Using Fourier acoustics to derive the Fresnel diffraction integral

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Given an arbitrary normal velocity distribution

$$\mathbf{u} = (x, y, 0, t) = u_0(x, y)e^{-i\omega t}\hat{e}_z \,.$$

Follow the recipe for the calculation of the field. The 2D spatial Fourier transform of the source condition, and its mapping to a pressure source, is

$$\begin{aligned} U_0(k_x, k_y, 0) &= \mathcal{F}_{xy}\{u_0(x, y)\} \,, \\ P_0(k_x, k_y, 0) &= \rho_0 c_0 \frac{k}{k_z} U(k_x, k_y, 0) \,, \end{aligned}$$

where now k_z is not $\sqrt{k^2 - k_x^2 - k_y^2}$ but is rather approximated by its binomial expansion:

$$k_z \simeq k \left(1 - \frac{k_x^2 + k_y^2}{2k^2}\right) \,. \tag{1}$$

Then the solution of the paraxial equation is

$$\begin{aligned} p_\omega(x, y, z) &= \mathcal{F}_{xy}^{-1}\{P_0(k_x, k_y)e^{ik_z z}\} \\ &= \rho_0 c_0 k \, \mathcal{F}_{xy}^{-1}\{U_0(k_x, k_y)e^{ik_z z}/k_z\} \\ &= \rho_0 c_0 k \, \mathcal{F}_{xy}^{-1}\{U_0(k_x, k_y)\} * * \mathcal{F}_{xy}^{-1}\{e^{ik_z z}/k_z\} \\ &= \rho_0 c_0 k \, u(x, y) * * \frac{e^{ikr}}{i2\pi r} \,, \end{aligned}$$

where the convolution theorem has been used to arrive at the third line above, and where the result from class, $\mathcal{F}_{xy}\{e^{ikr}/r\} = i2\pi e^{ik_z|z|}/k_z$, has been used to evaluate that line. Noting that $r \simeq z[1 + (x^2 + y^2)/2z^2]$ (rather than $\sqrt{x^2 + y^2 + z^2}$) in the Fresnel approximation, the above becomes

$$p_\omega(x, y, z) = \rho_0 c_0 k \, u(x, y) * * \frac{e^{ikz[1+(x^2+y^2)/2z^2]}}{i2\pi z} \,,$$

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Writing the convolution explicitly results in

$$
\begin{aligned}
p_\omega(x,y,z) &= -\frac{ik\rho_0 c_0}{2\pi}\frac{e^{ikz}}{z}\iint_{-\infty}^{\infty} u(x_0,y_0)e^{ikz[(x-x_0)^2+(y-y_0)^2]/2z^2}dx_0dy_0 \\
&= -\frac{ik\rho_0 c_0}{2\pi}\frac{e^{ikz}}{z}\iint_{-\infty}^{\infty} u(x_0,y_0)e^{ik[(x-x_0)^2+(y-y_0)^2]/2z}dx_0dy_0 \,. \qquad (2)
\end{aligned}
$$

Equation (2) is precisely the Fresnel diffraction integral, which is traditionally derived by taking the Fresnel approximation of the Rayleigh integral.