Radiation force exerted by progressive waves on a string in terms of polarizability

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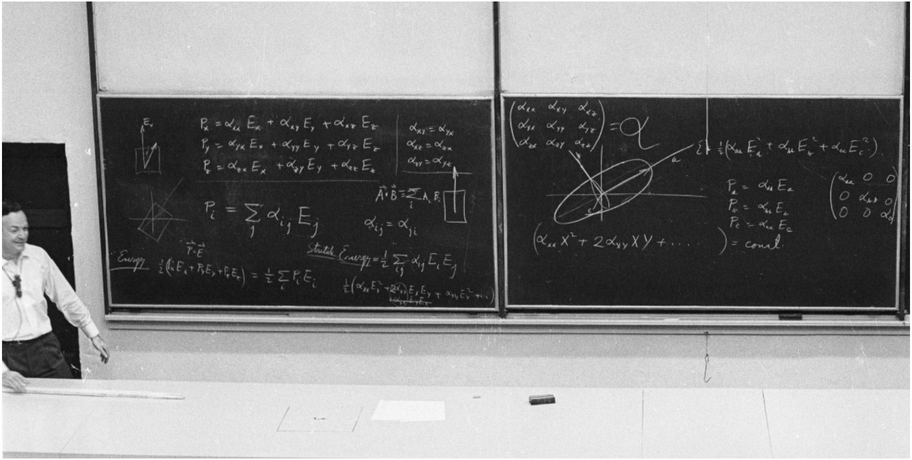
Electric polarizability



From *The Feynman Lectures*, Volume II, Ch. 31. California Institute of Technology, 1962.

Electric polarizability

Summary In anisotropic system, electric polarization is proportional to Electric field but not necessarily in same direction:

$$P_x = \alpha_{xx} E_x + \alpha_{xy} E_y + \alpha_{xz} E_z$$
$$P_y = \alpha_{yx} E_x + \alpha_{yy} E_y + \alpha_{yz} E_z$$
$$P_z = \alpha_{zx} E_x + \alpha_{zy} E_y + \alpha_{zz} E_z.$$

written $P_i = \sum_j \alpha_{ij} E_j$ ($i + j$ go x, y, z)

The set of 9 coef's α_{ij} form a tensor (2^{nd} rank).

(They change when you change coordinate axes)

In this case $\alpha_{ij} = \alpha_{ji}$ + α is a symmetrical tensor. For sym. tensor, axes (principle axes) can always be found so α_{xy} etc $= 0$, only $\alpha_{xx}, \alpha_{yy}, \alpha_{zz} \neq 0$.

Examples Electric conductivity. Moment of inertia.

Stress. Eg. S_{xz} = X-Component of force/per unit area across plane $\perp$ to the z axis

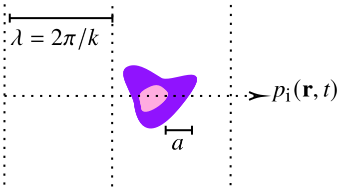
Strain.

There are tensors with more than two indices (no. of indices is called "rank")

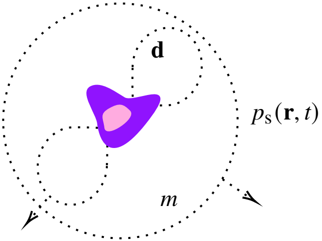
Unit Tensor: $\delta_{ij} = 0$ if $i \neq j$, $= 1$ if $i = j$ (Also called Kronecker delta)

From *The Feynman Lectures*, Volume II, Ch. 31. California Institute of Technology, 1962.

Acoustic polarizability



(a) incident acoustic waves, $ka \ll 1$



(b) scattered acoustic waves

- *Polarizabilities* describe a scatterer's response to incident fields.[1]
- The scattered monopole strength m and dipole moment $\mathbf{d}$ are given by

$$m = -\beta_0 \alpha_\mathrm{m} p_\mathrm{i} - i c_0^{-1} \boldsymbol{\alpha}_\mathrm{c} \cdot \mathbf{v}_\mathrm{i}\,, \tag{1a}$$

$$\mathbf{d} = -i c_0^{-1} \boldsymbol{\alpha}_\mathrm{c} p_\mathrm{i} + \rho_0 \underline{\boldsymbol{\alpha}}_\mathrm{d} \cdot \mathbf{v}_\mathrm{i}\,. \tag{1b}$$

- No exact formulas for α_m, $\underline{\boldsymbol{\alpha}}_\mathrm{d}$, and $\boldsymbol{\alpha}_\mathrm{c}$ in terms of material properties
- Approximations must be made to obtain explicit formulas.

[1] C. F. Sieck, A. Alù, and M. R. Haberman. *Phys. Rev. B.* 96 (2017), 104303.

Closed-form expressions for acoustic polarizability

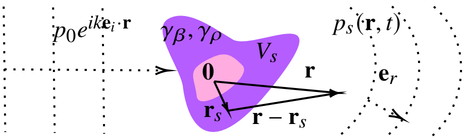
- Consider scatterer with material properties $\gamma_\beta(\mathbf{r}) = \beta_\mathrm{s}(\mathbf{r})/\beta_0 - 1$ and $\gamma_\rho(\mathbf{r}) = 1 - \rho_0/\rho_\mathrm{s}(\mathbf{r})$, where β = compressibility and ρ = density.
- The Born and subwavelength approximations yield the scattered wave,

$$\tilde{p}_\mathrm{s}(\mathbf{r}) = p_0 \frac{e^{ikr}}{r}\Phi\,, \quad \Phi(\mathbf{k}_\mathrm{s}) = \frac{k^2}{4\pi}\left[\alpha_\mathrm{m} + i\boldsymbol{\alpha}_\mathrm{c}\cdot(\mathbf{e}_\mathrm{i} - \mathbf{e}_r) + \alpha_\mathrm{d}\mathbf{e}_\mathrm{i}\cdot\mathbf{e}_r\right]. \tag{2}$$

- The *acoustic polarizabilities* appearing in Eq. (2) are

$$\alpha_\mathrm{m} = \int_{V_\mathrm{s}} \gamma_\beta(\mathbf{r}_\mathrm{s})\,\mathrm{d}V_\mathrm{s}\,, \quad \alpha_\mathrm{d} = \int_{V_\mathrm{s}} \gamma_\rho(\mathbf{r}_\mathrm{s})\mathrm{d}V_\mathrm{s}\,, \tag{3a}$$

$$\boldsymbol{\alpha}_\mathrm{c} = k\left[\int_{V_\mathrm{s}} \gamma_\beta(\mathbf{r}_\mathrm{s})\mathbf{r}_\mathrm{s}\,\mathrm{d}V_\mathrm{s} + \mathbf{e}_\mathrm{i}\cdot\mathbf{e}_r\int_{V_\mathrm{s}} \gamma_\rho(\mathbf{r}_\mathrm{s})\mathbf{r}_\mathrm{s}\,\mathrm{d}V_\mathrm{s}\right]. \tag{3b}$$



Motivation to study waves on a string

- Waves on a string can elucidate acoustic radiation force.[2]
- Scattering of waves on a string is easier than acoustic scattering.[3]
- **Objective:** Study scattering and radiation force for waves on a string using *polarizability*.

[2]Lord Rayleigh. *Philos. Mag.* 3 (1902), 338–346.

[3]P. M. Morse and K. U. Ingard. Princeton, New Jersey: Princeton University Press, 1968.

Outline

Momentum conservation

Polarizability

Uniform scatterer

Inhomogeneous scatterer

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Equation of motion

- The kinetic and potential energy densities are

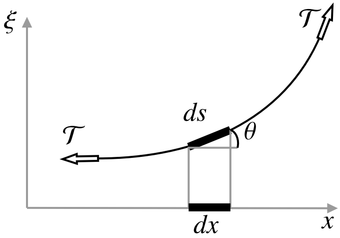
$$T = \tfrac{1}{2}\rho_0(\partial\xi/\partial t)^2\,, \quad U = \tfrac{1}{2}\mathcal{T}(\partial\xi/\partial x)^2\,, \quad \partial\xi/\partial x \ll 1\,. \tag{4}$$

- Insertion of the Lagrangian $L = T - U$ in the Euler-Lagrange equation

$$\frac{\partial L}{\partial \xi} - \frac{\partial}{\partial x}\frac{\partial L}{\partial \xi_x} - \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{\xi}} = 0$$

yields the linear wave equation,

$$\frac{\partial^2\xi}{\partial x^2} - \frac{1}{c_0^2}\frac{\partial^2\xi}{\partial t^2} = 0\,, \quad c_0 = \sqrt{\mathcal{T}/\rho_0}\,. \tag{5}$$



Momentum conservation

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- Multiplying the wave equation [Eq. (5)] by $\partial\xi/\partial x$ yields

$$\frac{\partial g}{\partial t} = \frac{\partial S}{\partial x}\,, \tag{6}$$

where

$$g \equiv I/c_0^2 = \text{momentum density} \tag{7}$$

$$I = -\mathcal{T}(\partial\xi/\partial t)(\partial\xi/\partial x) = \text{intensity} \tag{8}$$

$$S \equiv L - \rho_0(\partial\xi/\partial t)^2 = \text{radiation stress.} \tag{9}$$

- Equation (6) was presented without derivation by Morse and Ingard.[4]
- The integral form of Eq. (6) is

$$S(x_2) - S(x_1) - \frac{d}{dt}\int_{x_1}^{x_2} g\,dx = 0\,, \tag{10}$$

i.e., a wave unperturbed loses no momentum in passing from x_1 to x_2.

[4] P. M. Morse and K. U. Ingard. Princeton, New Jersey: Princeton University Press, 1968, pp. 100–106.

Radiation force

- For time-harmonic solutions, $\langle \frac{d}{dt} \int_{x_1}^{x_2} g \, dx \rangle = 0$.
- If an object between points x_1 and x_2 scatters the wave, then the difference in stress at x_1 and x_2 is the *radiation force*:

$$\boxed{F = \langle S(x_2) - S(x_1) \rangle \, .} \tag{11}$$

Quantity	**Waves on a string**	**Acoustic waves**
kinetic energy	$T = \frac{1}{2}\rho_0(\partial\xi/\partial t)^2$	$T = \frac{1}{2}\rho_0 v^2$
potential energy	$U = \frac{1}{2}\mathcal{T}(\partial\xi/\partial x)^2$	$U = \frac{1}{2}p^2/\rho_0 c_0^2$
intensity	$I = -\mathcal{T}(\partial\xi/\partial t)(\partial\xi/\partial x)$	$\mathbf{I} = p\mathbf{v}$
momentum dens.	$g = I/c_0^2$	$\mathbf{g} = \mathbf{I}/c_0^2$
radiation stress	$S = L - \rho_0(\partial\xi/\partial t)^2$	$\underline{\mathbf{S}} = \underline{\mathbf{I}}L - \rho_0\mathbf{v}\otimes\mathbf{v}$
energy cons.	$\partial I/\partial x + \partial E/\partial t = 0$	$\boldsymbol{\nabla}\cdot\mathbf{I} + \partial E/\partial t = 0$
momentum cons.	$\partial g/\partial t = \partial S/\partial x$	$\partial\mathbf{g}/\partial t = \boldsymbol{\nabla}\cdot\underline{\mathbf{S}}$
radiation force	$F = \langle S(x_2) - S(x_1)\rangle$	$\mathbf{F} = \oint \langle\underline{\mathbf{S}}\rangle\cdot d\mathbf{A}$

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Scattering from an extended mass on a string

- For $\xi(x,t) = \mathrm{Re}[\tilde{\xi}(x)e^{-i\omega t}]$, the inhomogeneous Helmholtz equation is[5]

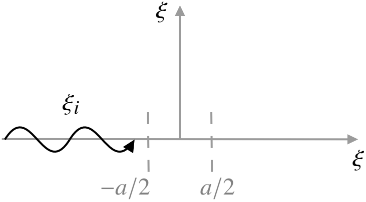
$$\frac{d^2\tilde{\xi}}{dx^2} + k^2\tilde{\xi}(x) = -k^2\mu(x)\tilde{\xi}(x)\,, \tag{12}$$

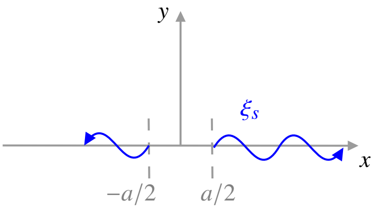
  where $\mu(x) = \rho_s(x)/\rho_0 - 1$ and $\rho_s(x)$ = density of scatterer.

- The solution of Eq. (12) is *implicit*:

$$\tilde{\xi}(x) = \tilde{\xi}_{\mathrm{i}}(x) + \tilde{\xi}_{\mathrm{s}}(x)\,, \tag{13a}$$

$$\tilde{\xi}_{\mathrm{s}}(x) = \tfrac{1}{2}ik\int_{-a/2}^{a/2} e^{ik|x-x_{\mathrm{s}}|}\mu(x_{\mathrm{s}})\tilde{\xi}(x_{\mathrm{s}})\,dx_{\mathrm{s}}\,. \tag{13b}$$





[5] P. M. Morse and K. U. Ingard. Princeton, New Jersey: Princeton University Press, 1968, Eq. (4.5.11) and p. 160.

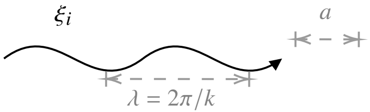
Two approximations

1. **Born**: If $|\mu| \ll 1$, then $|\xi_s| \ll |\xi_i|$, making Eq. (13) explicit:

$$\tilde{\xi}_{\mathrm{s}}(x) = \begin{cases} \frac{1}{2} ik\xi_0 e^{-ikx} \int_{-a/2}^{a/2} e^{2ikx_{\mathrm{s}}} \mu(x_{\mathrm{s}})\, dx_{\mathrm{s}}\,, & x \le -a/2\,, \\ \frac{1}{2} ik\xi_0 e^{ikx} \int_{-a/2}^{a/2} \mu(x_{\mathrm{s}})\, dx_{\mathrm{s}}\,, & x \ge a/2\,. \end{cases} \tag{14}$$

2. **Subwavelength**: Since $ka \sim |k_s x_{\mathrm{s}}| \ll 1$, the complex exponentials in the integrand of the first line of Eq. (14) can be Taylor expanded to linear order:

$$\begin{aligned} \tilde{\xi}_{\mathrm{s}}(x) &= \frac{1}{2} ik\xi_0 e^{-ikx} \int_{-a/2}^{a/2} (1 + 2ikx_{\mathrm{s}})\mu(x_{\mathrm{s}})\, dx_{\mathrm{s}}\,, \quad x \le -a/2 \\ &= \frac{1}{2} ik\xi_0 e^{-ikx} \left[\int_{-a/2}^{a/2} \mu(x_{\mathrm{s}})\, dx_{\mathrm{s}} + 2ik \int_{-a/2}^{a/2} \mu(x_{\mathrm{s}}) x_{\mathrm{s}}\, dx_{\mathrm{s}} \right]. \end{aligned} \tag{15}$$



Scattered field in terms of polarizabilities

- The approximations made in Eqs. (14) and (15) yield

$$\tilde{\xi}_s(x) = \begin{cases} \frac{1}{2} ik\xi_0 e^{-ikx}(\alpha_0 + i\alpha_1)\,, & x \le -a/2 \\ \frac{1}{2} ik\xi_0 e^{ikx}\alpha_0\,, & x \ge a/2 \end{cases} \tag{16}$$

where the polarizabilities (dimensions of length) are

$$\boxed{\begin{aligned} \alpha_0 &= \int_{-a/2}^{a/2} \mu(x_s)\, dx_s && (17a) \\ \alpha_1 &= 2k \int_{-a/2}^{a/2} \mu(x_s) x_s\, dx_s && (17b) \end{aligned}}$$

- In quasistatic limit and/or in the absence of asymmetry, $\alpha_1 \to 0$.
- Combining Eqs. (13) and (16) yields the total field:[6]

$$\tilde{\xi}(x) = \begin{cases} \xi_0[e^{ikx} + \frac{1}{2} ike^{-ikx}(\alpha_0 + i\alpha_1)]\,, & x \le -a/2\,, \\ \xi_0 e^{ikx}(1 + \frac{1}{2} ik\alpha_0)\,, & x > a/2\,. \end{cases} \tag{18}$$

[6] P. M. Morse and K. U. Ingard. Princeton, New Jersey: Princeton University Press, 1968, compare to Eq. (4.5.15).

Radiation force in terms of polarizabilities

- Evaluating $\langle S \rangle$ in Eq. (11) yields the radiation force:

$$\boxed{F = \langle S(a/2) - S(-a/2) \rangle = \tfrac{1}{8}\mathcal{T} k^4 \xi_0^2 \alpha_1^2 \,.} \tag{19}$$

- Check dimensions: $k^4 \xi_0^2 \alpha_1^2$ is dimensionless and $\mathcal{T}$ has units of force.

- Recall from Eq. (17b) that $\alpha_1 \propto k$. To leading order, $F \propto k^6$.

- Since α_0 does not appear in Eq. (19), symmetric Born-subwavelength scatterers on a string experience zero radiation force at leading order.

- Since $\langle S \rangle$ is independent of x on either side of the scatterer,

$$F = \langle S(x_2) - S(x_1) \rangle \,, \tag{20}$$

where x_1 and x_2 are any points behind/ahead of scatterer, respectively.

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Uniform scatterer with $\mu = \rho_1/\rho_0 - 1$

- From Eqs. (17), the polarizabilities are $\alpha_1 = 0$ and

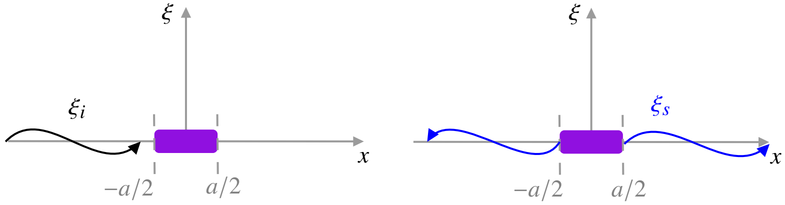
$$\alpha_0 = \int_{-a/2}^{a/2} \mu(x_\mathrm{s})\, dx_\mathrm{s} = a\rho_1/\rho_0\,. \tag{21}$$

- From Eq. (19) (Born and subwavelength approximations),

$$F = 0\,. \tag{22}$$

- Meanwhile, according to Born scattering [Eq. (14)],[7]

$$\tilde{\xi}_\mathrm{s} = \begin{cases} \frac{1}{2} i\xi_0 e^{-ikx}(\rho_1/\rho_0)\sin ka, & x \le -a/2 \\ \frac{1}{2} i\xi_0 e^{ikx}(\rho_1/\rho_0)ka\,, & x \ge a/2\,. \end{cases} \tag{23}$$



[7] P. M. Morse and K. U. Ingard. Princeton, New Jersey: Princeton University Press, 1968, Eqs. (4.5.29).

Uniform scatterer with $\mu = \rho_1/\rho_0 - 1$

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- Nondimensionalizing the forces to $\mathcal{T}(\xi_0/a)^2$ yields

$$\frac{F}{\mathcal{T}(\xi_0/a)^2} = 0 \qquad \text{(Born subwavelength)}$$

$$\frac{F}{\mathcal{T}(\xi_0/a)^2} = -\tfrac{1}{8}(ka)^2(\rho_1/\rho_0)\left[(ka)^2 - \sin^2 ka\right] \qquad \text{(Born)}$$

$$\simeq -\tfrac{1}{24}(\rho_1/\rho_0)\left[(ka)^6 - \tfrac{1}{12}(ka)^8\right].$$

where $\rho_1/\rho_0 \lesssim 0.4$.[8] *To leading order, the force is negative!*

- Born-subwavelength predicts zero force on scatterer because magnitudes of fields ahead and behind scatterer are equal

- Pulling forces have been studied extensively in acoustics.[9]

[8] P. M. Morse and K. U. Ingard. Princeton, New Jersey: Princeton University Press, 1968, p. 161.

[9] P. L. Marston. *J. Acoust. Soc. Am.* 120 (2006), 3518–3524; X.-D. Fan and L. Zhang. *J. Acoust. Soc. Am.* 150 (2021), 102–110; Y. Li et al. *Chinese Phys. B* 29 (2020), 054302.

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Inhomogeneous scatterer with $\mu(x) = (2x/a)\rho_1/\rho_0$

- From Eqs. (17), the polarizabilities are $\alpha_0 = 0$ and

$$\alpha_1 = 4(k/a)(\rho_1/\rho_0)\int_{-a/2}^{a/2} x_s^2\, dx_s = \tfrac{1}{3}ka^2\rho_1/\rho_0\,. \tag{25}$$

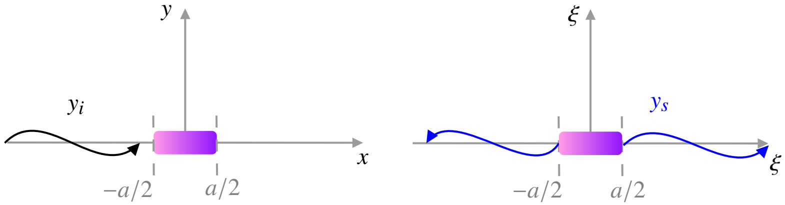
- From Eq. (19) (Born and subwavelength approximations),

$$F = \tfrac{1}{72}\mathcal{T}(\xi_0/a)^2(\rho_1/\rho_0)^2(ka)^6\,. \tag{26}$$

- Meanwhile, according to Born scattering [Eq. (14)],

$$\tilde{\xi}_s = \begin{cases} -\tfrac{1}{2}\xi_0 e^{-ikx}\frac{\rho_1}{\rho_0}\left(\frac{\sin ka}{ka} - \cos ka\right), & x \le -a/2 \\ 0\,, & x \ge a/2\,. \end{cases} \tag{27}$$

- Equation (27) shows that scattered field is direction-dependent.



Inhomogeneous scatterer with $\mu(x) = (2x/a)\rho_1/\rho_0$

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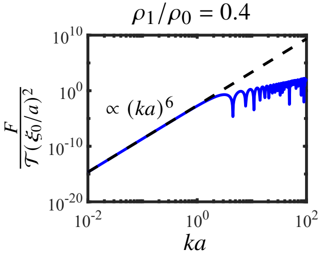
- Nondimensionalizing the forces to $\mathcal{T}(\xi_0/a)^2$ yields

$$\frac{F}{\mathcal{T}(\xi_0/a)^2} = \tfrac{1}{72}(\rho_1/\rho_0)^2(ka)^6 \qquad \text{(Born subwavelength)}$$

$$\frac{F}{\mathcal{T}(\xi_0/a)^2} = \tfrac{1}{8}(\rho_1/\rho_0)^2\left(\sin ka - ka\cos ka\right)^2. \qquad \text{(Born)}$$

- The force is in the direction of the incident wave.
- Magnitude of force is $3\times$ weaker than that on homogeneous scatterer.



Conclusion

Summary

- Derived the **momentum conservation equation** at $O(\xi^2)$
- Solved the **linear scattering problem** for extended scatterers
- Obtained the **radiation force** in terms of **polarizabilities**
- Compared the results to **Born scattering** for two examples

Future work

- Radiation force due to **standing waves** on a string
- Radiation force due to progressive waves in **2D** (water waves)

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Prof. M. R. Haberman

Prof. M. F. Hamilton

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