WAVES IN LAYERED MEDIA

Second Edition

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CHAPTER IV

REFLECTION AND REFRACTION OF SPHERICAL WAVES

IN the theory of propagation of electromagnetic and acoustic waves, it is usually necessary to take into account the finite distance of the source of the waves, both from the receiver and from the boundaries of the medium. The classical, and simplest, problem of such a nature is the problem of the field of a point source located at a finite distance from a plane interface between two homogeneous media. In other words, this problem is one of the reflection and refraction of a spherical wave. The present chapter will be devoted to it. The first to treat this problem for electromagnetic waves fairly completely was Sommerfeld [219]. Subsequently, there appeared the fundamental studies of Weyl [255], Fock (see [77, Chapter 23], Leontovich [138], Leontovich and Fock [139], and Baños [8].

We shall follow basically our own work [19, 20]. With the help of the method set forth in them, we can study the cases that have been considered to date by various methods from a unified point of view (for example, the case of a source on the interface and of an elevated source, the case of moderate and large conductivity of one of the media, the acoustic case, and so on). The problem of the refraction of spherical waves can be solved by the same method. The basic method involves the use of further development of the proposition of Weyl [255] on the expansion of the spherical waves in terms of plane waves. We shall consider only the case of a harmonic wave. The theory of the reflection of a spherical pulse has been developed, for example, in the work of Town [238]. Several features of the reflection from a plane interface of a pulse emitted from a linear source were studied in the work of Pierce [183]. In particular, the characteristics of the entry of reflected and refracted waves were analyzed in detail and compared with those which had been found earlier in the approximation of geometrical acoustics by Friedrichs and Keller [80]. Several theoretical and experimental results relative to the reflection of acoustic waves can be found in Ingard [107], Lawhead and Rudnick [137], and Rudnick [198].

26.2. *Elementary source in acoustics*

In acoustics the simplest source is a pulsating sphere of small radius. The acoustic pressure or acoustical potential of this source will also be expressed in the form of a spherical wave of the kind indicated above. If we again limit ourselves to the sinusoidal mode of the source and assume that the radius of the sphere is small in comparison with the wavelength, then the acoustic potential at a distance R from the sphere will be expressed by the formula [81]

$$\psi = (V_0/4\pi R)e^{i(kR-\omega t)},$$

where $V_0 = 4\pi r_0^2 v_0$ is the so-called volume velocity of the source and is equal to the product of the surface of the area of the sphere and the amplitude of the velocity of its surface.

In what follows we shall study the reflection and refraction of a spherical wave without regard to whether an electromagnetic or acoustic field is the case in point.

26.3. *Decomposition of a spherical wave into plane waves*

The difficulty of the problem of the reflection and refraction of a spherical wave on a plane interface between two media is due to the difference between the symmetries of the wave and the form of the boundary. While the wave has spherical symmetry, the boundary is a plane. It is therefore natural to solve the problem by expanding the spherical wave in plane waves, the more so that the theory of the reflection and refraction of plane waves is now well known to us.

Omitting the factor $e^{-i\omega t}$ everywhere, and also the factors which characterize the power of the source, we write down the spherical wave in the form e^{ikR}/R. Furthermore, assuming for the time being that the source is located at the origin, we have $R = (x^2 + y^2 + z^2)^{1/2}$.

In the plane $z = 0$ the field of the spherical wave will have the form e^{ikr}/r, where $r = (x^2 + y^2)^{1/2}$. We expand this field in a double Fourier integral in the variables x and y:

$$e^{ikr}/r = \iint_{-\infty}^{\infty} A(k_x, k_y)e^{i(k_x x + k_y y)}\, dk_x\, dk_y. \tag{26.11}$$

For the determination of $A(k_x, k_y)$, we have, according to the well-known formula,

$$(2\pi)^2 A(k_x, k_y) = \iint_{-\infty}^{\infty} (e^{ikr}/r)e^{-i(k_x x + k_y y)}\, dx\, dy. \tag{26.12}$$

We now transform to polar coordinates and define

$$k_x = q\cos\psi, \qquad k_y = q\sin\psi, \qquad q = (k_x^2 + k_y^2)^{1/2},$$
$$x = r\cos\phi, \qquad y = r\sin\phi, \qquad dx\,dy = r\,dr\,d\phi. \tag{26.13}$$

We then obtain

$$(2\pi)^2 A(k_x, k_y) = \int_0^{2\pi} d\phi \int_0^{\infty} e^{ir[k - q\cos(\psi-\phi)]}\,dr.$$

The integral over r is elementary. Moreover, if we assume that there is some absorption in the medium, no matter how small, i.e., that k has a positive imaginary part, then the substitution of the upper limit yields zero and we obtain

$$(2\pi)^2 A(k_x, k_y) = i\int_0^{2\pi} \frac{d\phi}{k - q\cos(\psi-\phi)} = \frac{i}{k}\int_0^{2\pi} \frac{d\delta}{1-(q/k)\cos\delta}.$$

Substituting the value of the tabular integral, we find

$$A(k_x, k_y) = \frac{i}{2\pi}\frac{1}{(k^2-q^2)^{1/2}} = \frac{i}{2\pi(k^2-k_x^2-k_y^2)^{1/2}}. \tag{26.14}$$

Thus

$$\frac{e^{ikr}}{r} = \frac{i}{2\pi}\iint_{-\infty}^{\infty} \frac{e^{i(k_x x + k_y y)}}{(k^2-k_x^2-k_y^2)^{1/2}}\,dk_x\,dk_y. \tag{26.15}$$

Just as was done in Section 14 for the bounded beam, the latter expression, which describes the field in the xy plane, is easily "continued" in space. As is known, each Fourier component here will correspond to a plane wave in space. From the formal side, it suffices, in such a "continuation," to add the following term in the exponent under the integral:

$$\pm\, ik_z z, \qquad \text{where} \qquad k_z \equiv (k^2-k_x^2-k_y^2)^{1/2}. \tag{26.16}$$

The plus sign corresponds to points lying in the half-space $z > 0$ and to waves propagating in the direction of positive z. The minus sign corresponds to points for which $z < 0$. Thus

$$z \geqslant 0, \qquad \frac{e^{ikR}}{R} = \frac{i}{2\pi}\iint_{-\infty}^{\infty} e^{i(k_x x + k_y y + k_z z)}\frac{dk_x\,dk_y}{k_z},$$
$$z \leqslant 0, \qquad \frac{e^{ikR}}{R} = \frac{i}{2\pi}\iint_{-\infty}^{\infty} e^{i(k_x x + k_y y - k_z z)}\frac{dk_x\,dk_y}{k_z}. \tag{26.17}$$

The validity of the "continuation" carried out here is based on the fact that the right side of the latter expression satisfies the wave equation (since the integrand satisfies it) and gives the correct value of the field for $z = 0$.

Equation (26.17) also represents the expansion of a spherical wave into plane waves. The exponent under the integral is that of a plane wave, the direction of propagation of which is given by the values of the components of the wave vector k_x, k_y, k_z.

The integration in (26.17) over the horizontal components k_x and k_y of the wave vector $\mathbf{k}$ can be replaced by integration over the angles θ and ϕ, which characterize the direction of propagation of each of the plane waves (see Fig. 26.1). Here

$$k_x = k \sin\theta \cos\phi, \qquad k_y = k \sin\theta \sin\phi, \qquad k_z = k \cos\theta. \tag{26.18}$$

We shall carry out the integration over ϕ between the limits zero and 2π. Integration over θ cannot be restricted to the real values of this angle. According to (26.16), k_z changes from the value $k_z = k$ for $k_x = k_y = 0$ to $k_z \to i\infty$ for $k_x \to \pm\infty$ or $k_y \to \pm\infty$.

Since we have $\cos\theta = k_z/k$ from (26.18), θ will change here from $\theta = 0$ to $\theta = \pi/2 - i\infty$. The path of integration over θ is chosen in the form of the contour Γ_0 shown in Fig. 26.2.

With the help of the formulas for the transformation of variables, we obtain $dk_x\, dk_y\, dk_z = k_0 \sin\theta\, d\theta\, d\phi$.

As a result, the expansion (26.17) can also be written in the form

$$z \geqslant 0, \qquad \frac{e^{ikR}}{R} = \frac{ik}{2\pi}\int_0^{\pi/2-i\infty}\int_0^{2\pi} e^{i(k_x x + k_y y + k_z z)} \sin\theta\, d\theta\, d\phi,$$

$$z \leqslant 0, \qquad \frac{e^{ikR}}{R} = \frac{ik}{2\pi}\int_0^{\pi/2-i\infty}\int_0^{2\pi} e^{i(k_x x + k_y y - k_z z)} \sin\theta\, d\theta\, d\phi, \tag{26.19}$$

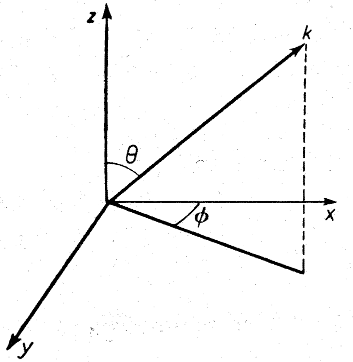

Fig. 26.1. The spatial position of the vector $\mathbf{k}$ and the orientation of the angles of θ and ϕ.

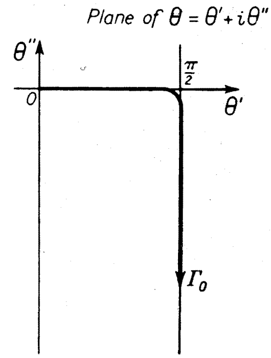

Fig. 26.2. The path of integration in the complex plane.

where k_x, k_y, and k_z are expressed in terms of θ and ϕ with the help of (26.18).

We thus see that in the expansion of a spherical wave, in addition to the usual waves of all possible directions in the angular ranges $0 \leqslant \phi \leqslant 2\pi$, $0 \leqslant \theta \leqslant /2$, there are also waves corresponding to the complex angles θ. Waves of such a type, also known as inhomogeneous waves, were considered in Section 1. At the points $\theta = \pi/2 - ia$, which correspond to the contour of integration Γ_0 (see Fig. 26.2), where a is a real positive quantity, these waves are propagated with a reduced wavelength along some direction in the xy plane (given by the angle ϕ) and fall off exponentially in amplitude in the z direction.

It is impossible to obtain the field which would have the required singularity as $R \to 0$ by superposition of ordinary plane waves only. However, it is not difficult to understand from graphic considerations how such a singularity is obtained with the use of inhomogeneous waves.

Setting $\theta = \pi/2 - ia$, we obtain the following from (26.18) for the components of the wave vector of the inhomogeneous waves:

$$k_x = k \cos\phi \cosh a, \qquad k_y = k \sin\phi \cosh a, \qquad k_z = i \sinh a. \tag{26.20}$$

As $a \to \infty$, we get $k_x \to \infty \cos\phi$, $k_y \to \infty \sin\phi$, $k_z \to i\infty$. This means that we have waves propagating in the horizontal plane (the xy plane) with a wavelength approaching zero and simultaneously attenuating in the vertical direction with an attenuation coefficient approaching infinity.

At $x = y = z = 0$ the superposition of an infinite number of these waves (the integral (26.19)) gives an infinite value for the field. In moving away from this point, we obtain finite values, either because of attenuation (for $z \neq 0$) or because of phase interference (for $x \neq 0$ or $y \neq 0$).

It should be noted that the direction of the coordinate axes in the decomposition of a spherical wave into plane waves given above can be chosen arbitrarily. We can therefore decompose the spherical wave into plane waves in such a way that the inhomogeneous waves appearing in this decomposition attenuate not in the direction of the z axis but in any other direction given in advance.

Finally, we note that we shall frequently replace the angle of incidence θ by its complementary angle of inclination χ in what follows. Replacing θ in (26.19) by $\theta = \pi/2 - \chi$ gives (for $z \geqslant 0$)

$$\frac{e^{ikR}}{R} = -\frac{ik}{2\pi}\int_{\pi/2}^{i\infty}\int_0^{2\pi} e^{i(k_x x + k_y y + k_z z)} \cos\chi \, d\chi \, d\phi, \tag{26.21}$$

where $k_x = k\cos\chi\cos\phi$, $k_y = k\cos\chi\sin\phi$, $k_z = k\sin\chi$.

The path of integration over χ runs from $\chi = \pi/2$ along the real axis to $\alpha = 0$ and then along the imaginary axis to $\chi = i\infty$.

Here we have considered the decomposition into plane waves of a harmonic spherical wave, the time dependence of the field of which is given by the factor $e^{-i\omega t}$. A similar decomposition is given in Poritsky [187] for a spherical wave of the form $(1/R)F(ct - R)$ with an arbitrary function F.

26.4. *Reflected wave in the integral representation*

Let a spherical wave be radiated at the point 0 at a distance z_0 from the interface (Fig. 26.3). In consideration of reflection from the interface, the total field will be

$$\psi = R^{-1}e^{ikR} + \psi_{\text{refl}}, \tag{26.22}$$

where ψ_{refl} is the reflected wave, the analysis of which is now our problem.

We shall assume in what follows that the origin of the rectangular set of coordinates is placed at the interface between the media (Fig. 26.3). The decomposition of a spherical wave incident on the boundary into plane waves will be written here in the form of Eq. (26.19), where we have $z - z_0$ in place of z.

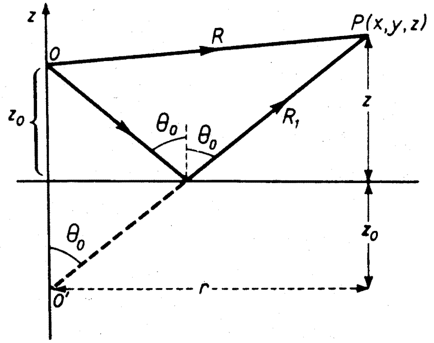

Fig. 26.3. The positions of the source O and the point of observation P with respect to the interface.

The reflected wave ψ_{refl} can obviously be represented in the form of the superposition of plane waves obtained from the reflection of the plane waves into which the original spherical wave was decomposed. In the reflection of each plane wave, its amplitude should be multiplied by the reflection coefficient $V(\theta)$, where θ is the angle of incidence of the wave; moreover, the phase change of the wave in its passage from the source to the boundary and then back to the point of observation should be taken into account.

Since the projection of the path traveled by this wave along the coordinate axes is x, y, and $z + z_0$, the expression for the reflected plane wave will be

$$V(\theta) \exp i\left[k_x x + k_y y + k_z(z + z_0)\right], \tag{26.23}$$

where k_x, k_y, and k_z are given as before by Eqs. (26.18).

Integrating over all plane waves, we get the expression ψ_{refl} in a form analogous to (26.19):

$$\psi_{\text{refl}} = (ik/2\pi)\int_0^{\pi/2 - i\infty}\int_0^{2\pi} e^{ik[x \sin\theta\cos\phi + y\sin\theta\sin\phi + (z+z_0)\cos\theta]}$$
$$\times V(\theta)\sin\theta\, d\theta\, d\phi. \tag{26.24}$$

Integration over θ and ϕ is performed in the same limits as in the expression (26.19) for the primary spherical wave.

The latter formula can be written in another form, taking it into account that the integral over ϕ reduces to the Bessel function of zero order. Actually, we set $x = r\cos\phi_1, y = r\sin\phi_1$. We then have in (26.24)

$$\int_0^{2\pi} e^{ik(x\cos\phi + y\sin\phi)\sin\theta}\, d\phi = \int_0^{2\pi} e^{ikr\sin\theta\cos(\phi-\phi_1)}\, d\phi = 2\pi J_0(u), \tag{26.25}$$

where $u = kr\sin\theta$. As a result, (26.24) is written as

$$\psi_{\text{refl}} = ik\int_0^{\pi/2 - i\infty} J_0(u) e^{ik(z+z_0)\cos\theta} V(\theta)\sin\theta\, d\theta. \tag{26.26}$$

The latter expression is also conveniently transformed by changing the limits of integration and replacing the Bessel function by Hankel functions. For this we note that

$$J_0(u) = \tfrac{1}{2}\left[H_0^{(1)}(u) + H_0^{(2)}(u)\right],$$

where $H_0^{(1)}$ and $H_0^{(2)}$ are the Hankel functions of first and second kind.

Substituting the last expression in (26.26), we divide the integral into two integrals. In this case we replace θ by $-\theta$ in the one in which $H_0^{(2)}$ appears, and take it into account that (see Watson [252, p. 89]) $H_0^{(2)}(e^{-\pi i}u) = -H_0^{(1)}(u)$ and that $V(-\theta) = V(\theta)$. We then obtain two integrals with identical integrand expressions, but one is taken in the limits from 0 to $\pi/2 - i\infty$, while the limits of the second will be from $-\pi/2 + i\infty$ to 0. Combining the two integrals into one, taken in the limits from $-\pi/2 + i\infty$ to $\pi/2 - i\infty$ along the path Γ_1 (see Fig. 28.1 below), we get

$$\psi_{\text{refl}} = (ik/2)\int_{-\pi/2+i\infty}^{\pi/2-i\infty} H_0^{(1)}(u)e^{ik(z+z_0)\cos\theta}V(\theta)\sin\theta\, d\theta. \quad (26.27)$$

Using this formula, we can calculate the wave reflected not only from the interface between two inhomogeneous media but also from any inhomogeneous layer, substituting the corresponding reflection coefficient $V(\theta)$†.

The expressions for the reflection coefficients of acoustic and electromagnetic waves from the boundary between two inhomogeneous media were calculated in Sections 2 and 4, respectively.

§ 27. Method of Steepest Descent. Reference Integrals

In the following, we shall be interested in the analysis of a reflected wave given by the integral (26.27) in the wave zone, i.e., at distances from the source that are large in comparison with the wavelength. It is then possible here to represent this in such a fashion that the fundamental role is played by only the plane waves, the direction of which is close to the direction of the ray $O'P$ (Fig. 26.3), which corresponds to reflection according to the laws of geometrical optics.

27.1. *Method of steepest descent*

The convenient mathematical technique here is the so-called method of steepest descent. It serves to estimate the values of integrals of the form

$$I = \int_C e^{\rho f(\zeta)}F(\zeta)\, d\zeta \quad (27.1)$$

for large values of the parameter ρ. Here $f(\zeta)$ and $F(\zeta)$ are certain, in general rather arbitrary, analytic functions of the complex variable ζ; C is the path of integration in the ζ plane, which in a special case may involve only real values of ζ.

The general theory of the method of steepest descent will be found in the literature [34, 50, 72, 111, 161, 203]. Roughly, this method reduces to the following. The path of integration in the complex plane can be deformed, within certain limits, without changing the value of the integral.

†Based on the method outlined here, Voit [246] considered the problem of reflection of a spherical wave in the transition from a fixed medium to a moving one.