

# Radiation force on inhomogeneous subwavelength scatterers due to progressive waves

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188<sup>th</sup> ASA & 25<sup>th</sup> ICA Meeting  
New Orleans, Louisiana



The University of Texas at Austin  
**Walker Department  
of Mechanical Engineering**  
*Cockrell School of Engineering*

# Outline

Introduction

Westervelt's formulation in the far field

Linear scattering problem

Examples

# Outline

## Introduction

Westervelt's formulation in the far field

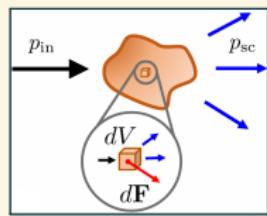
Linear scattering problem

Examples

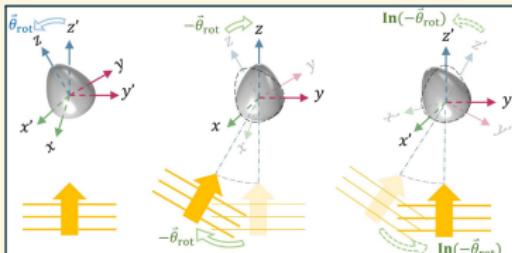
# Motivation

## Particle manipulation of asymmetric objects

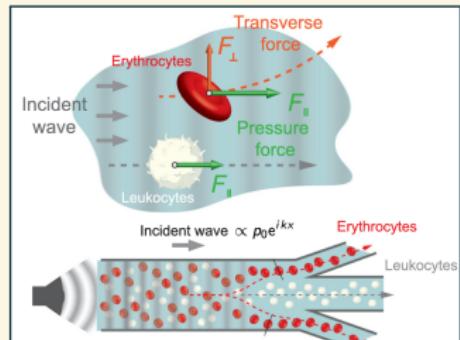
- ▶ Forces and torques due to standing waves have been investigated.<sup>1</sup>
- ▶ Jerome et al.<sup>2</sup> used the Born approximation to calculate radiation force and torque on subwavelength objects in primarily standing wave fields.
- ▶ Forces due to progressive waves have only recently garnered interest.<sup>3</sup>



T. S. Jerome.  
PhD thesis. University  
of Texas at Austin,  
2022



T. Tang, Y. Zhang, B. Dong, and L. Huang.  
*J. Acoust. Soc. Am.* 156 (2024),  
pp. 2767–2782



M. Smagin, I. Toftul, K. Y. Bliokh, and  
M. Petrov. *Phys. Rev. Appl.* 22 (2024),  
pp. 1–20

<sup>1</sup>E. B. Lima and G. T. Silva. *J. Acoust. Soc. Am.* 150 (2021), pp. 376–384.

<sup>2</sup>T. S. Jerome and M. F. Hamilton. *J. Acoust. Soc. Am.* 150 (2021), pp. 3417–3427.

<sup>3</sup>T. Tang and L. Huang. *J. Sound Vib.* 532 (2022), pp. 1–19.

# Conservation of momentum at quadratic order

- ▶ Acoustic radiation force is a consequence of  $O(\epsilon^2)$  momentum conservation:

$$\frac{\partial \mathbf{g}}{\partial t} = \nabla \cdot \underline{\mathbf{T}} \quad (1)$$

- ▶ The momentum density is

$$\mathbf{g} = p\mathbf{v}/c_0^2 \quad (2)$$

- ▶ In free space, the instantaneous acoustic radiation stress tensor is

$$\underline{\mathbf{T}} \equiv L\underline{\mathbf{I}} - \rho_0 \mathbf{v} \otimes \mathbf{v}. \quad (3)$$

- ▶ The Lagrangian density is

$$L = \frac{1}{2}\rho_0 v^2 - \frac{p^2}{2\rho_0 c_0^2}, \quad (4)$$

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# Westervelt's formulation in the far field



P. J. Westervelt (left) and R. V. Khokhlov (right)  
Copenhagen, 1973  
Photograph by D. T. Blackstock

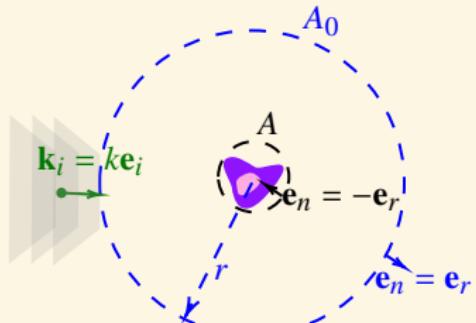
# Westervelt's formulation in the far field



P. J. Westervelt (left) and M. F. Hamilton (right)  
1984 International Symposium on Nonlinear Acoustics  
Kobe, Japan

# Westervelt's formulation in the far field

- The volume integral of  $\nabla \cdot \langle \mathbf{T} \rangle$  enclosing an object is the radiation force<sup>4</sup>



$$\mathbf{F} = \int_V \nabla \cdot \langle \mathbf{T} \rangle dV = \oint_A \langle \mathbf{T} \rangle \cdot d\mathbf{A}$$



P. J. Westervelt

- Invoking the far-field approximation and energy conservation yields

$$F_{\parallel} = \frac{p_0^2}{2\rho_0 c_0^2} \oint |\Phi_s(\mathbf{k}_s)|^2 (1 - \cos \psi_0) d\Omega_0, \quad (5a)$$

$$F_{\perp} = -\frac{p_0^2}{2\rho_0 c_0^2} \oint |\Phi_s(\mathbf{k}_s)|^2 \mathbf{e}_m \cdot \mathbf{e}_r d\Omega_0, \quad d\Omega_0 = r_0^{-2} dA_0. \quad (5b)$$

where  $\Phi_s$  = scattered directivity,  $\cos \psi = \mathbf{k}_s \cdot \mathbf{k}_i / k^2$ , and  $\Omega$  = solid angle.

<sup>4</sup>P. J. Westervelt. *J. Acoust. Soc. Am.* 29 (1957), pp. 26–29.

# Linear scattering problem

*[F]or the calculation of the average force correct up to terms of the second order in the velocity, it is sufficient to find the solution of the linear scattering problem.<sup>5</sup>*



Lev Petrovich Gor'kov

G. Boebinger, S. Iordansky, D. Pines, and L. Pitaevskii. *Physics Today* 70 (2017), pp. 68–69

<sup>5</sup>L. P. Gor'kov. *Sov. Phys. Dokl.* 6 (1962), pp. 773–775.

# Outline

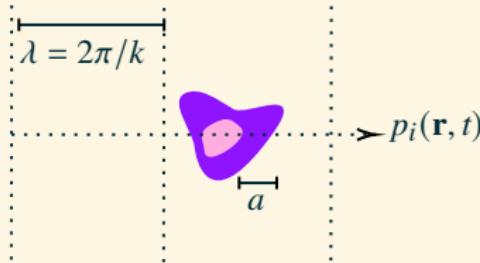
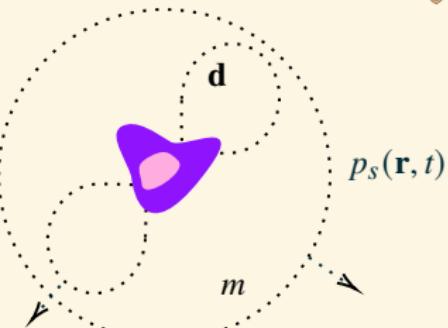
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# Acoustic polarizability

(a) incident acoustic waves,  $ka \ll 1$ 

(b) scattered acoustic waves

- ▶ Polarizabilities describe the heterogeneity's response to local fields:<sup>6</sup>
  - ▶  $\alpha_m$  relates  $p_i \leftrightarrow m$
  - ▶  $\underline{\alpha}_d$  relates  $\mathbf{v}_i \leftrightarrow \mathbf{d}$
  - ▶  $\alpha_c$  relates  $\mathbf{v}_i \leftrightarrow m$  and  $p_i \leftrightarrow \mathbf{d}$
- ▶ The scattered monopole strength  $m$  and dipole moment  $\mathbf{d}$  are given by

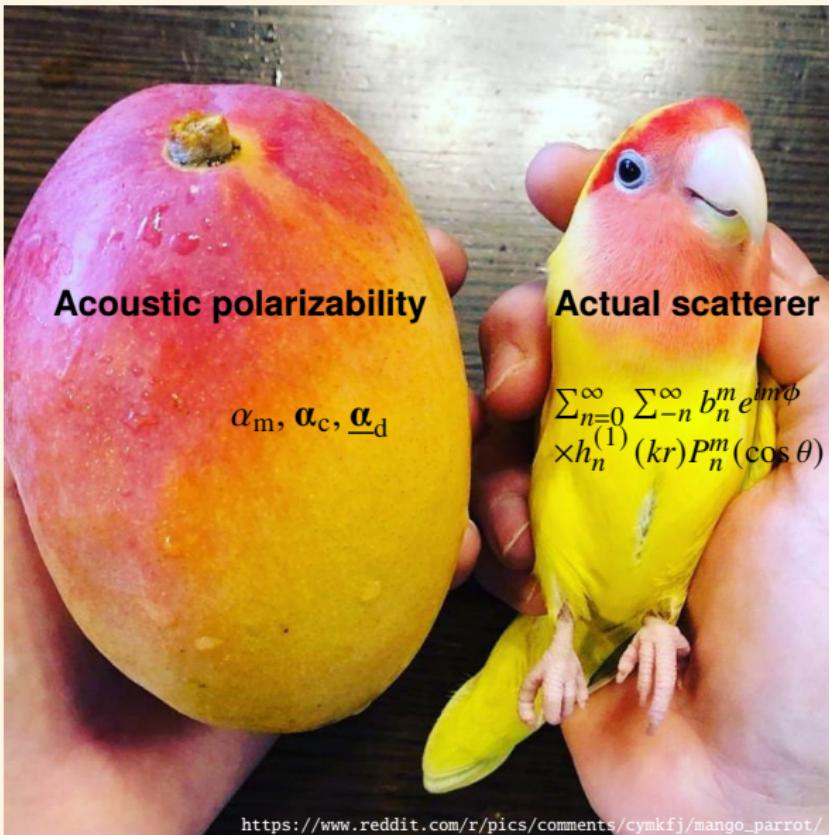
$$m = -\beta_0 \alpha_m p_i - i c_0^{-1} \alpha_c \cdot \mathbf{v}_i , \quad (6a)$$

$$\mathbf{d} = -i c_0^{-1} \alpha_c p_i + \rho_0 \underline{\alpha}_d \cdot \mathbf{v}_i . \quad (6b)$$

- ▶ Objective: Obtain a general approach to calculate  $\alpha_m$ ,  $\alpha_c$ , and  $\underline{\alpha}_d$ .

<sup>6</sup>C. F. Sieck, A. Alù, and M. R. Haberman. *Phys. Rev. B*. 96 (2017), pp. 1–20

# Multipole expansion to dipole order



# Scattering of sound from heterogeneities

- The equation of state in heterogeneous media is<sup>7</sup>

$$\frac{\partial \rho'}{\partial t} = \frac{1}{c^2} \frac{\partial p}{\partial t} - \mathbf{v} \cdot \nabla \rho(\mathbf{r}). \quad (7)$$

- The linearized mass conservation equation combined with Eq. (7) yields

$$\nabla \cdot \mathbf{v} = -\beta(\mathbf{r}) \frac{\partial p}{\partial t}. \quad (8)$$

- Linearizing the momentum equation yields

$$\frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{\rho(\mathbf{r})} \nabla p. \quad (9)$$

- Combining Eqs. (8) and (9) and assuming time-harmonic solutions yields

$$\nabla^2 \tilde{p}_\omega + k^2 \tilde{p}_\omega = k^2 f_1 \tilde{p}_\omega + \nabla \cdot \left( \frac{3f_2}{2+f_2} \nabla \tilde{p}_\omega \right). \quad (10)$$

where the contrast factors are<sup>8</sup>

$$f_1(\mathbf{r}) = 1 - \frac{\beta(\mathbf{r})}{\beta_0}, \quad f_2(\mathbf{r}) = \frac{2[\rho(\mathbf{r}) - \rho_0]}{2\rho(\mathbf{r}) + \rho_0}. \quad (11)$$

<sup>7</sup>A. D. Pierce. 2nd edition. Woodbury, New York: Acoustical Society of America, 1989.

<sup>8</sup>L. P. Gor'kov. Sov. Phys. Dokl. 6 (1962), pp. 773–775.

# Scattering of sound from heterogeneities

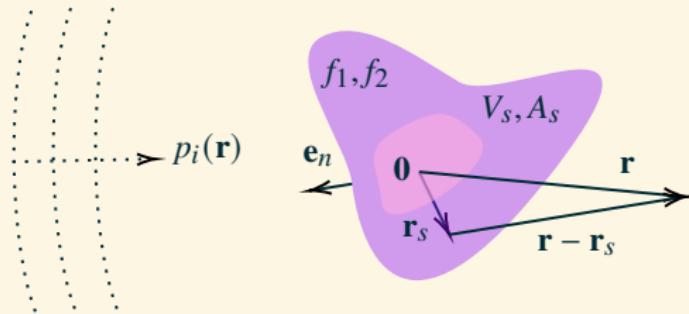
- The Helmholtz-Kirchhoff integral theorem solves Eq. (10),<sup>9</sup>

$$\tilde{p}_{\omega}(\mathbf{r}) = \tilde{p}_{i,\omega}(\mathbf{r}) + \tilde{p}_{s,\omega}(\mathbf{r}),$$

$$\tilde{p}_{s,\omega}(\mathbf{r}) = \int_{V_s} \left[ k^2 f_1(\mathbf{r}_s) \tilde{p}_{\omega}(\mathbf{r}_s) g(\mathbf{r}|\mathbf{r}_s) - \frac{3f_2(\mathbf{r}_s)}{2+f_2(\mathbf{r}_s)} \nabla_s \tilde{p}_{\omega}(\mathbf{r}_s) \cdot \nabla_s g(\mathbf{r}|\mathbf{r}_s) \right] dV_s.$$

- The origin is defined by the *centroid*,

$$\mathbf{0} \equiv \frac{\int_{V_s} \mathbf{r}_s dV_s}{\int_{V_s} dV_s}. \quad (12)$$



<sup>9</sup>P. M. Morse and K. U. Ingard. McGraw-Hill, 1968.

# Three approximations

1. **Far-field:** If  $r \gg a$ , then in terms of  $\mathbf{k}_s = k\mathbf{e}_r$ ,

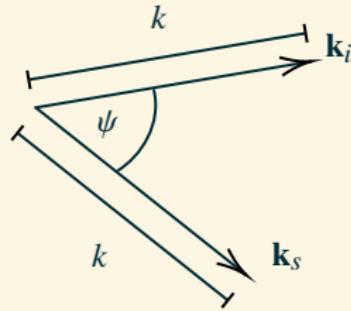
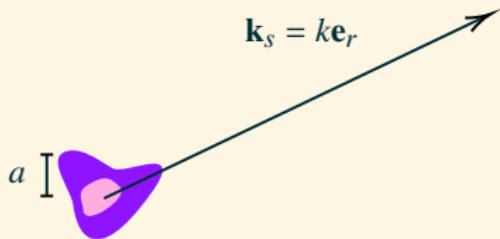
$$g(\mathbf{r}|\mathbf{r}_s) \simeq \frac{e^{ikr}}{4\pi r} e^{-i\mathbf{k}_s \cdot \mathbf{r}_s}, \quad \nabla_s g \simeq -i\mathbf{k}_s g. \quad (13)$$

2. **Subwavelength:** Since  $ka \sim |\mathbf{k}_s \cdot \mathbf{r}_s| \ll 1$ , one can approximate

$$e^{-i\mathbf{k}_s \cdot \mathbf{r}_s} \simeq 1 - i\mathbf{k}_s \cdot \mathbf{r}_s, \quad (14)$$

and therefore  $p_i(\mathbf{r}) = p_0 e^{i\mathbf{k}_i \cdot \mathbf{r}} \simeq p_0 (1 + i\mathbf{k}_i \cdot \mathbf{r})$ .

3. **Born:** If  $f_1, f_2 \ll 1$ , then  $|p_s| \ll |p_i|$ , and the problem becomes explicit.



# Solution of scattering problem to dipole order

- ▶ Identifying

$$\alpha_m = - \int_{V_s} f_1(\mathbf{r}_s) dV_s, \quad (15a)$$

$$\underline{\alpha}_d = \underline{\mathbf{I}} \int_{V_s} \frac{3f_2(\mathbf{r}_s)}{2+f_2(\mathbf{r}_s)} dV_s \equiv \alpha_d \underline{\mathbf{I}}, \quad (15b)$$

$$\underline{\alpha}_c = -k \left[ \int_{V_s} f_1(\mathbf{r}_s) \mathbf{r}_s dV_s - \cos \psi \int_{V_s} \frac{3f_2(\mathbf{r}_s)}{2+f_2(\mathbf{r}_s)} \mathbf{r}_s dV_s \right] \quad (15c)$$

yields the scattered far field:

$$p_s(\mathbf{r}) = p_0 \frac{e^{ikr}}{r} \Phi_s, \quad \Phi_s(\mathbf{k}_s) = \frac{k^2}{4\pi} [\alpha_m + i\underline{\alpha}_c \cdot (\mathbf{e}_i - \mathbf{e}_r) + \alpha_d \cos \psi]. \quad (16)$$

- ▶ Thus using

$$|\Phi_s|^2 = \frac{k^4}{16\pi^2} \{ \alpha_m^2 + 2\alpha_m \alpha_d \cos \psi + [\underline{\alpha}_c \cdot (\mathbf{e}_i - \mathbf{e}_r)]^2 + \alpha_d^2 \cos^2 \psi \} \quad (17)$$

with Eq. (5) gives the radiation force in terms of the polarizabilities.

# Distinction from Rayleigh scattering

$$p_s(\mathbf{r}) = p_0 \frac{e^{ikr}}{r} \Phi_s, \quad \Phi_s(\mathbf{k}_s) = \frac{k^2}{4\pi} [\alpha_m + i\alpha_c \cdot (\mathbf{e}_i - \mathbf{e}_r) + \alpha_d \cos \psi]$$

*Any complex exponential representing a phase shift within the scattering body or on its surface may be approximated as having unit value.<sup>10</sup>*



A. D. Pierce (left) and J. H. Ginsberg (right)  
J. H. Ginsberg. *Acoustics Today* 11 (2015), pp. 10–16

<sup>10</sup>J. H. Ginsberg. Vol. 2. Springer, 2018.

# Distinction from Rayleigh scattering

$$e^{-i\mathbf{k}_s \cdot \mathbf{r}_s} \simeq 1$$



$$e^{-i\mathbf{k}_s \cdot \mathbf{r}_s} \simeq 1 - i\mathbf{k}_s \cdot \mathbf{r}_s$$



# Outline

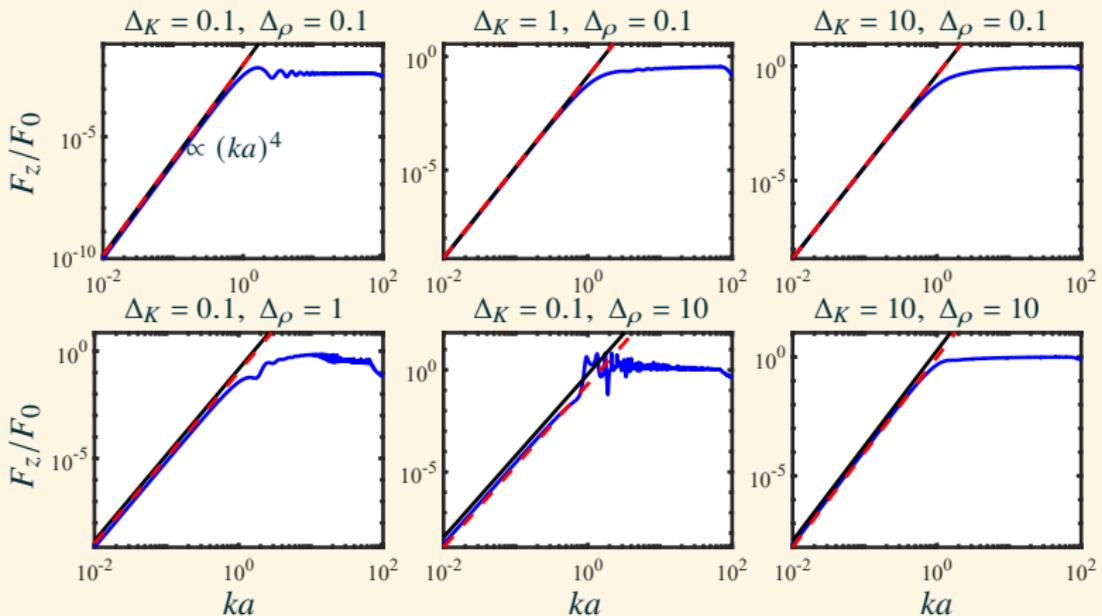
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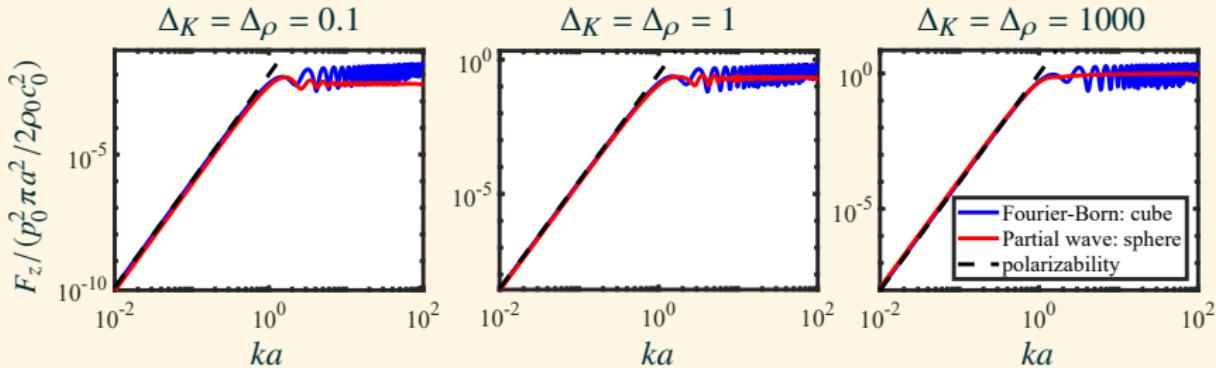
# Example 1: Homogeneous sphere<sup>11</sup>



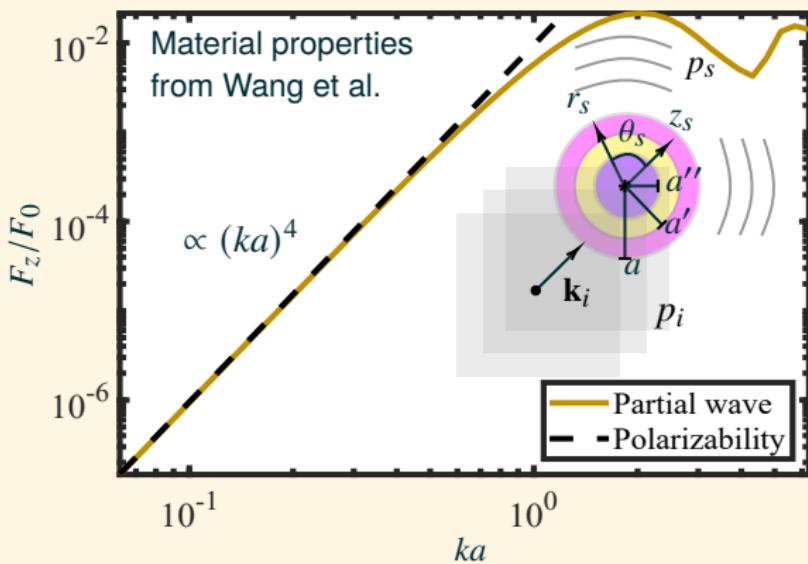
Comparison of the **polarizability formulation** (??), **Gor'kov's result** (??), and the **exact solution based on spherical wave expansions** (??), for values of  $\Delta_K = (K_s - K_0)/K_0$  and  $\Delta_\rho = (\rho_s - \rho_0)/\rho_0$  ranging two orders of magnitude.

<sup>11</sup>The forces are normalized to  $F_0 = p_0^2 \pi a^2 / 2\rho_0 c_0^2$ .

## Example 2: Homogeneous cube

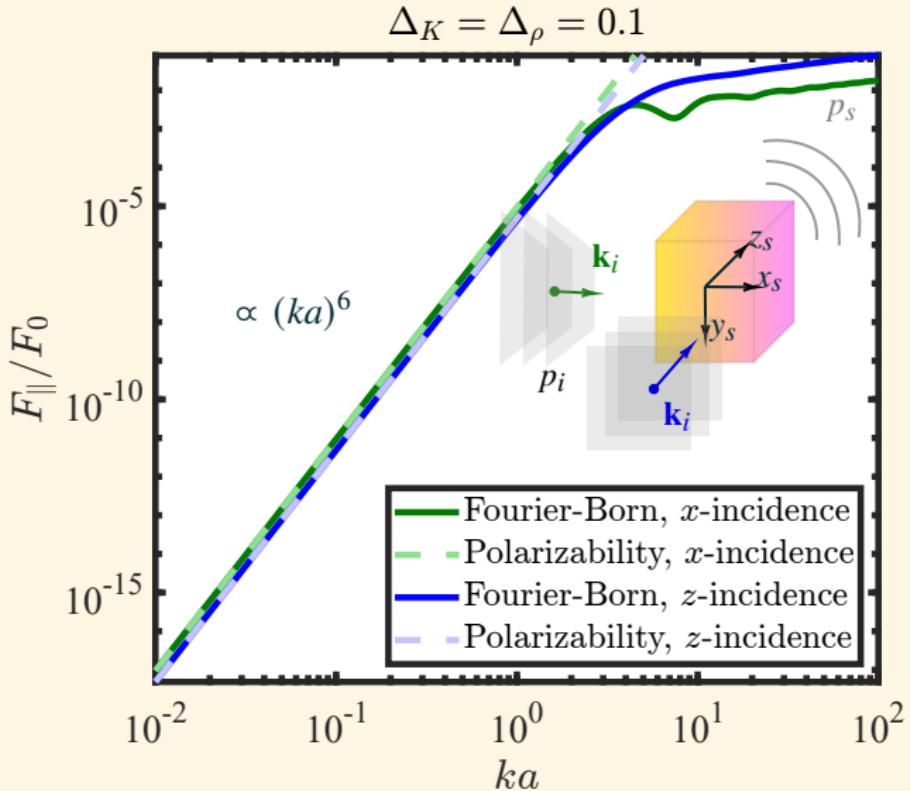


Comparison of the radiation force given by the **polarizability formulation** [dashed line, Eq. (??)] to the forces on a homogeneous **cube** [Eq. (??)] and **sphere** of equal volume [red curves, Eq. (??)] for  $\Delta_K = (K_s - K_0)/K_0$  and  $\Delta_\rho = (\rho_s - \rho_0)/\rho_0$  ranging four orders of magnitude. For  $ka \ll 1$  in all cases, the force on the cube converges to that on the sphere.

Example 3: Spherically symmetric nucleated cell<sup>12</sup>

<sup>12</sup>Y.-Y. Wang et al. *J. Appl. Phys.* 122 (2017), pp. 1–6.

# Example 4: Antisymmetric inhomogeneous cube



# Conclusion

## Summary

- ▶ Solved **linear scattering problem** at  $O[(ka)^3]$  in Born approximation
- ▶ Calculated acoustic radiation force due to progressive waves on **homogeneous and inhomogeneous cubes and spheres**
- ▶ Compared results to **partial wave expansions** and **Fourier-Born scattering**

## Future work

- ▶ Calculate radiation torque
- ▶ Develop ray theory for calculation of high-frequency asymptote

## Acknowledgments

- ▶ T. S. Jerome for the exact solutions on the homogeneous and layered spheres
- ▶ Chester M. McKinney Graduate Fellowship in Acoustics at ARL:UT



Dr. T. S. Jerome



Prof. M. R. Haberman



Prof. M. F. Hamilton

# References I

-  E. B. Lima and G. T. Silva. "Mean acoustic fields exerted on a subwavelength axisymmetric particle". *J. Acoust. Soc. Am.* 150 (2021), pp. 376–384.
-  T. S. Jerome and M. F. Hamilton. "Born approximation of acoustic radiation force and torque on inhomogeneous objects". *J. Acoust. Soc. Am.* 150 (2021), pp. 3417–3427.
-  T. Tang and L. Huang. "An efficient semi-analytical procedure to calculate acoustic radiation force and torque for axisymmetric irregular bodies". *J. Sound Vib.* 532 (2022), pp. 1–19.
-  T. S. Jerome. "Acoustic radiation force and torque on nonspherical objects". PhD thesis. University of Texas at Austin, 2022.
-  T. Tang, Y. Zhang, B. Dong, and L. Huang. "Computation of acoustic scattered fields and derived radiation force and torque for axisymmetric objects at arbitrary orientations". *J. Acoust. Soc. Am.* 156 (2024), pp. 2767–2782.
-  M. Smagin, I. Toftul, K. Y. Bliokh, and M. Petrov. "Acoustic lateral recoil force and stable lift of anisotropic particles". *Phys. Rev. Appl.* 22 (2024), pp. 1–20.
-  P. J. Westervelt. "Acoustic radiation pressure". *J. Acoust. Soc. Am.* 29 (1957), pp. 26–29.

## References II

-  L. P. Gor'kov. "On the forces acting on a small particle in an acoustical field in an ideal fluid". *Sov. Phys. Dokl.* 6 (1962), pp. 773–775.
-  G. Boebinger, S. Iordansky, D. Pines, and L. Pitaevskii. "Lev Petrovich Gor'kov". *Physics Today* 70 (2017), pp. 68–69.
-  C. F. Sieck, A. Alù, and M. R. Haberman. "Origins of Willis coupling and acoustic bianisotropy in acoustic metamaterials through source-driven homogenization". *Phys. Rev. B.* 96 (2017), pp. 1–20.
-  A. D. Pierce. *Acoustics: An Introduction to its Physical Principles and Applications*. 2nd edition. Woodbury, New York: Acoustical Society of America, 1989.
-  P. M. Morse and K. U. Ingard. *Theoretical Acoustics*. McGraw-Hill, 1968.
-  J. H. Ginsberg. *Acoustics: A Textbook for Engineers and Physicists*. Vol. 2. Springer, 2018.
-  — . "Allan D. Pierce: A Celebration of a Career in Acoustics in Commemoration of His Retirement as Editor-in Chief of the Acoustical Society of America". *Acoustics Today* 11 (2015), pp. 10–16.

## References III

-  Y.-Y. Wang et al. "Influences of the geometry and acoustic parameter on acoustic radiation forces on three-layered nucleate cells". *J. Appl. Phys.* 122 (2017), pp. 1–6.
-  J. M. Izen. *Comet NEOWISE*. 2020.
-  T. G. Wang and C. P. Lee, "Radiation Pressure and Acoustic Levitation," M. F. Hamilton and D. T. Blackstock, editors. *Nonlinear Acoustics, 3rd edition*. Cham, Switzerland: Springer, 2024.
-  T. B. A. Senior. "Low-frequency scattering". *J. Acoust. Soc. Am.* 53 (1973), pp. 742–747.
-  Lord Rayleigh. "On the transmission of light through an atmosphere containing small particles in suspension, and on the origin of the blue of the sky". *Philos. Mag.* 47 (1899), pp. 375–384.
-  X. Su and A. N. Norris. "Retrieval method for the bianisotropic polarizability tensor of Willis acoustic scatterers". *Phys. Rev. B* 98 (2018), pp. 1–8.
-  S. Sepehrirahnama, S. Oberst, Y. K. Chiang, and D. A. Powell. "Acoustic radiation force and radiation torque beyond particles: Effects of nonspherical shape and Willis coupling". *Phys. Rev. E* 104 (2021), pp. 1–11.

## References IV

-  L. Quan, Y. Ra'di, D. L. Sounas, and A. Alù. "Maximum Willis Coupling in Acoustic Scatterers". *Phys. Rev. Lett.* 120 (2018), pp. 1–7.
-  Y. A. Ilinskii, E. A. Zabolotskaya, B. C. Treweek, and M. F. Hamilton. "Acoustic radiation force on an elastic sphere in a soft elastic medium". *J. Acoust. Soc. Am.* 144 (2018), pp. 568–576.
-  C. A. Gokani, T. S. Jerome, M. R. Haberman, and M. F. Hamilton. "Born approximation of acoustic radiation force used for acoustofluidic separation". *Proc. Mtgs. Acoust.* 48 (2022), pp. 1–10.

# Notation I

Symbol	Description	Dimensions
$a$	characteristic size of scatterer	m
$c_0$	speed of sound	$\text{m s}^{-1}$
$\mathbf{F}$	force	$\text{kg m s}^{-2}$
$i$	complex unit	1
$\mathbf{k}$	wavenumber	$\text{m}^{-1}$
$p$	acoustic pressure	$\text{kg m}^{-1} \text{s}^{-2}$
$\mathbf{R}$	separation vector, $\mathbf{R} = \mathbf{r} - \mathbf{r}'$	m
$\mathbf{r}$	position vector	m
$\mathbf{v}$	particle velocity	$\text{m s}^{-1}$
$\rho_0$	ambient mass density	$\text{kg m}^{-3}$
$\underline{\mathbf{T}}$	acoustic radiation stress tensor	$\text{kg m}^{-1} \text{s}^{-2}$
$\underline{\mathbf{S}}$	instantaneous Poynting vector	$\text{kg s}^{-3}$
$\omega$	angular frequency, $\omega = 2\pi f$	$\text{s}^{-1}$

COMET Neowise<sup>13</sup><sup>13</sup> J. M. Izen, 2020