Ray propagation for linear sound speed gradient

Background

In class, we claimed that in an environment where the sound speed varies linearly with depth, the fastest path between two points is defined by the arc of a circle. Several methods were suggested as a way to show this (Taylor expanding the analytical travel time, numerically finding the minimum travel time, etc.). A direct way to show that the circular path minimizes the travel time is by the calculus of variations. Interestingly, Snell's law is "built into" the calculus of variations. Below are the relevant parameters:

- x = horizontal coordinate, which is positive in the rightward direction
- z = depth, which is positive in the downward direction
- $c(z) = c_0 + mz$ = linear sound speed profile
- $\mathrm{d}s$ = differential arc length

Calculus of variations approach

The Euler equation is

$$\frac{\partial f}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}z}\frac{\partial f}{\partial x'} = 0 \tag{1}$$

The time taken for the sound to travel along some path is

$$t = \int \frac{\mathrm{d}s}{c(z)} = \int \frac{\mathrm{d}s}{c_0 + mz} \tag{2}$$

Note that the differential $\mathrm{d}s$ can be written as $\mathrm{d}s = (\mathrm{d}x^2 + \mathrm{d}z^2)^{1/2}$. When multiplied and divided by $\mathrm{d}z$, this becomes $(x'^2 + 1)^{1/2}\,\mathrm{d}z$. Making this substitution, equation (2) becomes

$$t = \int \frac{(x'^2 + 1)^{1/2}}{c_0 + mz}\,\mathrm{d}z = \int f(z)\,\mathrm{d}z$$

The function $f(z)$ is now used in equation (1):

$$\frac{\partial}{\partial x}\frac{(x'^2+1)^{\frac{1}{2}}}{c_0 + mz} - \frac{\mathrm{d}}{\mathrm{d}z}\frac{\partial}{\partial x'}\frac{(x'^2+1)^{\frac{1}{2}}}{c_0 + mz} = 0 \tag{3}$$

The first term in equation (3) vanishes, giving

$$\frac{\mathrm{d}}{\mathrm{d}z}\frac{\partial}{\partial x'}\frac{(x'^2+1)^{\frac{1}{2}}}{c_0 + mz} = 0$$

Integrating over z introduces a constant, A:

$$\frac{\partial}{\partial x'}\frac{(x'^2+1)^{\frac{1}{2}}}{c_0 + mz} = A$$

Taking the derivative with respect to x', squaring the result, and denoting $(c_0 + mz)^2 = c^2(z)$, solving for x',

$$\begin{aligned}
\frac{x'(x'^2+1)^{-\frac{1}{2}}}{c_0+mz} &= A \\
\frac{x'^2(x'^2+1)^{-1}}{c^2(z)} &= A^2 \\
\frac{x'^2}{c^2(z)} &= A^2(x'^2+1) = A^2 + A^2x'^2 \\
x'^2 &= A^2c^2(z) + A^2c^2(z)x'^2 \\
(1-A^2c^2(z))x'^2 &= A^2c^2(z) \\
x' = \frac{\mathrm{d}x}{\mathrm{d}z} &= \frac{Ac(z)}{(1-A^2c^2(z))^{\frac{1}{2}}} \qquad \text{(separable)}
\end{aligned}$$

Integrating (separable) over z,

$$x = \int \frac{Ac(z)}{(1-A^2c^2(z))^{\frac{1}{2}}}\,\mathrm{d}z$$

Making the change of variable $z \mapsto y = c_0 + mz$, $\mathrm{d}z \mapsto \frac{\mathrm{d}y}{m}$,

$$\begin{aligned}
x &= \int \frac{Ay}{(1-A^2y^2)^{\frac{1}{2}}}\frac{\mathrm{d}y}{m} \\
&= \frac{A}{m}\sqrt{1-A^2y^2}\left(-\frac{1}{A^2}\right) \\
&= -\frac{1}{mA}\sqrt{1-A^2c^2}
\end{aligned}$$

Rearranging,

$$\begin{aligned}
1 &= -m^2A^2x^2 + A^2(c_0-mz)^2 \\
\frac{1}{A^2} &= m^2(z^2 - 2c_0z/m - x^2) + c_0^2
\end{aligned}$$

Completing the square,

$$\begin{aligned}
\frac{1}{A_2} &= m^2\left(\left(z+\frac{c_0}{m}\right)^2 + x^2 - \left(\frac{c_0}{m}\right)^2\right) + c_0^2 \\
\frac{1}{m^2A^2} &= \left(z+\frac{c_0}{m}\right)^2 + x^2 \qquad \text{(circle)}
\end{aligned}$$

The above equation describes a circle of radius $(mA)^{-1}$ ($[1/\mathrm{s}]^{-1}[\mathrm{s/m}]^{-1} = [\mathrm{m}]$) and vertical displacement $-c_0/m$ ($[\mathrm{m/s}][1/\mathrm{s}]^{-1} = [\mathrm{m}]$).

In conclusion, the Euler equation therefore predicts a circular ray path by minimizing the function f, which is proportional to the travel time t. Therefore, the circular ray path is the path that minimizes the travel time.