Effects of increasing orbital number on the field transformation in focused vortex beams

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5pPA, General Topics in Physical Acoustics
186th ASA Meeting, Ottawa

Our paper on this topic

JASA ARTICLE

Paraxial and ray approximations of acoustic vortex beams

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(Received 19 January 2024; revised 27 March 2024; accepted 27 March 2024; published online 22 April 2024)
[Editor: Philip L. Marston] Pages: 2707–2723

Outline

1. Introduction
2. Ray theory
3. Gaussian vortex source
4. Uniform vortex source
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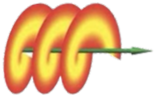
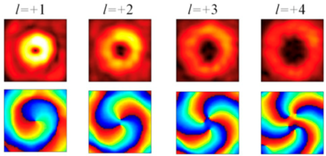
What is an acoustic vortex beam?

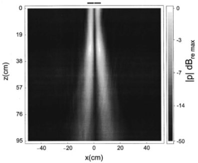
Characterized by...

- helical wavefronts
- orbital number $\ell =$ number of equiphase wavefronts in $\perp$ plane
- zero pressure on axis





Shi et al. 2017, *P. Natl. Acad. Sci. U.S.A.*

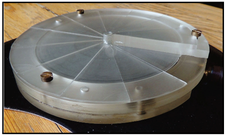


Hefner and Marston 1999, *J. Acoust. Soc. Am.*

What is an acoustic vortex beam?

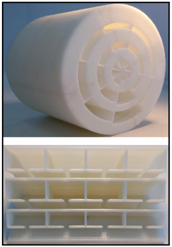
Generated by. . .

- phase plates
- transducer arrays
- metasurfaces



M. E. Terzi et al. 2017, *Moscow University Physics Bulletin*

Marzo, Caleap, and Drinkwater 2018, *Phys. Rev. Lett.*



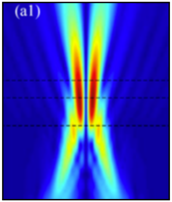
Jiang et al. 2016, *Phys. Rev. Lett.*

Focused vortex beams

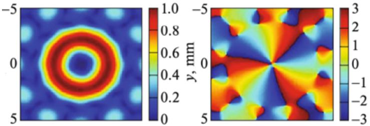
- Focused beams used for particle manipulation
- Magnitudes of vortex beam fields are axisymmetric
- In geometric focal plane $z = d$, field forms toroidal ring
- Weak spherical focusing studied in paraxial approximation,

$$i2k\frac{\partial q}{\partial z} + \nabla_\perp^2 q = 0\,, \quad p = qe^{ikz} \tag{1}$$

Baresch, Thomas, and Marchiano 2016, *Phys. Rev. Lett.*



Zhou et al. 2020, *J. Appl. Phys.*



M. E. Terzi et al. 2017, *Moscow University Physics Bulletin*

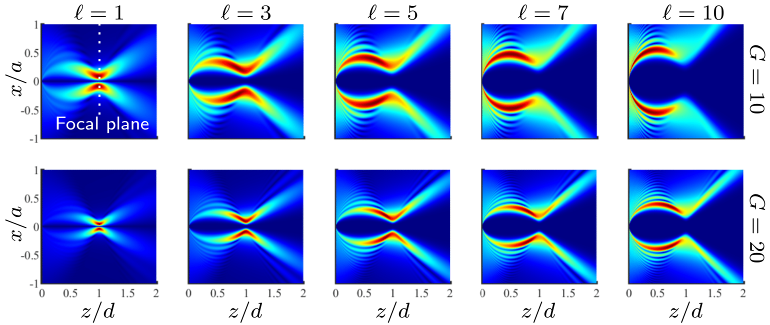
Prefocal shadow zone

- Magnitude of solution of Eq. (1) for focused Gaussian vortex source is

$$|q(r,z)| = \sqrt{8\pi}\,\frac{p_0 z}{kr^2}\left|\chi^{3/2}e^{-\chi}\left[I_{(\ell-1)/2}(\chi) - I_{(\ell+1)/2}(\chi)\right]\right|, \qquad (2)$$

$$\chi(r,z) = \frac{\frac{1}{8}(kar/z)^2}{1 - i(ka^2/2z)(1 - z/d)}$$

- For moderate values of focusing gain G, Eq. (2) reveals movement of vortex ring out of focal plane $z = d$ as ℓ increases



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Focused vortex source condition

- To explain field transformation with increasing ℓ, appeal to *ray theory*: $ka \to \infty$
- In homogeneous media, rays travel in straight lines
- In the vicinity of source, pressure field is

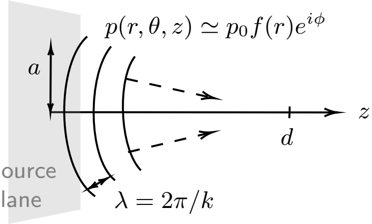
$$p(r,\theta,z) \simeq p_0 f(r) e^{i\phi}\,, \quad z \simeq 0$$

where $f(r)$ = axisymmetric amplitude distribution in source plane

- Phase accounts for focusing, helical wavefronts, and traveling wave motion:

$$\phi(r,\theta,z) = -kr^2/2d + \ell\theta + kz$$

Sunbeams



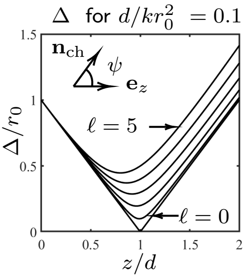
Wave normal $\mathbf{n}$ and annular channel radius Δ

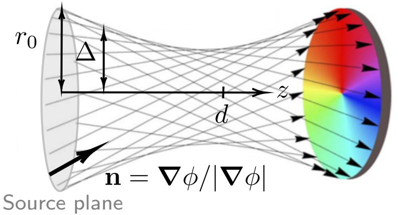
- Wave normal in source plane at distance r_0 from origin is

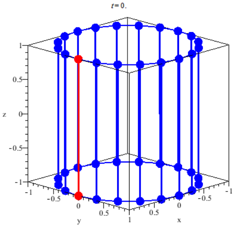
$$\mathbf{n} = \frac{\boldsymbol{\nabla}\phi}{|\boldsymbol{\nabla}\phi|} = \frac{-(r_0/d)\,\mathbf{e}_r + (\ell/kr_0)\,\mathbf{e}_\theta + \mathbf{e}_z}{\sqrt{(r_0/d)^2 + (\ell/kr_0)^2 + 1}}$$

- Radius of circle formed by family of rays emanating from $r = r_0$ in source plane is

$$\boxed{\Delta(r_0, z) = r_0 \left[(1 - z/d)^2 + (\ell d/kr_0^2)^2 (z/d)^2\right]^{1/2}}$$







Adapted from Richard et al. 2020, *New J. Phys.*

Pressure field according to ray theory

- Pressure field predicted by ray theory:

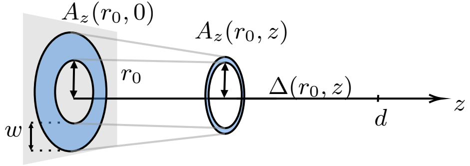
$$P(\Delta, z) = p_0 f(r_0) \sqrt{A(r_0, 0)/A(r_0, z)} \tag{3}$$

- Area of annular ray channel is

$$A(r_0, z) = A_z(r_0, z) \cos\psi(r_0, z) = 2\pi w \frac{\Delta(r_0, z)|\partial\Delta/\partial r_0|}{\sqrt{1 + (\partial\Delta/\partial z)^2}} \tag{4}$$

- Inserting Eq. (4) into Eq. (3) and calculating $\partial\Delta/\partial r_0$ and $\partial\Delta/\partial z$ yields

$$\boxed{P(\Delta, z) = p_0\, f(r_0) \left[\frac{\cos\psi(r_0, 0)/\cos\psi(r_0, z)}{|(1 - z/d)^2 - (\ell d/k r_0^2)^2 (z/d)^2|} \right]^{1/2}}$$



Caustics

- Caustics occur when cross-sectional area vanishes, i.e.,

$$A_z(r_0, z) = 2\pi r_0 w \left|(1 - z/d)^2 - (\ell d/k r_0^2)^2 (z/d)^2\right| = 0 \qquad (5)$$

- Substitution of roots of Eq. (5) into $\Delta(r_0, z)$ gives caustic coordinates:

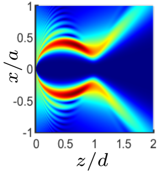
$$\boxed{\Delta_c(z) = \sqrt{(2\ell d/k)(z/d)|1 - z/d|}} \qquad (6)$$

- Squaring Eq. (6), notating $G = ka^2/2d$, and identifying

$$\Delta_c^2 = x_c^2 + y_c^2\,, \quad a_c^2 = \ell a^2/4G\,, \quad c_c^2 = d^2/4$$

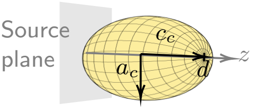
reveals that prefocal caustic is a spheroid of volume $V_c = \frac{1}{6}\ell\lambda d^2$:

$$\boxed{\frac{x_c^2 + y_c^2}{a_c^2} + \frac{(z_c - d/2)^2}{c_c^2} = 1\,, \quad 0 \le z_c \le d}$$

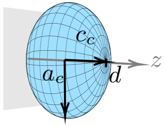


$G = 10$
$\ell = 5$

Prolate for $(d/a)^2 G > \ell$



Oblate for $(d/a)^2 G < \ell$



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Paraxial field $q(r,z)$, ray paths Δ, and caustics Δ_c

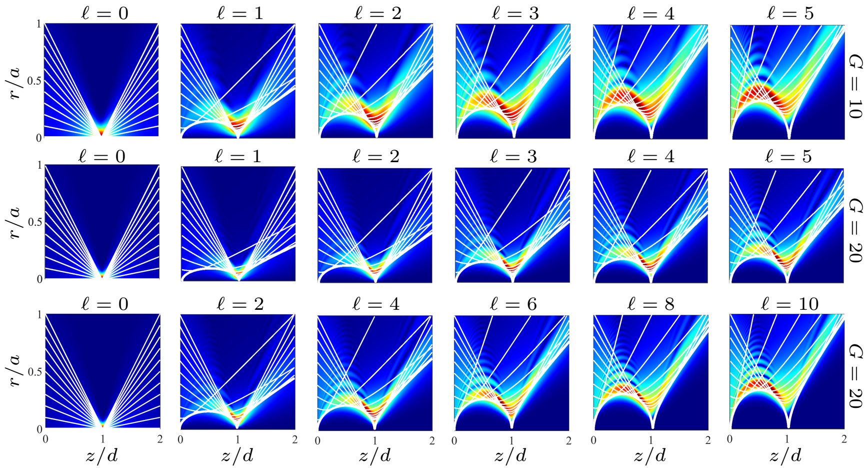


Figure 1: Overlays of caustics Δ_c (thick lines) and ray channels Δ (thin lines) on color plots of the amplitude of the paraxial field $|q(r,z)|$ due to a focused Gaussian vortex source. Rays emanating from $r > a$ have been suppressed for visual clarity.

Ray pressure field $P(\Delta, z)$

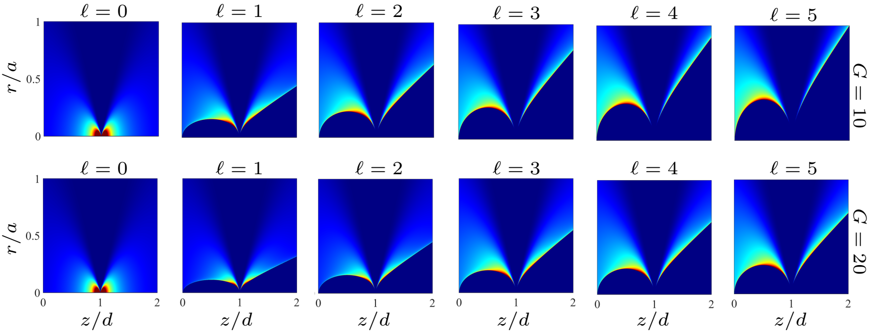


Figure 2: Color plots for the ray field $P(\Delta, z)$ due to a focused Gaussian vortex source.

Unfocused limit, $d \to \infty$

- For unfocused vortex beams, $d \to \infty$, for which Δ, Δ_c, and P reduce to

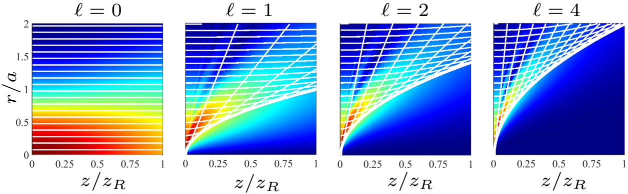
$$\Delta(r_0, z) = r_0[1 + (\ell z/kr_0^2)^2]^{1/2}\,, \quad \text{ray channel radius}$$

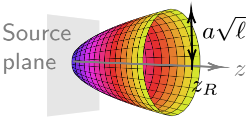
$$\Delta_c(z) = \sqrt{2\ell z/k}\,, \quad \text{caustic radius}$$

$$P(\Delta, z) = p_0 f(r_0)\left[\frac{\cos\psi(r_0,0)/\cos\psi(r_0,z)}{|1-(\ell z/kr_0^2)^2|}\right]^{1/2}$$

- Caustic surface is a paraboloid, where $z_R = ka^2/2$:

$$\frac{z}{z_R} = \frac{x_c^2 + y_c^2}{\ell a^2}\,, \quad z \geq 0$$





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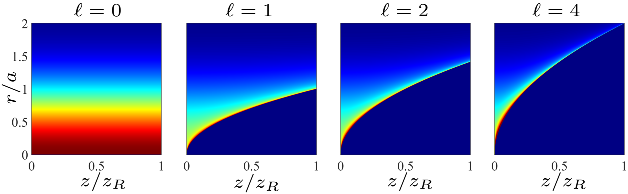

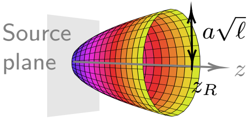

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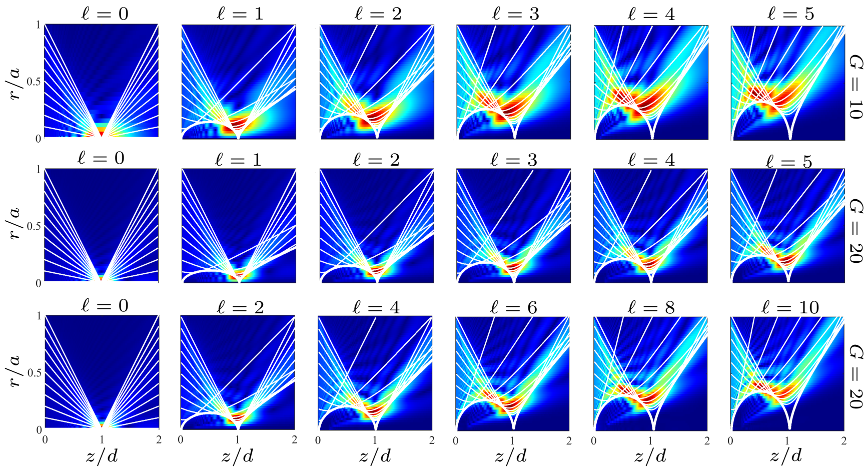

Figure 3: Overlays of caustics Δ_c (thick lines) and ray channels Δ (thin lines) on color plots of the amplitude of the paraxial field $|q(r, z)|$ due to a uniform focused vortex source.

Unfocused limit, $d \to \infty$

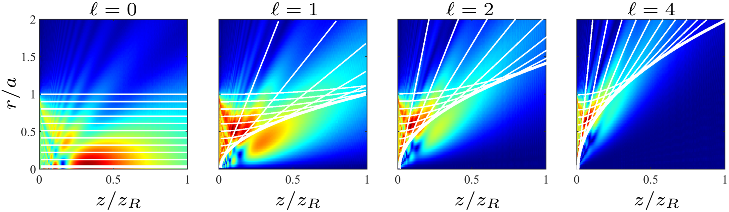

Figure 4: Overlays of caustics Δ_c (thick lines) and ray channels Δ (thin lines) on color plots of the amplitude of the paraxial field $|q(r,z)|$ due to a uniform unfocused vortex source.

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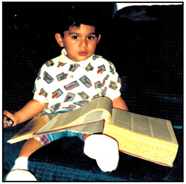
$[\mathrm{T}]\hbar\mathbf{an}\cdot\mathbf{k}\ \ Y_0\mathbf{u}\ \ f(0)\mathbf{r}\ \ \ell iSte^{ni}\mathbf{n}G!$

Summary

- Developed ray theory to describe field transformation associated with increase in orbital number
- Showed that shadow zone in prefocal region is bounded by a spheroidal surface
- Calculated pressure field from ray theory

Acknowledgments

- Chester M. McKinney Graduate Fellowship in Acoustics at the Applied Research Laboratories



Chirag Gokani

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Pressure field from ray theory (extra)

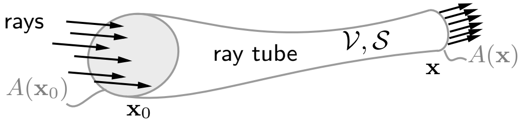
- Inserting $p(\mathbf{x}) = P(\mathbf{x}, \omega) e^{i\omega\tau(\mathbf{x})}$ into $\nabla^2 p + k^2 p = 0$ yields[1]

$$\nabla^2 P + i\omega[2\boldsymbol{\nabla} P \cdot \boldsymbol{\nabla}\tau + P\nabla^2\tau] - \omega^2 P[(\boldsymbol{\nabla}\tau)^2 - c^{-2}] = 0\,. \qquad (8.5.1)$$

- In limit that $\omega \to \infty$, it is necessary for $(\boldsymbol{\nabla}\tau)^2 = c^{-2}$ and

$$\boldsymbol{\nabla} \cdot (P^2 \boldsymbol{\nabla}\tau) = 0 \qquad (8.5.3b)$$

- Areas $A(\mathbf{x}_0)$ and $A(\mathbf{x})$ define the end caps of a *ray tube*



- Integrating Eq. (8.5.3b) over $\mathcal{V}$ and applying Gauss's theorem yields

$$\oint_{\mathcal{S}} (P^2 \boldsymbol{\nabla}\tau) \cdot \mathbf{n}\, dS = 0 \implies \boxed{P(\mathbf{x}) = P(\mathbf{x}_0)\sqrt{A(\mathbf{x}_0)/A(\mathbf{x})}\,.}$$

[1]Equation numbers refer to those in *Acoustics*, A. D. Pierce 2019.