Green's Functions

Definition:

$$\mathcal{L}\{g(\vec{r}|\vec{r}_0)\} = -\delta(\vec{r}-\vec{r}_0) \qquad (1)$$

where

$\mathcal{L}$ = linear operator
(e.g., $\mathcal{L} = \nabla^2 + k^2$)
g = Green's function

Consider general inhomo. eqn.

$$\mathcal{L}\{\phi(\vec{r})\} = -f(\vec{r}) \qquad (2)$$

Let

$$f(\vec{r}) = \iiint f(\vec{r}_0)\,\delta(\vec{r}-\vec{r}_0)\,dV_0$$

Then

$$\mathcal{L}\{\phi(\vec{r})\} = -\iiint f(\vec{r}_0)\,\delta(\vec{r}-\vec{r}_0)\,dV_0$$

$$= \iiint f(\vec{r}_0)\,\mathcal{L}\{g(\vec{r}|\vec{r}_0)\}\,dV_0, \text{ from (1)}$$

$$= \mathcal{L}\left\{\iiint f(\vec{r}_0)\,g(\vec{r}|\vec{r}_0)\,dV_0\right\}, \text{ linearity}$$

Particular sol'n of (2) is thus

$$\phi(\vec{r}) = \iiint f(\vec{r}_0)\,g(\vec{r}|\vec{r}_0)\,dV_0$$

where

$g(\vec{r}|\vec{r}_0)$ = free space Green's fn.

is defined by (1). g satisfies $\mathcal{L}\{g(\vec{r}|\vec{r}_0)\}=0$ everywhere but $\vec{r}=\vec{r}_0$.

1D (plane waves, strings, etc.)

$$\frac{\partial^2 g}{\partial x^2} + k^2 g = -\delta(x-x_0) \qquad (1)$$

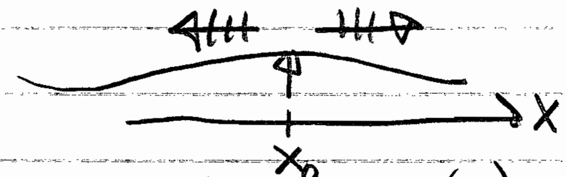
Sol'n must be of form

$$g(x|x_0) = A e^{ik(x-x_0)}, \quad x > x_0$$
$$= A e^{-ik(x-x_0)}, \quad x < x_0$$

or

$$g(x|x_0) = A e^{ik|x-x_0|}, \quad x \neq x_0 \qquad (2)$$

$\int_{x_0-\varepsilon}^{x_0+\varepsilon} (1)\, dx$:



$$\left.\frac{\partial g}{\partial x}\right|_{x_0-\varepsilon}^{x_0+\varepsilon} + k^2 \int_{x_0-\varepsilon}^{x_0+\varepsilon} g\, dx = -1 \qquad (3)$$

where $\left.\frac{\partial g}{\partial x}\right|_{x_0-\varepsilon}^{x_0+\varepsilon} = ikAe^{ik\varepsilon} - (-ik)Ae^{ik\varepsilon}$

$$\rightarrow 2ikA, \quad \varepsilon \to 0$$

$$\int_{x_0-\varepsilon}^{x_0+\varepsilon} g\, dx = \left(\int_{x_0-\varepsilon}^{x_0} + \int_{x_0}^{x_0+\varepsilon} \right) g\, dx$$

$$= \frac{A}{ik}\left(1 - e^{ik\varepsilon}\right) + \left(\frac{A}{-ik}\right)\left(e^{-ik\varepsilon} - 1\right)$$
$$\rightarrow 0, \quad \varepsilon \to 0$$

So $A = -\frac{1}{2ik}$ and

$$\boxed{g(x|x_0) = \frac{i}{2k} e^{ik|x-x_0|}}$$

<u>3D</u> (point source in free space)

$$\nabla^2 g + k^2 g = -\delta(\vec{r}-\vec{r}_0) \qquad (1)$$

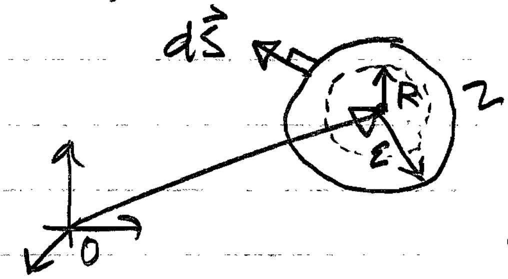
Solutions must have form

$$g(\vec{r}|\vec{r}_0) = A\frac{e^{ikR}}{R}$$

where

$$R = |\vec{r}-\vec{r}_0| = \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}$$

Integrate (1) over volume of sphere, radius ε, centered at $\vec{r}=\vec{r}_0$:



$S = 4\pi\varepsilon^2$
$V = \frac{4}{3}\pi\varepsilon^3$
$dV = 4\pi R^2 dR$

$$\iiint \nabla^2 g\, dV + k^2 \iiint g\, dV = -1$$

where

$$\iiint g\, dV = \int_0^\varepsilon A\frac{e^{ikR}}{R}4\pi R^2 dR$$

$$= 4\pi A\int_0^\varepsilon e^{ikR} R\, dR$$

$$= 4\pi A\frac{e^{ikR}}{ik}\left(R-\frac{1}{ik}\right)\Big|_0^\varepsilon \to 0,\ \varepsilon\to 0$$

$$\iiint \nabla^2 g\, dV = \iiint \vec{\nabla}\cdot(\vec{\nabla}g)\, dV$$

$$= \iint (\vec{\nabla}g)\cdot d\vec{s} \quad \text{divergence (Gauss's) thm.}$$

$$= \iint \frac{\partial g}{\partial R} \, dS$$

$$= \iint \left(-\frac{1}{R} + ik\right) A \frac{e^{ikR}}{R} \, dS$$

$$= \left(-\frac{1}{\cancel{\varepsilon}} + ik\right) A \frac{e^{ik\varepsilon}}{\cancel{\varepsilon}} \cdot 4\pi \cancel{\varepsilon^2}$$

$$\to -4\pi A, \quad \varepsilon \to 0$$

thus
$$\boxed{g(\vec{r}|\vec{r}_0) = \frac{e^{ik|\vec{r}-\vec{r}_0|}}{4\pi|\vec{r}-\vec{r}_0|} = \frac{e^{ikR}}{4\pi R}}$$

<u>2D</u> (line source in free space, point source on membrane)

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) g + k^2 g = -\delta(x-x_0)\delta(y-y_0)$$

Equivalently,

$$\nabla^2 g + k^2 g = -f(x,y) \qquad (1)$$

where

$$f(x,y) = \delta(x-x_0)\delta(y-y_0)$$

<u>Use 3D sol'n of (1):</u>

$$g \to g(\vec{\rho}|\vec{\rho}_0) = \iiint f(x_1, y_1)\, g(\vec{r}|\vec{r}_1)\, dV_1$$

where $g(\vec{r}|\vec{r}_1) = \dfrac{e^{ik|\vec{r}-\vec{r}_1|}}{4\pi|\vec{r}-\vec{r}_1|}$

$$= \iiint \delta(x_1 - x_0)\,\delta(y_1 - y_0) \cdot \frac{e^{ik\sqrt{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2}}}{4\pi\sqrt{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2}}\,dV_1$$

$$= \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{e^{ik\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_1)^2}}}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_1)^2}}\,dz_1$$

Let

$$(x-x_0)^2 + (y-y_0)^2 = |\vec{\rho} - \vec{\rho}_0|^2, \quad k(z-z_1) = -t \qquad dz_1 = \frac{dt}{k}$$

Then

$$g(\vec{\rho}\,|\,\vec{\rho}_0) = \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{e^{i\sqrt{k^2|\vec{\rho}-\vec{\rho}_0|^2+t^2}}}{\sqrt{k^2|\vec{\rho}-\vec{\rho}_0|^2+t^2}}\,dt$$

Hankel fn of first kind, order 0:

$$H_0^{(1)}(z) = \frac{1}{i\pi}\int_{-\infty}^{\infty} \frac{e^{i\sqrt{z^2+t^2}}}{\sqrt{z^2+t^2}}\,dt$$

thus

$$\boxed{g(\vec{\rho}\,|\,\vec{\rho}_0) = \frac{i}{4} H_0^{(1)}(k|\vec{\rho}-\vec{\rho}_0|)}$$

Since

$$H_0^{(1)}(z) \sim \sqrt{\frac{2}{\pi z}}\, e^{i(z-\pi/4)}, \quad |z| \gg 1$$

$$g(\vec{\rho}\,|\,\vec{\rho}_0) \sim \frac{e^{i\pi/4}}{\sqrt{8\pi}} \frac{e^{ik|\vec{\rho}-\vec{\rho}_0|}}{\sqrt{|\vec{\rho}-\vec{\rho}_0|}}, \quad k|\vec{\rho}-\vec{\rho}_0| \gg 1$$