**Grad school skills seminar: multivariable calculus** **November 15, 2024**

Today's meeting is by no means a comprehensive review of multivariable calculus; I've simply picked a few topics from *Vector Calculus* (Marsden and Tromba) that I found relevant/challenging.

1. Suppose that a bat is flying in the circle $x = \cos t$, $y = \sin t$, and that the air temperature is given by the formula $T = x^2 e^y - xy^3$. Find $dT/dt$, the rate of change in temperature the bat might feel: (a) by the chain rule and (b) by expressing $T$ in terms of $t$ and differentiating.

2. Thermodynamics texts use the relationship
$$\left(\frac{\partial y}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)\left(\frac{\partial x}{\partial z}\right) = -1 \,. \tag{1}$$
Prove Eq. (1) by differentiating $F(x, y, z) = 0$, where $x = f(y, z)$, $y = g(x, z)$, and $z = h(x, y)$.

3. Show that $(\partial V/\partial T)(\partial T/\partial P)(\partial P/\partial V) = -1$ for (a) the ideal gas law $PV = nRT$ and (b) the van der Waals gas law $P = RT/(V - \beta) - \alpha/V^2$, where $n$, $R$, $\alpha$, and $\beta$ are constants.

4. A bat finds itself in a loud environment in which the acoustic pressure amplitude is $|p(x, y)| = 2x^2 - 4y^2$. If the bat is at $(x, y) = (-1, 2)$, in what direction should it fly to most rapidly get to a quieter position?

5. Calculate the gradient of $1/r^2$, where $r = \sqrt{x^2 + y^2 + z^2}$, and find the direction of fastest increase at the point $(x, y, z) = (1, 1, 1)$.

6. Calculate the 2nd-order Taylor approximation of $f(x, y) = \sin(xy)$ at the point $(x_0, y_0) = (1, \pi/2)$.

7. In Cartesian coordinates, the position $\mathbf{A}$ of the centroid is given by $\mathbf{A} = A_x\mathbf{e}_x + A_y\mathbf{e}_y + A_z\mathbf{e}_z$, where
$$A_x = \frac{\int x dV}{\int dV}\,, \quad A_y = \frac{\int y dV}{\int dV}\,, \quad A_z = \frac{\int z dV}{\int dV}\,, \tag{2a}$$
and where $dV = dx\,dy\,dz$. Calculate $\mathbf{A}$ for a homogeneous hemisphere bounded by $0 \le r \le a$, $0 \le \theta \le \pi/2$, and $0 \le \phi \le 2\pi$. Note that $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, and $z = r\cos\theta$.

8. Express $\mathbf{A}$ for the centroid of the spheroid in Item 7 in spherical coordinates, noting that
$$\mathbf{e}_x = \sin\theta\cos\phi\,\mathbf{e}_r + \cos\theta\cos\phi\,\mathbf{e}_\theta - \sin\phi\,\mathbf{e}_\phi\,, \tag{3a}$$
$$\mathbf{e}_y = \sin\theta\sin\phi\,\mathbf{e}_r + \cos\theta\sin\phi\,\mathbf{e}_\theta + \cos\phi\,\mathbf{e}_\phi\,, \tag{3b}$$
$$\mathbf{e}_z = \cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta\,. \tag{3c}$$
For what angles $\theta$ is $\mathbf{A}$ given purely by $\mathbf{e}_r$?

9. Prove Gauss's and Stokes's theorems.