4.1 Introduction

What is relativity?

Consider a train moving with a speed of 60 km/hour. The train is observed by three observers.

(i) The first observer is standing at the station.

(ii) The second observer is moving in the direction of the train with a velocity of 20 km/hour.

(iii) The third observer is moving with a velocity of 20 km/hour in the opposite direction of the train.

The observations of the three observers are different as follows :

(i) The first observer would observe the velocity of the train as 60 km/hour.

(ii) The second observer would observe the velocity of the train as (60 – 20 =) 40 km/hour.

(iii) The third observer would observe the velocity of the train as (60 + 20 =) 80 km/hour.

The oldest theory of Physics is the Classical Physics or Newtonian Physics that deals with the absolute motion of an object considering space and time to be absolute and two separate entities. However, this concept failed to explain the motion with high velocities, very close to the velocity of light.

The development of theory of relativity by Einstein in 1905 revolutionalized the old concepts. It discards the concept of absolute motion and deals with objects and observers moving with high velocities ($\sim$ c) and relative velocities with respect to each other. This theory was developed in two steps and thus are divided into two parts.

(i) Einstein's Classical Theory of Relativity based on Classical Physics, *i.e.*, Newtonian mechanics.

(ii) Einstein's Special Theory of Relativity applicable to all laws of Physics.

4.2 Einsten's Classical Theory of Relativity (Newtonian Theory of Relativity)

Einstein initially developed his theory of relativity for classical physics, *i.e.*, Newtonian Mechanics. This is called Einstein's classical theory of relativity.

4.2.1 : Frame of Reference

The motion of an object can be described only with the help of a coordinate system. The coordinate system in such cases is known as the frame of reference. There are two types of frame of reference.

(1) **Inertial frame of reference or unaccelerated frame**

A frame of reference is said to be inertial when objects in this frame obey Newton's law of inertia and other laws of Newtonian mechanics. In this frame an object is not acted upon by an external force. It is at rest or moves with a constant velocity.

(2) **Non inertial frame**

A frame of reference which is in an accelerated motion with respected to an inertial frame of reference is called a non-inertial frame of reference. In such frame an object even without an external force acting on it, is accelerated. In non-inertial frame the Newton's laws are not valid.

Example : A ball placed the floor of a train moves to the rear if the train accelerates forward even though no forces act on it. In this case, the train moves in an inertial frame of reference and the ball is in a non-inertial frame of reference.

4.2.2 : Galilean Transformations

The transformation from one inertial frame of reference to another is called Galilean transformation. Knowing the laws of motion of an object in a reference system S, the laws of motion of the same object in another reference system S' can be derived.

Let us consider a physical event. An event is something that happens without depending on the reference frame used to describe it. Suppose a collision of two particles occur at a point (x, y, z) at an instant of t secs. We describe this event by the coordinates (x, y, z, t) in one frame of reference, say, in a laboratory on the earth. The same event observed from a different reference frame, *e.g.*, from an aircraft flying overhead would also be specified by a set of four coordinates in space and time (x', y', z', t') which is different from the earlier set of (x, y, z, t).

Consider now two observers O and P, where P travels with a constant velocity 'v' with respect to O along their common X-X' axis. Here E is the event specified by coordinates (x, y, z, t) and (x', y', z', t') in frames S and S' respectively.

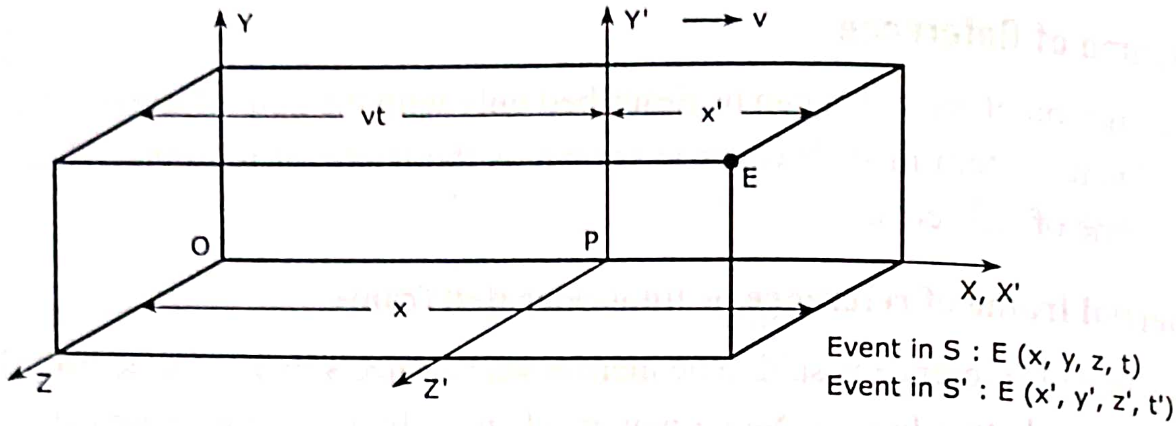

Fig. 4.1 : Frame of reference

(a) Galilean Coordinate Transformations

From Fig. 4.1, it is observed that

$$x' = x - vt, \;\; y' = y, \;\; z' = z \;\text{ and }\; t' = t \qquad \text{.................... (4.1}$$

These four equations are called Galilean coordinates transformations.

(b) Galilean Velocity Transformations

The velocity coordinates of the object in event E can be assigned as (u_x, u_y, u_z) an (u_x', u_y', u_z') in frame S and in frame S' respectively. Then from equation (4.1), it can b written as

$$u_x' = \frac{dx'}{dt'} = \frac{d}{dt}(x - vt)\frac{dt}{dt'} = \frac{dx}{dt} - v = u_x - v \qquad \text{as } \frac{dt}{dt'} = 1$$

Altogether, the Galilean velocity transformation are

$$u_x' = u_x - v, \quad u_y' = v_y, \quad u_z' = v_z \qquad \text{.................... (4.2}$$

(c) Galilean Acceleration Transformation

In inertial frames of reference S and S', the acceleration components remain th same. Thus,

$$a_x' = a_x, \quad a_y' = a_y, \quad a_z' = a_z \qquad \text{.................... (4.3}$$

4.3 Einstein's Special Theory of Relativity

Einstein observed that his Classical Theory of Relativity fails for very high spee (v ~ c) particles. This is due to the fact that in Newtonian mechanics, there is no limit, i principle, to the allowed speed of a particle. In 1905, he extended his Classical Theory o Relativity to include all the laws of Physics and Special Theory of Relativity was developed

The special theory of relativity deals with the problems in which one frame of reference moves with a constant linear velocity relative to another frame of reference.

4.3.1 : Postulates of Special Theory of Relativity

Einstein in his Special Theory of Relativity postulated that

(i) **All the fundamental laws of physics retain the same form in all the inertial frames of reference.**

(ii) **The velocity of light in free space is constant and is independent of the relative motion of the source and the observer in any frame of reference.**

4.3.2 : Einstein proved the following facts based on his theory of relativity

Let v be the velocity of a spaceship with respect to a given frame of reference where an observer makes his observations.

(a) All clocks on the spaceship will go slow by a factor $\sqrt{1-(v^2/c^2)}$.

(b) The mass of the spaceship increases by a factor $\left[1-(v^2/c^2)\right]^{-1/2}$.

(c) All objects on the spaceship will be contracted by a factor $\sqrt{1-(v^2/c^2)}$.

(d) The speed of a material object can never exceed the velocity of light.

(e) Mass and energy are interconvertible,

$$E = mc^2$$

(f) If two objects A and B are moving with velocities u and v respectively along the X-axis, the relative velocity of A with respect to B is given by

$$v_R = \frac{u-v}{1-(uv/c^2)}$$

Here, u and v are both comparable with the value of c.

4.3.3 : Lorentz Transformation of Space and Time

In Newtonian mechanics, the Galilean transformations expressed in equations (4.1), (4.2) and (4.3) relate the space and time coordinates in one inertial frame to those the other frame. However, these equations are not valid for cases where the object velocity v approaches the value of c, the velocity of light. The transformation equations apply for all velocities upto c and incorporate the invariance of the speed of light were developed in 1890 by Lorentz. These are known as Lorentz transformations.

Let us consider two inertial frames S and S' as shown in the Fig. 4.2. The frame S' moves with a velocity v with respect to S in the positive X direction.

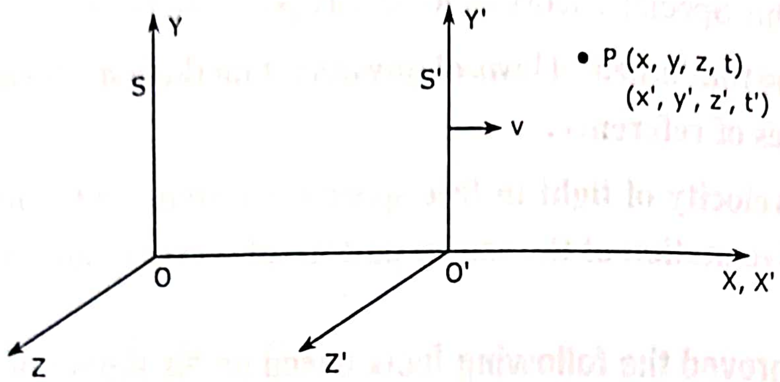

Fig. 4.2

Consider two observers O and O' situated at the origin in the frames S and S' respectively. Two coordinate systems coincide initially at the instant t = t' = 0. Suppose optical source is v kept at the common origin of the two frames. Let the source release a pulse at t = t' = 0 and at the same instant frame S' starts moving with a constant velocity v along +X direction, relative to frame S. This pulse reaches a point P with coordinates (x, y, z, t) and (x', y', z', t') in frames S and S' respectively.

Since S' is moving along +X direction with respect to S, the transformation equation of x and x' can be written as

$$x' = k(x - vt) \qquad \text{.....(4.4)}$$

where, k is the constant of proportionality.

The inverse relation can be written as,

$$x = k(x' + vt') \qquad \text{.....(4.5)}$$

Putting equation (4.4) in equation (4.5), we can write

$$x = k[k(x - vt) + vt']$$

$$\therefore \quad t' = kt - \frac{kx}{v}\left(1 - \frac{1}{k^2}\right) \qquad \text{.....(4.6)}$$

Now, according to the second postulate of relativity, the speed of light c remains constant. So the velocity of the light pulse spreading out from the common origin observed by observers O and O' should be the same

$$x = ct \qquad \text{.................... (4.7)}$$

$$x' = ct'$$

Substituting equation (4.7) in equation (4.4) and equation (4.5), we have

$$ct' = k(c - v)t \qquad \text{.................... (4.8)}$$

and

$$ct = k(c + v)t' \qquad \text{.................... (4.9)}$$

Multiplying equations (4.8) with equation (4.9), we have .

$$\therefore \qquad k^2 = \frac{c^2}{c^2 - v^2}$$

$$\therefore \qquad k = \pm\frac{1}{\sqrt{1-(v^2/c^2)}} \qquad \text{.................... (4.10)}$$

and

$$1 - \frac{1}{k^2} = \frac{v^2}{c^2}$$

Using equations (4.10) in equation (4.4), we have

$$x' = \frac{x - vt}{\sqrt{1-(v^2/c^2)}} \qquad \text{.................... (4.11)}$$

Substituting equations (4.10) and (4.11), we have

$$t' = \frac{t-(xv/c^2)}{\sqrt{1-(v^2/c^2)}} \qquad \text{.................... (4.12)}$$

Hence, if the frame S' moves with a velocity v in +X direction with respect to the frame S, the transformation equations are,

$$x' = \frac{x - vt}{\sqrt{1-(v^2/c^2)}}, \quad y' = y, \quad z' = z, \quad t' = \frac{t-(xv/c^2)}{\sqrt{1-(v^2/c^2)}} \qquad \text{.............. (4.13)}$$

On the otherhand, if the frame S moves with a velocity v in – X direction with respect to the frame S', we get the inverse transformation equations as

$$x = \frac{x' + vt'}{\sqrt{1-(v^2/c^2)}}, \quad y = y', \quad z = z', \quad t = \frac{t'-(x'v/c^2)}{\sqrt{1-(v^2/c^2)}} \qquad \text{.............. (4.14)}$$

If the speed of the moving frame is much smaller than the velocity of light, *i.e.*, if $v \ll c$, the Lorentz transformation equations reduce to Galilean transformation equations.

4.4 Time Dilation

The meaning of time dilation is extension of time. Time dilation is a difference in the elapsed time measured by two clocks due to a relative motion between them. To explain it let us consider two frames of reference S and S' with S' moving with a velocity v along X direction with respect to S as shown in Fig. 4.3. Imagine a gun placed at a fixed position P (x', y', z') in the frame S'. Suppose it fires two shots at instants t_1' and t_2' measured by the observer O' in the frame S'.

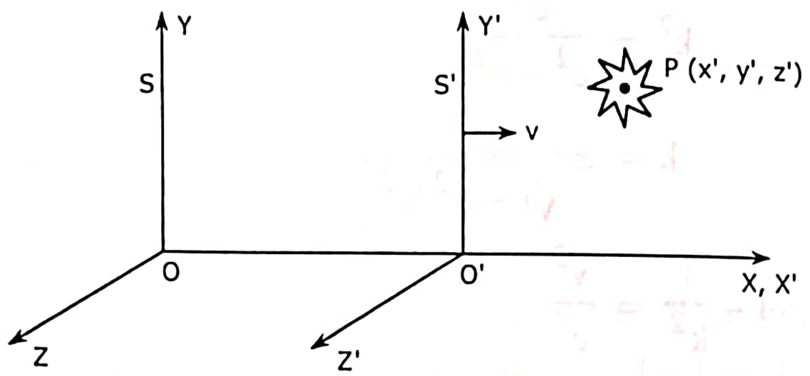

Fig. 4.3 : Time dilation

The time interval $(t_2' - t_1')$ of the two shots measured by O' at rest in the moving frame S' is called the proper time interval and is given by

$$T_O = t_2' - t_1' \qquad \text{.................... (4.15)}$$

As the motion between the two frames is relative, we may assume that the frame S is moving with velocity – v along the – X direction relative to frame S'. In frame S, the observer O who is at rest hears these two shots at different times t_1 and t_2.

The time interval appears to him is given by

$$t = t_2 - t_1 \qquad \text{.................... (4.16)}$$

From inverse Lorentz transformation equations, we get

$$t_1 = \frac{t_1' + (vx'/c^2)}{\sqrt{1-(v^2/c^2)}} \qquad \text{.................... (4.17)}$$

$$t_2 = \frac{t_2' + (vx'/c^2)}{\sqrt{1-(v^2/c^2)}} \qquad \text{.................... (4.18)}$$

Substituting equations (4.17) and (4.18) in equation (4.16), we get

$$T = \frac{t_2' - t_1'}{\sqrt{1-(v^2/c^2)}} \qquad \text{.................... (4.19)}$$

Using equation (4.15) in equation (4.19), we have

$$T = \frac{T_0}{\sqrt{1-(v^2/c^2)}} \qquad \text{.................... (4.20)}$$

which shows that $T > T_0$.

Here, T_o is called the **proper time which is defined as the time measured in the frame of reference in which the object is at rest.**

This verifies that the actual time interval in the moving frame appears to be lengthened by a factor $\frac{1}{\sqrt{v^2 - c^2}}$ when it is measured by an observer in the fixed frame, v being the relative velocity between the two frames.

4.5 Length Contraction

In classical mechanics the length of an object is independent of the velocity of the observer moving relative to the object. However, in the theory of relativity, the length of an object depends on the relative velocity between the observer and the object.

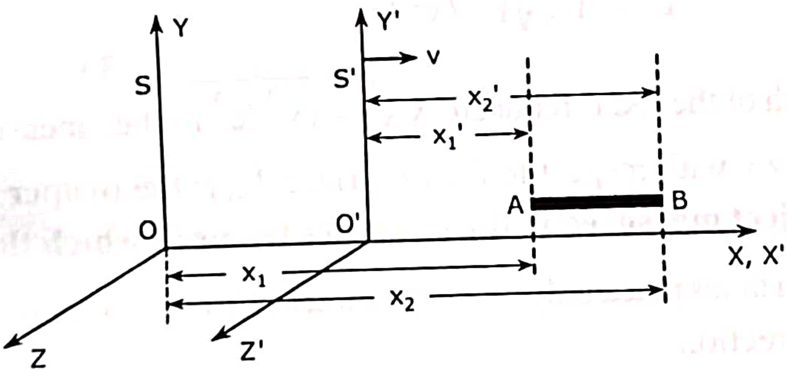

Fig. 4.4

To explain this, let us consider two inertial frames S and S' with S' moving with a velocity v in the X direction with respect to S.

Let a rod AB be at rest the moving frame S'. Its actual length is L_o at any instant as measured by the observer O' also at rest in the frame S'. So,

$$L_o = x_2' - x_1' \qquad \text{.................... (4.21)}$$

where, x_1' and x_2' are the x coordinates of the rod in frame S' as shown in the Fig. 4.4.

At the same time, the length of AB measured by an observer O in the stationary frame S is given by

$$L = x_2 - x_1 \qquad \text{.................... (4.22)}$$

x_1 and x_2 being the x coordinates of the rod in frame S.

From Lorentz transformation,

$$x_1' = \frac{x_1 - vt}{\sqrt{1-(v^2/c^2)}} \qquad \text{.................... (4.23)}$$

$$x_2' = \frac{x_2 - vt}{\sqrt{1-(v^2/c^2)}} \qquad \text{.................... (4.24)}$$

Substituting equations (4.23) and (4.24) in equation (4.21), we get the actual length as

$$L_o = \frac{x_2 - x_1}{\sqrt{1-(v^2/c^2)}} \qquad \text{.................... (4.25)}$$

Using equation (4.22) in equation (4.25), we have

$$L_o = \frac{L}{\sqrt{1-(v^2/c^2)}}$$

$$\therefore \qquad L = L_o\sqrt{1-(v^2/c^2)} \qquad \text{.................... (4.26)}$$

Thus, the length of the rod is reduced by $\sqrt{1-(v^2/c^2)}$ when measured by an observer moving with velocity v with respect to the rod. Here, L_o is the **proper length defined as the length of the object measured in the reference frame in which the object is at rest.**

The contraction takes place only along the direction of motion and remains unchanged in a perpendicular direction.

4.6 Einstein's Mass-Energy Relation

In classical mechanics, the mass of a particle is independent of its velocity but in Einstein's special theory of relativity, the mass of a moving object depends upon its velocity and is given by

$$m = \frac{m_o}{\sqrt{1-(v^2/c^2)}}$$

where, m_o is the rest mass and v is the velocity of the moving body and c is the velocity of light.

The increase in the energy of a particle by the applications of force may be estimated by the work done on it.

If a particle is displaced by a distance dx on the application of a force F, the kinetic energy dE generated and stored in it is given by the work done,

$$dE = dW = F\,dx \qquad \text{..................... (4.27)}$$

Now, the force is defined as the time rate of change of momentum of the particle, by Newton's second law. Hence,

$$F = \frac{d(mv)}{dt} \qquad \text{..................... (4.28)}$$

Where, m is the mass of the particle and v is its velocity with which it moves on the application of the force F.

Thus, combining equations (4.27) and (4.28), we get

$$dE = \frac{d(mv)}{dt} \cdot dx$$

$$dE = \frac{dx}{dt}\, d(mv) = v\,[m\,dv + v\,dm]$$

$$\therefore \quad dE = mv\,dv + v^2\,dm \qquad \text{..................... (4.29)}$$

Again, $$m = \frac{m_0}{\sqrt{1-(v^2/c^2)}}$$

So, $$m^2 = \frac{m_0^2}{1-(v^2/c^2)}$$

$$m^2 c^2 - m^2 v^2 = m_0^2 c^2 \qquad \text{..................... (4.30)}$$

Differentiating equation (4.30), with m_0 and c constants, we have

$$2m\,dm\,c^2 - 2m\,dm\,v^2 - 2v\,dv\,m^2 = 0$$

$$\therefore \quad dm\,c^2 = v^2\,dm + mv\,dv \qquad \text{..................... (4.31)}$$

Substituting equation (4.31) in equation (4.29), we get

$$dE = dm\,c^2 \qquad \text{..................... (4.32)}$$

Showing that the **change in kinetic energy is directly proportional to the change in mass of the particle.**

From equation (4.30), it is obvious that for a rest object $v = 0$ and mass $m = m_0$, the rest mass.

If the particle moves with a velocity v, its mass become m and its kinetic energy becomes E_k. Therefore, integrating equation (4.32), we get

$$\int_0^{E_k} dE = c^2 \int_{m_o}^{m} dm$$

$$\therefore \quad E_k = c^2 (m - m_o)$$

$$\therefore \quad E_k = mc^2 - m_o c^2 \quad \text{.......... (4.33)}$$

$$\text{or} \quad mc^2 = E_k + m_o c^2 \quad \text{.......... (4.34)}$$

Here, mc^2 is the total energy, $m_o c^2$ is the rest mass energy and E_k is its kinetic energy. Hence, we write

$$E = E_k + m_o c^2 \quad \text{.......... (4.35)}$$

$$\text{and} \quad \mathbf{E = mc^2} \quad \text{.......... (4.36)}$$

Equation (4.36) is known as ***Einstein's mass-energy relation.***

4.7 Important Points to Remembers

1. Space and Time transformation relations

	Galilean transformation	*Lorentz transformation*	*Inverse Lorentz transformation*
X - coordinates	$x' = x - vt$	$x' = \dfrac{x - vt}{\sqrt{1-(v^2/c^2)}}$	$x = \dfrac{x' - vt}{\sqrt{1-(v^2/c^2)}}$
Y - coordinates	$y' = y$	$y' = y$	$y = y'$
Z - coordinates	$z' = z$	$z' = z$	$z = z'$
Time coordinate	$t' = t$	$t' = \dfrac{t - (vx/c^2)}{\sqrt{1-(v^2/c^2)}}$	$t = \dfrac{t' + (vx'/c^2)}{\sqrt{1-(v^2/c^2)}}$

2. Time dilation : $T = \dfrac{T_o}{\sqrt{1-(v^2/c^2)}}$

3. Length contraction : $L = L_o \sqrt{1-(v^2/c^2)}$

4. Einstein's mass energy relation : $E = mc^2$ and $E_k = mc^2 - m_o c^2$.