This is Dr. Hamilton's treatment of Dr. Blackstock's chapter 4, sections A & B.

Section A

Assume a time-harmonic pressure and hence time-harmonic particle velocity:

$$p(x,t) = P(x)e^{j\omega t}$$

$$u(x,t) = U(x)e^{j\omega t}$$

Substitution into the Helmholtz equation, $\frac{\partial^2 p}{\partial x^2} - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2}$, yields

$$P = Ae^{-jkx} + Be^{jkx}$$

Dividing by the impedance gives

$$U = \frac{A}{Z_0}e^{-jkx} - \frac{B}{Z_0}e^{jkx}$$

Changing variables $x \to l - d$, where $d = 0$ is the location of the load and l is the length of the tube,

$$P = Ae^{-jkl}e^{jkd} + Be^{jkl}e^{-jkd}$$

Calling $P_i \equiv Ae^{-jkl}$ and $P_r \equiv Be^{jkl}$, and factoring out P_i,

$$P = (1 + R\ )\,e^{-2jkd}P_i e^{jkd}$$

Evaluating the above equation at the load ($x = d$),

$$P = (1 + R)P_i$$

Similarly,

$$U = (1 - R)\frac{P_i}{\rho_0 c_0}$$

The load impedance is then calculated to be

$$Z_n = \rho_0 c_0 \frac{1 + R}{1 - R}$$

This can be solved for R:

$$R = \frac{Z_n - \rho_0 c_0}{Z_n + \rho_0 c_0}$$

Section B

Again assuming the time-harmonic signals as in section A, where

$$P(x) = A\cos kx + B\sin kx$$

Using the momentum equation, $\rho_0 \frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} = 0$,

$$U(x) = -\frac{1}{j\omega\rho_0}\frac{\mathrm{d}P}{\mathrm{d}x} = \frac{A}{j\rho_0 c_0}\sin kx - \frac{B}{j\rho_0 c_0}\cos kx$$

The coefficients A and B are found by applying the boundary conditions.

At $x = L$,

$$\begin{aligned} Z_{\mathrm{n}} = \frac{P(L)}{U(L)} &= j\rho_0 c_0 \frac{A\cos kL + B\sin kL}{A\sin kL - B\cos kL} \\ &= j\rho_0 c_0 \frac{A/B + \tan kL}{A/B\tan kL - 1} \\ \implies \frac{A}{B} &= j\frac{Z_n/\rho_0 c_0 + j\tan kL}{1 + j(Z_n/\rho_0 c_0)\tan kL} \end{aligned}$$

Meanwhile, at $x = 0$,

$$\begin{aligned} Z_{\mathbf{in}} = \frac{P(0)}{U(0)} &= -j\rho_0 c_0 \frac{A}{B} \\ \implies \frac{A}{B} &= j\frac{Z_{\mathrm{in}}}{\rho_0 c_0} \end{aligned}$$

Setting the above two equations for $\frac{A}{B}$ equal and solving for $\frac{Z_{\mathrm{in}}}{\rho_0 c_0}$,

$$\frac{Z_{\mathrm{in}}}{\rho_0 c_0} = \frac{Z_{\mathrm{n}}/\rho_0 c_0 + j\tan kL}{1 + j(Z_n/\rho_0 c_0)\tan kL} \tag{1}$$

There are three special cases that follow from equation (1):

1. For $Z_n = 0$ (i.e., pressure release),

$$Z_{\mathrm{in}} = j\rho_0 c_0 \tan kL$$

2. For $|Z_n| \to \infty$ (i.e., rigid),

$$Z_{\mathrm{in}} = -j\rho_0 c_0 \cot kL$$

3. For $Z_n = \rho_0 c_0$ (i.e., impedance matching),

$$Z_{\mathrm{in}} = 1$$

10/4/21 lecuture–section B continued.

From last time, four equations were derived:

1. $P = A\cos kx + B\sin kx$
2. $U = \frac{A}{j\rho_0 c_0}\sin kx - \frac{B}{j\rho_0 c_0}\cos kx$
3. $\frac{A}{B} = j\frac{Z_{\text{in}}}{\rho_0 c_0}$
4. $\frac{Z_{\text{in}}}{\rho_0 c_0} = \frac{Z_n/\rho_0 c_0 + j\tan kl}{1+j(Z_n/\rho_0 c_0)\tan kl}$

We now consider (1) a velocity source and (2) a pressure source:

Velocity Source

The velocity source boundary condition is

$$u(x=0,t) = u_0 e^{j\omega t}$$

Applying this to our decomposition $u(x,t) = U(x)e^{j\omega t}$,

$$U(0) = u_0$$

Now applying the boundary condition to equation (2) above,

$$B = -j\rho_0 c_0 u_0$$

Substituting the above into equation (3) gives

$$A = Z_{\text{in}} u_0$$

With A and B determined, we can write our complete solutions for $P(x)$ and $U(x)$:

$$P(x)/\rho_0 c_0 u_0 = \frac{Z_{\text{in}}}{\rho_0 c_0}\cos kx - j\sin kx$$

$$U(x)/u_0 = \frac{Z_{\text{in}}}{j\rho_0 c_0}\sin kx + \cos kx$$

Pressure Source

Now, our boundary condition is

$$p(x=0,t) = p_0 e^{j\omega t}$$

Applying this to equation (1) above gives

$$A = p_0$$

and applying the boundary condition to (3) gives

$$B = \frac{\rho_0 c_0}{j Z_{\text{in}}} p_0$$

With A and B determined, our complete solutions for $P(x)$ and $U(x)$ are:

$$P(x)/p_0 = \cos kx + \frac{\rho_0 c_0}{j Z_{\text{in}}} \sin kx$$

$$U(x)/(p_0/\rho_0 c_0) = -j \sin kx + \frac{\rho_0 c_0}{Z_{\text{in}}} \sin kx$$