The Paraxial Approximation for a Focused Wave

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Chirag brought up an interesting point about the validity of the paraxial approximation for a focused wave. Here is my investigation...

Consider a spherical wave focused at coordinates $(r, z) = (0, d)$, given by

$$p = p_0 \frac{e^{-ikR}}{kR} e^{-i\omega t}, \qquad R = \sqrt{r^2 + (z-d)^2}. \tag{1}$$

We wish to investigate the validity of the paraxial approximation in the region $z \leq d$.

Letting $q = pe^{-ikz}$, the paraxial approximation is

$$\left| \frac{\partial^2 q}{\partial z^2} \right| \ll 2k \left| \frac{\partial q}{\partial z} \right|. \tag{2}$$

Using $\partial R / \partial z = (z-d)/R$, we calculate the derivatives as

$$\frac{\partial q}{\partial z} = \left[-ik - \frac{ik(z-d)}{R} - \frac{z-d}{R^2} \right] q, \tag{3}$$

$$\frac{\partial^2 q}{\partial z^2} = \left[-k^2 - \frac{ik + 2k^2(z-d)}{R} - \frac{1 - 2ik(z-d) + k^2(z-d)^2}{R^2} + \frac{3ik(z-d)^2}{R^3} + \frac{3(z-d)^2}{R^4} \right] q. \tag{4}$$

We now look at the behavior near the focus. Let $z = d - \epsilon$ and $r = 0$, so $R = \epsilon$. Making these substitutions, we obtain

$$\frac{\partial q}{\partial z} = \frac{q}{\epsilon}, \qquad \frac{\partial^2 q}{\partial z^2} = \frac{2q}{\epsilon^2}. \tag{5}$$

The relevant quantities for the paraxial approximation are

$$2k \left| \frac{\partial q}{\partial z} \right| = \frac{2k}{\epsilon} |q|, \qquad \left| \frac{\partial^2 q}{\partial z^2} \right| = \frac{2}{\epsilon^2} |q|. \tag{6}$$

For sufficiently small ϵ, this gives

$$\left| \frac{\partial^2 q}{\partial z^2} \right| \gg 2k \left| \frac{\partial q}{\partial z} \right|. \tag{7}$$

The paraxial approximation is violated!