Eigenfunctions of a Room

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This was written in response to a discussion with Chirag Gokani about the interpretation of eigenfunctions and eigenvalues in acoustics. I find the following example to be a helpful illustration of several properties.

1 Problem Setup

Consider an enclosed room with volume V and surface area S, which will be excited at a temporal angular frequency ω. The acoustic pressure field within the room obeys the Helmholtz equation

$$\nabla^2 p + k^2 p = 0, \tag{1}$$

where $k = \omega/c_0$. The impedance boundary condition at the walls can be expressed as

$$A(\omega)p|_S + B(\omega)\frac{\partial p}{\partial n}\bigg|_S = 0, \tag{2}$$

where $\hat{n}$ is the normal direction to the wall. For simplicity, assume this same boundary condition holds at all the walls.

2 A Useful Vector Identity

Before analyzing the eigenfunctions of the room, we first prove an identity that we will use later. We start with the divergence theorem,

$$\int_V \nabla \cdot \vec{F} dV = \int_S \vec{F} \cdot d\vec{S}. \tag{3}$$

Let $\vec{F} = g\nabla f$, so $\nabla \cdot \vec{F} = g\nabla^2 f + \nabla g \cdot \nabla f$, and (3) becomes

$$\int_V \left(g\nabla^2 f + \nabla g \cdot \nabla f\right) dV = \int_S g\nabla f \cdot d\vec{S} = \int_S g\frac{\partial f}{\partial n} dS. \tag{4}$$

This holds for any f and g, so we can interchange f and g to get

$$\int_V \left(f\nabla^2 g + \nabla f \cdot \nabla g\right) dV = \int_S f\nabla g \cdot d\vec{S} = \int_S f\frac{\partial g}{\partial n} dS. \tag{5}$$

Subtracting (4) from (5) yields

$$\boxed{\int_V \left(f\nabla^2 g - g\nabla^2 f\right) dV = \int_S \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right) dS}. \tag{6}$$

3 Orthogonality of Eigenfunctions

Now, we prove that the eigenfunctions (modes) of the room are orthogonal. Suppose the eigenfunction-eigenvalue pairs (p_m, k_m^2) and (p_n, k_n^2) satisfy

$$\nabla^2 p_m + k_m^2 p_m = 0, \tag{7}$$

$$A(\omega)p_m|_S + B(\omega)\frac{\partial p_m}{\partial n}\bigg|_S = 0, \tag{8}$$

$$\nabla^2 p_n + k_n^2 p_n = 0, \tag{9}$$

$$A(\omega)p_n|_S + B(\omega)\frac{\partial p_n}{\partial n}\bigg|_S = 0. \tag{10}$$

The coefficients in both boundary conditions are evaluated at the excitation frequency ω. These modes have resonance frequencies $\omega_m = c_0 k_m$ and $\omega_n = c_0 k_n$, respectively.[1] Choosing $f = p_m$ and $g = p_n$ in (6) gives

$$\int_V \left(p_m\nabla^2 p_n - p_n\nabla^2 p_m\right) dV = \int_S \left(p_m\frac{\partial p_n}{\partial n} - p_n\frac{\partial p_m}{\partial n}\right) dS. \tag{11}$$

Using (7) and (9) to eliminate the ∇^2 terms from the LHS, and using (8) and (10) to eliminate the $\partial/\partial n$ terms from the RHS, we obtain

$$(k_m^2 - k_n^2)\int_V p_m p_n dV = 0. \tag{12}$$

If $k_m^2 \neq k_n^2$, and if we normalize the eigenfunctions such that $\int_V p_n^2 dV = 1$, we obtain the orthogonality condition

$$\boxed{\int_V p_m p_n dV = \delta_{mn}}. \tag{13}$$

[1] To be more precise, the interpretation of ω_n as a resonance frequency of the room only works if the impedance of the walls is independent of frequency (as is the case for perfectly rigid walls). In general, if the room is excited at the frequency ω_n, the coefficients A and B in the boundary condition (10) are evaluated at ω_n rather than ω, and ω_n may not be an eigenfrequency of this different problem.

4 Solution for a Point Source Excitation

Suppose the room is excited by a harmonic point source of frequency ω located at $\vec{r}_0$. This introduces a term on the RHS of (1):

$$\nabla^2 p + k^2 p = -\delta\left(\vec{r} - \vec{r}_0\right). \tag{14}$$

We express the pressure as a superposition of eigenfunctions[2] as

$$p = \sum_m A_m p_m, \tag{15}$$

where each eigenfunction satisfies equations of the form of (7) and (8). We will solve for the weighting coefficients A_m. Substituting (15) into (14) yields

$$\sum_m A_m \left(\nabla^2 + k^2\right) p_m = -\delta\left(\vec{r} - \vec{r}_0\right). \tag{16}$$

Using (7) to eliminate the ∇^2 term, we get

$$\sum_m A_m \left(k^2 - k_m^2\right) p_m = -\delta\left(\vec{r} - \vec{r}_0\right). \tag{17}$$

Taking $\int_V (\cdots) p_n dV$ of both sides and exploiting the orthogonality property (13) yields[3]

$$A_n \left(k^2 - k_n^2\right) = -p_n\left(\vec{r}_0\right). \tag{18}$$

Therefore,

$$A_n = \frac{p_n\left(\vec{r}_0\right)}{k_n^2 - k^2} = \frac{c_0^2 \; p_n\left(\vec{r}_0\right)}{\omega_n^2 - \omega^2}, \tag{19}$$

and the pressure field in the room is

$$\boxed{p\left(\vec{r}, t\right) = e^{j\omega t} \sum_n \frac{c_0^2 \; p_n\left(\vec{r}_0\right) p_n\left(\vec{r}\right)}{\omega_n^2 - \omega^2}}. \tag{20}$$

5 Conclusions

This example illustrates a few interesting points. If a room with impedance boundary conditions at the walls is excited at a frequency ω, the resulting pressure field can be expressed as a sum of orthogonal eigenfunctions. These orthogonal eigenfunctions are the modes of the

[2]This is a leap of faith, as we have not proven that the eigenfunctions form a complete basis. I'm not immediately sure how to prove this.

[3]This only works in the absence of degeneracy, that is, if all eigenfunctions have distinct eigenvalues. What happens if, for example, there are two modes with the same resonance frequency? This occurs sometimes in rectangular rooms. I will sweep this under the rug for now.

room, and the associated eigenvalues correspond to the resonance frequencies of the modes (where, again, this interpretation is most clear if the impedance of the walls is independent of frequency). From the denominator of (20), we see that the n^{th} mode blows up when excited at its resonance frequency. However, in general, the modes comprising the solution are excited away from their resonance frequencies. Note that this analysis makes no assumptions about the room geometry or whether it is possible to solve for the eigenfunctions using separation of variables.