Radiation force on inhomogeneous subwavelength scatterers due to progressive waves

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Introduction

Westervelt's formulation in the far field

Linear scattering problem

Examples

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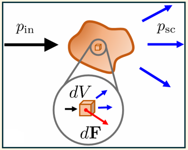
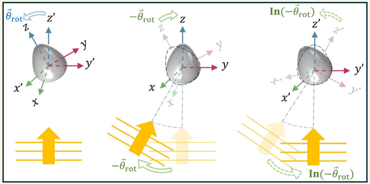
Motivation

Particle manipulation of asymmetric objects

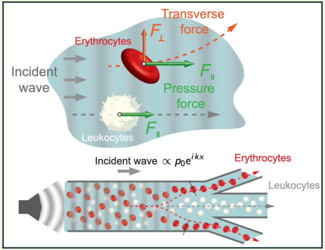
- Forces and torques due to standing waves have been investigated.[1]
- Jerome et al.[2] used the Born approximation to calculate radiation force and torque on subwavelength objects in primarily standing wave fields.
- Forces due to progressive waves have only recently garnered interest.[3]



T. S. Jerome. PhD thesis. University of Texas at Austin, 2022



T. Tang, Y. Zhang, B. Dong, and L. Huang. *J. Acoust. Soc. Am.* 156 (2024), pp. 2767–2782



M. Smagin, I. Toftul, K. Y. Bliokh, and M. Petrov. *Phys. Rev. Appl.* 22 (2024), pp. 1–20

[1] E. B. Lima and G. T. Silva. *J. Acoust. Soc. Am.* 150 (2021), pp. 376–384.
[2] T. S. Jerome and M. F. Hamilton. *J. Acoust. Soc. Am.* 150 (2021), pp. 3417–3427.
[3] T. Tang and L. Huang. *J. Sound Vib.* 532 (2022), pp. 1–19.

Conservation of momentum at quadratic order

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- Acoustic radiation force is a consequence of $O(\epsilon^2)$ momentum conservation:

$$\boxed{\frac{\partial \mathbf{g}}{\partial t} = \boldsymbol{\nabla} \cdot \underline{\mathbf{T}}} \tag{1}$$

- The momentum density is

$$\mathbf{g} = p\mathbf{v}/c_0^2 \tag{2}$$

- In free space, the instantaneous acoustic radiation stress tensor is

$$\underline{\mathbf{T}} \equiv L\underline{\mathbf{I}} - \rho_0 \mathbf{v} \otimes \mathbf{v}\,. \tag{3}$$

- The Lagrangian density is

$$L = \tfrac{1}{2}\rho_0 v^2 - \frac{p^2}{2\rho_0 c_0^2}\,, \tag{4}$$

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Westervelt's formulation in the far field

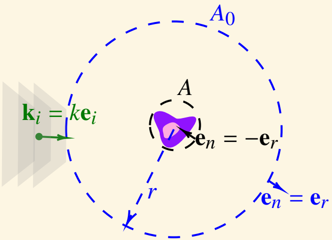
P. J. Westervelt (left) and R. V. Khokhlov (right)
Copenhagen, 1973
Photograph by D. T. Blackstock

Westervelt's formulation in the far field

P. J. Westervelt (left) and M. F. Hamilton (right)
1984 International Symposium on Nonlinear Acoustics
Kobe, Japan

Westervelt's formulation in the far field

- The volume integral of $\boldsymbol{\nabla} \cdot \langle \underline{\mathbf{T}} \rangle$ enclosing an object is the radiation force[4]



$$\mathbf{F} = \int_V \boldsymbol{\nabla} \cdot \langle \underline{\mathbf{T}} \rangle dV = \oint_A \langle \underline{\mathbf{T}} \rangle \cdot d\mathbf{A}$$

P. J. Westervelt

- Invoking the far-field approximation and energy conservation yields

$$F_{\parallel} = \frac{p_0^2}{2\rho_0 c_0^2} \oint |\Phi_s(\mathbf{k}_s)|^2 (1 - \cos\psi_0) d\Omega_0 \,, \tag{5a}$$

$$F_{\perp} = -\frac{p_0^2}{2\rho_0 c_0^2} \oint |\Phi_s(\mathbf{k}_s)|^2 \, \mathbf{e}_m \cdot \mathbf{e}_r \, d\Omega_0 \,, \quad d\Omega_0 = r_0^{-2} \, dA_0 \,. \tag{5b}$$

where Φ_s = scattered directivity, $\cos\psi = \mathbf{k}_s \cdot \mathbf{k}_i / k^2$, and Ω = solid angle.

[4] P. J. Westervelt. *J. Acoust. Soc. Am.* 29 (1957), pp. 26–29.

Linear scattering problem

[F]or the calculation of the average force correct up to terms of the second order in the velocity, it is sufficient to find the solution of the linear scattering problem.[5]

G. Boebinger, S. Iordansky, D. Pines, and L. Pitaevskii. *Physics Today* 70 (2017), pp. 68–69

[5] L. P. Gor'kov. *Sov. Phys. Dokl.* 6 (1962), pp. 773–775.

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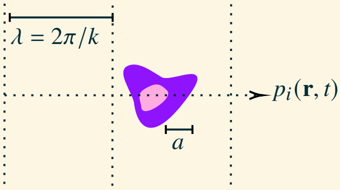
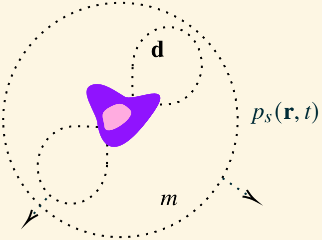
Acoustic polarizability



(a) incident acoustic waves, $ka \ll 1$



(b) scattered acoustic waves

- *Polarizabilities* describe the heterogeneity's response to local fields:[6]
  - α_m relates $p_i \leftrightarrow m$
  - $\underline{\boldsymbol{\alpha}}_d$ relates $\mathbf{v}_i \leftrightarrow \mathbf{d}$
  - $\boldsymbol{\alpha}_c$ relates $\mathbf{v}_i \leftrightarrow m$ and $p_i \leftrightarrow \mathbf{d}$
- The scattered monopole strength m and dipole moment $\mathbf{d}$ are given by

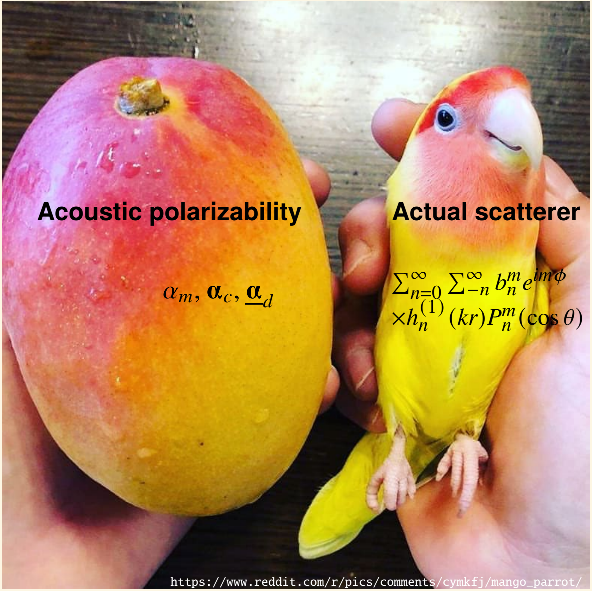
$$m = -\beta_0 \alpha_m p_i - i c_0^{-1} \boldsymbol{\alpha}_c \cdot \mathbf{v}_i \,, \tag{6a}$$

$$\mathbf{d} = -i c_0^{-1} \boldsymbol{\alpha}_c p_i + \rho_0 \underline{\boldsymbol{\alpha}}_d \cdot \mathbf{v}_i \,. \tag{6b}$$

- **Objective: Obtain a general approach to calculate α_m, $\boldsymbol{\alpha}_c$, and $\underline{\boldsymbol{\alpha}}_d$.**

[6] C. F. Sieck, A. Alù, and M. R. Haberman. *Phys. Rev. B.* 96 (2017), pp. 1–20

Multipole expansion to dipole order



Scattering of sound from heterogeneities

- The equation of state in heterogeneous media is[7]

$$\frac{\partial \rho'}{\partial t} = \frac{1}{c^2}\frac{\partial p}{\partial t} - \mathbf{v}\cdot\boldsymbol{\nabla}\rho(\mathbf{r})\,. \tag{7}$$

- The linearized mass conservation equation combined with Eq. (7) yields

$$\boldsymbol{\nabla}\cdot\mathbf{v} = -\beta(\mathbf{r})\frac{\partial p}{\partial t}\,. \tag{8}$$

- Linearizing the momentum equation yields

$$\frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{\rho(\mathbf{r})}\boldsymbol{\nabla}p\,. \tag{9}$$

- Combining Eqs. (8) and (9) and assuming time-harmonic solutions yields

$$\nabla^2\tilde{p}_\omega + k^2\tilde{p}_\omega = k^2 f_1\tilde{p}_\omega + \boldsymbol{\nabla}\cdot\left(\tfrac{3f_2}{2+f_2}\boldsymbol{\nabla}\tilde{p}_\omega\right)\,. \tag{10}$$

where the contrast factors are[8]

$$f_1(\mathbf{r}) = 1 - \frac{\beta(\mathbf{r})}{\beta_0}\,,\quad f_2(\mathbf{r}) = \frac{2[\rho(\mathbf{r})-\rho_0]}{2\rho(\mathbf{r})+\rho_0}\,. \tag{11}$$

[7] A. D. Pierce. 2nd edition. Woodbury, New York: Acoustical Society of America, 1989.
[8] L. P. Gor'kov. *Sov. Phys. Dokl.* 6 (1962), pp. 773–775.

Scattering of sound from heterogeneities

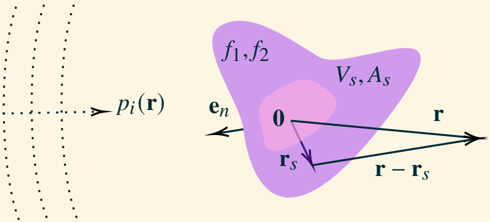
- The Helmholtz-Kirchhoff integral theorem solves Eq. (10),[9]

$$\tilde{p}_\omega(\mathbf{r}) = \tilde{p}_{i,\omega}(\mathbf{r}) + \tilde{p}_{s,\omega}(\mathbf{r}) \,,$$

$$\tilde{p}_{s,\omega}(\mathbf{r}) = \int_{V_s} \left[k^2 f_1(\mathbf{r}_s) \tilde{p}_\omega(\mathbf{r}_s) g(\mathbf{r}|\mathbf{r}_s) - \tfrac{3 f_2(\mathbf{r}_s)}{2 + f_2(\mathbf{r}_s)} \boldsymbol{\nabla}_s \tilde{p}_\omega(\mathbf{r}_s) \cdot \boldsymbol{\nabla}_s g(\mathbf{r}|\mathbf{r}_s) \right] dV_s \,.$$

- The origin is defined by the *centroid*,

$$\mathbf{0} \equiv \frac{\int_{V_s} \mathbf{r}_s \, dV_s}{\int_{V_s} dV_s} \,. \tag{12}$$



[9] P. M. Morse and K. U. Ingard. McGraw-Hill, 1968.

Three approximations

1. **Far-field:** If $r \gg a$, then in terms of $\mathbf{k}_s = k\mathbf{e}_r$,

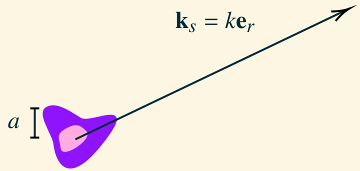
$$g(\mathbf{r}|\mathbf{r}_s) \simeq \frac{e^{ikr}}{4\pi r} e^{-i\mathbf{k}_s \cdot \mathbf{r}_s}\,, \quad \boldsymbol{\nabla}_s g \simeq -i\mathbf{k}_s g\,. \tag{13}$$

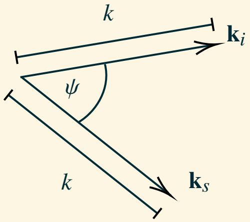
2. **Subwavelength:** Since $ka \sim |\mathbf{k}_s \cdot \mathbf{r}_s| \ll 1$, one can approximate

$$e^{-i\mathbf{k}_s \cdot \mathbf{r}_s} \simeq 1 - i\mathbf{k}_s \cdot \mathbf{r}_s\,, \tag{14}$$

   and therefore $p_i(\mathbf{r}) = p_0 e^{i\mathbf{k}_i \cdot \mathbf{r}} \simeq p_0(1 + i\mathbf{k}_i \cdot \mathbf{r})$.

3. **Born:** If $f_1, f_2 \ll 1$, then $|p_s| \ll |p_i|$, and the problem becomes explicit.





Solution of scattering problem to dipole order

- Identifying

$$\alpha_m = -\int_{V_s} f_1(\mathbf{r}_s)\,dV_s\,, \tag{15a}$$

$$\underline{\boldsymbol{\alpha}}_d = \underline{\mathbf{I}}\int_{V_s} \frac{3f_2(\mathbf{r}_s)}{2+f_2(\mathbf{r}_s)}\,dV_s \equiv \alpha_d\underline{\mathbf{I}}\,, \tag{15b}$$

$$\boldsymbol{\alpha}_c = -k\left[\int_{V_s} f_1(\mathbf{r}_s)\mathbf{r}_s\,dV_s - \cos\psi\int_{V_s}\frac{3f_2(\mathbf{r}_s)}{2+f_2(\mathbf{r}_s)}\mathbf{r}_s\,dV_s\right] \tag{15c}$$

yields the scattered far field:

$$p_s(\mathbf{r}) = p_0\frac{e^{ikr}}{r}\Phi_s\,, \quad \Phi_s(\mathbf{k}_s) = \frac{k^2}{4\pi}\left[\alpha_m + i\boldsymbol{\alpha}_c\cdot(\mathbf{e}_i - \mathbf{e}_r) + \alpha_d\cos\psi\right]. \tag{16}$$

- Thus using

$$|\Phi_s|^2 = \frac{k^4}{16\pi^2}\left\{\alpha_m^2 + 2\alpha_m\alpha_d\cos\psi + [\boldsymbol{\alpha}_c\cdot(\mathbf{e}_i-\mathbf{e}_r)]^2 + \alpha_d^2\cos^2\psi\right\} \tag{17}$$

with Eq. (5) gives the radiation force in terms of the polarizabilities.

Distinction from Rayleigh scattering

$$p_s(\mathbf{r}) = p_0 \frac{e^{ikr}}{r}\Phi_s\,, \quad \Phi_s(\mathbf{k}_s) = \frac{k^2}{4\pi}\left[\alpha_m + i\boldsymbol{\alpha}_c \cdot (\mathbf{e}_i - \mathbf{e}_r) + \alpha_d \cos\psi\right]$$

Any complex exponential representing a phase shift within the scattering body or on its surface may be approximated as having unit value.[10]

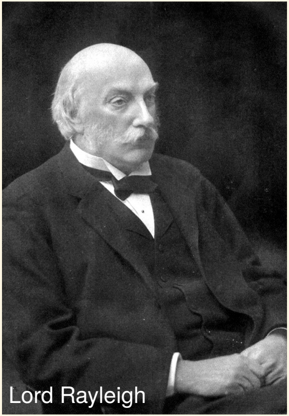
A. D. Pierce (left) and J. H. Ginsberg (right)
J. H. Ginsberg. *Acoustics Today* 11 (2015), pp. 10–16

[10] J. H. Ginsberg. Vol. 2. Springer, 2018.

Distinction from Rayleigh scattering

$$e^{-i\mathbf{k}_s\cdot\mathbf{r}_s} \simeq 1$$

$$e^{-i\mathbf{k}_s\cdot\mathbf{r}_s} \simeq 1 - i\mathbf{k}_s\cdot\mathbf{r}_s$$



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Example 1: Homogeneous sphere[11]

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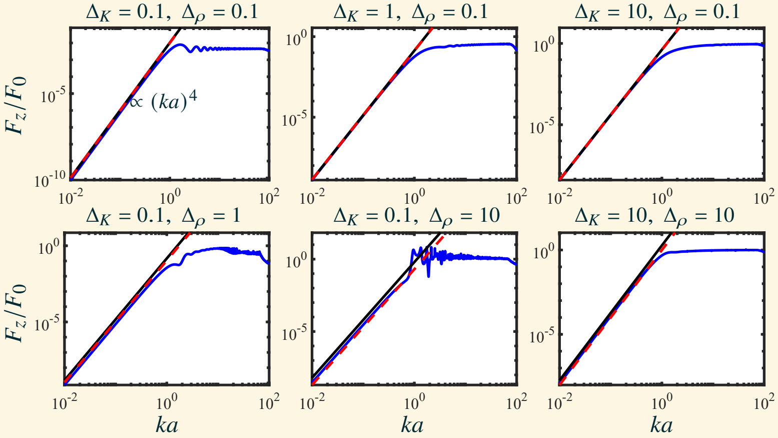
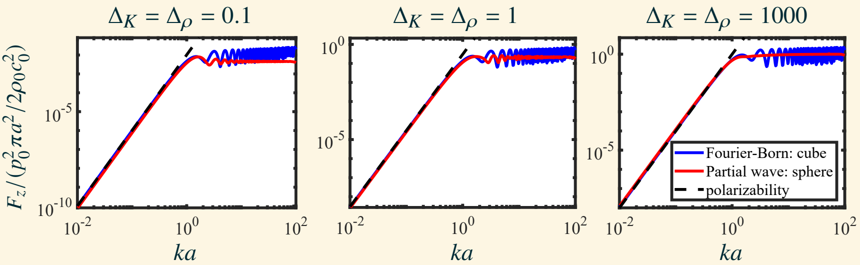


Comparison of the polarizability formulation, **Gor'kov's result**, and the exact solution based on spherical wave expansions, for values of $\Delta_K = (K_s - K_0)/K_0$ and $\Delta_\rho = (\rho_s - \rho_0)/\rho_0$ ranging two orders of magnitude.

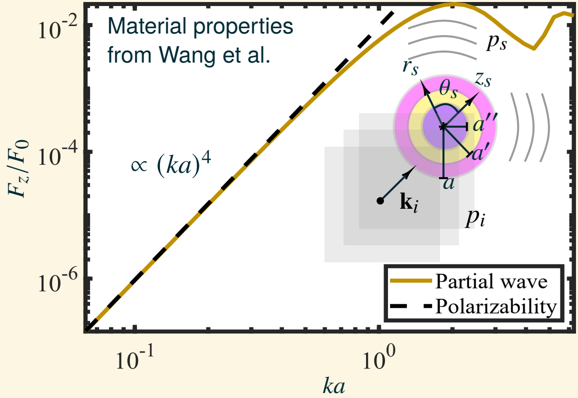
[11] The forces are normalized to $F_0 = p_0^2 \pi a^2 / 2\rho_0 c_0^2$.

Example 2: Homogeneous cube



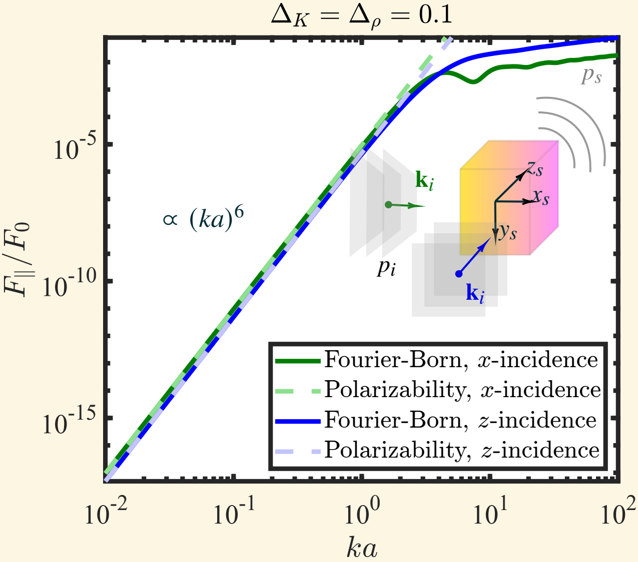
Comparison of the radiation force given by the **polarizability formulation** [dashed line] to the forces on a homogeneous cube and sphere of equal volume [red curves] for $\Delta_K = (K_s - K_0)/K_0$ and $\Delta_\rho = (\rho_s - \rho_0)/\rho_0$ ranging four orders of magnitude. For $ka \ll 1$ in all cases, the force on the cube converges to that on the sphere.

Example 3: Spherically symmetric nucleated cell[12]



[12] Y.-Y. Wang et al. *J. Appl. Phys.* 122 (2017), pp. 1–6.

Example 4: Antisymmetric inhomogeneous cube



Conclusion

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Summary

- Solved **linear scattering problem** at $O[(ka)^3]$ in Born approximation
- Calculated acoustic radiation force due to progressive waves on **homogeneous and inhomogeneous cubes and spheres**
- Compared results to **partial wave expansions** and **Fourier-Born scattering**

Future work

- Calculate radiation torque
- Develop ray theory for calculation of high-frequency asymptote

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- T. S. Jerome for the exact solutions on the homogeneous and layered spheres
- Chester M. McKinney Graduate Fellowship in Acoustics at ARL:UT

Dr. T. S. Jerome

Prof. M. R. Haberman

Prof. M. F. Hamilton

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Notation I

Symbol	Description	Dimensions
a	characteristic size of scatterer	m
c_0	speed of sound	m s^{-1}
$\mathbf{F}$	force	kg m s^{-2}
i	complex unit	1
$\mathbf{k}$	wavenumber	m^{-1}
p	acoustic pressure	kg m^{-1} s^{-2}
$\mathbf{R}$	separation vector, $\mathbf{R} = \mathbf{r} - \mathbf{r}'$	m
$\mathbf{r}$	position vector	m
$\mathbf{v}$	particle velocity	m s^{-1}
ρ_0	ambient mass density	kg m^{-3}
$\underline{\mathbf{T}}$	acoustic radiation stress tensor	kg m^{-1} s^{-2}
$\mathbf{S}$	instantaneous Poynting vector	kg s^{-3}
ω	angular frequency, $\omega = 2\pi f$	s^{-1}