

Notating $R \equiv |\mathbf{r} - \mathbf{r}_0|$, show that¹

$$g_\omega = \frac{1}{4\pi R} e^{ikR}$$

solves

$$\nabla^2 g_\omega(\mathbf{r}|\mathbf{r}_0) + k^2 g_\omega(\mathbf{r}|\mathbf{r}_0) = -\delta(\mathbf{r} - \mathbf{r}_0) \quad (1)$$

First, integrate both sides of equation (1) over a sphere of small radius a :

$$\iiint_V \nabla^2 g_\omega(\mathbf{r}|\mathbf{r}_0) dV + \iiint_V k^2 g_\omega(\mathbf{r}|\mathbf{r}_0) dV = - \iiint_V \delta(\mathbf{r} - \mathbf{r}_0) dV \quad (2)$$

Note that the right-hand-side of equation (2) is just -1 , by definition of the delta function. Meanwhile, the first integral on the left-hand-side of equation (2) can be written as

$$\begin{aligned} \iiint_V \nabla \cdot \nabla g_\omega dV &= \frac{1}{4\pi} \iiint_V \nabla \cdot \hat{R} \left(-\frac{1}{R^2} + \frac{ik}{R} \right) e^{ikR} dV \\ &= \iiint_V \left(-\frac{1}{R^2} + \frac{ik}{R} \right) e^{ikR} dS \\ &= \frac{4\pi a^2}{4\pi} \left(-\frac{1}{a^2} + \frac{ik}{a} \right) e^{ika} \\ &= (-1 + ika)e^{ika} \\ &= (-1 + ika) \left(1 + ika - \frac{(ka)^2}{2!} - i \frac{(ka)^3}{3!} + \frac{(ka)^4}{4!} + \dots \right) \\ &\rightarrow -1 \text{ as } a \rightarrow 0 \end{aligned}$$

The second integral on the left-hand-side of equation (2) is evaluated in spherical coordinates², and the resulting radial integral is evaluated by parts:

$$\begin{aligned} \frac{k^2}{4\pi} \iiint_V \frac{e^{ikR}}{R} dV &= \frac{k^2}{4\pi} \int_0^{2\pi} \int_0^\pi \int_0^a R e^{ikR} \sin \theta dR d\theta d\phi \\ &= k^2 \int_0^a R e^{ikR} dR \\ &= -ikae^{ika} + e^{ika} - 1 \\ &= -ikae^{ika} + 1 + ika - \frac{(ka)^2}{2!} - i \frac{(ka)^3}{3!} + \frac{(ka)^4}{4!} \dots - 1 \\ &\rightarrow 0 \text{ as } a \rightarrow 0 \end{aligned}$$

The left-hand-side and right-hand-side of equation (2) both equal -1 , showing that g_ω solves equation (1).

¹See Morse & Ingard, section 7.1

²Recall that the Jacobian in spherical coordinates is $R^2 \sin \theta$.