Green's functions in 1D and 2D*

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I need the expression for the Green's function in 1D for a scattering problem. For the 3D case, see my notes on my website. The 3D case is also covered in Morse & Ingard.

1D

The Green's function g in 1D satisfies the 1D inhomogeneous Helmholtz equation

$$\frac{\partial^2 g}{\partial x^2} + k^2 g = -\delta(x - x_0) \tag{1}$$

The solution must have the d'Alambert form

$$g(x|x_0) = Ae^{ik(x-x_0)}, \quad x > x_0$$
$$g(x|x_0) = Ae^{-ik(x-x_0)}, \quad x < x_0$$

which is more conveniently expressed as

$$g(x|x_0) = Ae^{ik|x-x_0|}, \quad x \neq x_0 \tag{2}$$

The task is to determine the constant A. To do so, equation (1) is integrated over x from $x_0 - \epsilon$ to $x_0 + \epsilon$:

$$\begin{aligned} &\int_{x_0-\epsilon}^{x_0+\epsilon} \frac{\partial^2 g}{\partial x^2} dx + \int_{x_0-\epsilon}^{x_0+\epsilon} k^2 g dx = -1 \\ \implies &\left.\frac{\partial g}{\partial x}\right|_{x_0-\epsilon}^{x_0+\epsilon} + k^2 \int_{x_0-\epsilon}^{x_0+\epsilon} g dx = -1 \end{aligned} \tag{3}$$

Noting that

$$\begin{aligned} \left.\frac{\partial g}{\partial x}\right|_{x_0-\epsilon}^{x_0+\epsilon} &= 2ikAe^{ik\epsilon} \\ &\to 2ikA \text{ as } \epsilon \to 0 \end{aligned}$$

*Based on Dr. Mark F. Hamilton's lecture notes, Wave Phenomena, UT Austin and that

$$\begin{aligned}\int_{x_0-\epsilon}^{x_0+\epsilon} g dx &= \int_{x_0-\epsilon}^{x_0} g dx + \int_{x_0}^{x_0+\epsilon} g dx \\ &= -\frac{A}{ik}(1 - e^{ik\epsilon}) + \frac{A}{ik}(e^{ik\epsilon} - 1) \\ &\to 0 \text{ as } \epsilon \to 0,\end{aligned}$$

equation (3) becomes

$$2ikA = -1 \implies A = \frac{i}{2k}$$

So the Green's function in 1D is

$$\boxed{g(x|x_0) = \frac{i}{2k} e^{ik|x-x_0|}}$$

2D