\sin^2\frac{3\pi}{8} + \sin^2\frac{\pi}{8}\right)}{\sin^2\frac{\pi}{8}\sin^2\frac{3\pi}{8}} + 3 = \frac{16}{\sin^2\frac{\pi}{4}} + 3 = 32 + 3 = 35$$

111. We have, $\dfrac{4+\sec 20^\circ}{\text{cosec } 20^\circ}$

$$= \frac{\sin 20^\circ}{\cos 20^\circ}(4\cos 20^\circ + 1)$$
$$= \frac{2\sin 40^\circ + \sin 20^\circ}{\cos 20^\circ}$$
$$= \frac{\sin 40^\circ + (\sin 40^\circ + \sin 20^\circ)}{\cos 20^\circ}$$
$$= \frac{\sin 40^\circ + 2\sin 30^\circ \cos 10^\circ}{\cos 20^\circ}$$
$$= \frac{\sin 40^\circ + \sin 80^\circ}{\cos 20^\circ}$$
$$= \frac{2\sin 60^\circ \cos 20^\circ}{\cos 20^\circ} = 2\times\frac{\sqrt{3}}{2} = \sqrt{3}$$

Hence, square of the value of expression = 3

112. $A + B + C = \pi$ ...(i)

$$\frac{\sin A}{3} = \frac{\cos B}{3} = \frac{\tan C}{2} \quad \text{...(ii)}$$

$\Rightarrow \sin A = \cos B \Rightarrow A + B = \dfrac{\pi}{2}$ (rejected)

Or $A - B = \dfrac{\pi}{2}$ ...(iii)

$\Rightarrow 2A + C = \dfrac{3\pi}{2}$ [from Eqs. (i) and (ii)]

Now $\dfrac{\sin A}{\tan C} = \dfrac{3}{2}$ [from Eq. (ii)

$\Rightarrow \dfrac{\sin A}{\tan\left(\dfrac{3\pi}{2} - 2A\right)} = \dfrac{3}{2}$ [from Eq. (iii)]

$\Rightarrow 2\sin A = 3\cot 2A$

$\Rightarrow 2\sin A = \dfrac{3\cdot(2\cos^2 A - 1)}{2\sin A\cos A}$

$\Rightarrow 4\cos A(1-\cos^2 A) = 3(2\cos^2 A - 1)$

$\Rightarrow 4\cos^3 A + 6\cos^2 A - 4\cos A - 3 = 0$

Put $\cos A = -\dfrac{1}{2}$

$\Rightarrow (2\cos A + 1)(2\cos^2 A + 2\cos A - 3) = 0$

$\Rightarrow \cos A = -\dfrac{1}{2}$,

$$\cos A = \frac{-2 \pm \sqrt{4+24}}{4} = -1 \pm \sqrt{7} \text{ (rejected)}$$

$\Rightarrow A = \dfrac{2\pi}{3}, B = \dfrac{\pi}{6}, C = \dfrac{\pi}{6}$

$\therefore \dfrac{\sin A}{\cos 2A} + \dfrac{\cos B}{\cot 2B} + \dfrac{\tan C}{\cot 2C} = \dfrac{1}{2} + \dfrac{1}{2} + 1 = 2$

113. $f(\theta) = \dfrac{1}{1+g(\theta)}$ ...(i)

Given, $2f(\alpha) - g(\beta) = 1$

$2f(\alpha) = 1 + g(\beta) = \dfrac{1}{f(\beta)}$ [from Eq. (i)]

$2f(\alpha)f(\beta) = 1$ ...(ii)

Now, $2f(\beta) - g(\alpha) = 2f(\beta) + 1 - (1 + g(\alpha))$

$$= 2f(\beta) + 1 - \frac{1}{f(\alpha)}$$
$$= \frac{2f(\alpha)f(\beta) - 1}{f(\alpha)} + 1 = 1 \quad \text{[from Eq. (ii)]}$$

114. As we know that $\dfrac{1-\sin x}{\cos x} = \dfrac{\cos x}{1+\sin x}$

$$\log_{\left|\frac{1-\sin x}{\cos x}\right|}\left|\frac{1+\sin x}{\cos x}\right| = -1$$

Now, series is

$$\text{Let } S = 1 - \frac{x}{2} - \frac{x^2}{4} - \ldots = 1 - \frac{\frac{x}{2}}{1-\frac{x}{2}}$$
$$= 1 - \frac{-x}{2-x} = \frac{2-x-x}{2-x} = \frac{2(1-x)}{(2-x)} = \frac{k(1-x)}{(2-x)}$$

Thus, $k = 2$

115. From the second relation $9x\sin^3\theta = 5y\cos^3\theta$.

$\Rightarrow \dfrac{\cos^3\theta}{9x} = \dfrac{\sin^3\theta}{5y} = k^3$ (say)

$\Rightarrow \cos\theta = k(9x)^{\frac{1}{3}}$ and $\sin\theta = k(5y)^{\frac{1}{3}}$

Squaring and adding, we get

$$1 = \cos^2\theta + \sin^2\theta = k^2\left[(9x)^{\frac{2}{3}} + (5y)^{\frac{2}{3}}\right]$$

and $\dfrac{9x}{k(9x)^{\frac{1}{3}}} + \dfrac{5y}{k(5y)^{\frac{1}{3}}} = 56$ (From Ist relation)

$\Rightarrow (9x)^{\frac{2}{3}} + (5y)^{\frac{2}{3}} = 56k$

$\Rightarrow \left[(9x)^{\frac{2}{3}} + (5y)^{\frac{2}{3}}\right]^2 = (56)^2k^2 = \dfrac{(56)^2}{(9x)^{\frac{2}{3}} + (5y)^{\frac{2}{3}}}$

$\Rightarrow \left[(9x)^{\frac{2}{3}} + (5y)^{\frac{2}{3}}\right]^3 = (56)^2 = 3136.$

116. $A > \dfrac{\pi}{2} \Rightarrow B + C < \dfrac{\pi}{2}$

$\Rightarrow \tan(B+C) > 0 \Rightarrow \dfrac{\tan B + \tan C}{1 - \tan B\tan C} > 0$

$\Rightarrow \tan B\tan C < 1$ as $\tan B > 0, \tan C > 0$

$\Rightarrow [x] = 2635 - 1 = 2634$

117. $\because \cot\left(7\dfrac{1^\circ}{2}\right) = \dfrac{1+\cos 15^\circ}{\sin 15^\circ}$

$$= \frac{1 + \frac{\sqrt{3}+1}{2\sqrt{2}}}{\frac{\sqrt{3}-1}{2\sqrt{2}}} = \frac{2\sqrt{2}+\sqrt{3}+1}{\sqrt{3}-1}$$
$$= \frac{(2\sqrt{2}+\sqrt{3}+1)(\sqrt{3}+1)}{2}$$

$$= \frac{2\sqrt{6} + 2\sqrt{2} + 3 + \sqrt{3} + \sqrt{3} + 1}{2}$$
$$= \sqrt{6} + \sqrt{2} + 2 + \sqrt{3}$$
$$= \sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{6}$$

and $4 \cos 36° = 4\left(\frac{\sqrt{5}+1}{4}\right) = \sqrt{5} + 1 = \sqrt{5} + \sqrt{1}$

Hence, $4 \cos 36° + \cot\left(7\frac{1°}{2}\right)$
$$= \sqrt{1} + \sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{5} + \sqrt{6}$$

$\therefore$ $n_1 = 1, n_2 = 2, n_3 = 3, n_4 = 4, n_5 = 5$ and $n_6 = 6$

$\therefore$ $\sum_{i=1}^{6} n_i^2 = n_1^2 + n_2^2 + n_3^2 + n_4^2 + n_5^3 + n_6^2$
$$= 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2$$
$$= 91$$

118. $\because \prod_{r=1}^{4} \sin(rA) = \sin A \sin 2A \sin 3A \sin 4A$
$$= \sin A \cdot 2 \sin A \cos A \cdot (3 \sin A - 4 \sin^3 A \cdot 2 \sin 2A \cos 2A)$$
$$= 2 \sin^2 A \cos A \cdot \sin A(3 - 4 \sin^2 A)$$
$$\cdot\, 4 \sin A \cos A \cdot (1 - 2 \sin^2 A)$$
$$= 8x^2(1-x)(3-4x)(1-2x)$$
$$= 24x^2 - 104x^3 + 144x^4 - 64x^5$$

On comparing, we get $a = 24, b = -104, c = 144, d = -64$
$10a - 7b + 15c - 5d$
$$= 10 \times 24 - 7 \times -104 + 15 \times 144 - 5 \times -64$$
$$= 240 + 728 + 2160 + 320 = 3448$$

119. Let $x + 5 = 14 \cos\theta$ and $y - 12 = 14 \sin\theta$

$\therefore x^2 + y^2 = (14 \cos\theta - 5)^2 + (14 \sin\theta + 12)^2$
$$= 196 + 25 + 144 + 28\,(12 \sin\theta - 5 \cos\theta)$$
$$= 365 + 28\,(12 \sin\theta - 5 \cos\theta)$$

$\therefore$ $\sqrt{x^2 + y^2}\Big|_{\min} = \sqrt{365 - 28 \times 13} = \sqrt{365 - 364} = 1$

120. $\because 12° \times 5 = 60°$

Let $12° = \theta$

$\therefore$ $5\theta = 60°$

$\Rightarrow$ $3\theta + 2\theta = 60°$

$\therefore$ $\cos(3\theta + 2\theta) = \cos 60°$

$\Rightarrow$ $\cos 3\theta \cos 2\theta - \sin 3\theta \sin 2\theta = \frac{1}{2}$

$\Rightarrow (4 \cos^3\theta - 3 \cos\theta)(2 \cos^2\theta - 1) - (3 \sin\theta - 4 \sin^3\theta)$
$$2 \sin\theta \cos\theta = \frac{1}{2}$$

Let $\cos\theta = x$

$\therefore$ $(4x^3 - 3x)(2x^2 - 1) - 2x(3 - 4(1 - x^2))(1 - x^2) = \frac{1}{2}$

$\Rightarrow$ $(8x^5 - 10x^3 + 3x) - (2x - 2x^3)(4x^2 - 1) = \frac{1}{2}$

$\Rightarrow$ $(16x^5 - 20x^3 + 6x) - (4x - 4x^3)(4x^2 - 1) - 1 = 0$

$\Rightarrow$ $32x^5 - 40x^3 + 10x - 1 = 0$

$\Rightarrow$ $\left(x - \frac{1}{2}\right)(32x^4 + 16x^3 - 32x^2 - 16x + 2) = 0$

but $x \neq \frac{1}{2}$,

$\therefore$ $16x^4 + 8x^3 - 16x^2 - 8x + 1 = 0$

$\therefore$ Degree is 4.

121. From conditional identities, we have
$$\frac{\sin 2A + \sin 2B + \sin 2C}{\sin A + \sin B + \sin C}$$
$$= \frac{4 \sin A \sin B \sin C}{4 \cos\left(\frac{A}{2}\right) \cos\left(\frac{B}{2}\right) \cos\left(\frac{C}{2}\right)}$$
$$= 8 \sin\left(\frac{A}{2}\right) \sin\left(\frac{B}{2}\right) \sin\left(\frac{C}{2}\right) \Rightarrow k = 8$$

and $3k^3 + 2k^2 + k + 1 = 1536 + 128 + 8 + 1 = 1673$

122. $x = \cot\frac{11\pi}{8} = \cot\left(\pi + \frac{3\pi}{8}\right) = \cot\frac{3\pi}{8} = \sqrt{2} - 1$

$\Rightarrow$ $(x + 1)^2 = 2$

$\therefore$ $x^2 + 2x - 1 = 0$

Now, $f(x) = x^4 + 4x^3 + 2x^2 - 4x + 7$
$$= x^2(x^2 + 2x - 1) + 2x^3 + 3x^2 - 4x + 7$$
$$= 0 + 2x^3 + 3x^2 - 4x + 7$$
$$= 2x(x^2 + 2x - 1) - x^2 - 2x + 7 = -x^2 - 2x + 7$$
$$= -(x^2 + 2x - 1) + 6 = 0 + 6 = 6$$

123. $\frac{\sqrt{(\sin A)}}{\sqrt{(\sin B)} + \sqrt{(\sin C)} - \sqrt{(\sin A)}} = \frac{\sqrt{a}}{\sqrt{b} + \sqrt{c} - \sqrt{a}}$

Now, $\sqrt{b} + \sqrt{c} - \sqrt{a} = \frac{(\sqrt{b} + \sqrt{c} - \sqrt{a})(\sqrt{b} + \sqrt{c} + \sqrt{a})}{(\sqrt{b} + \sqrt{c} + \sqrt{a})}$
$$= \frac{(\sqrt{b} + \sqrt{c})^2 - a}{(\sqrt{b} + \sqrt{c} + \sqrt{a})} = \frac{(b + c - a) + 2\sqrt{bc}}{(\sqrt{b} + \sqrt{c} + \sqrt{a})} > 0$$

Hence, $\sqrt{b} + \sqrt{c} - \sqrt{a} > 0$

Now, let $\sqrt{b} + \sqrt{c} - \sqrt{a} = x, \sqrt{c} + \sqrt{a} - \sqrt{b} = y$

and $\sqrt{a} + \sqrt{b} - \sqrt{c} = z$

$\therefore$ $\frac{\sqrt{\sin A}}{\sqrt{\sin B} + \sqrt{\sin C} - \sqrt{\sin A}} = \frac{y + z}{2x}$

$\Rightarrow$ $\sum \frac{\sqrt{(\sin A)}}{\sqrt{(\sin B)} + \sqrt{(\sin C)} - \sqrt{(\sin A)}}$
$$= \frac{1}{2}\left\{\frac{y}{x} + \frac{z}{x}\right\} + \frac{1}{2}\left\{\frac{z}{y} + \frac{x}{y}\right\} + \frac{1}{2}\left\{\frac{x}{z} + \frac{y}{z}\right\}$$
$$= \frac{1}{2}\left(\frac{x}{y} + \frac{y}{x}\right) + \frac{1}{2}\left(\frac{y}{z} + \frac{z}{y}\right) + \frac{1}{2}\left(\frac{z}{x} + \frac{x}{z}\right)$$
$$\geq 1 + 1 + 1 \quad (\because \text{AM} \geq \text{GM})$$

$2020 \sum \frac{\sqrt{(\sin A)}}{\sqrt{(\sin B)} + \sqrt{(\sin C)} - \sqrt{(\sin A)}} \leq 6060$

$\therefore$ Minimum value is 6060.

124. We have, $\sin\theta(1+\sin^2\theta)=1-\sin^2\theta$

$\Rightarrow \quad \sin\theta(2-\cos^2\theta)=\cos^2\theta$

Squaring both sides, we get

$$\sin^2\theta(2-\cos^2\theta)^2=\cos^4\theta$$

$\Rightarrow \quad (1-\cos^2\theta)(4-4\cos^2\theta+\cos^4\theta)=\cos^4\theta$

$\Rightarrow \quad -\cos^6\theta+5\cos^4\theta-8\cos^2\theta+4=\cos^4\theta$

$\therefore \quad \cos^6\theta-4\cos^4\theta+8\cos^2\theta=4$

125. $16\left(\cos\theta-\cos\frac{\pi}{8}\right)\left(\cos\theta-\cos\frac{3\pi}{8}\right)\left(\cos\theta-\cos\frac{5\pi}{8}\right)\left(\cos\theta-\cos\frac{7\pi}{8}\right)$

$$=16\left(\cos\theta-\cos\frac{\pi}{8}\right)\left(\cos\theta-\cos\frac{7\pi}{8}\right)\times\left(\cos\theta-\cos\frac{3\pi}{8}\right)\left(\cos\theta-\cos\frac{5\pi}{8}\right)$$

$$=16\left(\cos\theta-\cos\frac{\pi}{8}\right)\left(\cos\theta+\cos\frac{\pi}{8}\right)\times\left(\cos\theta-\cos\frac{3\pi}{8}\right)\left(\cos\theta+\cos\frac{3\pi}{8}\right)$$

$$=16\left(\cos^2\theta-\cos^2\frac{\pi}{8}\right)\left(\cos^2\theta-\cos^2\frac{3\pi}{8}\right)$$

$$=16\left(\cos^2\theta-\cos^2\frac{\pi}{8}\right)\left(\cos^2\theta-\sin^2\frac{\pi}{8}\right)$$

$$=16\left(\cos^4\theta-\cos^2\theta+\sin^2\frac{\pi}{8}\cos^2\frac{\pi}{8}\right)$$

$$=16\left(\cos^4\theta-\cos^2\theta+\frac{1}{8}\right)$$

$$=16\left(-\cos^2\theta\sin^2\theta+\frac{1}{8}\right)=16\left(\frac{-\sin^2 2\theta}{4}+\frac{1}{8}\right)$$

$$=16\left(\frac{1-2\sin^2 2\theta}{8}\right)=\frac{16\cos4\theta}{8}=2\cos4\theta$$

$\therefore \quad \lambda=2$

126. $2k\cos\cos40^\circ=\dfrac{1}{\sin20^\circ}+\dfrac{1}{\sqrt{3}\cos20^\circ}$

$$=\frac{\sqrt{3}\cos20^\circ+\sin20^\circ}{\sqrt{3}\sin20^\circ\cos20^\circ}$$

$$=\frac{\frac{\sqrt{3}}{2}\cos20^\circ+\frac{1}{2}\sin20^\circ}{\frac{\sqrt{3}}{4}\sin40^\circ}$$

$$=\frac{\sin60^\circ\cos20^\circ+\cos60^\circ\sin20^\circ}{\left(\frac{\sqrt{3}}{4}\right)\sin40^\circ}$$

$$=\left(\frac{4}{\sqrt{3}}\right)2\cos40^\circ$$

$\Rightarrow \quad 2k^2=16$

so $18k^4+162k^2+369=1745$

127. $\tan82\frac{1}{2}^\circ=\cot7\frac{1}{2}^\circ=\dfrac{\cos7\frac{1}{2}^\circ}{\sin7\frac{1}{2}^\circ}$

On multiplying numerator and denominator by $2\cos7\frac{1}{2}^\circ$, we get

$$\tan82\frac{1}{2}^\circ=\frac{2\cos^2 7\frac{1}{2}^\circ}{2\sin7\frac{1}{2}^\circ\cos7\frac{1}{2}^\circ}=\frac{1+\cos15^\circ}{\sin15^\circ}$$

$$=\frac{1+\cos(45^\circ-30^\circ)}{\sin(45^\circ-30^\circ)}=\frac{1+\frac{\sqrt{3}+1}{2\sqrt{2}}}{\frac{\sqrt{3}-1}{2\sqrt{2}}}$$

$$=\frac{2\sqrt{2}+\sqrt{3}+1}{\sqrt{3}-1}\times\frac{\sqrt{3}+1}{\sqrt{3}+1}$$

$$=\frac{2\sqrt{2}(\sqrt{3}+1)+(\sqrt{3}+1)^2}{2}=\frac{2\sqrt{2}(\sqrt{3}+1)+(4+2\sqrt{3})}{2}$$

$$=\sqrt{2}(\sqrt{3}+1)+(2+\sqrt{3})=\sqrt{6}+\sqrt{2}+\sqrt{4}+\sqrt{3}$$

$$=\sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{6}=(\sqrt{3}+\sqrt{2})(\sqrt{2}+1)$$

128. LHS $=\dfrac{2-m(\sin2\alpha+\sin2\beta)}{1-m(\sin2\alpha+\sin2\beta)+m^2\sin2\alpha\sin2\beta}$

$$=\frac{2-2m\sin(\alpha+\beta)\cos(\alpha-\beta)}{1-2m\sin(\alpha+\beta).\cos(\alpha-\beta)+4m^2\sin\alpha\cos\alpha\sin\beta\cos\beta}$$

$$=\frac{2\{1-\cos^2(\alpha-\beta)\}}{1-2m\sin(\alpha+\beta).\cos(\alpha-\beta)+4m^2\sin\alpha\cos\alpha\sin\beta\cos\beta}$$

[using $m\sin(\alpha+\beta)=\cos(\alpha-\beta)$]

$$=\frac{2\sin^2(\alpha-\beta)}{1-2\cos^2(\alpha-\beta)+m^2[\sin(\alpha+\beta)+\sin(\alpha-\beta)][\sin(\alpha+\beta)-\sin(\alpha-\beta)]}$$

$$=\frac{2\sin^2(\alpha-\beta)}{1-2\cos^2(\alpha-\beta)+m^2\sin^2(\alpha+\beta)-m^2\sin^2(\alpha-\beta)}$$

$$=\frac{2\sin^2(\alpha-\beta)}{1-\cos^2(\alpha-\beta)-m^2\sin^2(\alpha-\beta)}$$

$$=\frac{2\sin^2(\alpha-\beta)}{\sin^2(\alpha-\beta)-m^2\sin^2(\alpha-\beta)}=\frac{2}{1-m^2}$$

129. Given $\tan\frac{1}{4}(\beta+\gamma-2\alpha).\tan\frac{1}{4}(\gamma+\alpha-\beta)\tan\frac{1}{4}(\alpha+\beta-\gamma)=1$,

where $\alpha+\beta+\gamma=\pi$

$\Rightarrow \quad \tan\left(\frac{\pi-2\alpha}{4}\right)\tan\left(\frac{\pi-2\beta}{4}\right)\tan\left(\frac{\pi-2\gamma}{4}\right)=1$

$\Rightarrow \quad \left(1-\tan\frac{\alpha}{2}\right)\left(1-\tan\frac{\beta}{2}\right)\left(1-\tan\frac{\gamma}{2}\right)$

$$=\left(1+\tan\frac{\alpha}{2}\right)\left(1+\tan\frac{\beta}{2}\right)\left(1+\tan\frac{\gamma}{2}\right)$$

$\Rightarrow \quad \tan\frac{\alpha}{2}+\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}+\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}=0$

$\Rightarrow \quad \tan\frac{\alpha}{2}+\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}=-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2} \quad$ ...(i)

Also, $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}+\tan\frac{\beta}{2}\tan\frac{\gamma}{2}+\tan\frac{\gamma}{2}\tan\frac{\alpha}{2}=1$ ...(ii)

On squaring Eq. (i) and using Eq. (ii); $\left\{\because \frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}=\frac{\pi}{2}\right\}$

$$\tan^2\frac{\alpha}{2}+\tan^2\frac{\beta}{2}+\tan^2\frac{\gamma}{2}+2=\tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}\tan^2\frac{\gamma}{2} \quad ...(iii)$$

The equation to be prove is

$$1+\cos\alpha+\cos\beta+\cos\gamma=0$$

$$\Rightarrow 1+\frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}+\frac{1-\tan^2\frac{\beta}{2}}{1+\tan^2\frac{\beta}{2}}+\frac{1-\tan^2\frac{\gamma}{2}}{1+\tan^2\frac{\gamma}{2}}=0$$

$$\Rightarrow \frac{2}{1+\tan^2\frac{\alpha}{2}}+\frac{2\left(1-\tan^2\frac{\beta}{2}\tan^2\frac{\gamma}{2}\right)}{\left(1+\tan^2\frac{\beta}{2}\right)\left(1+\tan^2\frac{\gamma}{2}\right)}=0$$

$$\Rightarrow \left(1+\tan^2\frac{\alpha}{2}\right)\left(1-\tan^2\frac{\beta}{2}\tan^2\frac{\gamma}{2}\right)+\left(1+\tan^2\frac{\beta}{2}\right)\left(1+\tan^2\frac{\gamma}{2}\right)=0$$

$$\Rightarrow \left(1+\tan^2\frac{\alpha}{2}-\tan^2\frac{\beta}{2}\tan^2\frac{\gamma}{2}-\tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}\tan^2\frac{\gamma}{2}\right)+\left(1+\tan^2\frac{\beta}{2}+\tan^2\frac{\gamma}{2}+\tan^2\frac{\beta}{2}\tan^2\frac{\gamma}{2}\right)=0$$

$$\Rightarrow 2+\tan^2\frac{\alpha}{2}+\tan^2\frac{\beta}{2}+\tan^2\frac{\gamma}{2}=\tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}\tan^2\frac{\gamma}{2} \quad ...(iv)$$

From Eq. (iii) and (iv)

$$1+\cos\alpha+\cos\beta+\cos\gamma=0$$

130. We have, $\sin^4 x+\cos^4 x\le\sin^2 x+\cos^2 x$,

as $|\sin x|\le 1$ and $|\cos x|\le 1$

$\Rightarrow a\le 1$ ...(i)

Next, $\sin^4 x+\cos^4 x=a$

$\Rightarrow (\sin^2 x+\cos^2 x)^2-2\sin^2 x\cos^2 x=a$

$\Rightarrow 1-\frac{1}{2}\sin^2 2x=a$

$\Rightarrow \frac{1}{2}\sin^2 2x=1-a$

$\Rightarrow 1-a\le\frac{1}{2}$ $\left[\because \frac{1}{2}\sin^2 x\le\frac{1}{2}\right]$

$\Rightarrow a\ge\frac{1}{2}$ ...(ii)

From Eqs. (i) and (ii),

$$\frac{1}{2}\le a\le 1$$

131. Let $a\sec\theta-b\tan\theta=x$

So, $a^2\sec^2\theta=(x+b\tan\theta)^2$

$\Rightarrow a^2(1+\tan^2\theta)=x^2+2bx\tan\theta+b^2\tan^2\theta$

$\Rightarrow \tan^2\theta(a^2-b^2)-2bx\tan\theta+(a^2-x^2)=0$

$\Rightarrow \left(\tan\theta-\frac{bx}{a^2-b^2}\right)^2=\frac{a^2(x^2+b^2-a^2)}{(a^2-b^2)^2}$

Thus, $x^2+(b^2-a^2)\ge 0$

$\Rightarrow x^2\ge a^2-b^2$

Thus, the minimum value of x is $\sqrt{a^2-b^2}$, which is attained at $\theta=\sin^{-1}\left(\frac{b}{a}\right)$.

132. We can write

$(b\tan\gamma-c\tan\beta)^2+(c\tan\alpha-a\tan\gamma)^2+(a\tan\beta-b\tan\alpha)^2$

$=(a^2+b^2+c^2)(\tan^2\alpha+\tan^2\beta+\tan^2\gamma)-(a\tan\alpha+b\tan\beta+c\tan\gamma)^2$

The minimum value of the LHS being zero, that of

$$(a^2+b^2+c^2)(\tan^2\alpha+\tan^2\beta+\tan^2\gamma)-k^2\ge 0$$

$$\Rightarrow \tan^2\alpha+\tan^2\beta+\tan^2\gamma\ge\frac{k^2}{a^2+b^2+c^2}$$

Hence, minimum value of $\tan^2\alpha+\tan^2\beta+\tan\gamma$ is $\left(\frac{k^2}{a^2+b^2+c^2}\right)$.

133. Here, $\frac{x}{y}=\frac{\tan(\theta+\alpha)}{\tan(\theta+\beta)}$. By componendo and dividendo

$$\frac{x+y}{x-y}=\frac{\tan(\theta+\alpha)+\tan(\theta+\beta)}{\tan(\theta+\alpha)-\tan(\theta+\beta)}=\frac{\sin(2\theta+\alpha+\beta)}{\sin(\alpha-\beta)}$$

$$\therefore \frac{x+y}{x-y}\sin^2(\alpha-\beta)=\sin(2\theta+\alpha+\beta)\cdot\sin(\alpha-\beta)$$

$$\frac{x+y}{x-y}\cdot\sin^2(\alpha-\beta)=\frac{1}{2}\{\cos 2(\theta+\beta)-\cos 2(\theta+\alpha)\} \quad ...(i)$$

Similarly,

$$\frac{y+z}{y-z}\cdot\sin^2(\beta-\gamma)=\frac{1}{2}\{\cos 2(\theta+\gamma)-\cos 2(\theta+\beta)\} \quad ...(ii)$$

and $\frac{z+x}{z-x}\cdot\sin^2(\gamma-\alpha)=\frac{1}{2}\{\cos 2(\theta+\alpha)-\cos 2(\theta+\gamma)\}$ ...(iii)

From Eqs. (i), (ii) and (iii), we get

$$\Sigma\frac{x+y}{x-y}\cdot\sin^2(\alpha-\beta)=0$$

134. $f(x)$ may be written as $f(x)=\sum_{k=1}^{n}\frac{1}{2^{k-1}}\cos(a_k+x)$

$$=\sum_{k=1}^{n}\frac{1}{2^{k-1}}(\cos a_k\cos x-\sin a_k\sin x)$$

$$=\left(\sum_{k=1}^{n}\frac{1}{2^{k-1}}\cdot\cos a_k\right)\cdot\cos x\left(-\sum_{k=1}^{n}\frac{1}{2^{k-1}}\cdot\sin a_k\right)\cdot\sin x$$

where, $A=\sum_{k=1}^{n}\frac{1}{2^{k-1}}\cos a_k$, $B=\sum_{k=1}^{n}\frac{1}{2^{k-1}}\sin a_k$. Now, A and B both cannot be zero, for if they were then $f(x)$ would vanish identically.

Now,

$$f(x_1)=A\cos x_1-B\sin x_1=0$$
$$f(x_2)=A\cos x_2-B\sin x_2=0$$

$\Rightarrow \tan x_1=\frac{A}{B}$ and $\tan x_2=\frac{A}{B}$

$\Rightarrow \tan x_1=\tan x_2\Rightarrow x_2-x_1=m\pi$.

135. $x = \tan(n\theta + \alpha) - \tan(n\theta + \beta)$

$$= \frac{\sin(n\theta+\alpha)}{\cos(n\theta+\alpha)} - \frac{\sin(n\theta+\beta)}{\cos(n\theta+\beta)}$$

$$= \frac{\sin(n\theta+\alpha-n\theta-\beta)}{\cos(n\theta+\alpha)\cos(n\theta+\beta)} = \frac{2\sin(\alpha-\beta)}{\cos(2n\theta+\alpha+\beta)+\cos(\alpha-\beta)}$$

$$\Rightarrow \cos(2n\theta+\alpha+\beta)+\cos(\alpha-\beta) = \frac{2\sin(\alpha-\beta)}{x} \quad \text{...(i)}$$

Again $y = \cot(n\theta+\alpha) - \cot(n\theta+\beta)$

$$= \frac{\cos(n\theta+\alpha)}{\sin(n\theta+\alpha)} - \frac{\cos(n\theta+\beta)}{\sin(n\theta+\beta)} = \frac{\sin(n\theta+\beta-n\theta-\alpha)}{\sin(n\theta+\alpha)\sin(n\theta+\beta)}$$

$$\Rightarrow \quad y = \frac{\sin(\beta-\alpha)}{\cos(\alpha-\beta)-\cos(2n\theta+\alpha+\beta)}$$

$$\Rightarrow \cos(\alpha-\beta)-\cos(2n\theta+\alpha+\beta) = \frac{2\sin(\beta-\alpha)}{y} \quad \text{...(ii)}$$

On adding Eqs. (i) and (ii), we get

$$2\cos(\alpha-\beta) = \frac{2\sin(\alpha-\beta)}{x} + \frac{2\sin(\beta-\alpha)}{y}$$

$$\Rightarrow \quad \cot(\alpha-\beta) = \frac{1}{x} - \frac{1}{y}$$

136. $\{\sin(\alpha-\beta)+\cos(\alpha+2\beta)\cdot\sin\beta\}^2 = 4\cos\alpha\cdot\sin\beta\cdot\sin(\alpha+\beta)$

$$\Rightarrow \{\sin\alpha\cos\beta - \sin\beta\cos\alpha + (\cos\alpha\cos2\beta - \sin\alpha\sin2\beta)\sin\beta\}^2$$
$$= 4\cos\alpha\sin\beta\sin(\alpha+\beta)$$

$$\Rightarrow \{\tan\alpha - \tan\beta + \cos2\beta\cdot\tan\beta - \sin2\beta\cdot\tan\alpha\tan\beta\}^2$$
$$= 4\tan\beta(\tan\alpha+\tan\beta) \quad \{\because \text{dividing by } \cos^2\alpha.\cos^2\beta\}$$

$\Rightarrow$

$$\{\tan\alpha\cdot\cos2\beta - \tan\beta + \cos2\beta\cdot\tan\beta\}^2 = 4\tan\beta\{\tan\alpha+\tan\beta\}$$

$$\Rightarrow \{(\tan\alpha+\tan\beta)\cdot\cos2\beta - \tan\beta\}^2 = 4\tan\beta(\tan\alpha+\tan\beta) \text{ ...(i)}$$

If $$(\tan\alpha+\tan\beta) = \frac{\tan\beta}{x} \quad \text{...(ii)}$$

Eq. (i) becomes;

$$\left\{\frac{\tan\beta}{x}\cdot\cos2\beta - \tan\beta\right\}^2 = 4\tan\beta\cdot\frac{\tan\beta}{x}$$

$$\Rightarrow \quad (\cos2\beta - x)^2 = 4x$$

$$\Rightarrow \quad \cos^2 2\beta + x^2 - 2x\cos2\beta = 4x$$

$$\Rightarrow \quad x^2 - 2x(\cos2\beta+2) + \cos^2 2\beta = 0$$

$$\Rightarrow \quad x = (\cos2\beta+2) \pm 2\sqrt{1+\cos2\beta}$$

$$\Rightarrow \quad x = \cos2\beta + 2 \pm 2\sqrt{2\cos^2\beta}$$

$$\Rightarrow \quad x = 2\cos^2\beta - 1 + 2 \pm 2\sqrt{2}\cos\beta$$
$$= (\sqrt{2}\cos\beta \pm 1)^2$$

$$\Rightarrow \quad \tan\alpha+\tan\beta = \frac{\tan\beta}{x} = \frac{\tan\beta}{(\sqrt{2}\cos\beta-1)^2} \quad [\text{since, } x<1]$$

137. Let
$$\Delta = \begin{vmatrix} \sin A & \sin B & \sin C \\ \cos A & \cos B & \cos C \\ \cos^3 A & \cos^3 B & \cos^3 C \end{vmatrix}$$

$$= \cos A\cos B\cos C \begin{vmatrix} \tan A & \tan B & \tan C \\ 1 & 1 & 1 \\ \cos^2 A & \cos^2 B & \cos^2 C \end{vmatrix}$$

$$= \cos A\cos B\cos C \begin{vmatrix} \tan A & \tan B - \tan A & \tan C - \tan A \\ 1 & 0 & 0 \\ \cos^2 A & \cos^2 B - \cos^2 A & \cos^2 A - \cos^2 A \end{vmatrix}$$

$\left[\text{since, } \tan B - \tan A = -\dfrac{\sin(A-B)}{\cos A\cos B},\right.$

$\left.\cos^2 B - \cos^2 A = \sin(A-B)\sin(A+B)\right]$

$$\therefore \quad \Delta = -\cos A\cos B\cos C \begin{vmatrix} -\dfrac{\sin(A-B)}{\cos A\cos B} & -\dfrac{\sin(A-C)}{\cos A\cos C} \\ \sin(A-B)\sin(A+B) & \sin(A-C)\sin(A+C) \end{vmatrix}$$

$$= \cos A\cos B\cos C\cdot \frac{\sin(A-B).\sin(A-C)}{\cos A\cos B\cos C}\begin{vmatrix} \cos C & \cos B \\ \sin(A+B) & \sin(A+C) \end{vmatrix}$$

$$= -\sin(B-C)\sin(C-A)\sin(A-B) = 0$$

If $B = C$ or $C = A$ or $A = B$

Hence, ΔABC is an isosceles.

138. Here, $$\frac{\sqrt{\sin A}}{\sqrt{\sin B}+\sqrt{\sin C}-\sqrt{\sin A}} = \frac{\sqrt{a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}}$$

Now, $$\sqrt{b}+\sqrt{c}-\sqrt{a} = \frac{(\sqrt{b}+\sqrt{c}-\sqrt{a})(\sqrt{b}+\sqrt{c}+\sqrt{a})}{(\sqrt{b}+\sqrt{c}+\sqrt{a})}$$

$$= \frac{b+c-a+2\sqrt{bc}}{\sqrt{b}+\sqrt{c}+\sqrt{a}} > 0$$

Hence, $\sqrt{b}+\sqrt{c}-\sqrt{a} = 0$

Let $\sqrt{b}+\sqrt{c}-\sqrt{a} = x$, $\sqrt{c}+\sqrt{a}-\sqrt{b} = y$, $\sqrt{a}+\sqrt{b}-\sqrt{c} = z$

$$\therefore \quad \frac{\sqrt{\sin A}}{\sqrt{\sin B}+\sqrt{\sin C}-\sqrt{\sin A}} = \frac{y+z}{2x}$$

$$\Rightarrow \Sigma = \frac{\sqrt{\sin A}}{\sqrt{\sin B}+\sqrt{\sin C}-\sqrt{\sin A}}$$

$$= \frac{1}{2}\left\{\frac{y}{x}+\frac{z}{x}\right\} + \frac{1}{2}\left\{\frac{y}{z}+\frac{z}{y}\right\} + \frac{1}{2}\left\{\frac{z}{x}+\frac{x}{y}\right\}$$

which is greater than or equal to 3, as each term $\left(\dfrac{1}{2}\left(\dfrac{x}{y}+\dfrac{y}{x}\right)\text{etc.}\right)$ is greater than or equal to 1.

(using AM $\geq$ GM)

Now, equality hold if and only if,

$$\frac{x}{y} = \frac{y}{x}, \frac{y}{z} = \frac{z}{y}$$

and $$\frac{z}{x} = \frac{x}{y} \quad \text{i.e. } x = y = z$$

$\Rightarrow a = b = c$ i.e. triangle is equilateral.

139. $2(\cos(\alpha-\beta)+\cos(\beta-\gamma)+\cos(\gamma-\alpha))+3 = 0$

$$\Rightarrow 2(\cos(\alpha+\theta-(\beta+\theta)) + \cos(\beta+\theta-(\gamma+\theta))$$
$$+ \cos(\gamma+\theta-(\alpha+\theta))) + 3 = 0$$

$$\Rightarrow 2(\cos(\alpha+\theta).\cos(\beta+\theta) + \sin(\alpha+\theta).\sin(\beta+\theta) + \ldots + \ldots))$$
$$+ \{(\sin^2(\alpha+\theta)+\cos^2(\alpha+\theta)) + (\sin^2(\beta+\theta)+\cos^2(\beta+\theta))$$
$$+ (\sin^2(\gamma+\theta)+\cos^2(\gamma+\theta))\} = 0$$

$\Rightarrow\ (\sin(\gamma+\theta)+\sin(\beta+\theta)+\sin(\alpha+\theta))^2+(\cos(\alpha+\theta)$
$+\cos(\beta+\theta)+\cos(\gamma+\theta))^2=0$

which is only possible if;

$$\sin(\alpha+\theta)+\cos(\beta+\theta)+\sin(\gamma+\theta)=0 \quad \text{...(i)}$$
$$\cos(\alpha+\theta)+\cos(\beta+\theta)+\cos(\gamma+\theta)=0 \quad \text{...(ii)}$$

From Eq. (ii), we get

$$d(\cos(\alpha+\theta)+\cos(\beta+\theta)+\cos(\gamma+\theta))=0$$
$$\Rightarrow\ \sin(\alpha+\theta)\cdot d\alpha+\sin(\beta+\theta)\cdot d\beta+\sin(\gamma+\theta)\cdot d\gamma=0$$
$$\Rightarrow\ \frac{d\alpha}{\sin(\beta+\theta)\cdot\sin(\gamma+\theta)}+\frac{d\beta}{\sin(\alpha+\theta)\cdot\sin(\gamma+\theta)}+\frac{d\gamma}{\sin(\alpha+\theta)\cdot\sin(\beta+\theta)}=0$$

140. The quadratic equation,

$4^{\sec^2\alpha}x^2+2x+\left(\beta^2-\beta+\frac{1}{2}\right)=0$ have real roots

$$\Rightarrow\ \text{Discriminant}=4-4\cdot4^{\sec^2\alpha}\left(\beta^2-\beta+\frac{1}{2}\right)\geq0$$
$$\Rightarrow\ 4^{\sec^2\alpha}\left(\beta^2-\beta+\frac{1}{2}\right)\leq1$$
$$\left[\text{but } 4^{\sec^2\alpha}\geq4, \beta^2-\beta+\frac{1}{2}=\left(\beta+\frac{1}{2}\right)^2+\frac{1}{4}\geq\frac{1}{4}\right]$$

i.e. the equation will be satisfied only when $4^{\sec^2\alpha}=4$ and

$$\beta^2-\beta+\frac{1}{2}=\frac{1}{4}$$
$$\Rightarrow\ \sec^2\alpha=1 \text{ and } \left(\beta-\frac{1}{2}\right)^2=0$$
$$\Rightarrow\ \cos^2\alpha=1 \text{ and } \beta=\frac{1}{2}$$
$$\Rightarrow\ \alpha=n\pi \text{ and } \beta=\frac{1}{2}$$
$$\cos\alpha+\cos^{-1}\beta=\cos n\pi+\cos^{-1}\left(\frac{1}{2}\right)$$
$$=1+\frac{\pi}{3}, \text{ when } n \text{ is an even integer.}$$
$$=-1+\frac{\pi}{3}, \text{ when } n \text{ is an odd integer.}$$

i.e. values of $\cos\alpha+\cos^{-1}\beta$ is $\frac{\pi}{3}-1, \frac{\pi}{3}+1$.

141. $f(\theta)=1-(a\cos\theta+b\sin\theta)-(A\cos2\theta+B\sin2\theta)$

$$\Rightarrow\ f(\theta)=1-\sqrt{a^2+b^2}\cos(\theta-\alpha)-\sqrt{A^2+B^2}\cos(2\theta-\beta)$$

Now, $f\left(\alpha+\frac{\pi}{4}\right)=1-\frac{\sqrt{a^2+b^2}}{\sqrt{2}}-\sqrt{A^2+B^2}\cos\left(\frac{\pi}{2}+2\alpha-\beta\right)$

$$=1-\frac{\sqrt{a^2+b^2}}{\sqrt{2}}+\sqrt{A^2+B^2}\sin(2\alpha-\beta) \quad \text{...(i)}$$

and $f\left(\alpha-\frac{\pi}{4}\right)=1-\frac{\sqrt{a^2+b^2}}{\sqrt{2}}-\sqrt{A^2+B^2}\cos\left(2\alpha-\beta-\frac{\pi}{2}\right)$

$$=1-\frac{\sqrt{a^2+b^2}}{\sqrt{2}}-\sqrt{A^2+B^2}\sin(2\alpha-\beta) \quad \text{...(ii)}$$

On adding Eqs. (i) and (ii),

$$f\left(\alpha+\frac{\pi}{4}\right)+f\left(\alpha-\frac{\pi}{4}\right)=2-2\frac{\sqrt{a^2+b^2}}{\sqrt{2}}\geq0$$
$$\Rightarrow\ \sqrt{a^2+b^2}\leq\sqrt{2}$$
$$\Rightarrow\ a^2+b^2\leq2$$

Similarly putting $\theta=\beta$ and $\beta+\pi$. We have,

$$f(\beta)+f(\beta+\pi)=2-2\sqrt{A^2+B^2}\geq0$$
$$\Rightarrow\ \sqrt{A^2+B^2}\leq1 \Rightarrow A^2+B^2\leq1$$

142. Clearly θ_1, θ_0 are roots of ; $\frac{\cos\theta}{\cos\theta_2}+\frac{\sin\theta}{\sin\theta_2}=1$

$$\Rightarrow\ \frac{\cos\theta}{\cos\theta_2}=1-\frac{\sin\theta}{\sin\theta_2}\Rightarrow\frac{\cos^2\theta}{\cos^2\theta_2}=1+\frac{\sin^2\theta}{\sin^2\theta_2}-\frac{2\sin\theta}{\sin\theta_2}$$
$$\Rightarrow\ \sin^2\theta\left(\frac{1}{\sin^2\theta_2}+\frac{1}{\cos^2\theta_2}\right)-\frac{2\sin\theta}{\sin\theta_2}+\left(1-\frac{1}{\cos^2\theta_2}\right)=0$$

The roots of the equation are θ_0 and θ_1.

$$\Rightarrow\ \sin\theta_0\cdot\sin\theta_1=\frac{1-\frac{1}{\cos^2\theta_2}}{\frac{1}{\sin^2\theta_2}+\frac{1}{\cos^2\theta_2}}$$
$$=(\cos^2\theta_2-1)\cdot\sin^2\theta_2=-\sin^4\theta_2$$
$$\Rightarrow\ \frac{\sin\theta_0\cdot\sin\theta_1}{\sin^2\theta_2}=-\sin^2\theta_2 \quad \text{...(i)}$$

Similarly, taking a quadratic in $\cos\theta$, we get

$$\Rightarrow\ \frac{\cos\theta_0\cdot\cos\theta_1}{\sin^2\theta_2}=-\cos^2\theta_2 \quad \text{...(ii)}$$

On adding Eqs. (i) and (ii), we get

$$\frac{\sin\theta_0\sin\theta_1}{\sin^2\theta_2}+\frac{\cos\theta_0\cos\theta_1}{\cos^2\theta_2}=-1$$

143. Let the given expression be E, then E can be written as,

$$E=\sum_{k=1}^{n-1}{}^nC_k \cos kx\cdot\cos(n+k)x+\sum_{k=1}^{n-1}{}^nC_k\sin(n-k)x.\sin(2n-k)x$$

or $$E=\sum_{k=1}^{n-1}{}^nC_k\cos kx \cos(n+k)x+\sum_{k=1}^{n-1}{}^nC_k\sin(k)x\cdot\sin(n+k)x$$

[replacing k by $(n-k)$ in the second]

Sum and using ${}^nC_k={}^nC_{n-k}$

$$E=\sum_{k=1}^{n-1}{}^nC_k(\cos kx\cos(n+k)x+\sin kx\cdot\sin(n+k)x)$$
$$=\sum_{k=1}^{n-1}{}^nC_k\cos nx$$
$$=\cos nx\{({}^nC_0+{}^nC_1+\ldots+{}^nC_n)-{}^nC_0-{}^nC_n\}$$
$$=\cos nx\{2^n-2\}$$
$$\therefore\ E=(2^n-2)\cos nx$$

144. It is evident from the inequality that,

$$|\sqrt{1+\sin 2x}-\sqrt{1-\sin 2x}| \le \sqrt{2} \ \forall \ x \in [0, 2\pi]$$

as $|\sqrt{1+\sin 2x}-\sqrt{1-\sin 2x}| \le \sqrt{1+\sin 2x} \le \sqrt{2}$

Now,

$2\cos x \le |\sqrt{1+\sin 2x}-\sqrt{1-\sin 2x}|$ holds for all x for which $\cos x \le 0$.

$$\Rightarrow \quad x \le \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \quad \text{...(i)}$$

Now, if $\cos x > 0$

Then, $4\cos^2 x \le 1+\sin 2x + 1 - \sin 2x - \sqrt{1-\sin^2 2x}$

$\Rightarrow \quad 2 + 2\cos 2x \le 2 - 2|\cos x|$

$\Rightarrow \quad |\cos 2x| \le -\cos 2x$

$\Rightarrow \quad \cos 2x \le 0$

$$\Rightarrow \quad x \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right] \cup \left[\frac{5\pi}{4}, \frac{7\pi}{4}\right] \quad \text{...(ii)}$$

Hence, from Eqs. (i) and (ii)

$$x \in \left[\frac{\pi}{4}, \frac{7\pi}{4}\right]$$

145. The given equation can be rewritten as, $x^2 - 3 = 3\left[\sin\left(x - \frac{\pi}{6}\right)\right]$

Here, right hand side can take only the values $-3, 0, 3$.

Case I When $x^2 - 3 = -3 \Rightarrow x = 0$

At $x = 0$, $\left[\sin\left(x - \frac{\pi}{6}\right)\right] = -1$, so $x = 0$ is a solution.

Case II When $x^2 - 3 = 0 \Rightarrow x = \pm\sqrt{3}$

Now at $x = \sqrt{3}$, $\left[\sin\left(x - \frac{\pi}{6}\right)\right] = 0 \Rightarrow x = \sqrt{3}$

But at $x = -\sqrt{3}$, $\left[\sin\left(x - \frac{\pi}{6}\right)\right] = -1$, hence $x = -\sqrt{3}$ is not a solution.

Case III When $x^2 - 3 = 3 \Rightarrow x = \pm\sqrt{6}$

But $\left[\sin\left(\pm\sqrt{6} - \frac{\pi}{6}\right)\right] \ne 1 \Rightarrow x = \pm\sqrt{6}$ is not a solution.

Hence, the given equation has only two solutions $x = 0$ and $\sqrt{3}$.

146. $\sum_{r=0}^{n} {}^nC_r a^r b^{n-r} \cos(rB - (n-r)A)$

$= \text{real part of } \sum_{r=0}^{n} {}^nC_r a^r b^{n-r} e^{i\{rB-(n-r)A\}}$

Now, $\sum_{r=0}^{n} {}^nC_r a^r b^{n-r} e^{i\{rB-(n-r)A\}}$

$= \sum_{r=0}^{n} {}^nC_r (ae^{iB})^r (be^{-iA})^{n-r} = (ae^{iB} + be^{-iA})^n$

$= (a\cos B + ia\sin B + b\cos A - bi\sin A)^n$

$= \{(a\cos B + b\cos A) + i(a\sin B - b\sin A)\}^n$

$= \{C + i\cdot 0\}^n = C^n$

147. Let $z^5 + 1 = 0 \Rightarrow z^5 = -1 = (\cos(2r+1) + i\sin(2r+1)\pi)$

$\Rightarrow \quad z = e^{i\left(\frac{2r+1}{5}\right)\pi}, r = 0, 1, 2, 3, 4$

$\Rightarrow$ Roots of $z^k + 1 = 0$ are $e^{i\pi/5}, e^{i3\pi/5}, e^{i\pi}, e^{i7\pi/5}, e^{i9\pi/5}$. Clearly $e^{i7\pi/5}, e^{i9\pi/5}$ and $e^{i3\pi/5}, e^{i7\pi/5}$ are pairwise conjugate.

$\Rightarrow z^5 + 1 = (z - e^{i\pi})(z - e^{i\pi/5})(z - e^{-i\pi/5})(z - e^{-i3\pi/5})(z - e^{i7\pi/5})$

$$\Rightarrow z^5 + 1 = (z+1)\left(z^2 - 2z\cos\frac{\pi}{5} + 1\right)\left(z^2 - 2z\cos\frac{3\pi}{5} + 1\right) \quad \text{...(i)}$$

It is required factorisation of $z^5 + 1$.

$$\text{Now,} \quad \frac{z^5+1}{z+1} = 1 - z + z^2 - z^2 + z^4 \quad \text{...(ii)}$$

$\Rightarrow \quad 1 - z + z^2 - z^3 + z^4 =$

$$\left(z^2 - 2z\cos\frac{\pi}{5} + 1\right)\left(z^2 - 2z\cos\frac{3\pi}{5} + 1\right)$$

[using Eqs. (i) and (ii)}

On dividing both side by z^2

$$\left(z^2 + \frac{1}{z^2}\right) - \left(z + \frac{1}{z}\right) + 1$$

$$= \left(z + \frac{1}{z} - 2\cos\frac{\pi}{5}\right)\left(z + \frac{1}{z} - 2\cos\frac{3\pi}{5}\right)$$

Let $z = e^{i\theta}$

$$\Rightarrow \quad z^2 + \frac{1}{z^2} = 2\cos 2\theta, \ z + \frac{1}{z} = 2\cos\theta$$

$$\Rightarrow \ 2\cos 2\theta - 2\cos\theta + 1 = 4\left(\cos\theta - \cos\frac{\pi}{5}\right)\left(\cos\theta - \cos\frac{3\pi}{5}\right)$$

Putting $\theta = 0$, we get

$$\frac{1}{4} = \left(1 - \cos\frac{\pi}{5}\right)\left(1 - \cos\frac{3\pi}{5}\right)$$

$$\Rightarrow \quad \frac{1}{4} = 2\sin^2\frac{\pi}{10}\cdot 2\sin^2\frac{3\pi}{10}$$

$$\Rightarrow \quad \sin\frac{\pi}{10}\cdot\sin\frac{3\pi}{10} = \frac{1}{4}$$

$$\Rightarrow \quad 4\sin\frac{\pi}{10}\cdot\cos\left(\frac{\pi}{2} - \frac{3\pi}{10}\right) = 1$$

$$\Rightarrow \quad 4\sin\frac{\pi}{10}\cdot\cos\frac{\pi}{5} = 1$$

148. Let $\theta = \frac{(2n+1)\pi}{7}$, where $n = 0, 1, 2, 3, 4, 5, 6$.

Then, $4\theta = (2n+1)\pi - 3\theta$

$\Rightarrow \quad \cos 4\theta = -\cos 3\theta$

$\Rightarrow \quad 2\cos^2 2\theta - 1 = -(4\cos^3\theta - 3\cos\theta)$

$\Rightarrow \quad 2(2\cos^2\theta - 1)^2 - 1 = -4\cos^3\theta + 3\cos\theta$

$\Rightarrow \quad 2(2x^2 - 1)^2 - 1 = -4x^3 + 3x \quad$ [put $x = \cos\theta$]

$\Rightarrow \quad 8x^4 + 4x^3 - 8x^2 - 3x + 1 = 0$

$\Rightarrow \quad (x+1)(8x^3 - 4x^2 - 4x + 1) = 0$

The roots of this equation are,

$$\cos\frac{\pi}{7}, \cos\frac{3\pi}{7}, \cos\frac{5\pi}{7}, \cos\frac{7\pi}{7}, \cos\frac{9\pi}{7}, \cos\frac{11\pi}{7}, \cos\frac{13\pi}{7}$$

$\therefore$ The roots of $8x^3 - 4x^2 - 4x + 1 = 0$

$$\text{are} \quad \cos\frac{\pi}{7}, \cos\frac{3\pi}{7}, \cos\frac{5\pi}{7} \quad \text{...(i)}$$

Put $x = \frac{1}{y}$ in Eq. (i) (i.e. $y = \sec\theta$), then

$\sec\frac{\pi}{7}, \sec\frac{3\pi}{7}, \sec\frac{5\pi}{7}$ are the roots of the equation.

$$\frac{8}{y^3} - \frac{4}{y^2} - \frac{4}{y} + 1 = 0$$

$$\Rightarrow \quad y^3 - 4y^2 - 4y + 8 = 0$$

$$\Rightarrow \quad \sec\frac{\pi}{7} + \sec\frac{3\pi}{7} + \sec\frac{5\pi}{7} = 4$$

Again putting $\frac{1}{x^2} = y$ in Eq. (i)

(i.e. $y = \sec^2\theta$)

$$\frac{8}{y^{3/2}} - \frac{4}{y} - \frac{4}{y^{1/2}} + 1 = 0$$

$$\Rightarrow \quad 8 - 4y^{1/2} - 4y + y^{3/2} = 0$$

$$\Rightarrow \quad y^{1/2}(y-4) = 4(y-2)$$

$$\Rightarrow \quad y(y-4)^2 = 16(y-2)^2$$

$$\Rightarrow \quad y^3 - 24y^2 + 80y - 64 = 0 \qquad \text{...(ii)}$$

Hence, the roots are

$$\sec^2\frac{\pi}{7}, \sec^2\frac{3\pi}{7}, \sec^2\frac{5\pi}{7}$$

Now, putting $y = 1 + z$, (i.e. $z = \tan^2\theta$)

We have,

$$(1+z)^3 - 24(1+z)^2 + 80(1+z) - 64 = 0$$

$$\Rightarrow \quad z^3 - 21z^2 + 35z - 7 = 0$$

whose roots are $\tan^2\frac{\pi}{7}, \tan^2\frac{3\pi}{7}, \tan^2\frac{5\pi}{7}$.

149. We have, $2(\cos\beta - \cos\alpha) + \cos\alpha\cos\beta = 1$

or $4(\cos\beta - \cos\alpha) + 2\cos\alpha\cos\beta = 2$

$$\Rightarrow \quad 1 - \cos\alpha + \cos\beta - \cos\alpha\cos\beta = 3 + 3\cos\alpha - 3\cos\beta - 3\cos\alpha\cos\beta$$

$$\Rightarrow \quad (1-\cos\alpha)(1+\cos\beta) = 3(1+\cos\alpha)(1-\cos\beta)$$

$$\Rightarrow \quad \frac{(1-\cos\alpha)}{(1+\cos\alpha)} = \frac{3(1-\cos\beta)}{1+\cos\beta}$$

$$\Rightarrow \quad \tan^2\frac{\alpha}{2} = 3\tan^2\frac{\beta}{2}$$

$$\therefore \quad \tan\frac{\alpha}{2} \pm \sqrt{3}\tan\frac{\beta}{2} = 0$$

150. Here, $x^2 - 2x\sec\theta + 1 = 0$ has roots α_1 and β_1.

$$\therefore \quad \alpha_1, \beta_1 = \frac{2\sec\theta \pm \sqrt{4\sec^2\theta - 4}}{2\times 1} = \frac{2\sec\theta \pm 2|\tan\theta|}{2}$$

Since, $\theta \in \left(-\frac{\pi}{6}, -\frac{\pi}{12}\right)$,

i.e. $\theta \in$ IV quadrant $= \frac{2\sec\theta \mp 2\tan\theta}{2}$

$\therefore \quad \alpha_1 = \sec\theta - \tan\theta$ and $\beta_1 = \sec\theta + \tan\theta$

[as $\alpha_1 > \beta_1$]

and $x^2 + 2x\tan\theta - 1 = 0$ has roots α_2 and β_2.

i.e. $\alpha_2, \beta_2 = \frac{-2\tan\theta \pm \sqrt{4\tan^2\theta + 4}}{2}$

$\therefore \quad \alpha_2 = -\tan\theta + \sec\theta$

and $\beta_2 = -\tan\theta - \sec\theta$ [as $\alpha_2 > \beta_2$]

Thus, $\alpha_1 + \beta_2 = -2\tan\theta$

151. Here, $\displaystyle\sum_{k=1}^{13} \frac{1}{\sin\left\{\frac{\pi}{4} + \frac{(k-1)\pi}{6}\right\}\sin\left(\frac{\pi}{4} + \frac{k\pi}{6}\right)}$

Converting into differences, by multiplying and dividing by $\sin\left[\left(\frac{\pi}{4} + \frac{k\pi}{6}\right) - \left\{\frac{\pi}{4} + \frac{(k-1)\pi}{6}\right\}\right]$, i.e. $\sin\left(\frac{\pi}{6}\right)$.

$$\therefore \sum_{k=1}^{13} \frac{\sin\left[\left(\frac{\pi}{4} + k\frac{\pi}{6}\right) - \left\{\frac{\pi}{4} + (k-1)\frac{\pi}{6}\right\}\right]}{\sin\frac{\pi}{6}\left\{\sin\left\{\frac{\pi}{4} + (k-1)\frac{\pi}{6}\right\}\sin\left(\frac{\pi}{4} + k\frac{\pi}{6}\right)\right\}}$$

$$= 2\sum_{k=1}^{13} \frac{\left[\begin{array}{l}\sin\left(\frac{\pi}{4} + \frac{k\pi}{6}\right)\cos\left\{\frac{\pi}{4} + (k-1)\frac{\pi}{6}\right\} \\ -\sin\left\{\frac{\pi}{4} + (k-1)\frac{\pi}{6}\right\}\cos\left(\frac{\pi}{4} + \frac{k\pi}{6}\right)\end{array}\right]}{\sin\left\{\frac{\pi}{4} + (k-1)\frac{\pi}{6}\right\}\sin\left(\frac{\pi}{4} + k\frac{\pi}{6}\right)}$$

$$= 2\sum_{k=1}^{13}\left[\cot\left\{\frac{\pi}{4} + (k-1)\frac{\pi}{6}\right\} - \cot\left(\frac{\pi}{4} + k\frac{\pi}{6}\right)\right]$$

$$= 2\left[\left\{\cot\left(\frac{\pi}{4}\right) - \cot\left(\frac{\pi}{4} + \frac{\pi}{6}\right)\right\} + \left\{\cot\left(\frac{\pi}{4} + \frac{\pi}{6}\right) - \cot\left(\frac{\pi}{4} + \frac{2\pi}{6}\right)\right\} + \ldots + \left\{\cot\left(\frac{\pi}{4} + 12\frac{\pi}{6}\right) - \cot\left(\frac{\pi}{4} + 13\frac{\pi}{6}\right)\right\}\right]$$

$$= 2\left\{\cot\frac{\pi}{4} - \cot\left(\frac{\pi}{4} + 13\frac{\pi}{6}\right)\right\}$$

$$= 2\left[1 - \cot\left(\frac{29\pi}{12}\right)\right] = 2\left[1 - \cot\left(2\pi + \frac{5\pi}{12}\right)\right]$$

$$= 2\left[1 - \cot\frac{5\pi}{12}\right] \qquad \left[\because \cot\frac{5\pi}{12} = (2-\sqrt{3})\right]$$

$$= 2(1 - 2 + \sqrt{3})$$

$$= 2(\sqrt{3} - 1)$$

152. $f(\cos 4\theta) = \frac{2}{2 - \sec^2\theta}$ ...(i)

At $\cos 4\theta = \frac{1}{3} \Rightarrow 2\cos^2 2\theta - 1 = \frac{1}{3}$

$$\Rightarrow \quad \cos^2 2\theta = \frac{2}{3} \Rightarrow \cos 2\theta = \pm\sqrt{\frac{2}{3}} \qquad \text{...(ii)}$$

$$\therefore \quad f(\cos 4\theta) = \frac{2\cdot\cos^2\theta}{2\cos^2\theta - 1} = \frac{1 + \cos 2\theta}{\cos 2\theta}$$

$$\Rightarrow \quad f\left(\frac{1}{3}\right) = 1 \pm \sqrt{\frac{3}{2}} \qquad \text{[from Eq. (ii)]}$$

153. Given equations can be written as

$$x\sin 3\theta - \frac{\cos 3\theta}{y} - \frac{\cos 3\theta}{z} = 0 \quad \text{...(i)}$$

$$x\sin 3\theta - \frac{2\cos 3\theta}{y} - \frac{2\sin 3\theta}{z} = 0 \quad \text{...(ii)}$$

and $x\sin 3\theta - \frac{2}{y}\cos 3\theta - \frac{1}{z}(\cos 3\theta + \sin 3\theta) = 0$ ...(iii)

Eqs. (ii) and (iii), implies

$$2\sin 3\theta = \cos 3\theta + \sin 3\theta \Rightarrow \sin 3\theta = \cos 3\theta$$

$\therefore \quad \tan 3\theta = 1$

$\Rightarrow \quad 3\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}$ or $\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{9\pi}{12}$

154. For $0 < \theta < \frac{\pi}{2}$

$$\sum_{m=1}^{6} \operatorname{cosec}\left(\theta + \frac{(m-1)\pi}{4}\right)\operatorname{cosec}\left(\theta + \frac{m\pi}{4}\right) = 4\sqrt{2}$$

$$\Rightarrow \sum_{m=1}^{6} \frac{1}{\sin\left(\theta + \frac{(m-1)\pi}{4}\right)\sin\left(\theta + \frac{m\pi}{4}\right)} = 4\sqrt{2}$$

$$\Rightarrow \sum_{m=1}^{6} \frac{\sin\left[\theta + \frac{m\pi}{4} - \left(\theta + \frac{(m-1)\pi}{4}\right)\right]}{\sin\frac{\pi}{4}\left\{\sin\left(\theta + \frac{(m-1)\pi}{4}\right)\sin\left(\theta + \frac{m\pi}{4}\right)\right\}} = 4\sqrt{2}$$

$$\Rightarrow \sum_{m=1}^{6} \frac{\cot\left(\theta + \frac{(m-1)\pi}{4}\right) - \cot\left(\theta + \frac{m\pi}{4}\right)}{1/\sqrt{2}} = 4\sqrt{2}$$

$$\Rightarrow \sum_{m=1}^{6} \left[\cot\left(\theta + \frac{(m-1)\pi}{4}\right) - \cot\left(\theta + \frac{m\pi}{4}\right)\right] = 4$$

$$\Rightarrow \cot(\theta) - \cot\left(\theta + \frac{\pi}{4}\right) + \cot\left(\theta + \frac{\pi}{4}\right) - \cot\left(\theta + \frac{2\pi}{4}\right)$$

$$+ \dots + \cot\left(\theta + \frac{5\pi}{4}\right) - \cot\left(\theta + \frac{6\pi}{4}\right) = 4$$

$$\Rightarrow \cot\theta - \cot\left(\frac{3\pi}{2} + \theta\right) = 4$$

$$\Rightarrow \cot\theta + \tan\theta = 4$$

$$\Rightarrow \tan^2\theta - 4\tan\theta + 1 = 0$$

$$\Rightarrow (\tan\theta - 2)^2 - 3 = 0$$

$$\Rightarrow (\tan\theta - 2 + \sqrt{3})(\tan\theta - 2 - \sqrt{3}) = 0$$

$$\Rightarrow \tan\theta = 2 - \sqrt{3}$$

or $\tan\theta = 2 + \sqrt{3}$

$$\Rightarrow \theta = \frac{\pi}{12};\ \theta = \frac{5\pi}{12} \quad \left[\because \theta \in \left(0, \frac{\pi}{2}\right)\right]$$

155. $\frac{\sin^4 x}{2} + \frac{\cos^4 x}{3} = \frac{1}{5}$

$$\Rightarrow \frac{\sin^4 x}{2} + \frac{(1 - \sin^2 x)^2}{3} = \frac{1}{5}$$

$$\Rightarrow \frac{\sin^4 x}{2} + \frac{1 + \sin^4 x - 2\sin^2 x}{3} = \frac{1}{5}$$

$$\Rightarrow 5\sin^4 x - 4\sin^2 x + 2 = \frac{6}{5}$$

$$\Rightarrow 25\sin^4 x - 20\sin^2 x + 4 = 0$$

$$\Rightarrow (5\sin^2 x - 2)^2 = 0$$

$$\Rightarrow \sin^2 x = \frac{2}{5}$$

$$\cos^2 x = \frac{3}{5}, \tan^2 x = \frac{2}{3}$$

$$\therefore \frac{\sin^8 x}{8} + \frac{\cos^8 x}{27} = \frac{1}{125}$$

156. As when $\theta \in \left(0, \frac{\pi}{4}\right)$, $\tan\theta < \cot\theta$

Since, $\tan\theta < 1$ and $\cot\theta > 1$

$\therefore \quad (\tan\theta)^{\cot\theta} < 1$ and $(\cot\theta)^{\tan\theta} > 1$

$\therefore$ $t_4 > t_1$ which only holds in (b).

Therefore, (b) is the answer.

157. Since, $\cos(\alpha - \beta) = 1$

$\Rightarrow \quad \alpha - \beta = 2n\pi$

But $\quad -2\pi < \alpha - \beta < 2\pi \quad$ [as $\alpha, \beta \in (-\pi, \pi)$]

$\therefore \quad \alpha - \beta = 0$ ...(i)

Given, $\cos(\alpha + \beta) = \frac{1}{e}$

$\Rightarrow \cos 2\alpha = \frac{1}{e} < 1$, which is true for four values of α.

[as $-2\pi < 2\alpha < 2\pi$]

158. Given, $5(\tan^2 x - \cos^2 x) = 2\cos 2x + 9$

$$\Rightarrow 5\left(\frac{1 - \cos 2x}{1 + \cos 2x} - \frac{1 + \cos 2x}{2}\right) = 2\cos 2x + 9$$

Put $\cos 2x = y$, we have

$$5\left(\frac{1 - y}{1 + y} - \frac{1 + y}{2}\right) = 2y + 9$$

$$\Rightarrow 5(2 - 2y - 1 - y^2 - 2y) = 2(1 + y)(2y + 9)$$

$$\Rightarrow 5(1 - 4y - y^2) = 2(2y + 9 + 2y^2 + 9y)$$

$$\Rightarrow 5 - 20y - 5y^2 = 22y + 18 + 4y^2$$

$$\Rightarrow 9y^2 + 42y + 13 = 0$$

$$\Rightarrow 9y^2 + 3y + 39y + 13 = 0$$

$$\Rightarrow 3y(3y + 1) + 13(3y + 1) = 0$$

$$\Rightarrow (3y + 1)(3y + 13) = 0$$

$$\Rightarrow y = -\frac{1}{3}, -\frac{13}{3}$$

$$\therefore \cos 2x = -\frac{1}{3}, -\frac{13}{3}$$

$$\Rightarrow \cos 2x = -\frac{1}{3} \quad \left[\because \cos 2x \neq -\frac{13}{3}\right]$$

Now, $\cos 4x = 2\cos^2 2x - 1 = 2\left(-\frac{1}{3}\right)^2 - 1$

$$= \frac{2}{9} - 1 = -\frac{7}{9}$$

159. $f_k(x) = \frac{1}{k}(\sin^k x + \cos^k x)$, where $x \in R$ and $k \geq 1$

Now, $f_4(x) - f_6(x)$

$$= \frac{1}{4}(\sin^4 x + \cos^4 x) - \frac{1}{6}(\sin^6 x + \cos^6 x)$$
$$= \frac{1}{4}(1 - 2\sin^2 x \cdot \cos^2 x) - \frac{1}{6}(1 - 3\sin^2 x \cdot \cos^2 x)$$
$$= \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$$

160. Given expression is

$$\frac{\tan A}{1 - \cot A} + \frac{\cot A}{1 - \tan A} = \frac{\sin A}{\cos A} \times \frac{\sin A}{\sin A - \cos A} + \frac{\cos A}{\sin A} \times \frac{\cos A}{\cos A - \sin A}$$
$$= \frac{1}{\sin A - \cos A}\left\{\frac{\sin^3 A - \cos^3 A}{\cos A \sin A}\right\}$$
$$= \frac{\sin^2 A + \sin A \cos A + \cos^2 A}{\sin A \cos A}$$
$$= \frac{1 + \sin A \cos A}{\sin A \cos A}$$
$$= 1 + \sec A \operatorname{cosec} A$$

161. **Given** A ΔPQR such that

$$3 \sin P + 4 \cos Q = 6 \quad \text{...(i)}$$
$$4 \sin Q + 3 \cos P = 1 \quad \text{...(ii)}$$

On squaring and adding Eqs. (i) and (ii) both sides we get

$$(3 \sin P + 4 \cos Q)^2 + (4 \sin Q + 3 \cos P)^2 = 36 + 1$$
$$\Rightarrow 9(\sin^2 P + \cos^2 P) + 16(\sin^2 Q + \cos^2 Q) + 2 \times 3 \times 4(\sin P \cos Q + \sin Q \cos P) = 37$$
$$\Rightarrow 24[\sin(P + Q)] = 37 - 25$$
$$\Rightarrow \sin(P + Q) = \frac{1}{2}$$

Since, P and Q are angles of ΔPQR, hence $0° < P, Q < 180°$.

$$\Rightarrow P + Q = 30° \text{ or } 150°$$
$$\Rightarrow R = 150° \text{ or } 30°$$

Hence, two cases arise here.

***Case* I** $R = 150°$

$$R = 150° \Rightarrow P + Q = 30°$$
$$\Rightarrow 0 < P, Q < 30°$$
$$\Rightarrow \sin P < \frac{1}{2}, \cos Q < 1$$
$$\Rightarrow 3 \sin P + 4 \cos Q < \frac{3}{2} + 4$$
$$\Rightarrow 3 \sin P + 4 \cos Q < \frac{11}{2} < 6$$

$\Rightarrow 3 \sin P + 4 \cos Q \Rightarrow 6$ is not possible.

***Case* II** $R = 30°$

Hence, $R = 30°$ is the only possibility.

162. $A = \sin^2 x + \cos^4 x$

$$\Rightarrow A = 1 - \cos^2 x + \cos^4 x$$
$$= \cos^4 x - \cos^2 x + \frac{1}{4} + \frac{3}{4}$$
$$= \left(\cos^2 x - \frac{1}{2}\right)^2 + \frac{3}{4} \quad \text{...(i)}$$

where, $0 \leq \left(\cos^2 x - \frac{1}{2}\right)^2 \leq \frac{1}{4}$ ...(ii)

$$\therefore \frac{3}{4} \leq A \leq 1$$

163. $\cos(\alpha + \beta) = \frac{4}{5} \Rightarrow \alpha + \beta \in$ Ist quadrant

and $\sin(\alpha - \beta) = \frac{5}{13} \Rightarrow \alpha - \beta \in$ Ist quadrant

Now, $2\alpha = (\alpha + \beta) + (\alpha - \beta)$

$$\therefore \tan 2\alpha = \frac{\tan(\alpha + \beta) + \tan(\alpha - \beta)}{1 - \tan(\alpha + \beta)\tan(\alpha - \beta)} = \frac{\frac{3}{4} + \frac{5}{12}}{1 - \frac{3}{4} \cdot \frac{5}{12}} = \frac{56}{33}$$

164. $\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta) = -\frac{3}{2}$

$$\Rightarrow 2[\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta)] + 3 = 0$$
$$\Rightarrow 2[\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta)] + \sin^2\alpha + \cos^2\alpha + \sin^2\beta + \cos^2\beta + \sin^2\gamma + \cos^2\gamma = 0$$
$$\Rightarrow (\sin\alpha + \sin\beta + \sin\gamma)^2 + (\cos\alpha + \cos\beta + \cos\gamma)^2 = 0$$

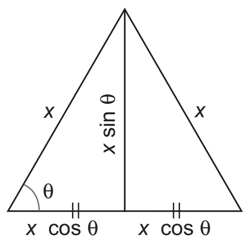
It is possible when,

$$\sin\alpha + \sin\beta + \sin\gamma = 0$$

and $\cos\alpha + \cos\beta + \cos\gamma = 0$

Hence, both statements A and B are true.

165. Area $= \frac{1}{2} \times$ Base $\times$ Altitude



$$= \frac{1}{2} \times (2x \cos\theta) \times (x \sin\theta) = \frac{1}{2}x^2 \sin 2\theta$$

[since, maximum value of $\sin 2\theta$ is 1]

$\therefore$ Maximum area $= \frac{1}{2}x^2$

166. Given, $\cos x + \sin x = \frac{1}{2}$

$$\therefore \frac{1 - \tan^2\frac{x}{2}}{1 + \tan^2\frac{x}{2}} + \frac{2\tan\frac{x}{2}}{1 + \tan^2\frac{x}{2}} = \frac{1}{2}$$

Let $\tan\frac{x}{2} = t \Rightarrow \frac{1 - t^2}{1 + t^2} + \frac{2t}{1 + t^2} = \frac{1}{2}$

$$\Rightarrow \quad 2(1 - t^2 + 2t) = 1 + t^2 \Rightarrow 3t^2 - 4t - 1 = 0$$

$$\Rightarrow \quad t = \frac{2 \pm \sqrt{7}}{3}$$

As $\quad 0 < x < \pi \Rightarrow 0 < \frac{x}{2} < \frac{\pi}{2}$

So, $\tan \frac{x}{2}$ is positive.

$$\therefore \quad t = \tan \frac{x}{2} = \frac{2 + \sqrt{7}}{3}$$

Now, $\quad \tan x = \dfrac{2 \tan \frac{x}{2}}{1 - \tan^2 \frac{x}{2}} = \dfrac{2t}{1 - t^2}$

$$\Rightarrow \quad \tan x = \frac{2\left(\frac{2 + \sqrt{7}}{3}\right)}{1 - \left(\frac{2 + \sqrt{7}}{3}\right)^2}$$

$$\Rightarrow \quad \tan x = \frac{-3(2 + \sqrt{7})}{1 + 2\sqrt{7}} \times \frac{1 - 2\sqrt{7}}{1 - 2\sqrt{7}}$$

$$\Rightarrow \quad \tan x = -\left(\frac{4 + \sqrt{7}}{3}\right)$$

167. Since, $\tan \frac{P}{2}$ and $\tan \frac{Q}{2}$ are the roots of equation

$$ax^2 + bx + c = 0$$

$$\therefore \quad \tan \frac{P}{2} + \tan \frac{Q}{2} = -\frac{b}{a} \quad \text{...(i)}$$

and $\quad \tan \frac{P}{2} \tan \frac{Q}{2} = \frac{c}{a}$

Also, $\quad \frac{P}{2} + \frac{Q}{2} + \frac{R}{2} = \frac{\pi}{2} \quad [\because P + Q + R = \pi]$

$$\Rightarrow \quad \frac{P + Q}{2} = \frac{\pi}{2} - \frac{R}{2}$$

$$\Rightarrow \quad \frac{P + Q}{2} = \frac{\pi}{4} \quad \left[\because \angle R = \frac{\pi}{2} \text{ (given)}\right]$$

$$\Rightarrow \quad \tan\left(\frac{P}{2} + \frac{Q}{2}\right) = 1$$

$$\Rightarrow \quad \frac{\tan \frac{P}{2} + \tan \frac{Q}{2}}{1 - \tan \frac{P}{2} \tan \frac{Q}{2}} = 1$$

$$\Rightarrow \quad \frac{-\frac{b}{a}}{1 - \frac{c}{a}} = 1$$

$$\Rightarrow \quad -\frac{b}{a} = 1 - \frac{c}{a} \quad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad -b = a - c$$

$$\Rightarrow \quad c = a + b$$

Alternate Solution

Since, $\quad \angle R = \frac{\pi}{2}$

$$\Rightarrow \quad \angle P + \angle Q = \frac{\pi}{2}$$

$$\Rightarrow \quad \frac{\angle P}{2} = \frac{\pi}{4} - \frac{\angle Q}{2}$$

$$\therefore \quad \tan \frac{P}{2} = \tan\left(\frac{\pi}{4} - \frac{Q}{2}\right)$$

$$= \frac{\tan \frac{\pi}{4} - \tan \frac{Q}{2}}{1 + \tan \frac{\pi}{4} \tan \frac{Q}{2}}$$

$$\Rightarrow \quad \tan \frac{P}{2} + \tan \frac{P}{2} \tan \frac{Q}{2} = 1 - \tan \frac{Q}{2}$$

$$\Rightarrow \quad \tan \frac{P}{2} + \tan \frac{Q}{2} = 1 - \tan \frac{P}{2} \tan \frac{Q}{2}$$

$$\Rightarrow \quad -\frac{b}{a} = 1 - \frac{c}{a}$$

$$\Rightarrow \quad -b = a - c$$

$$\Rightarrow \quad c = a + b$$