Analytical solutions for acoustic vortex beam radiation from planar and spherically focused circular pistons

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The University of Texas at Austin
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Outline

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Our publications on this work

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Paraxial and ray approximations of acoustic vortex beams

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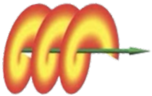
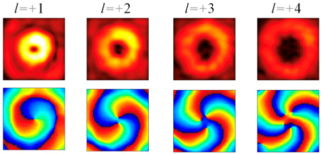
What is an acoustic vortex beam?

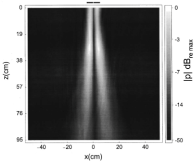
Characterized by...

- helical wavefronts
- orbital number ℓ = number of equiphase wavefronts in $\perp$ plane
- zero acoustic pressure on axis





C. Shi et al. *P. Natl. Acad. Sci. U.S.A.* 114 (2017), pp. 7250–7253

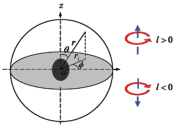


B. T. Hefner and P. L. Marston. *J. Acoust. Soc. Am.* 106 (1999), pp. 3313–3316

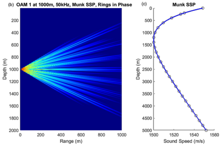
What is an acoustic vortex beam?

Used for...

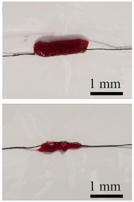
- particle manipulation
- underwater communications
- therapeutic biomedical ultrasound
- sound diffusion



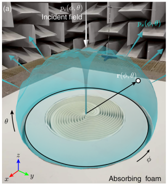
L. Zhang and P. L. Marston. *Phys. Rev. E* 84 (2011), pp. 1–5



M. E. Kelly and C. Shi. *JASA Express Lett.* 3 (2023), pp. 1–5



S. Guo et al. *Ultrasound Med. Biol.* 48 (2022), pp. 1907–1917

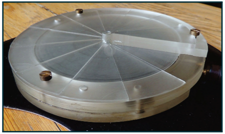


N. Jiménez, J. -P. Groby, and V. Romero-García. *Sci. Rep.* 11 (2021), pp. 1–13

What is an acoustic vortex beam?

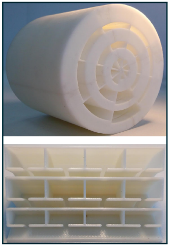
Generated by...

- phase plates
- transducer arrays
- metasurfaces



M. E. Terzi et al. *Moscow University Physics Bulletin* 72 (2017), pp. 61–67

A. Marzo, M. Caleap, and B. W. Drinkwater. *Phys. Rev. Lett.* 120 (2018), pp. 1–6



X. Jiang et al. *Phys. Rev. Lett.* 117 (2016), pp. 1–5

Previous analytical descriptions of vortex beams

- Bessel vortex beams are modes of the cylindrical wave equation:[1]

$$p(r,\theta,z,t) = p_0 J_\ell(k_r r) e^{i(\ell\theta + k_z z - \omega t)}\,, \quad k = \frac{\omega}{c_0} = \sqrt{k_r^2 + k_z^2}\,. \tag{1}$$

- Gaussian vortex beams are generalizations of Gaussian beams:[2]

$$\begin{aligned} p(r,\theta,z,t) &= \sqrt{8\pi}\left(\frac{p_0 z}{kr^2}\right)\chi^{3/2} e^{-\chi}\left[I_{(\ell-1)/2}(\chi) - I_{(\ell+1)/2}(\chi)\right] \\ &\quad \times\ e^{i[\ell\theta - (\ell+1)\pi/2 + kr^2/2z + k_z z - \omega t]}\,, \\ \chi(r,z) &= \frac{\frac{1}{8}(kar/z)^2}{1 - i(ka^2/2z)(1 - z/d)}\,. \end{aligned} \tag{2}$$

- Fields described by Eqs. (1) and (2) require infinite source conditions.
- Equation (1) implies infinite energy,[3] because $\int_0^\infty |J_\ell(k_r r)|^2 r\,dr \to \infty$.
- **Objective: derive solutions for vortex fields radiated by circular pistons**

[1] N. Jiménez et al. *Phys. Rev. E* 94 (2016), pp. 1–9.
[2] C. A. Gokani, M. R. Haberman, and M. F. Hamilton. *J. Acoust. Soc. Am.* 155 (2024), pp. 2707–2723.
[3] M. R. Lapointe. *Opt. Laser Technol.* 24 (1992), pp. 315–321.

Outline

Paraxial equation and its integral solution

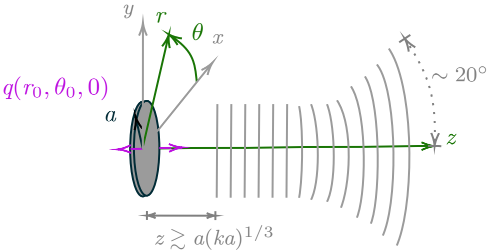
- For $p = qe^{i(kz-\omega t)}$ and $|\partial^2 q/\partial z^2| \ll 2k|\partial q/\partial z|$, $\nabla^2 p - c_0^{-2}\ddot{p} = 0$ reduces to

$$i2k\frac{\partial q}{\partial z} + \nabla_\perp^2 q = 0\,. \qquad (3)$$

- $\nabla_\perp^2$ is the Laplacian in the plane perpendicular to the z axis.
- Equation (3) is solved by the Fresnel diffraction integral:

$$q(r,\theta,z) = -\frac{ik}{2\pi z}\int_0^{2\pi}\int_0^{\infty} q(r_0,\theta_0,0)\, e^{i(k/2z)[r^2+r_0^2-2rr_0\cos(\theta_0-\theta)]}\, r_0\,dr_0\,d\theta_0\,,$$

where $q(r_0,\theta_0,0)$ is the prescribed pressure field in the plane $z=0$.



Vortex radiation from unfocused circular piston

- A circular vortex source condition is first considered:

$$q(r,\theta,0) = p_0 \, \mathrm{circ}(r/a) \, e^{i\ell\theta} \,, \tag{4}$$

  where $\mathrm{circ}(x) = 1$ for $0 \le x \le 1$ and 0 for $x > 1$.
- Insertion of Eq. (4) in the Fresnel diffraction integral leads to

$$q = -ikp_0 \frac{e^{i(ka^2/2z)r^2/a^2}}{z} e^{i\ell(\theta-\pi/2)} \int_0^a e^{i(ka^2/2z)r_0^2/a^2} J_\ell(krr_0/z) \, r_0 \, dr_0 \,. \tag{5}$$

- Using Watson's relation,[4] Eq. (5) reduces to

$$q = -\frac{ikp_0}{z} e^{i\ell(\theta-\pi/2)} \int_0^a J_\ell(krr_0/z) \, r_0 \, dr_0 \tag{6}$$

  for $z \gg z_R$, where $z_R = ka^2/2$ is the Rayleigh distance.

[4] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd ed. Sec. 2.2, Eq. (5).

Vortex radiation from unfocused circular piston

- Taking the integral in Eq. (6) leads to an analytical solution:

$$\boxed{q_\ell(r,\theta,z) = -ip_0\,\frac{z}{kr^2}\,e^{i\ell(\theta-\pi/2)}\,F_\ell(kar/z)\,, \quad z \gg z_R\,,} \tag{7}$$

where[5]

$$F_\ell(\xi) = \int_0^\xi J_\ell(t)\,t\,dt = \xi\,\frac{\Gamma(\ell/2+1)}{\Gamma(\ell/2)}\sum_{k=0}^{\infty}\frac{(\ell+2k+1)\Gamma(\ell/2+k)}{\Gamma(\ell/2+2+k)}\,J_{\ell+2k+1}(\xi)\,. \tag{8}$$

- Equation (8) equals the following closed-form expressions for $1 \le \ell \le 4$:

$$F_0(\xi) = \xi J_1(\xi)\,, \qquad \ell = 0 \tag{9a}$$

$$F_1(\xi) = \tfrac{\pi}{2}\xi\left[\mathbf{H}_0(\xi)J_1(\xi) - \mathbf{H}_1(\xi)J_0(\xi)\right]\,, \qquad \ell = 1 \tag{9b}$$

$$F_2(\xi) = 2 - 2J_0(\xi) - \xi J_1(\xi)\,, \qquad \ell = 2 \tag{9c}$$

$$F_3(\xi) = \left[\tfrac{3\pi}{2}\xi\,\mathbf{H}_0(\xi) - 8\right]J_1(\xi) + \left[4\xi - \tfrac{3\pi}{2}\xi\,\mathbf{H}_1(\xi)\right]J_0(\xi)\,, \qquad \ell = 3 \tag{9d}$$

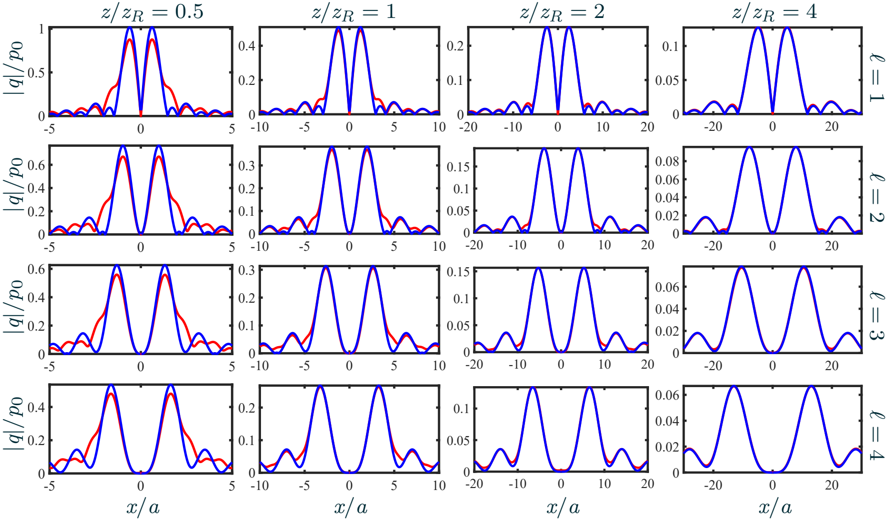
$$F_4(\xi) = 4 - 8J_1(\xi)/\xi - 4J_2(\xi) - \xi J_3(\xi)\,, \qquad \ell = 4 \tag{9e}$$

[5] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, 4th ed. Item 11.1.1.

Verification of Eq. (7)

- The validity of Eq. (7) is assessed by comparison to

$$q(x, y, z) = \mathcal{F}_{xy}^{-1}\left\{e^{ik_z z}\mathcal{F}_{xy}[q(x, y, 0)]\right\}\,, \quad k_z = k - (k_x^2 + k_y^2)/2k\,. \qquad (10)$$



Vortex radiation from focused circular piston

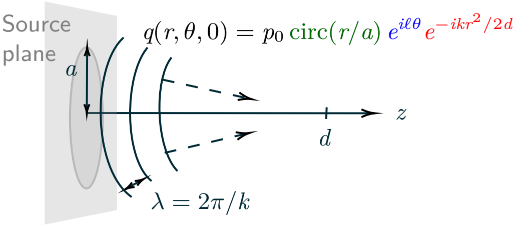
- To describe spherical focusing at a geometric focal length d, source condition (4) is multiplied by $\exp(-ikr^2/2d)$:

$$q(r,\theta,0) = p_0 \, \mathrm{circ}(r/a) \, e^{i\ell\theta} \, e^{-ikr^2/2d} \,, \tag{11}$$

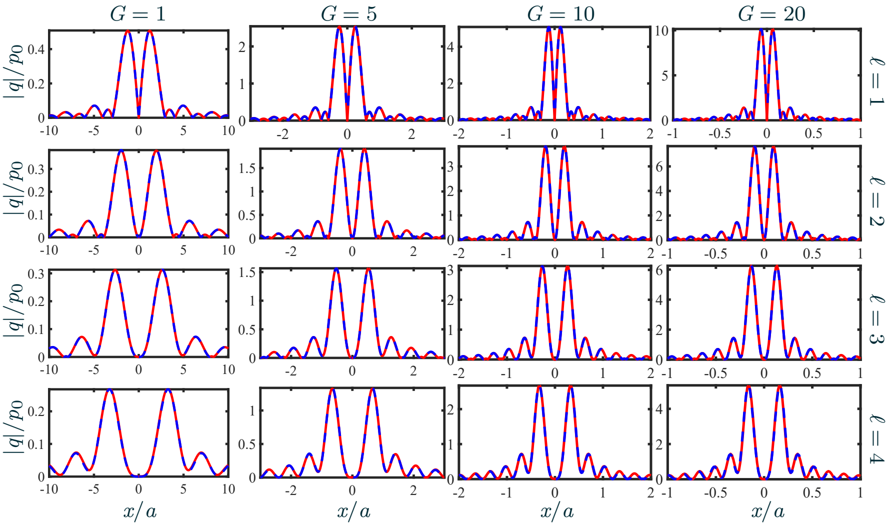
- An analytical solution of Eq. (3) is available at $z = d$:

$$\boxed{q_\ell(r,\theta,d) = -ip_0 \frac{d}{kr^2} e^{ikr^2/2d} e^{i\ell(\theta-\pi/2)} F_\ell(kar/d) \,,} \tag{12}$$

where $F_\ell(\xi)$ is given by Eq. (8).



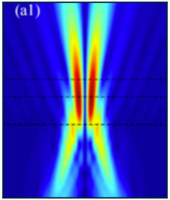
Verification of Eq. (12): analytical, Fourier



Outline

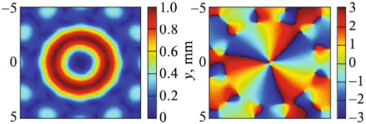
Vortex ring radius

- The magnitudes of vortex beam fields are axisymmetric.
- In the far field of Eq. (7), the field is conical.
- In the geometric focal plane $z = d$ of Eq. (12), the field forms toroidal ring.
- Equations (7) and (12) can be used to find the radius of these features.



C. Zhou et al. *J. Appl. Phys.* 128 (2020), pp. 1–12

D. Baresch, J. -L. Thomas, and R. Marchiano. *Phys. Rev. Lett.* 116 (2016), pp. 1–6



M. E. Terzi et al. *Moscow University Physics Bulletin* 72 (2017), pp. 61–67

Vortex ring radius

- Maximizing Eqs. (7) and (12) in r amounts to solving

$$\frac{d|\xi^{-1}F_\ell(\xi)|}{d\xi} = 0\,,$$

where $\xi = kar/z$ (unfocused) and $\xi = kar/d$ (focused).

- Using Eq. (8) for F_ℓ and taking the derivative yields[6]

$$\sum_{k=0}^{\infty} \frac{(\ell+2k+1)\Gamma(\ell/2+k)}{\Gamma(\ell/2+2+k)} \left[\frac{J_{\ell+2k}(\xi) - J_{\ell+2k+2}(\xi)}{2\xi} - \frac{J_{\ell+2k+1}(\xi)}{\xi^2} \right] = 0\,. \tag{13}$$

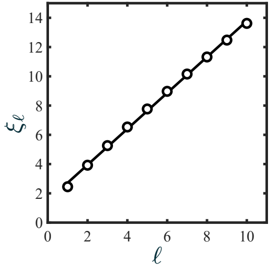
- The roots ξ_ℓ of Eq. (13) are fit to a line: $\xi_\ell = 1.23\ell + 1.49$.
- Solving $\xi_\ell = kar_\ell/z$ and $\xi_\ell = kar_\ell/d$ for r_ℓ yields

$$r_\ell = \frac{\xi_\ell z}{ka}\,, \quad z \gg z_R\,, \tag{14a}$$

$$= \frac{\xi_\ell d}{ka}\,, \quad z = d\,. \tag{14b}$$

[6] I. S. Gradshteyn and I. M. Ryzhik. New York: Academic Press, 1980, Item 8.471-2.

Vortex ring radius



Comparison of roots ξ_ℓ (circles) with least-squares fit $\xi_\ell = 1.23\ell + 1.49$ (line)

Outline

A simplified diffraction integral

- The angular spectrum method is equivalent to the first Rayleigh integral:

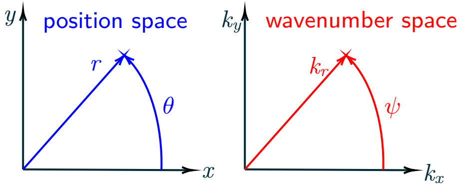
$$p(r,\theta,z) = \rho_0 c_0 k \mathcal{F}_{2\mathrm{D}}^{-1}\{\mathcal{F}_{2\mathrm{D}}\{u_z(r,\theta)\} e^{ik_z z}/k_z\}\,, \tag{15}$$

where

$$\mathcal{F}_{2\mathrm{D}}\{f(r,\theta)\} = g(k_r,\psi) = \int_0^{2\pi}\int_0^{\infty} f(r,\theta)\, e^{-ik_r r\cos(\theta-\psi)}\, r\, dr\, d\theta\,, \tag{16a}$$

$$\mathcal{F}_{2\mathrm{D}}^{-1}\{g(k_r,\psi)\} = f(r,\theta) = \int_0^{2\pi}\int_0^{\infty} g(k_r,\psi)\, e^{ik_r r\cos(\theta-\psi)}\, k_r\, dk_r\, d\psi\,. \tag{16b}$$

- Directly evaluating Eq. (15) can suffer from aliasing and discretization error.[7]



[7] J. Blackmore, R. O. Cleveland, and J. Mobley. *J. Acoust. Soc. Am.* 144 (2018), pp. 2947–2951.

A simplified diffraction integral

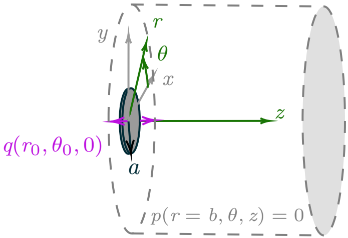
- In light of F_ℓ given by Eq. (8) and Watson's relation[8]

$$J_n(\beta) = \frac{1}{2\pi}\int_\alpha^{2\pi+\alpha} e^{i(n\phi-\beta\sin\phi)}\,d\phi\,, \tag{17}$$

  Eq. (15) reduces to

$$p/\rho_0 c_0 u_0 = e^{i\ell\theta}\int_0^\infty (k/k_z)F_\ell(k_r a)J_\ell(k_r r)e^{ik_z z}\,dk_r/k_r\,. \tag{18}$$

- Equation (18) can be derived by considered the limiting case of a piston in a tube, amounting to an alternative theory of diffraction.



[8] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd ed. Sec. 2.2, Eq. (5).

A simplified diffraction integral

- In terms of the dimensionless parameters

$$P \equiv p/\rho_0 c_0 u_0 \,, \quad R \equiv r/a \,, \quad Z \equiv z/z_R \,, \quad K \equiv ka \,, \tag{19}$$

where z_R is the Rayleigh distance $ka^2/2$, and where

$$k_z/k = \sqrt{1-(\zeta/K)^2}\,, \quad \zeta \equiv k_r a \,, \tag{20}$$

Eq. (18) becomes

$$\boxed{P = e^{i\ell\theta} \int_0^\infty \frac{F_\ell(\zeta) J_\ell(\zeta R)}{\zeta\sqrt{1-(\zeta/K)^2}} \, e^{iK^2 Z\sqrt{1-(\zeta/K)^2}/2} \, d\zeta \,.} \tag{21}$$

- Equation (21) is equivalent to and easier to evaluate than Eq. (15).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A simplified diffraction integral

- Equation (21) can be evaluated analytically for $R = \ell = 0$:

$$P(Z) = \int_0^\infty \frac{J_1(\zeta)}{\sqrt{1-(\zeta/K)^2}}\, e^{iK^2 Z\sqrt{1-(\zeta/K)^2}/2}\, d\zeta\,. \tag{22}$$

- Equation (22) evaluates to[9]

$$P(Z) = -iK\, I_{1/2}[-i\chi_-(Z)]\, K_{1/2}[-i\chi_+(Z)]\,, \tag{23}$$

where I_ν and K_ν are the modified Bessel functions of order ν, and where

$$\chi_\pm(Z) = \tfrac{K}{2}\Big[\sqrt{1+(KZ/2)^2} \pm KZ/2\Big]. \tag{24}$$

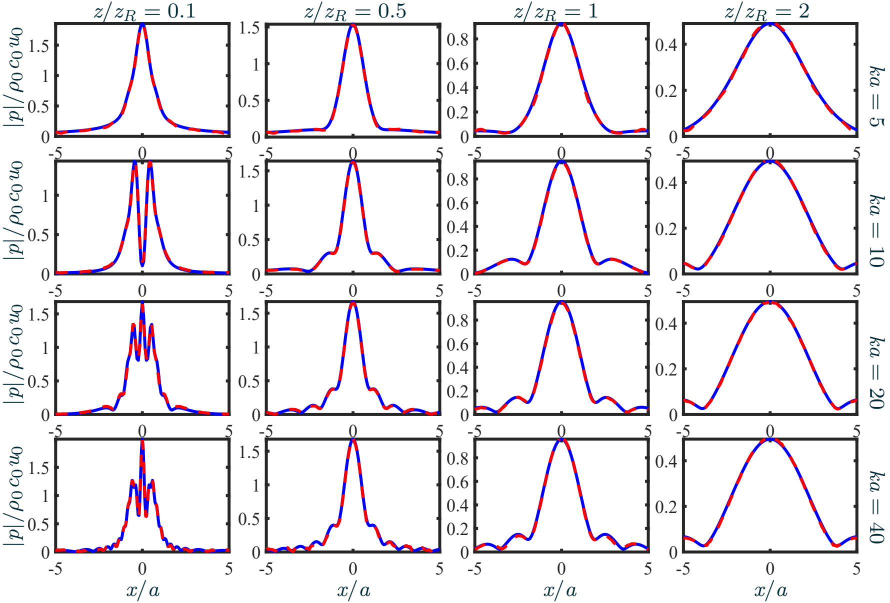
- Bessel function identities reduce Eq. (23) to

$$P(Z) = -2i\sin[\chi_-(Z)]\, e^{i\chi_+(Z)}\,, \tag{25}$$

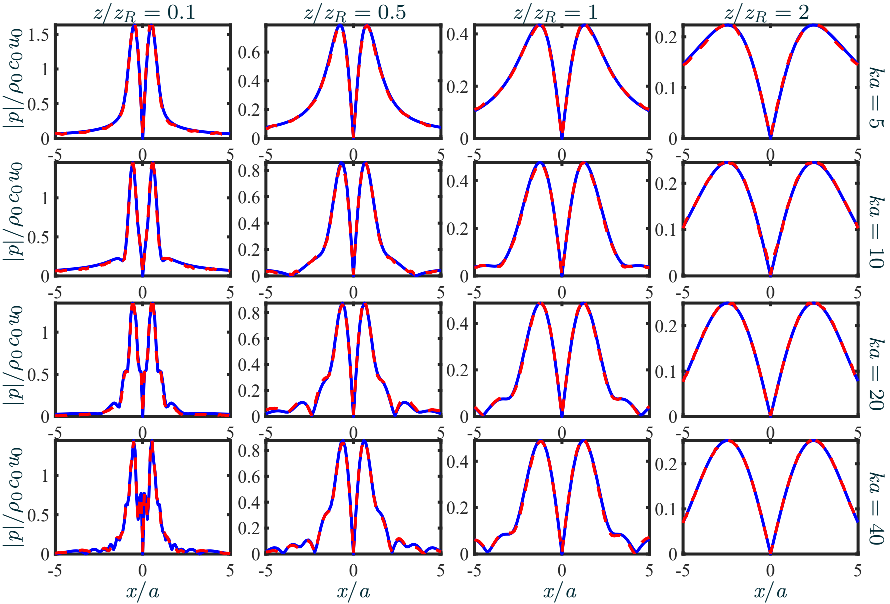
recovering the on-axis pressure radiated by a planar circular piston.[10]

[9] I. S. Gradshteyn and I. M. Ryzhik. New York: Academic Press, 1980, Item 6.637-1.
[10] A. D. Pierce. Secs. 8.2 and 8.5. Cham, Switzerland: Springer, 2019, Eq. (5.7.3).

Equation (21) for $\ell = 0$: semi-analytical, Fourier



Equation (21) for $\ell = 1$: semi-analytical, Fourier



Outline

Thank you for listening!

Summary

- Solved paraxial equation for planar and focused circular vortex sources
- Calculated the ring radius for both solutions
- Derived simplified diffraction integral equivalent to Rayleigh integral

Further reading

- "Paraxial and ray approximations of acoustic vortex beams," J. Acoust. Soc. Am. **155**, 2707–2723 (2024).
- "Analytical solutions for acoustic vortex beam radiation from planar and spherically focused circular pistons," JASA Express Lett. **4**, 124001 (2024).

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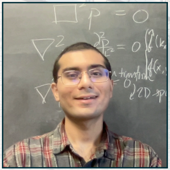
- Chester M. McKinney Graduate Fellowship in Acoustics at the Applied Research Laboratories



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Prof. Michael R. Haberman

Prof. Mark F. Hamilton

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Notation I

Symbol	Description	Dimensions
a	source radius	m
c_0	speed of sound	m s^{-1}
d	focal length	m
G	focusing gain $ka^2/2d$	1
i	complex unit	1
$\mathbf{k}$	wavenumber	m^{-1}
ℓ	orbital number	1
ρ_0	ambient mass density	kg m^{-3}
p	acoustic pressure	kg m^{-1} s^{-2}
q	paraxial pressure	kg m^{-1} s^{-2}
$\mathbf{R}$	separation vector $\mathbf{R} = \mathbf{r} - \mathbf{r}'$	m
$\mathbf{r}$	position vector	m
$\mathbf{v}$	particle velocity	m s^{-1}
z_R	Rayleigh distance, $ka^2/2$	m
ω	angular frequency, $\omega = 2\pi f$	s^{-1}