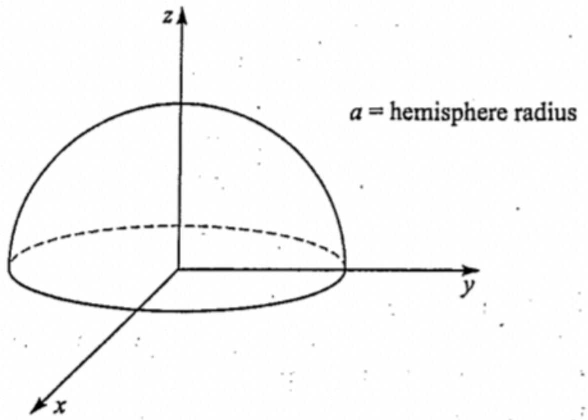
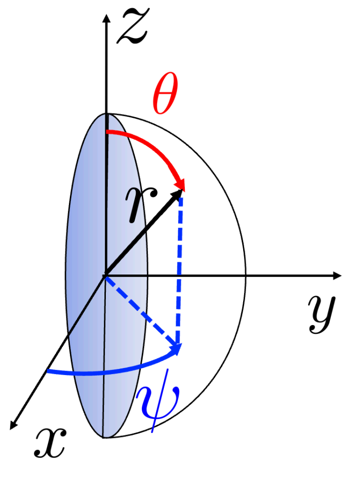
Problem from *Fundamentals of Physical Acoustics* by D. T. Blackstock.

10–13. A certain sound field inside a hollow hemisphere (radius a, all surfaces rigid) has the property that the pressure is zero along the z axis. Find the lowest two eigenfrequencies for this field, and identify their corresponding eigenfunctions ϕ_{lmn}.



Solving the wave equation for the orientation above involves eigenfunctions that feature azimuthal symmetry (i.e., no dependence on ψ), but it is difficult to satisfy the boundary conditions (specifically, that the base is rigid at $\theta = \pi/2$ and that $p = 0$ on for $\theta = 0$). Instead, the hemisphere is rotated 90° about the x-axis, as shown below:



Start with the most general form of solution for sound in a spherical enclosure. The time dependence is suppressed, and the brackets $\{\dots\}$ denote "linear combinations of … "

$$R(r)\Theta(\theta)\Psi(\psi) = \begin{Bmatrix} j_n(kr) \\ n_n(kr) \end{Bmatrix} P_n^m(\cos\theta) \begin{Bmatrix} \cos(m\psi) \\ \sin(m\psi) \end{Bmatrix} . \tag{1}$$

The Neumann functions are thrown out because they diverge at $r = 0$, which is a point in space in which the sound is finite, so equation (1) therefore becomes

$$R(r)\Theta(\theta)\Psi(\psi) = j_n(kr)P_n^m(\cos\theta)\begin{Bmatrix}\cos(m\psi)\\ \sin(m\psi)\end{Bmatrix}, \tag{2}$$

The boundary conditions are

$$\left.\frac{\partial}{\partial r}p(r,\theta,\psi)\right|_{r=a} = 0 \tag{i}$$

$$\left.\frac{\partial}{\partial \psi}p(r,\theta,\psi)\right|_{\psi=0} = 0 \tag{ii}$$

$$\left.\frac{\partial}{\partial \psi}p(r,\theta,\psi)\right|_{\psi=\pi} = 0 \tag{iii}$$

$$p(r,\pi/2,0) = 0 \tag{iv}$$

Boundary condition (i) describes curved hemisphere being rigid, and boundary conditions (ii) and (iii) describes the base of the hemisphere (now corresponding to the $y = 0$ plane) being rigid. Boundary condition (iv) corresponds to the requirement that $p = 0$ along axis of symmetry of the hemisphere (now corresponding to the y-axis). Setting boundary condition (i) equal to equation (2) gives the condition

$$j_n'(ka) = 0 \qquad \Longrightarrow \qquad k_{nl} = \frac{x_{nl}'}{a}, \tag{3}$$

where x_{nl}' is the lth root of the nth order derivative of the spherical Bessel function given by table 10.1 in Blackstock's book. Meanwhile, setting boundary condition (ii) equal to equation (2) gives

$$\left.\frac{\partial}{\partial\psi}(A\cos m\psi + B\sin m\psi)\right|_{\psi=0} = 0 \qquad \Longrightarrow \qquad B = 0.$$

Next, invoking boundary condition (iii) determines the values of m:

$$\left.\frac{\partial}{\partial\psi}(A\cos m\psi)\right|_{\psi=\pi} = 0 \qquad \Longrightarrow \qquad \sin m\pi = 0 \qquad \Longrightarrow \qquad m = 0, 1, 2\ldots$$

Finally, invoking boundary condition (iv) gives

$$P_n^m[\cos(\pi/2)]\cos(m\pi/2) = 0$$

Note that $\cos(m\pi/2)$ is ± 1 for m even and 0 for m odd. Therefore, $P_n^m(0) = 0$ by the zero product property. At this juncture, the following footnote in Blackstock's book is noted:

> [4] The odd, even properties of $P_n^m(z)$ are worth noting. Equation B–32 shows that P_n^m is an odd function of z if $m+n$ is an odd number, an even function if $m+n$ is an even number. This information is useful, for example, when one analyzes the sound field in a hemispherical enclosure.

$P_n^m(0) = 0$ is guaranteed for an odd function $P_n^m(z)$. Therefore, $m + n =$ odd. The eigenfunctions are therefore given by

$$\boxed{\begin{aligned} p_{nlm} &= j_n(k_{nl}r)P_n^m(\cos\theta)\left\{\cos(m\psi)\right\}, \qquad (4) \\ \text{where} \quad m &= 0, 1, 2\ldots \quad \text{and } m + n = \text{ odd} \end{aligned}}$$

The lowest two nonzero[1] eigenfrequencies are found to be proportional to the underlined values of $x'_{nl} = x'_{11}$ and $x'_{nl} = x'_{21}$ below,

Table 10.1 Tables of Zeros of Spherical Bessel Functions

	Roots $x_{n\ell}$ of $j_n(x) = 0$					Roots $x'_{n\ell}$ of $j'_n(x) = 0$				
ℓ	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$
1	π	4.493	5.763	6.988	8.183	0	$\underline{2.082}$	$\underline{3.342}$	4.514	5.647
2	2π	7.725	9.095	10.417	11.705	4.493	5.940	7.290	8.578	9.840
3	3π	10.904	12.323	13.698	15.040	7.725	9.206	10.614	11.973	13.296
4	4π	14.066	15.515	16.924	18.301	10.904	12.405	13.846	15.245	16.609
5	5π	17.221	18.689	20.122	21.525	14.066	15.579	17.043	18.468	19.862

i.e., the lowest two eigenfrequencies are

$$\boxed{f_{11} = \frac{2.082c_0}{2\pi a}, \qquad f_{21} = \frac{3.342c_0}{2\pi a}}$$

[1] Let's talk more about the "dc" solution for corresponding to $x'_{nl} = x'_{01}$