Analytical solutions for acoustic vortex beam radiation from planar and spherically focused circular pistons

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The University of Texas at Austin
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of Mechanical Engineering
Cockrell School of Engineering

Outline

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Our publications on this work

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Analytical solutions for acoustic vortex beam radiation from planar and spherically focused circular pistons

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Paraxial and ray approximations of acoustic vortex beams

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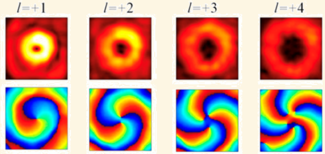
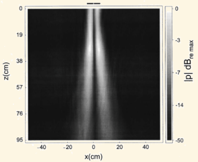
What is an acoustic vortex beam?

Characterized by...

- helical wavefronts
- orbital number ℓ = number of equiphase wavefronts in $\perp$ plane
- zero acoustic pressure on axis



C. Shi et al. *P. Natl. Acad. Sci. U.S.A.* 114 (2017), pp. 7250–7253

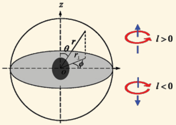


B. T. Hefner and P. L. Marston. *J. Acoust. Soc. Am.* 106 (1999), pp. 3313–3316

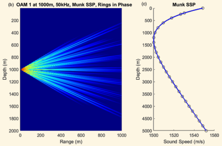
What is an acoustic vortex beam?

Used for...

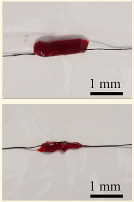
- particle manipulation
- underwater communications
- therapeutic biomedical ultrasound
- sound diffusion



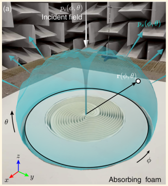
L. Zhang and P. L. Marston. *Phys. Rev. E* 84 (2011), pp. 1–5



M. E. Kelly and C. Shi. *JASA Express Lett.* 3 (2023), pp. 1–5



S. Guo et al. *Ultrasound Med. Biol.* 48 (2022), pp. 1907–1917

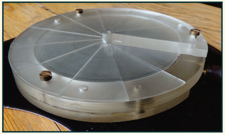


N. Jiménez, J. -P. Groby, and V. Romero-García. *Sci. Rep.* 11 (2021), pp. 1–13

What is an acoustic vortex beam?

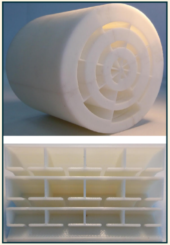
Generated by...

- phase plates
- transducer arrays
- metasurfaces



M. E. Terzi et al. *Moscow University Physics Bulletin* 72 (2017), pp. 61–67

A. Marzo, M. Caleap, and B. W. Drinkwater. *Phys. Rev. Lett.* 120 (2018), pp. 1–6



X. Jiang et al. *Phys. Rev. Lett.* 117 (2016), pp. 1–5

Previous analytical descriptions of vortex beams

- Bessel vortex beams are modes of the cylindrical wave equation:[1]

$$p(r,\theta,z,t) = p_0 J_\ell(k_r r) e^{i(\ell\theta + k_z z - \omega t)}\,, \quad k = \frac{\omega}{c_0} = \sqrt{k_r^2 + k_z^2}\,. \tag{1}$$

- Gaussian vortex beams are generalizations of Gaussian beams:[2]

$$\begin{aligned} p(r,\theta,z,t) &= \sqrt{8\pi}\left(\frac{p_0 z}{kr^2}\right)\chi^{3/2} e^{-\chi}\left[I_{(\ell-1)/2}(\chi) - I_{(\ell+1)/2}(\chi)\right] \\ &\quad \times e^{i[\ell\theta - (\ell+1)\pi/2 + kr^2/2z + k_z z - \omega t]}\,, \\ \chi(r,z) &= \frac{\frac{1}{8}(kar/z)^2}{1 - i(ka^2/2z)(1 - z/d)}\,. \end{aligned} \tag{2}$$

- Fields described by Eqs. (1) and (2) require infinite source conditions.
- Equation (1) implies infinite energy,[3] because $\int_0^\infty |J_\ell(k_r r)|^2 r\,dr \to \infty$.
- **Objective: derive solutions for vortex fields radiated by circular pistons**

[1] N. Jiménez et al. *Phys. Rev. E* 94 (2016), pp. 1–9.
[2] C. A. Gokani, M. R. Haberman, and M. F. Hamilton. *J. Acoust. Soc. Am.* 155 (2024), pp. 2707–2723.
[3] M. R. Lapointe. *Opt. Laser Technol.* 24 (1992), pp. 315–321.

Outline

Paraxial equation and its integral solution

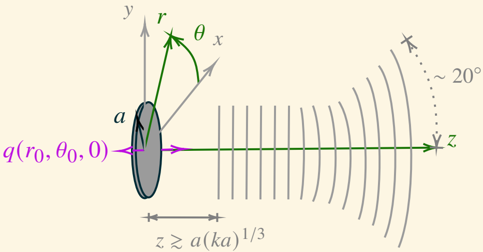
- For $p = qe^{i(kz-\omega t)}$ and $|\partial^2 q/\partial z^2| \ll 2k|\partial q/\partial z|$, $\nabla^2 p - c_0^{-2}\ddot{p} = 0$ reduces to

$$i2k\frac{\partial q}{\partial z} + \nabla_\perp^2 q = 0\,. \qquad (3)$$

- $\nabla_\perp^2$ is the Laplacian in the plane perpendicular to the z axis.
- Equation (3) is solved by the Fresnel diffraction integral:

$$q(r,\theta,z) = -\frac{ik}{2\pi z}\int_0^{2\pi}\int_0^{\infty} q(r_0,\theta_0,0)e^{i(k/2z)[r^2+r_0^2-2rr_0\cos(\theta_0-\theta)]}\, r_0 dr_0 d\theta_0\,,$$

where $q(r_0,\theta_0,0)$ is the prescribed pressure field in the plane $z=0$.



Vortex radiation from unfocused circular piston

- A circular vortex source condition is first considered:

$$q(r,\theta,0) = p_0\,\mathrm{circ}(r/a)e^{i\ell\theta}, \tag{4}$$

where $\mathrm{circ}(x) = 1$ for $0 \le x \le 1$ and 0 for $x > 1$.

- Insertion of Eq. (4) in the Fresnel diffraction integral leads to

$$q = -ikp_0 \frac{e^{i(ka^2/2z)r^2/a^2}}{z} e^{i\ell(\theta-\pi/2)} \int_0^a e^{i(ka^2/2z)r_0^2/a^2} J_\ell(krr_0/z)\, r_0 dr_0\,. \tag{5}$$

- Using Watson's relation,[4] Eq. (5) reduces to

$$q = -\frac{ikp_0}{z} e^{i\ell(\theta-\pi/2)} \int_0^a J_\ell(krr_0/z)\, r_0 dr_0 \tag{6}$$

for $z \gg z_R$, where $z_R = ka^2/2$ is the Rayleigh distance.

[4] G. N. Watson. Cambridge, UK: Cambridge University Press, 1944, Sec. 2.2, Eq. (5).

Vortex radiation from unfocused circular piston

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- Taking the integral in Eq. (6) leads to an analytical solution:

$$\boxed{q_\ell(r,\theta,z) = -ip_0\frac{z}{kr^2}e^{i\ell(\theta-\pi/2)}F_\ell(kar/z)\,, \quad z \gg z_R\,,} \tag{7}$$

where[5]

$$F_\ell(\xi) = \int_0^\xi J_\ell(t)t\,dt = \xi\,\frac{\Gamma(\ell/2+1)}{\Gamma(\ell/2)}\sum_{k=0}^{\infty}\frac{(\ell+2k+1)\Gamma(\ell/2+k)}{\Gamma(\ell/2+2+k)}J_{\ell+2k+1}(\xi). \tag{8}$$

- Equation (8) equals the following closed-form expressions for $0 \le \ell \le 4$:[6]

$$F_0(\xi) = \xi J_1(\xi)\,, \qquad \ell = 0 \tag{9a}$$

$$F_1(\xi) = \tfrac{\pi}{2}\xi\left[\mathbf{H}_0(\xi)J_1(\xi) - \mathbf{H}_1(\xi)J_0(\xi)\right]\,, \qquad \ell = 1 \tag{9b}$$

$$F_2(\xi) = 2 - 2J_0(\xi) - \xi J_1(\xi)\,, \qquad \ell = 2 \tag{9c}$$

$$F_3(\xi) = \left[\tfrac{3\pi}{2}\xi\,\mathbf{H}_0(\xi) - 8\right]J_1(\xi) + \left[4\xi - \tfrac{3\pi}{2}\xi\,\mathbf{H}_1(\xi)\right]J_0(\xi)\,, \qquad \ell = 3 \tag{9d}$$

$$F_4(\xi) = 4 - 8J_1(\xi)/\xi - 4J_2(\xi) - \xi J_3(\xi)\,. \qquad \ell = 4 \tag{9e}$$

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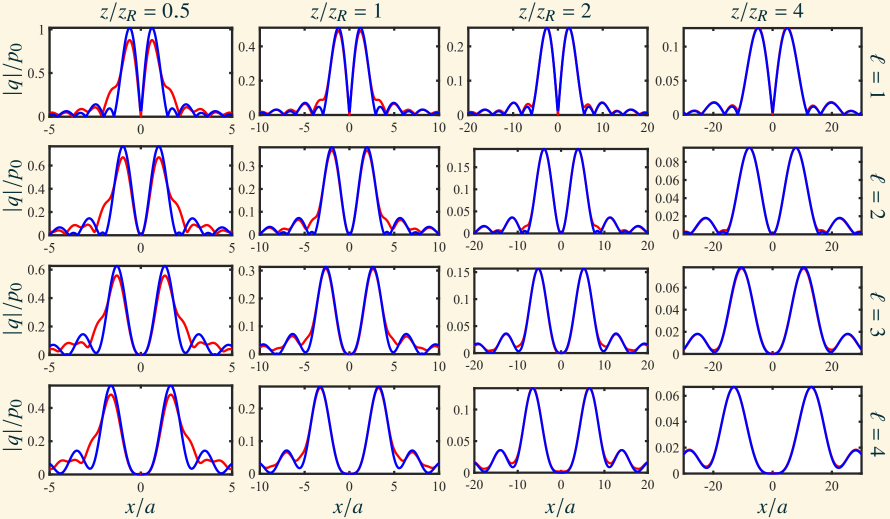
[5] M. Abramowitz and I. A. Stegun, editors. New York: Dover Publications, 1972, Item 11.1.1.
[6] C. A. Gokani, M. R. Haberman, and M. F. Hamilton. *JASA Express Lett.* 4 (2024), pp. 1–7.

Verification of Eq. (7)

- The validity of Eq. (7) is assessed by comparison to

$$q(x,y,z) = \mathcal{F}_{xy}^{-1}\left\{e^{ik_z z}\mathcal{F}_{xy}[q(x,y,0)]\right\}, \quad k_z = k - (k_x^2 + k_y^2)/2k\,. \tag{10}$$



Vortex radiation from focused circular piston

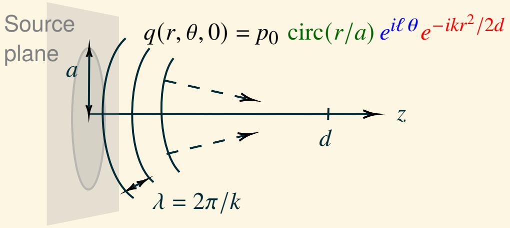
- To describe spherical focusing at a geometric focal length d, source condition (4) is multiplied by $\exp(-ikr^2/2d)$:

$$q(r,\theta,0) = p_0 \, \mathrm{circ}(r/a) \, e^{i\ell\theta} e^{-ikr^2/2d} \,, \qquad (11)$$

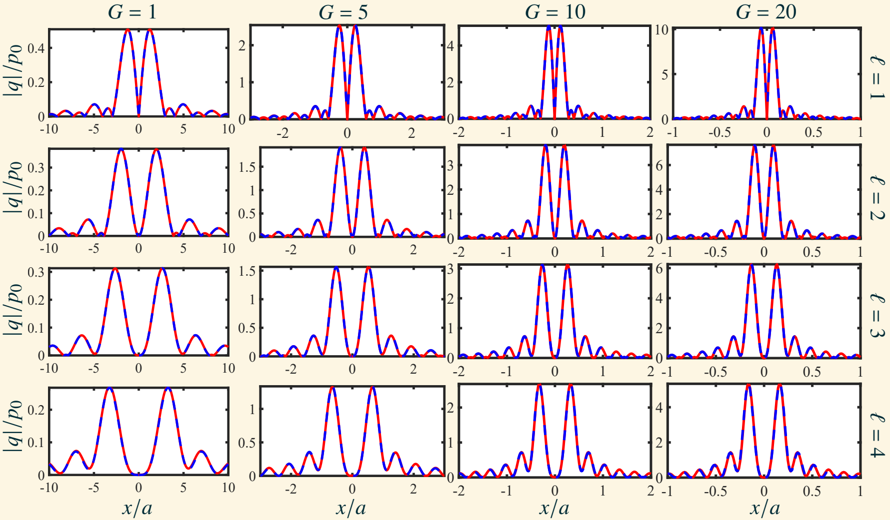
- An analytical solution of Eq. (3) is available at $z = d$:

$$\boxed{q_\ell(r,\theta,d) = -ip_0 \frac{d}{kr^2} e^{ikr^2/2d} e^{i\ell(\theta-\pi/2)} F_\ell(kar/d) \,,} \qquad (12)$$

where $F_\ell(\xi)$ is given by Eq. (8).



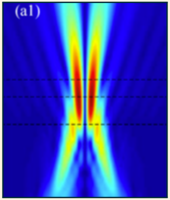
Verification of Eq. (12): analytical, Fourier



Outline

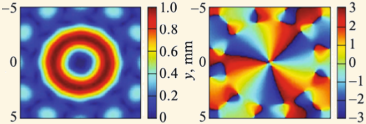
Vortex ring radius

- The magnitudes of vortex beam fields are axisymmetric.
- In the far field of Eq. (7), the field is conical.
- In the geometric focal plane $z = d$ of Eq. (12), the field forms toroidal ring.
- Equations (7) and (12) can be used to find the radius of these features.



C. Zhou et al. *J. Appl. Phys.* 128 (2020), pp. 1–12

D. Baresch, J. -L. Thomas, and R. Marchiano. *Phys. Rev. Lett.* 116 (2016), pp. 1–6



M. E. Terzi et al. *Moscow University Physics Bulletin* 72 (2017), pp. 61–67

Vortex ring radius

- Maximizing Eqs. (7) and (12) in r amounts to solving

$$\frac{d|\xi^{-1}F_\ell(\xi)|}{d\xi} = 0\,,$$

where $\xi = kar/z$ (unfocused) and $\xi = kar/d$ (focused).

- Using Eq. (8) for F_ℓ and taking the derivative yields[7]

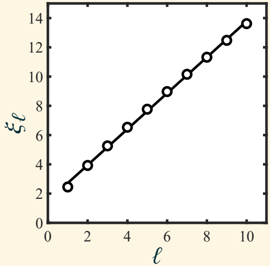
$$\sum_{k=0}^{\infty} \frac{(\ell+2k+1)\Gamma(\ell/2+k)}{\Gamma(\ell/2+2+k)} \left[\frac{J_{\ell+2k}(\xi) - J_{\ell+2k+2}(\xi)}{2\xi} - \frac{J_{\ell+2k+1}(\xi)}{\xi^2}\right] = 0\,. \qquad (13)$$

- The roots ξ_ℓ of Eq. (13) are fit to a line: $\xi_\ell = 1.23\ell + 1.49$.
- Solving $\xi_\ell = kar_\ell/z$ and $\xi_\ell = kar_\ell/d$ for r_ℓ yields

$$\boxed{\begin{aligned} r_\ell &= \frac{\xi_\ell z}{ka}\,, \quad z \gg z_R\,, \quad \text{far field,} && (14a)\\ &= \frac{\xi_\ell d}{ka}\,, \quad z = d\,, \quad \text{focal plane.} && (14b)\end{aligned}}$$

[7] I. S. Gradshteyn and I. M. Ryzhik. New York: Academic Press, 1980, Item 8.471-2.

Vortex ring radius



Comparison of roots ξ_ℓ (circles) with least-squares fit $\xi_\ell = 1.23\ell + 1.49$ (line)

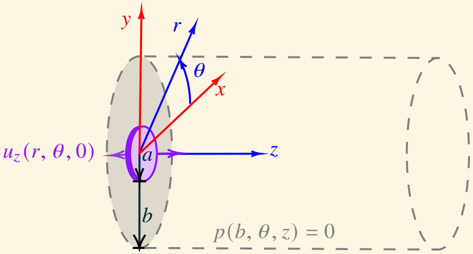
Outline

Alternative approach to diffraction theory

- The Rayleigh integral

$$p(\mathbf{r}) = -\frac{ik\rho_0 c_0}{2\pi}\int_{A_0} u_z(\mathbf{r}_0)\,\frac{e^{ik|\mathbf{r}-\mathbf{r}_0|}}{|\mathbf{r}-\mathbf{r}_0|}\,dA_0 \tag{15}$$

  is the starting-point for the study of diffraction from circular pistons.
- Equation (15) is traditionally derived from the Helmholtz-Kirchhoff integral.[8]
- Consider a piston placed concentrically within a tube of radius b.



[8] A. D. Pierce. Cham, Switzerland: Springer, 2019, Eqs. (5.2.6) and (5.7.3).

Alternative approach to diffraction theory

- The solution of the Helmholtz equation for that scenario is

$$p(r,\theta,z) = \sum_{n=1}^{\infty} A_{\ell n} J_\ell(\alpha_{\ell n} r/b) e^{i(\ell\theta + \beta_{\ell n} z)}\,, \tag{16a}$$

$$A_{\ell n} = \frac{2k\rho_0 c_0 u_0}{\alpha_{\ell n}^2 \beta_{\ell n}} \frac{F_\ell(\alpha_{\ell n} a/b)}{J_{\ell+1}^2(\alpha_{\ell n})}\,, \quad \beta_{\ell n} = \sqrt{k^2 - k_r^2}\,, \quad k_r = \alpha_{\ell n}/b\,, \tag{16b}$$

  where $\alpha_{\ell n}$ is the n^{th} root of J_ℓ.
- The ratio $\alpha_{\ell n}/b$ is vanishingly small except for large n, for which[9]

$$\alpha_{\ell n} \approx \pi(n - 1/4 + \ell/2), \quad n \gg 1\,. \tag{17}$$

- Defining $\zeta = \alpha_{\ell n} a/b \approx \frac{a}{b}\pi(n - 1/4 + \ell/2)$ for $n \gg 1$ sets $\Delta\zeta = \pi a/b$:

$$p = \rho_0 c_0 u_0 e^{i\ell\theta} \sum_{n=1}^{\infty} \frac{\Delta\zeta}{\zeta} \frac{F_\ell(\zeta) J_\ell(\zeta r/a)}{\sqrt{1 - (\zeta/ka)^2}} e^{ikz\sqrt{1-(\zeta/ka)^2}}\,. \tag{18}$$

- As $b \to \infty$, Eq. (18) tends to

$$\boxed{p = \rho_0 c_0 u_0 e^{i\ell\theta} k \int_0^\infty F_\ell(k_r a) J_\ell(k_r r) \frac{e^{ik_z z}}{k_z} \frac{dk_r}{k_r}\,. \quad \text{(exact)}} \tag{19}$$

[9] I. S. Gradshteyn and I. M. Ryzhik. New York: Academic Press, 1980, Eq. (8.547).

Alternative approach to diffraction theory

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- Using Watson's relation[10] and the 2D Fourier transforms

$$\mathcal{F}_{\mathrm{2D}}\{f(r,\theta)\} = g(k_r,\psi) = \int_0^{2\pi}\int_0^{\infty} f(r,\theta)e^{-ik_r r\cos(\theta-\psi)}\, r\,dr\,d\theta\,, \tag{20a}$$

$$\mathcal{F}_{\mathrm{2D}}^{-1}\{g(k_r,\psi)\} = f(r,\theta) = \int_0^{2\pi}\int_0^{\infty} g(k_r,\psi)e^{ik_r r\cos(\theta-\psi)}\, k_r\,dk_r\,d\psi\,, \tag{20b}$$

  Eq. (19) recovers the angular spectrum method,

$$p(r,\theta,z) = \rho_0 c_0 k\, \mathcal{F}_{\mathrm{2D}}^{-1}\{\mathcal{F}_{\mathrm{2D}}\{u_z(r,\theta)\}e^{ik_z z}/k_z\}\,. \tag{21}$$

- Since $\mathcal{F}_{\mathrm{2D}}\{e^{ikr}/r\} = i2\pi e^{ik_z|z|}/k_z$,[11] where $r=\sqrt{x^2+y^2+z^2}$, Eq. (21) becomes

$$p = -\frac{ik\rho_0 c_0}{2\pi}\, u(x,y) ** \frac{e^{ikr}}{r}\,. \tag{22}$$

- By the definition of the convolution operation, Eq. (22) recovers Eq. (15).

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[10] G. N. Watson. Cambridge, UK: Cambridge University Press, 1944, Sec. 2.2, Eq. (5).

[11] L. T. Brekhovskikh, translated by R. T. Beyer. Academic Press, 1980, pp. 227–234.

Dimensionless form of Eq. (19)

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- In terms of the dimensionless parameters

$$P \equiv p/\rho_0 c_0 u_0\,, \quad R \equiv r/a\,, \quad Z \equiv z/z_R\,, \quad K \equiv ka\,, \tag{23}$$

where z_R is the Rayleigh distance $ka^2/2$, and where

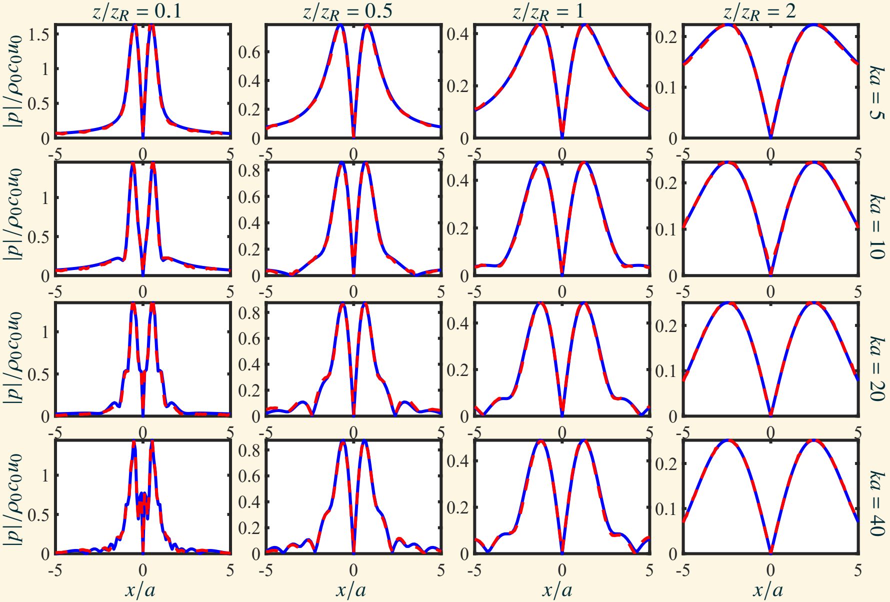
$$k_z/k = \sqrt{1-(\zeta/K)^2}\,, \quad \zeta \equiv k_r a\,, \tag{24}$$

Eq. (19) becomes

$$P = e^{i\ell\theta} \int_0^\infty \frac{F_\ell(\zeta) J_\ell(\zeta R)}{\zeta\sqrt{1-(\zeta/K)^2}} e^{iK^2 Z\sqrt{1-(\zeta/K)^2}/2}\, d\zeta\,. \tag{25}$$

- Equation (25) is equivalent to and easier to evaluate than Eq. (21).

Equation (25) for $\ell = 1$: semi-analytical, Fourier



On-axis pressure of baffled circular piston

- Equation (25) can be evaluated analytically for $R = \ell = 0$:

$$P(Z) = \int_0^\infty \frac{J_1(\zeta)}{\sqrt{1-(\zeta/K)^2}}\, e^{iK^2 Z\sqrt{1-(\zeta/K)^2}/2} d\zeta \,. \tag{26}$$

- Equation (26) evaluates to[12]

$$P(Z) = -iK\, I_{1/2}[-i\chi_-(Z)]\, K_{1/2}[-i\chi_+(Z)]\,, \tag{27}$$

where I_ν and K_ν are the modified Bessel functions of order ν, and where

$$\chi_\pm(Z) = \tfrac{K}{2}\Big[\sqrt{1+(KZ/2)^2} \pm KZ/2\Big]. \tag{28}$$

- Bessel function identities reduce Eq. (27) to

$$P(Z) = -2i\sin[\chi_-(Z)]e^{i\chi_+(Z)}\,, \tag{29}$$

recovering the on-axis pressure radiated by a planar circular piston.[13]

[12] I. S. Gradshteyn and I. M. Ryzhik. New York: Academic Press, 1980, Item 6.637-1.
[13] A. D. Pierce. Cham, Switzerland: Springer, 2019, Eq. (5.7.3).

Outline

Thank you for listening!

Summary

- Solved paraxial equation for planar and focused circular vortex sources
- Calculated the ring radius for both solutions
- Derived alternative theory of diffraction from circular pistons

Further reading

- "Paraxial and ray approximations of acoustic vortex beams," J. Acoust. Soc. Am. **155**, 2707–2723 (2024).
- "Analytical solutions for acoustic vortex beam radiation from planar and spherically focused circular pistons," JASA Express Lett. **4**, 124001 (2024).

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Prof. Mark F. Hamilton

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Notation I

Symbol	Description	Dimensions
a	source radius	m
c_0	speed of sound	m s^{-1}
d	focal length	m
G	focusing gain $ka^2/2d$	1
i	complex unit	1
$\mathbf{k}$	wavenumber	m^{-1}
ℓ	orbital number	1
ρ_0	ambient mass density	kg m^{-3}
p	acoustic pressure	kg m^{-1} s^{-2}
q	paraxial pressure	kg m^{-1} s^{-2}
$\mathbf{R}$	separation vector $\mathbf{R} = \mathbf{r} - \mathbf{r}'$	m
$\mathbf{r}$	position vector	m
$\mathbf{v}$	particle velocity	m s^{-1}
z_R	Rayleigh distance, $ka^2/2$	m
ω	angular frequency, $\omega = 2\pi f$	s^{-1}

Equation (25) for $\ell = 0$: semi-analytical, Fourier