Reciprocity, passivity, and causality in fully coupled acousto-electrodynamic media

Chirag A. Gokani
Samuel P. Wallen
Michael R. Haberman

Applied Research Laboratories
Walker Department of Mechanical Engineering
University of Texas at Austin

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Outline

Background

1. Piezoelectricity
2. Electrodynamic and acoustic bianisotropy
3. Acousto-electrodynamic constitutive relations

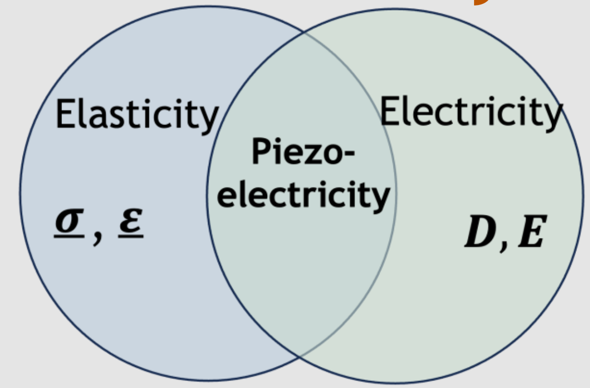
Derivations for acousto-electrodynamic media

1. Reciprocity
2. Passivity
3. Causality

Closing

1. Verification of source-driven constitutive relations
2. Interpretation

Piezoelectricity



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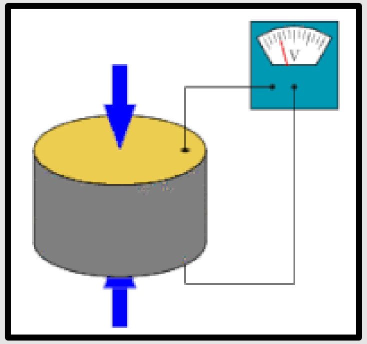
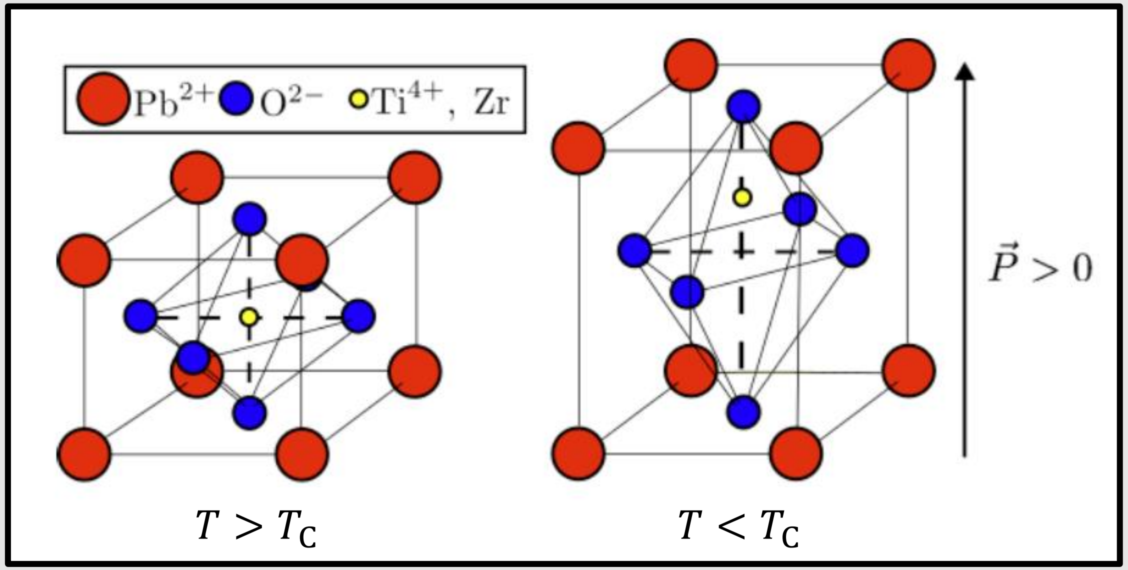


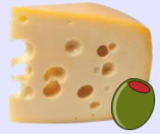


Etymology

πιέζω

piézō

"to squeeze or press"



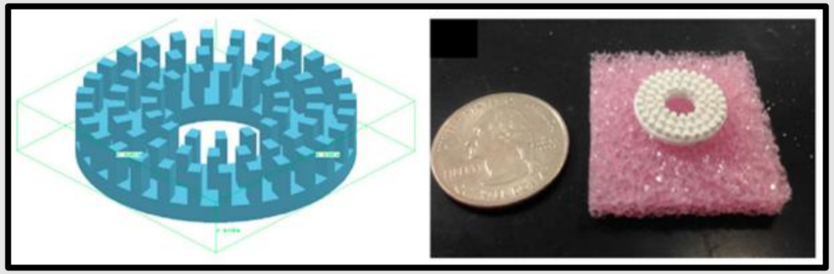
Piezoelectric constitutive relations

compliance, stress, electric field

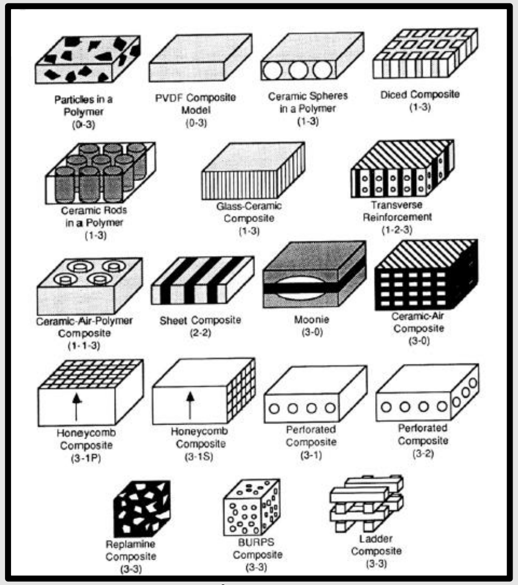
$$\text{strain} \rightarrow \underline{\varepsilon} = \mathcal{S} : \underline{\sigma} + \tilde{d}^{\mathsf{T}} \cdot E$$

$$\text{Electric displacement} \rightarrow D = \tilde{d} : \underline{\sigma} + \underline{\epsilon}^{\mathsf{T}} \cdot E$$

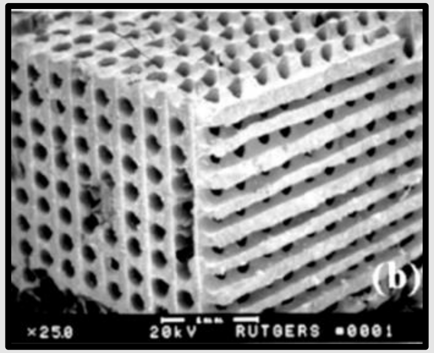
coupling, permittivity

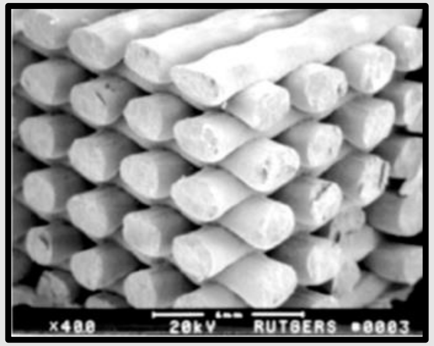


Smirnov, et al., Ceram. Int. (2021)



Smirnov, et al., Ceram. Int. (2021)





Electrodynamic and acoustic bianisotropy

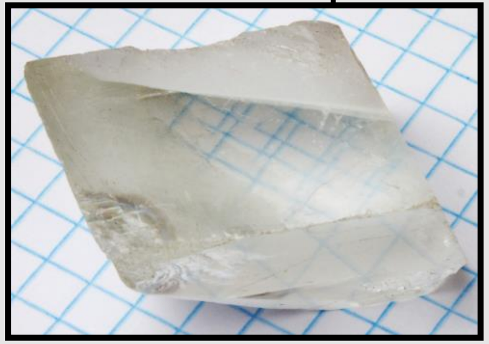
$$\boldsymbol{D} = \epsilon \boldsymbol{E}$$
$$\boldsymbol{B} = \mu \boldsymbol{H}$$

Isotropic

Hubble deep field, NASA

$$\boldsymbol{D} = \underline{\epsilon} \boldsymbol{E}$$
$$\boldsymbol{B} = \underline{\mu} \boldsymbol{H}$$

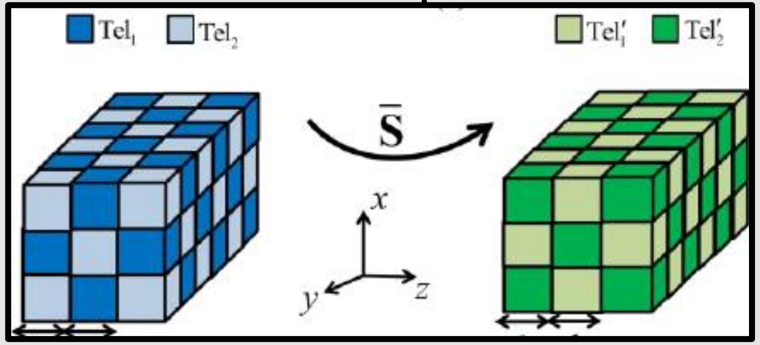
Anisotropic



Birefringence, Wikipedia

$$\boldsymbol{D} = \epsilon \boldsymbol{E} + \zeta \boldsymbol{H}$$
$$\boldsymbol{B} = \xi \boldsymbol{E} + \mu \boldsymbol{H}$$

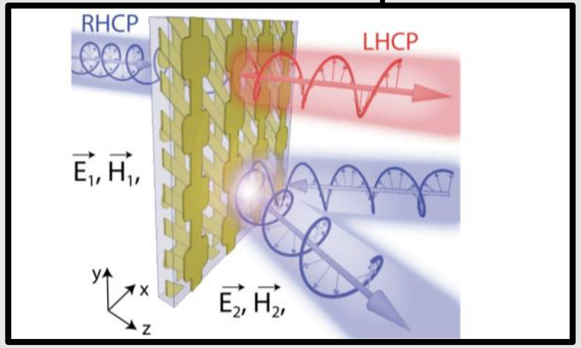
Biisotropic



Prudencio et al., IEEE (2014)

$$\boldsymbol{D} = \underline{\epsilon} \boldsymbol{E} + \underline{\zeta} \boldsymbol{H}$$
$$\boldsymbol{B} = \underline{\xi} \boldsymbol{E} + \underline{\mu} \boldsymbol{H}$$

Bianisotropic

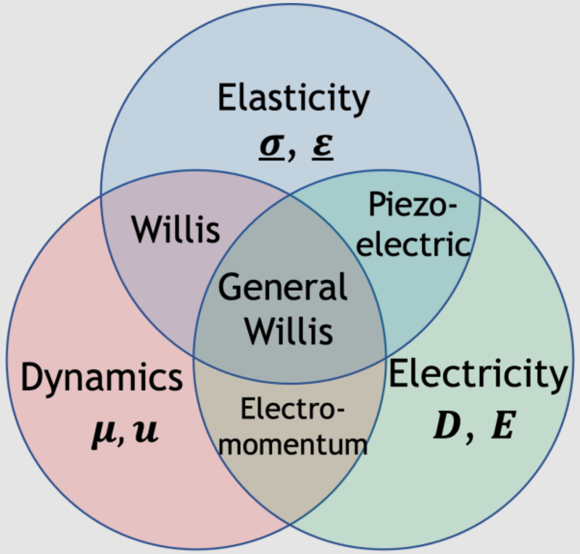


C. Pfeiffer et al., PRL (2014)

$$\varepsilon = -\beta p$$
$$\boldsymbol{\mu} = \rho \boldsymbol{u}$$

Isotropic

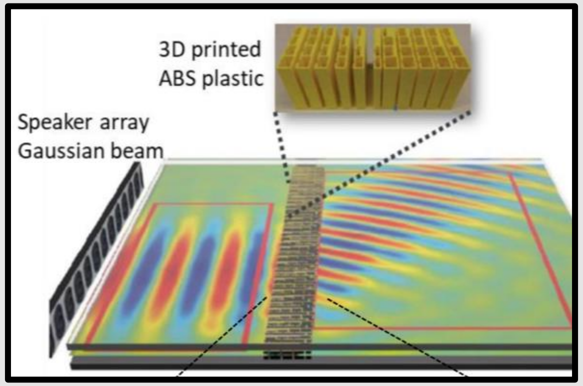
Meyerson Symphony Center



Pernas-Salomón and Shmuel, J. Mech. Phys. Solids **134**, 103770 (2020)

$$\varepsilon = -\beta p + \boldsymbol{\gamma} \cdot \boldsymbol{u}$$
$$\boldsymbol{\mu} = -\boldsymbol{\eta} p + \underline{\rho} \boldsymbol{u}$$

Bianisotropic



J. Li et al., Nat. Comm. (2018)

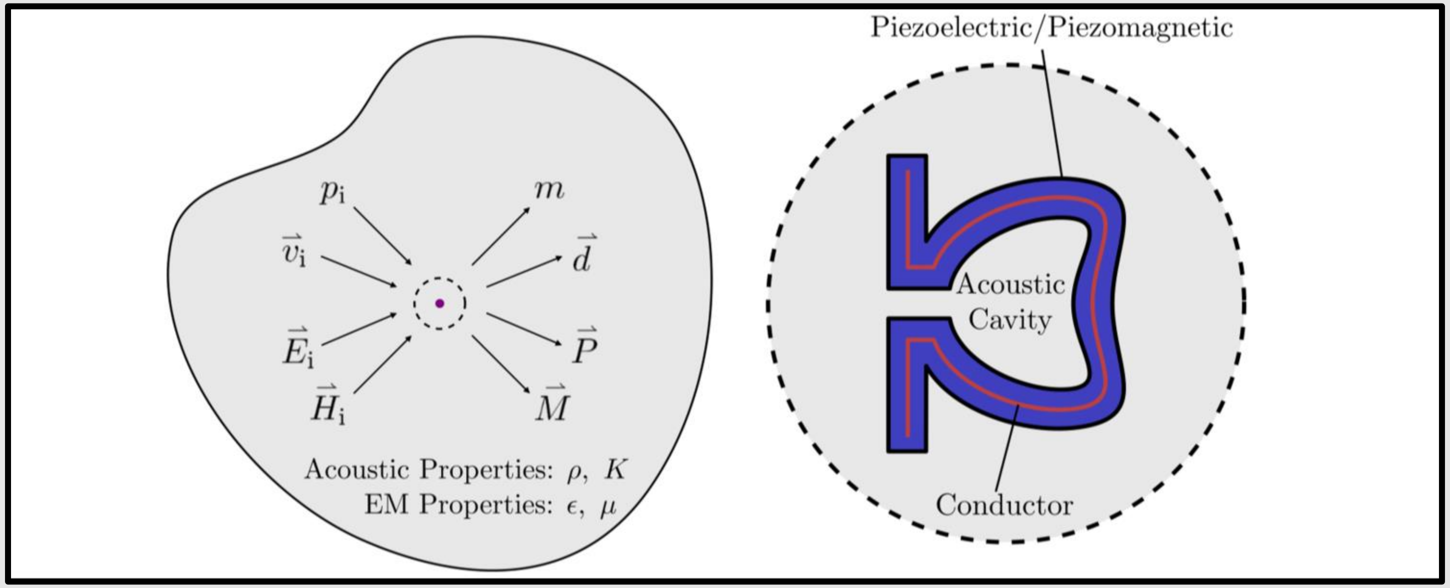
Acousto-electrodynamic constitutive relations

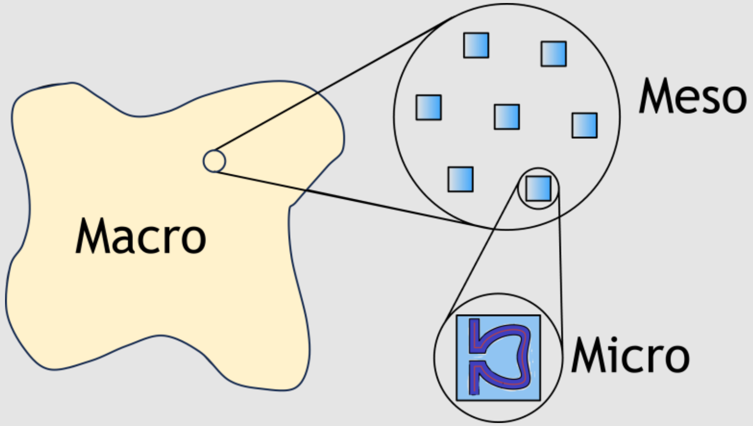
$$\begin{pmatrix} \varepsilon \\ \boldsymbol{\mu} \\ \boldsymbol{D} \\ \boldsymbol{B} \end{pmatrix} = \begin{pmatrix} -\beta & \boldsymbol{\gamma}^{\mathsf{T}} & \boldsymbol{b}^{\mathsf{T}} & \boldsymbol{d}^{\mathsf{T}} \\ -\boldsymbol{\eta} & \underline{\boldsymbol{\rho}} & \underline{\boldsymbol{v}} & \underline{\boldsymbol{m}} \\ \boldsymbol{c} & \underline{\boldsymbol{w}} & \underline{\boldsymbol{\epsilon}} & \underline{\boldsymbol{\xi}} \\ \boldsymbol{e} & \underline{\boldsymbol{n}} & \underline{\boldsymbol{\zeta}} & \underline{\boldsymbol{\mu}} \end{pmatrix} \begin{pmatrix} p \\ \boldsymbol{u} \\ \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}$$

Components of constitutive block matrix

1. acoustic bianisotropy
2. electromagnetic bianisotropy
3. piezoelectricity
4. piezomagnetism
5. **electromomentum**
6. **magnetomomentum**

"[O]ne can think about metasurfaces **simultaneously controlling and transforming waves of different nature**, which can lead to the creation of **multiphysics** bianisotropic metasurfaces."
—V. S. Asadchy et al., Nanophotonics **7**, 6 (2018)





Acousto-electrodynamic constitutive relations

$$\begin{pmatrix} \varepsilon \\ \boldsymbol{\mu} \\ \boldsymbol{D} \\ \boldsymbol{B} \end{pmatrix} = \begin{pmatrix} -\beta & \boldsymbol{\gamma}^\mathsf{T} & \boldsymbol{b}^\mathsf{T} & \boldsymbol{d}^\mathsf{T} \\ -\boldsymbol{\eta} & \underline{\boldsymbol{\rho}} & \underline{\boldsymbol{v}} & \underline{\boldsymbol{m}} \\ \boldsymbol{c} & \underline{\boldsymbol{w}} & \underline{\boldsymbol{\epsilon}} & \underline{\boldsymbol{\xi}} \\ \boldsymbol{e} & \underline{\boldsymbol{n}} & \underline{\boldsymbol{\zeta}} & \underline{\boldsymbol{\mu}} \end{pmatrix} \begin{pmatrix} p \\ \boldsymbol{u} \\ \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}$$

"[O]ne can think about metasurfaces **simultaneously controlling and transforming waves of different nature**, which can lead to the creation of **multiphysics** bianisotropic metasurfaces."
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Components of constitutive block matrix
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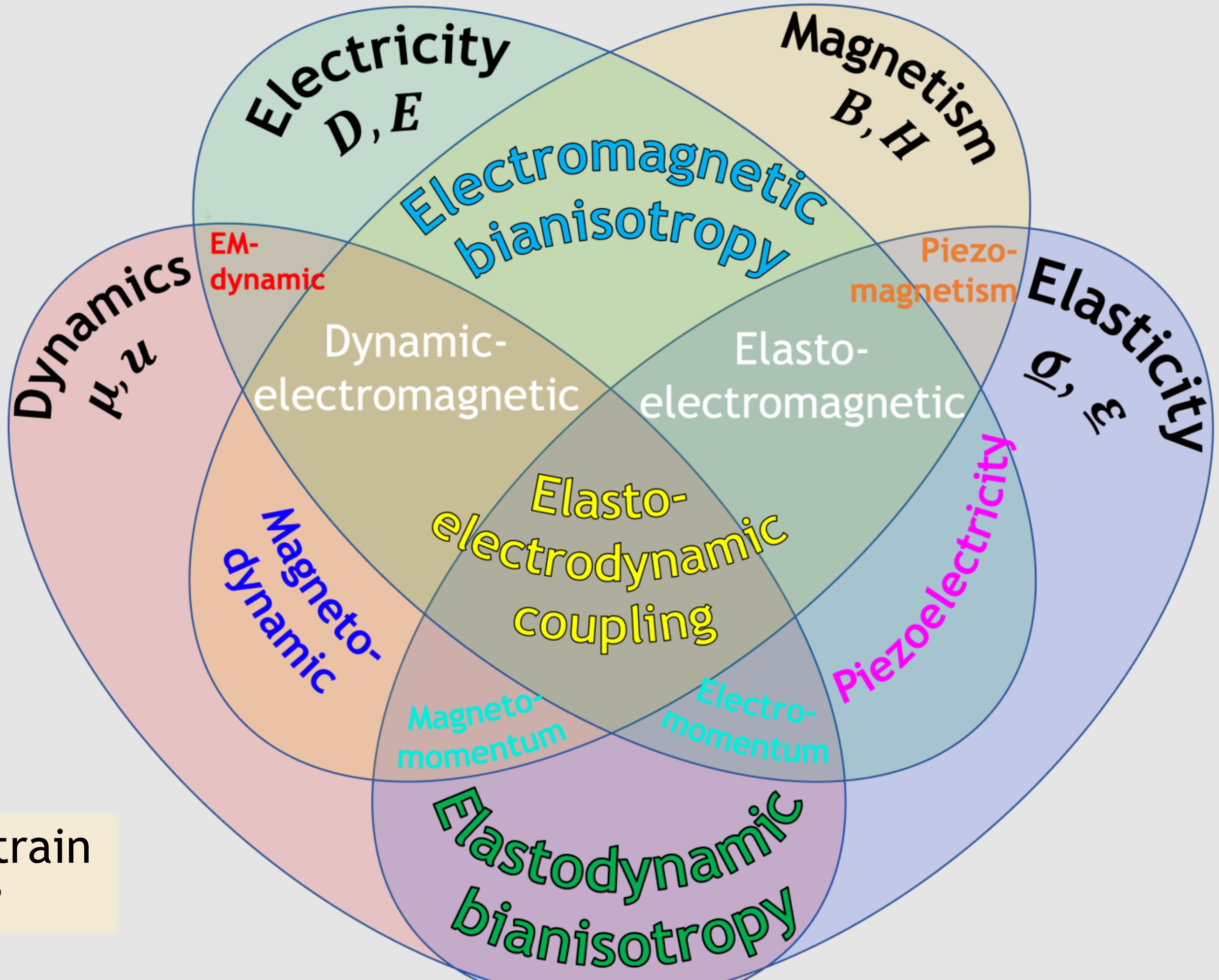

How do **reciprocity**, **passivity**, and **causality** constrain the acousto-electrodynamic constitutive relations?

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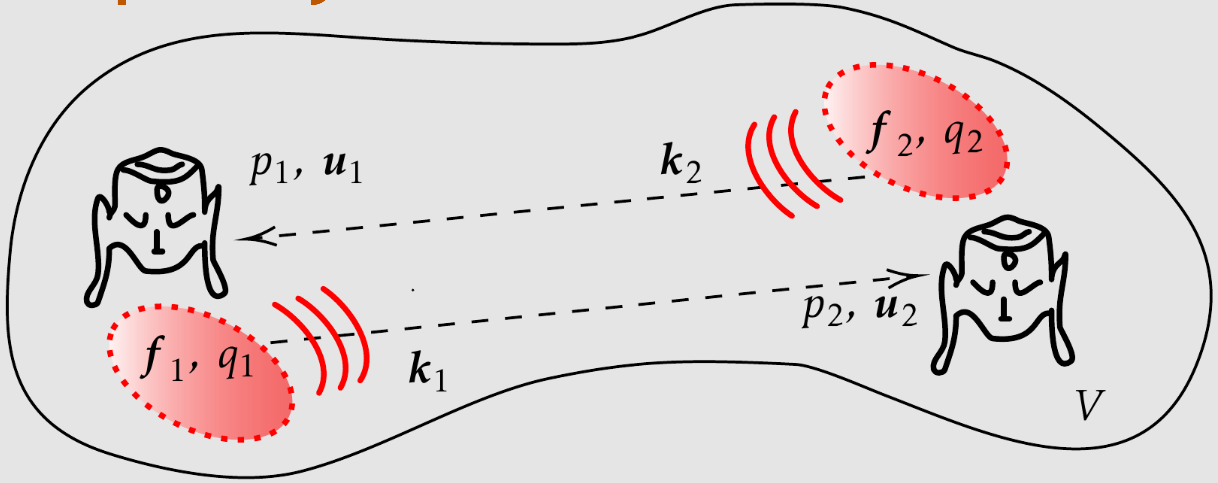
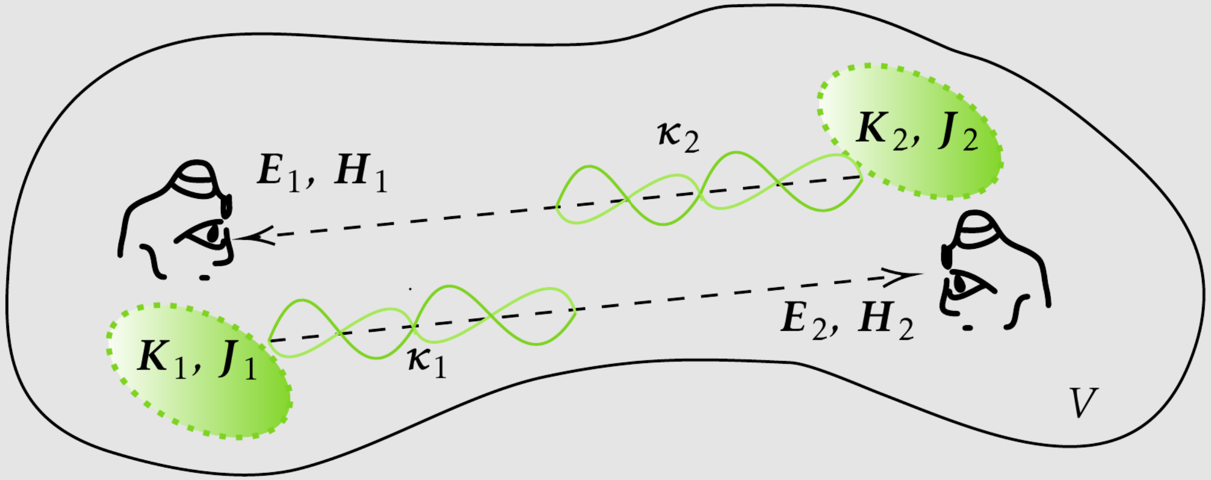
Reciprocity





Governing equations for 1

$$\begin{aligned}
\nabla p_1 &= i\omega\boldsymbol{\mu}_1 + \boldsymbol{f}_1\\
\nabla\cdot\boldsymbol{u}_1 &= -i\omega\varepsilon_1 + q_1\\
\nabla\times\boldsymbol{E}_1 &= i\omega\boldsymbol{B}_1 - \boldsymbol{K}_1\\
\nabla\times\boldsymbol{H}_1 &= -i\omega\boldsymbol{D}_1 + \boldsymbol{J}_1
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{u}_2\cdot\nabla p_1 &= i\omega\boldsymbol{u}_2\cdot\boldsymbol{\mu}_1 + \boldsymbol{u}_2\cdot\boldsymbol{f}_1\\
p_2\nabla\cdot\boldsymbol{u}_1 &= -i\omega p_2\varepsilon_1 + p_2 q_1\\
\boldsymbol{H}_2\cdot(\nabla\times\boldsymbol{E}_1) &= i\omega\boldsymbol{H}_2\cdot\boldsymbol{B}_1 - \boldsymbol{H}_2\cdot\boldsymbol{K}_1\\
\boldsymbol{E}_2\cdot(\nabla\times\boldsymbol{H}_1) &= -i\omega\boldsymbol{E}_2\cdot\boldsymbol{D}_1 + \boldsymbol{E}_2\cdot\boldsymbol{J}_1
\end{aligned}$$

$$\left\{\begin{aligned}
\nabla\cdot(p_1\boldsymbol{u}_2) &= i\omega(\boldsymbol{u}_2\cdot\boldsymbol{\mu}_1 - p_1\varepsilon_2) + \boldsymbol{u}_2\cdot\boldsymbol{f}_1 + p_1 q_2\\
\nabla\cdot(p_2\boldsymbol{u}_1) &= i\omega(\boldsymbol{u}_1\cdot\boldsymbol{\mu}_2 - p_2\varepsilon_1) + \boldsymbol{u}_1\cdot\boldsymbol{f}_2 + p_2 q_1
\end{aligned}\right.$$

$$\left\{\begin{aligned}
\nabla\cdot(\boldsymbol{E}_1\times\boldsymbol{H}_2) &= i\omega(\boldsymbol{H}_2\cdot\boldsymbol{B}_1 + \boldsymbol{E}_1\cdot\boldsymbol{D}_2) - \boldsymbol{E}_1\cdot\boldsymbol{J}_2 - \boldsymbol{H}_2\cdot\boldsymbol{K}_1\\
\nabla\cdot(\boldsymbol{E}_2\times\boldsymbol{H}_1) &= i\omega(\boldsymbol{H}_1\cdot\boldsymbol{B}_2 + \boldsymbol{E}_2\cdot\boldsymbol{D}_1) - \boldsymbol{E}_2\cdot\boldsymbol{J}_1 - \boldsymbol{H}_1\cdot\boldsymbol{K}_2
\end{aligned}\right.$$

$$+$$

Governing equations for 2

$$\begin{aligned}
\nabla p_2 &= i\omega\boldsymbol{\mu}_2 + \boldsymbol{f}_2\\
\nabla\cdot\boldsymbol{u}_2 &= -i\omega\varepsilon_2 + q_2\\
\nabla\times\boldsymbol{E}_2 &= i\omega\boldsymbol{B}_2 - \boldsymbol{K}_2\\
\nabla\times\boldsymbol{H}_2 &= -i\omega\boldsymbol{D}_2 + \boldsymbol{J}_2.
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{u}_1\cdot\nabla p_2 &= i\omega\boldsymbol{u}_1\cdot\boldsymbol{\mu}_2 + \boldsymbol{u}_1\cdot\boldsymbol{f}_2\\
p_1\nabla\cdot\boldsymbol{u}_2 &= -i\omega p_1\varepsilon_2 + p_1 q_2\\
\boldsymbol{H}_1\cdot(\nabla\times\boldsymbol{E}_2) &= i\omega\boldsymbol{H}_1\cdot\boldsymbol{B}_2 - \boldsymbol{H}_1\cdot\boldsymbol{K}_2\\
\boldsymbol{E}_1\cdot(\nabla\times\boldsymbol{H}_2) &= -i\omega\boldsymbol{E}_1\cdot\boldsymbol{D}_2 + \boldsymbol{E}_1\cdot\boldsymbol{J}_2
\end{aligned}$$

Add, integrate over all space, and apply divergence theorem

$$\begin{aligned}
\nabla\cdot(p_1\boldsymbol{u}_2 - p_2\boldsymbol{u}_1) &= \boldsymbol{u}_2\cdot\boldsymbol{f}_1 - \boldsymbol{u}_1\cdot\boldsymbol{f}_2 + p_1 q_2 - p_2 q_1\\
&\quad + i\omega(\boldsymbol{u}_2\cdot\boldsymbol{\mu}_1 - \boldsymbol{u}_1\cdot\boldsymbol{\mu}_2 - p_1\varepsilon_2 + p_2\varepsilon_1)\\
\nabla\cdot(\boldsymbol{E}_1\times\boldsymbol{H}_2 - \boldsymbol{E}_2\times\boldsymbol{H}_1) &= \boldsymbol{E}_2\cdot\boldsymbol{J}_1 - \boldsymbol{E}_1\cdot\boldsymbol{J}_2 + \boldsymbol{H}_1\cdot\boldsymbol{K}_2 - \boldsymbol{H}_2\cdot\boldsymbol{K}_1\\
&\quad + i\omega(\boldsymbol{H}_2\cdot\boldsymbol{B}_1 - \boldsymbol{H}_1\cdot\boldsymbol{B}_2 + \boldsymbol{E}_1\cdot\boldsymbol{D}_2 - \boldsymbol{E}_2\cdot\boldsymbol{D}_1)
\end{aligned}$$

Reciprocity

$$\cancel{\int_A (p_1\boldsymbol{u}_2 - p_2\boldsymbol{u}_1 + \boldsymbol{E}_1\times\boldsymbol{H}_2 - \boldsymbol{E}_2\times\boldsymbol{H}_1)\cdot\hat{\boldsymbol{n}}dA} \rightarrow = 0 \text{ by the Sommerfeld radiation condition}$$

$$= \cancel{\int_V (\boldsymbol{u}_2\cdot\boldsymbol{f}_1 - \boldsymbol{u}_1\cdot\boldsymbol{f}_2 + p_1q_2 - p_2q_1 + \boldsymbol{E}_2\cdot\boldsymbol{J}_1 - \boldsymbol{E}_1\cdot\boldsymbol{J}_2 + \boldsymbol{H}_1\cdot\boldsymbol{K}_2 - \boldsymbol{H}_2\cdot\boldsymbol{K}_1)dV} \rightarrow = 0 \text{ by reciprocity}$$

$$-i\omega\int_V \underbrace{(\boldsymbol{u}_1\cdot\boldsymbol{\mu}_2 - \boldsymbol{u}_2\cdot\boldsymbol{\mu}_1 + p_1\varepsilon_2 - p_2\varepsilon_1 + \boldsymbol{H}_1\cdot\boldsymbol{B}_2 - \boldsymbol{H}_2\cdot\boldsymbol{B}_1 - \boldsymbol{E}_1\cdot\boldsymbol{D}_2 + \boldsymbol{E}_2\cdot\boldsymbol{D}_1)}_{=0}dV$$

Acousto-electrodynamic constitutive relations

$$\begin{pmatrix}\varepsilon\\ \boldsymbol{\mu}\\ \boldsymbol{D}\\ \boldsymbol{B}\end{pmatrix} = \begin{pmatrix}-\beta & \boldsymbol{\gamma}^{\mathsf{T}} & \boldsymbol{b}^{\mathsf{T}} & \boldsymbol{d}^{\mathsf{T}}\\ -\boldsymbol{\eta} & \underline{\boldsymbol{\rho}} & \underline{\boldsymbol{v}} & \underline{\boldsymbol{m}}\\ \boldsymbol{c} & \underline{\boldsymbol{w}} & \underline{\boldsymbol{\epsilon}} & \underline{\boldsymbol{\xi}}\\ \boldsymbol{e} & \underline{\boldsymbol{n}} & \underline{\boldsymbol{\zeta}} & \underline{\boldsymbol{\mu}}\end{pmatrix}\begin{pmatrix}p\\ \boldsymbol{u}\\ \boldsymbol{E}\\ \boldsymbol{H}\end{pmatrix}$$

$$\boldsymbol{\gamma}_1 = \boldsymbol{\gamma}^{\mathrm{e}} + \boldsymbol{\gamma}^{\mathrm{o}}$$
$$\boldsymbol{\gamma}_2 = \boldsymbol{\gamma}^{\mathrm{e}} - \boldsymbol{\gamma}^{\mathrm{o}}$$
$$\boldsymbol{\eta}_1 = \boldsymbol{\eta}^{\mathrm{e}} + \boldsymbol{\eta}^{\mathrm{o}}$$
$$\boldsymbol{\eta}_2 = \boldsymbol{\eta}^{\mathrm{e}} - \boldsymbol{\eta}^{\mathrm{o}}$$
$$\underline{\boldsymbol{\zeta}}_1 = \underline{\boldsymbol{\zeta}}^{\mathrm{e}} + \underline{\boldsymbol{\zeta}}^{\mathrm{o}}$$
$$\underline{\boldsymbol{\zeta}}_2 = \underline{\boldsymbol{\zeta}}^{\mathrm{e}} - \underline{\boldsymbol{\zeta}}^{\mathrm{o}}$$

etc....

$\underline{\boldsymbol{\rho}} = \underline{\boldsymbol{\rho}}^{\mathsf{T}}$ ← dynamic density ∈ Sym[1]

$\underline{\boldsymbol{\mu}} = \underline{\boldsymbol{\mu}}^{\mathsf{T}}$ ← permeability ∈ Sym[2]

$\underline{\boldsymbol{\epsilon}} = \underline{\boldsymbol{\epsilon}}^{\mathsf{T}}$ ← permittivity ∈ Sym[2]

All constitutive properties are generally *dispersive*, i.e., effective properties can change dramatically if sub-wavelength resonances exist in the microstructure.

$\boldsymbol{\gamma} = \boldsymbol{\eta}^{\mathrm{o}} - \boldsymbol{\eta}^{\mathrm{e}}$ ← acoustic bianisotropy vector[1]

$\underline{\boldsymbol{\xi}} = (\underline{\boldsymbol{\zeta}}^{\mathrm{o}})^{\mathsf{T}} - (\underline{\boldsymbol{\zeta}}^{\mathrm{e}})^{\mathsf{T}}$ ← electromagnetic bianisotropy tensor[2]

$\underline{\boldsymbol{v}} = (\underline{\boldsymbol{w}}^{\mathrm{o}})^{\mathsf{T}} - (\underline{\boldsymbol{w}}^{\mathrm{e}})^{\mathsf{T}}$ ← electromomentum coupling tensor[3]

$\underline{\boldsymbol{m}} = -(\underline{\boldsymbol{n}}^{\mathrm{o}})^{\mathsf{T}} + (\underline{\boldsymbol{n}}^{\mathrm{e}})^{\mathsf{T}}$ ← magnetomomentum coupling tensor

$\boldsymbol{b} = \boldsymbol{c}^{\mathrm{o}} - \boldsymbol{c}^{\mathrm{e}}$ = "***d***," conventional piezoelectric constant in strain-charge form[3]

$\boldsymbol{d} = -\boldsymbol{e}^{\mathrm{o}} + \boldsymbol{e}^{\mathrm{e}}$ ← piezomagnetic coupling vector

[1]Sieck et al., PRB **96**, 104303 (2017)
[2]Kong, Proc. IEEE **60**, 9 (1972)
[3]Pernas-Salomón and Shmuel, J. Mech. Phys. Solids **134**, 103770 (2020)

Passivity

Lossless, passive condition

$$\Re\mathfrak{e}(\nabla\cdot\boldsymbol{I}+\nabla\cdot\boldsymbol{S})=0$$

Time-avg. Poynting vectors

$$\boldsymbol{I}=p\boldsymbol{u}^*/2,\ \boldsymbol{S}=\boldsymbol{E}\times\boldsymbol{H}^*/2$$

Time-harmonic governing eqs.

$$\nabla p=i\omega\boldsymbol{\mu} \longleftarrow \text{momentum}$$
$$\nabla\cdot\boldsymbol{u}=-i\omega\varepsilon \longleftarrow \text{continuity}$$
$$\nabla\times\boldsymbol{E}=i\omega\boldsymbol{B} \longleftarrow \text{Faraday}$$
$$\nabla\times\boldsymbol{H}=-i\omega\boldsymbol{D} \longleftarrow \text{Ampere}$$

Acousto-electrodynamic constitutive relations

$$\begin{pmatrix}\varepsilon\\ \boldsymbol{\mu}\\ \boldsymbol{D}\\ \boldsymbol{B}\end{pmatrix}=\begin{pmatrix}-\beta & \boldsymbol{\gamma}^\top & \boldsymbol{b}^\top & \boldsymbol{d}^\top\\ -\boldsymbol{\eta} & \underline{\boldsymbol{\rho}} & \underline{\boldsymbol{v}} & \underline{\boldsymbol{m}}\\ \boldsymbol{c} & \underline{\boldsymbol{w}} & \underline{\boldsymbol{\epsilon}} & \underline{\boldsymbol{\xi}}\\ \boldsymbol{e} & \underline{\boldsymbol{n}} & \underline{\boldsymbol{\zeta}} & \underline{\boldsymbol{\mu}}\end{pmatrix}\begin{pmatrix}p\\ \boldsymbol{u}\\ \boldsymbol{E}\\ \boldsymbol{H}\end{pmatrix}$$

Passivity renders the constitutive block matrix *Hermitian.*

substitute in constitutive relations

$$\frac{i\omega}{4}(\boldsymbol{\mu}\cdot\boldsymbol{u}^*-\boldsymbol{\mu}^*\cdot\boldsymbol{u}+\varepsilon^*p-\varepsilon p^*+\boldsymbol{B}\cdot\boldsymbol{H}^*-\boldsymbol{B}^*\cdot\boldsymbol{H}-\boldsymbol{D}^*\cdot\boldsymbol{E}+\boldsymbol{D}\cdot\boldsymbol{E}^*)=0$$

$$\mathfrak{Im}\,\beta=0 \longleftarrow \text{compressibility} \in \mathfrak{R}^1$$
$$\underline{\boldsymbol{\rho}}^\dagger=\underline{\boldsymbol{\rho}} \longleftarrow \text{dynamic density}^1$$
$$\boldsymbol{\eta}=\boldsymbol{\gamma}^* \longleftarrow \text{acous. bianisotropy}^1$$
$$\underline{\boldsymbol{\mu}}=\underline{\boldsymbol{\mu}}^\dagger \longleftarrow \text{permeability}^2$$
$$\underline{\boldsymbol{\epsilon}}=\underline{\boldsymbol{\epsilon}}^\dagger \longleftarrow \text{permittivity}^2$$
$$\underline{\boldsymbol{\zeta}}=\underline{\boldsymbol{\xi}}^\dagger \longleftarrow \text{EM bianisotropy}^2$$

$$\underline{\boldsymbol{w}}^\dagger=\underline{\boldsymbol{v}} \longleftarrow \text{electromomentum}^3$$
$$\underline{\boldsymbol{n}}^\dagger=\underline{\boldsymbol{m}} \longleftarrow \text{magnetomomentum}$$
$$\boldsymbol{c}=-\boldsymbol{b}^* \longleftarrow \text{piezoelectricity}^3$$
$$\boldsymbol{d}=-\boldsymbol{e}^* \longleftarrow \text{piezomagnetism}$$

Define $\tilde{p}=-p=\mathrm{tr}\,(\boldsymbol{\sigma})/3$

$$\begin{pmatrix}\varepsilon\\ \boldsymbol{\mu}\\ \boldsymbol{D}\\ \boldsymbol{B}\end{pmatrix}=\begin{pmatrix}\beta & \boldsymbol{\gamma}^\top & \boldsymbol{b}^\top & \boldsymbol{d}^\top\\ \boldsymbol{\gamma}^* & \underline{\boldsymbol{\rho}} & \underline{\boldsymbol{v}} & \underline{\boldsymbol{m}}\\ \boldsymbol{b}^* & \underline{\boldsymbol{v}}^\dagger & \underline{\boldsymbol{\epsilon}} & \underline{\boldsymbol{\xi}}\\ \boldsymbol{d}^* & \underline{\boldsymbol{m}}^\dagger & \underline{\boldsymbol{\xi}}^\dagger & \underline{\boldsymbol{\mu}}\end{pmatrix}\begin{pmatrix}\tilde{p}\\ \boldsymbol{u}\\ \boldsymbol{E}\\ \boldsymbol{H}\end{pmatrix}$$

What about a reciprocal ***and*** *passive acousto-electrodynamic medium?*

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[1]Sieck et al., PRB **96**, 104303 (2017)
[2]Kong, Proc. IEEE **60**, 9 (1972)
[3]Pernas-Salomón and Shmuel, J. Mech. Phys. Solids **134**, 103770 (2020)

Reciprocity *and* passivity

Hermitian tensors

$$\underline{\rho} = \underline{\rho}^{\dagger}$$
$$\underline{\epsilon} = \underline{\epsilon}^{\dagger}$$
$$\underline{\mu} = \underline{\mu}^{\dagger}$$

Lossless, passive, linear media

Real symmetric tensors

$$\underline{\rho} = \underline{\rho}^{\mathsf{T}}$$
$$\underline{\epsilon} = \underline{\epsilon}^{\mathsf{T}}$$
$$\underline{\mu} = \underline{\mu}^{\mathsf{T}}$$

Lossless reciprocal media

To satisfy reciprocity *and* passivity, $\underline{\boldsymbol{\rho}}$, $\underline{\boldsymbol{\epsilon}}$, and $\underline{\boldsymbol{\mu}}$ must be real and symmetric.

$\mathfrak{Im}\,(\cdot) = 0$ for lossless case

$$\mathrm{Im}(\underline{\rho}) \geq 0$$
$$\mathrm{Im}(\beta) \geq 0$$
$$\mathrm{Im}(\underline{\epsilon}) \geq 0$$
$$\mathrm{Im}(\underline{\mu}) \geq 0$$

Results from reciprocity

$$\underline{\rho} = \underline{\rho}^{\mathsf{T}}$$
$$\underline{\mu} = \underline{\mu}^{\mathsf{T}}$$
$$\underline{\epsilon} = \underline{\epsilon}^{\mathsf{T}}$$
$$\gamma = \eta^{\mathrm{o}} - \eta^{\mathrm{e}}$$
$$\underline{\xi} = (\underline{\zeta}^{\mathrm{o}})^{\mathsf{T}} - (\underline{\zeta}^{\mathrm{e}})^{\mathsf{T}}$$
$$\underline{\boldsymbol{v}} = (\underline{\boldsymbol{w}}^{\mathrm{o}})^{\mathsf{T}} - (\underline{\boldsymbol{w}}^{\mathrm{e}})^{\mathsf{T}}$$
$$\underline{\boldsymbol{m}} = -(\underline{\boldsymbol{n}}^{\mathrm{o}})^{\mathsf{T}} + (\underline{\boldsymbol{n}}^{\mathrm{e}})^{\mathsf{T}}$$
$$\boldsymbol{b} = \boldsymbol{c}^{\mathrm{o}} - \boldsymbol{c}^{\mathrm{e}}$$
$$\boldsymbol{d} = -\boldsymbol{e}^{\mathrm{o}} + \boldsymbol{e}^{\mathrm{e}}$$

Passive condition: $\mathfrak{Re}\,(\nabla \cdot \boldsymbol{I} + \nabla \cdot \boldsymbol{S}) \geq 0$ (equality for lossless case)

$$\frac{i\omega}{4}(\boldsymbol{\mu} \cdot \boldsymbol{u}^* - \boldsymbol{\mu}^* \cdot \boldsymbol{u} + \varepsilon^* p - \varepsilon p^* + \boldsymbol{B} \cdot \boldsymbol{H}^* - \boldsymbol{B}^* \cdot \boldsymbol{H} - \boldsymbol{D}^* \cdot \boldsymbol{E} + \boldsymbol{D} \cdot \boldsymbol{E}^*) \geq 0$$

- Substitute results from reciprocity into lossless, passive condition
- Group like terms and insert definitions following Sieck et al.[1]
- Note that $\mathfrak{Re}\; iz = -\mathfrak{Im}\; z$ for complex scalar, vector, or tensor z
- Since fields are arbitrary, require $\mathfrak{Im}\,(\text{all } \chi) = 0$ to satisfy equality

Definitions following Sieck[1]

$$\eta^{\mathrm{o}} = \chi^{\mathrm{o}}_{pu} \qquad \eta^{\mathrm{e}} = i\chi^{\mathrm{e}}_{pu}$$
$$\underline{\zeta}^{\mathrm{o}} = \underline{\chi}^{\mathrm{o}}_{EH} \qquad \underline{\zeta}^{\mathrm{e}} = i\underline{\chi}^{\mathrm{e}}_{EH}$$
$$\underline{\boldsymbol{n}}^{\mathrm{o}} = i\underline{\chi}^{\mathrm{o}}_{uH}, \qquad \underline{\boldsymbol{n}}^{\mathrm{e}} = \underline{\chi}^{\mathrm{e}}_{uH}$$
$$\boldsymbol{c}^{\mathrm{o}} = i\chi^{\mathrm{o}}_{pE}, \qquad \boldsymbol{c}^{\mathrm{e}} = \chi^{\mathrm{e}}_{pE}$$
$$\underline{\boldsymbol{w}}^{\mathrm{o}} = \underline{\chi}^{\mathrm{o}}_{uE}, \qquad \underline{\boldsymbol{w}}^{\mathrm{e}} = i\underline{\chi}^{\mathrm{e}}_{uE}$$
$$\boldsymbol{e}^{\mathrm{o}} = \chi^{\mathrm{o}}_{pH}, \qquad \boldsymbol{e}^{\mathrm{e}} = i\chi^{\mathrm{e}}_{pH}$$

$\mathfrak{Im}\,(\cdot) = 0$ for lossless case

$$\mathrm{Im}(\text{all } \chi) \geq 0$$

[1]Sieck et al., PRB **96**, 104303 (2017)

Causality

- Effect cannot precede cause
- Guaranteed for passive media
- Describes how β, $\boldsymbol{\rho}$, $\boldsymbol{\epsilon}$, and $\boldsymbol{\mu}$ depend on ω

$A(\omega)$ represents any constitutive quantity

$$A(\omega) = A'(\omega) + iA''(\omega)$$

(A can be a scalar, vector, or tensor)

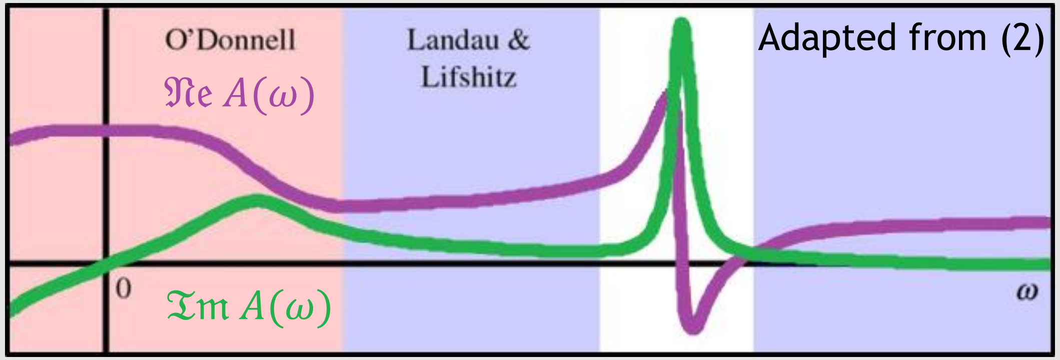
$A'(\infty)$ = frozen state

$(\omega \gg 2\pi/\tau)$

Kramers-Kronig relates $A''(\omega)$ to $A'(\omega) - A'(\infty)$[1,2]

$$A'(\omega) - A'(\infty) = \frac{2}{\pi}\mathcal{P}\int_0^\infty \frac{\lambda A''(\lambda)}{\lambda^2 - \omega^2} d\lambda$$

$$A''(\omega) = -\frac{2}{\pi}\mathcal{P}\int_0^\infty \frac{\omega A'(\lambda)}{\lambda^2 - \omega^2} d\lambda$$



$$\frac{\partial}{\partial \omega}\mathfrak{Re}\, A(\omega) < 0$$

$$\frac{\partial}{\partial \omega}\mathfrak{Re}\, A(\omega) > 0$$

Relevant case away from bandgaps and resonances

$\lambda = \omega$ dominates

$\lambda = \omega$ is negligible

Relaxation dominates[3]

$$A''(\omega) \simeq -\frac{\pi}{2}\omega\frac{\partial A'}{\partial \omega}$$

Loss is neglibible:[1] $A''(\omega) \simeq 0$

$$\frac{\partial A'}{\partial \omega} = \frac{4}{\pi}\int_0^\infty \frac{\lambda\omega}{(\lambda^2 - \omega^2)^2} A''(\lambda) d\lambda.$$

Recall that passivity requires that $A''(\omega) \geq 0$.

- Storage and loss moduli related by Kramers-Kronig relations
- Measured/predicted dispersion must obey

$$\frac{\partial \beta}{\partial \omega} \geq 0, \quad \frac{\partial \boldsymbol{\rho}}{\partial \omega} \geq 0, \quad \frac{\partial \boldsymbol{\epsilon}}{\partial \omega} \geq 0, \quad \frac{\partial \boldsymbol{\mu}}{\partial \omega} \geq 0.$$

[1] L. D. Landau and E. M. Lifshitz. Electrodynamics of continuous media (1960)
[2] M. B. Muhlestein et al., P. Roy Soc. A - Math Phy. (2016)
[3] M. O'Donnell et al., J. Acoust. Soc. Am. (1981)
R. Christensen, Theory of viscoelasticity: An Introduction (2012)

Outline

Background

1. Piezoelectricity
2. Electrodynamic and acoustic bianisotropy
3. Acousto-electrodynamic constitutive relations

Derivations for acousto-electrodynamic media

1. Reciprocity
2. Passivity
3. Causality

Closing

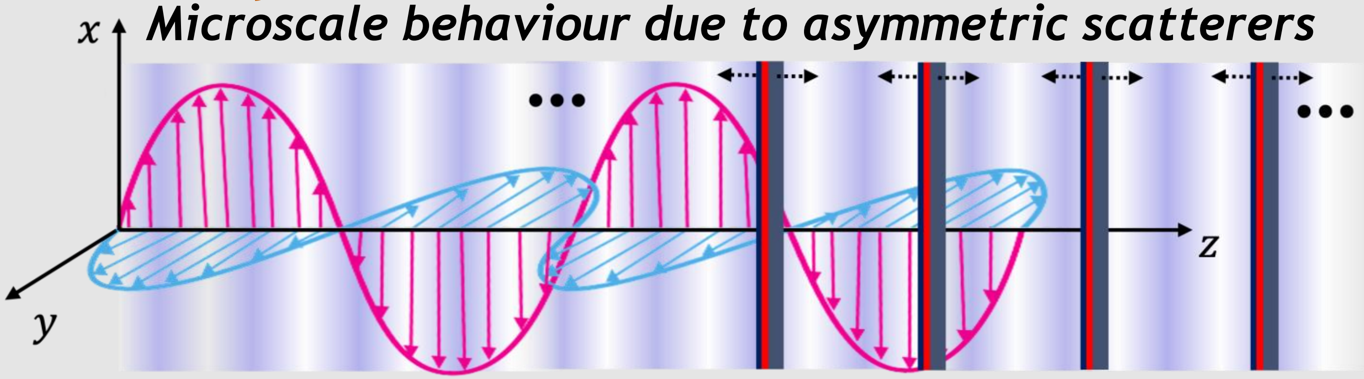
1. Verification of source-driven constitutive relations
2. Interpretation

Verification of source-driven homogenization theory

periodic | passive | identical

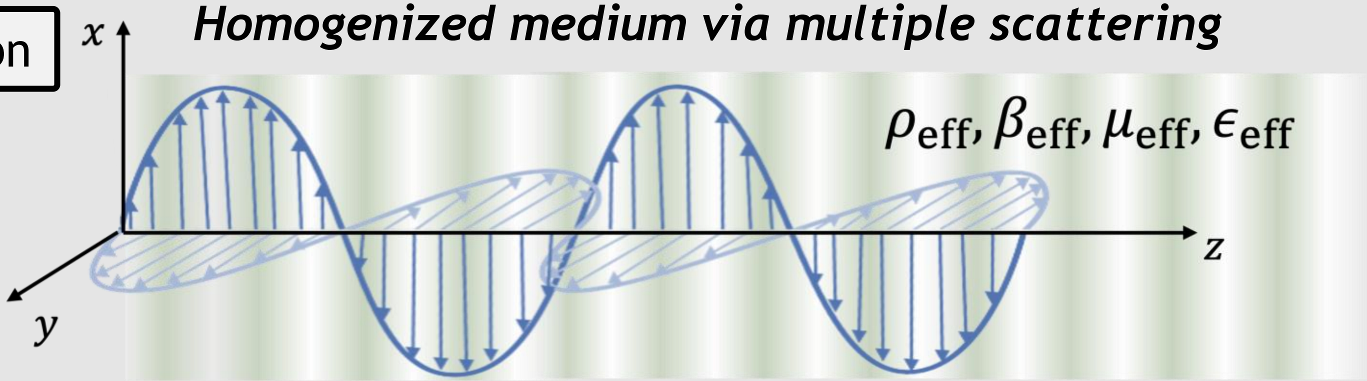
1D | reciprocal | $ka, \kappa a \ll 1$

Microscale behaviour due to asymmetric scatterers



Homogenization

Homogenized medium via multiple scattering



Effective constitutive relations obey the symmetries mandated by reciprocity and passivity

$$\begin{pmatrix} \varepsilon_{\text{eff}} \\ \mu_{v,\text{eff}} \\ D_{\text{eff}} \\ B_{\text{eff}} \end{pmatrix} = \begin{pmatrix} \boxed{-\beta_{\text{eff}}} & \boxed{\chi^{\text{o}}_{pu} - i\chi^{\text{e}}_{pu}} & \boxed{i\chi^{\text{o}}_{pE} - \chi^{\text{e}}_{pE}} & \boxed{-(\chi^{\text{o}}_{pH} - i\chi^{\text{e}}_{pH})} \\ \boxed{-(\chi^{\text{o}}_{pu} + i\chi^{\text{e}}_{pu})} & \boxed{\rho_{\text{eff}}} & \boxed{\chi^{\text{o}}_{uE} - i\chi^{\text{e}}_{uE}} & \boxed{-i\chi^{\text{o}}_{uH} + \chi^{\text{e}}_{uH}} \\ \boxed{i\chi^{\text{o}}_{pE} + \chi^{\text{e}}_{pE}} & \boxed{\chi^{\text{o}}_{uE} + i\chi^{\text{e}}_{uE}} & \boxed{\epsilon_{\text{eff}}} & \boxed{\chi^{\text{o}}_{EH}} \\ \boxed{\chi^{\text{o}}_{pH} + i\chi^{\text{e}}_{pH}} & \boxed{i\chi^{\text{o}}_{uH} + \chi^{\text{e}}_{uH}} & \boxed{\chi^{\text{o}}_{EH}} & \boxed{\mu_{\text{eff}}} \end{pmatrix} \begin{pmatrix} p_{\text{eff}} \\ u_{\text{eff}} \\ E_{\text{eff}} \\ H_{\text{eff}} \end{pmatrix}$$

C. A. Gokani et al., *J. Acoust. Soc. Am.* **153**, A120 (2023).

Thanks for listening!

Student award competition:

Summary

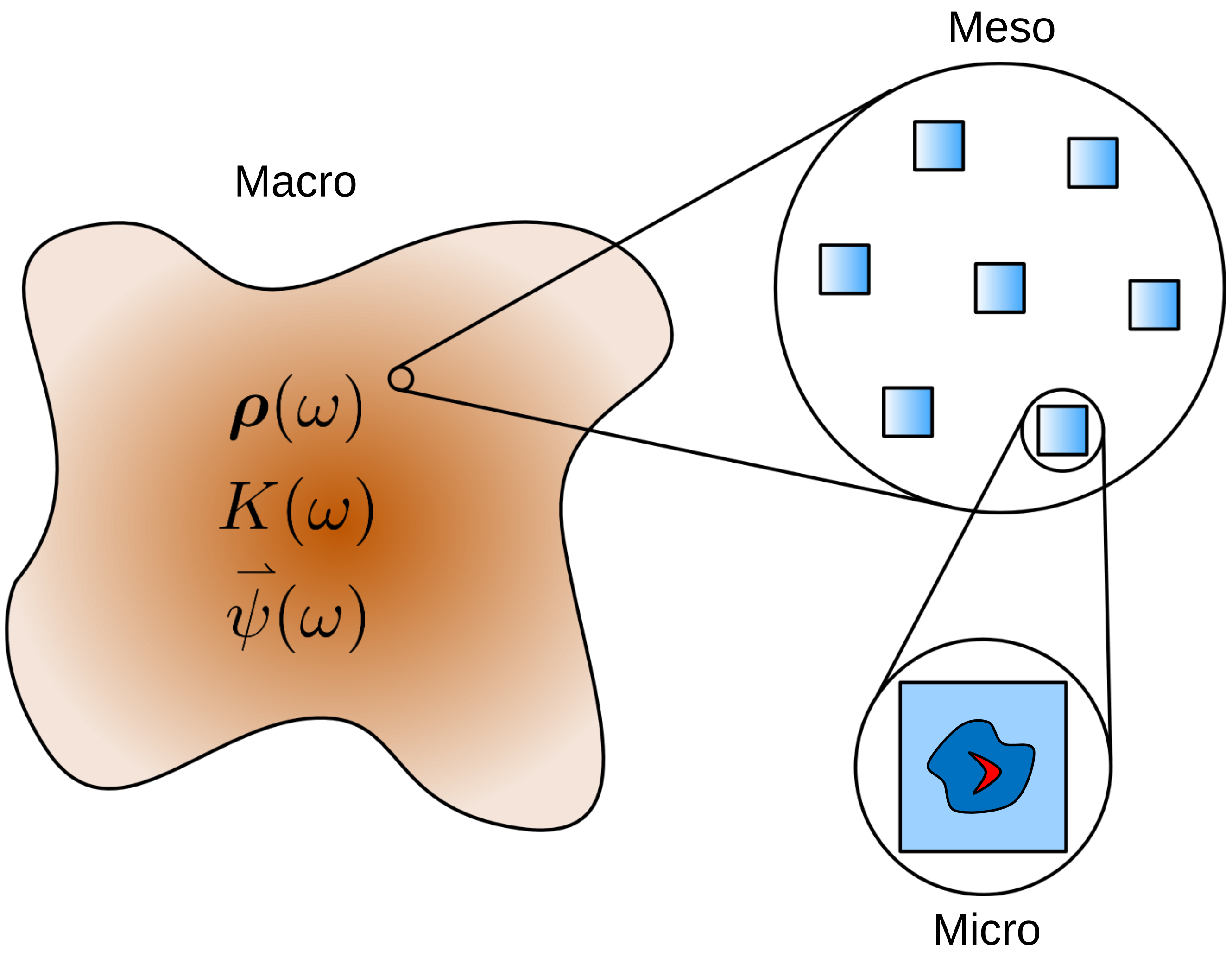
- Derived constraints on acousto-electrodynamic coupled media imposed by reciprocity, passivity, and causality
- Recovered relations for dynamic piezoelectricity, acoustic bianisotropy, and electrodynamic bianisotropy
- Showed existing source-driven homogenization theory for coupled acousto-electrodynamic media obeys reciprocity and passivity

Acknowledgments

ARL:UT Chester M. McKinney Graduate Fellowship in Acoustics
Defense Advanced Research Projects Agency (DARPA)

Extra slides

Homogenization



Constitutive Relations

$$p = K\varepsilon + \vec{\psi}\cdot\dot{\vec{v}}$$

$$\vec{\mu} = \vec{\psi}\dot{\varepsilon} + \boldsymbol{\rho}\cdot\vec{v}$$

Macroscale

↑ (Dynamic Homogenization Method)

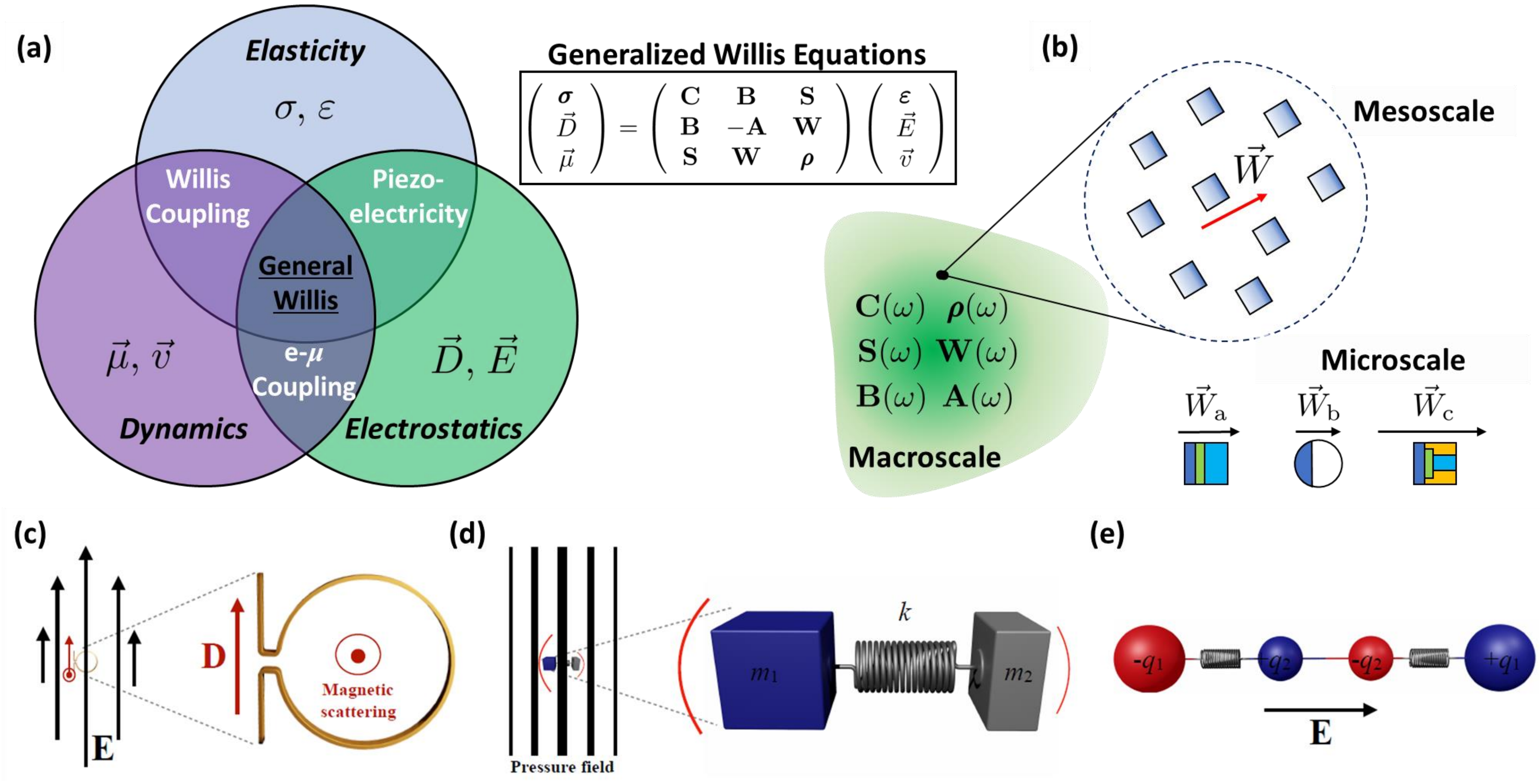
Mesoscale

↑ (Polarizability)

Microscale

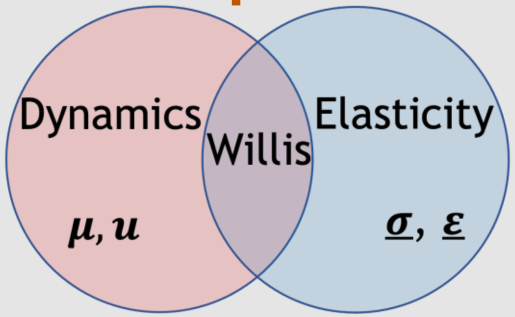
Sieck, C.F., Alu, A., Haberman, M.R.., Phys. Rev. B, **96**, 104303, (2017).

Electro-momentum Coupling

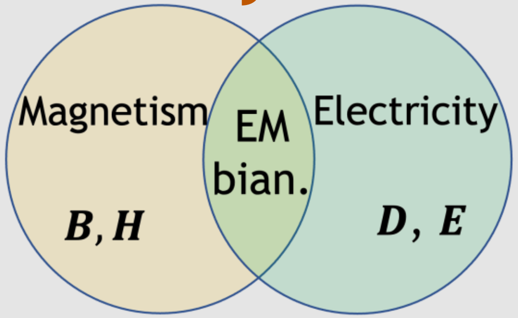


Pernas-Salomon and Shmuel, J. Mech. Phys. Solids 134, 103770 (2020)
Pernas-Salomon and Shmuel, Phys. Rev. Appl. 14, 064005 (2020)
Pernas-Salomon, Haberman, Norris and Shmuel, Wave Motion 106, 102797 (2021)

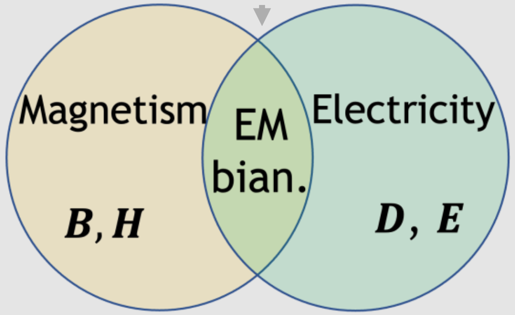
Developments in the study of bianisotropic media



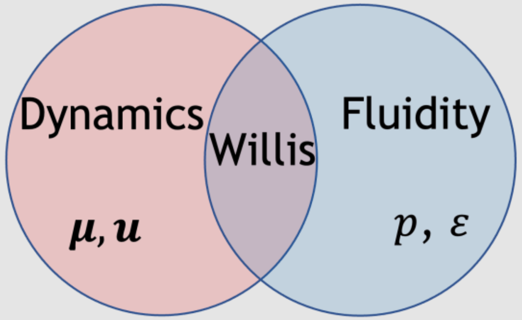
Willis, 1981



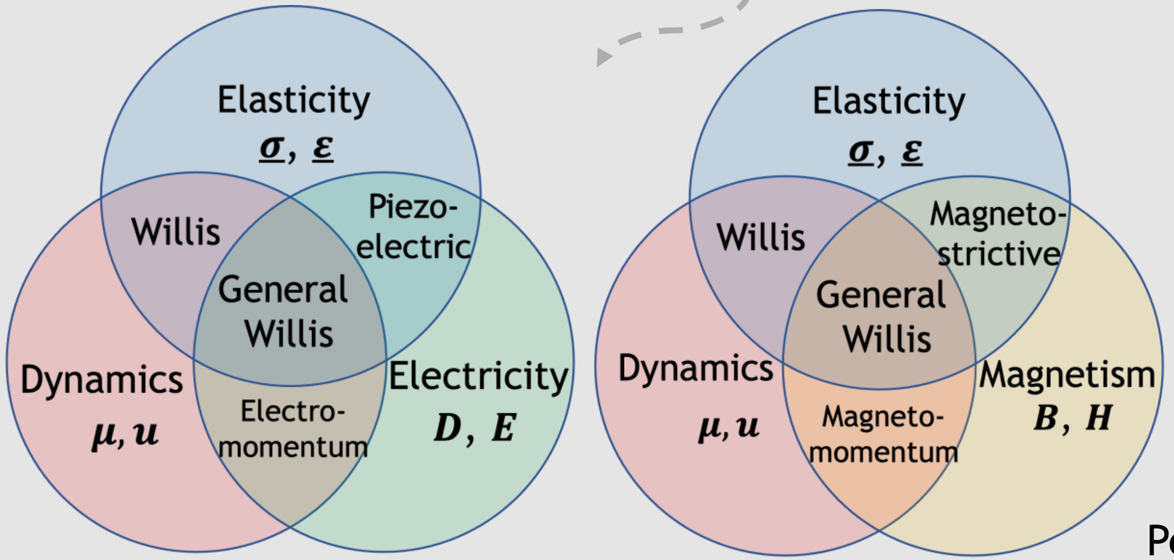
Cheng & Kong, 1968



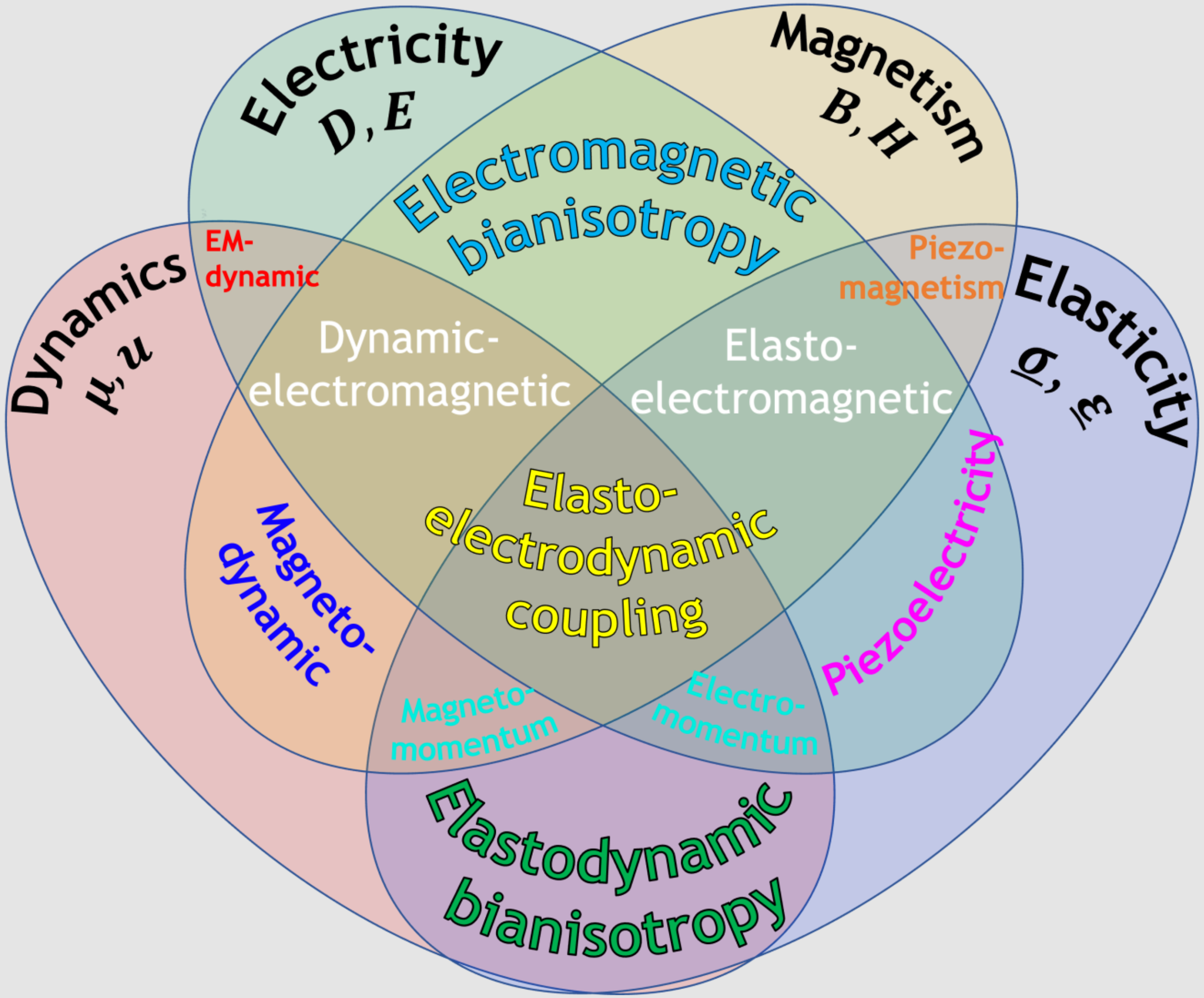
Alu, 2011



Sieck et al., 2017

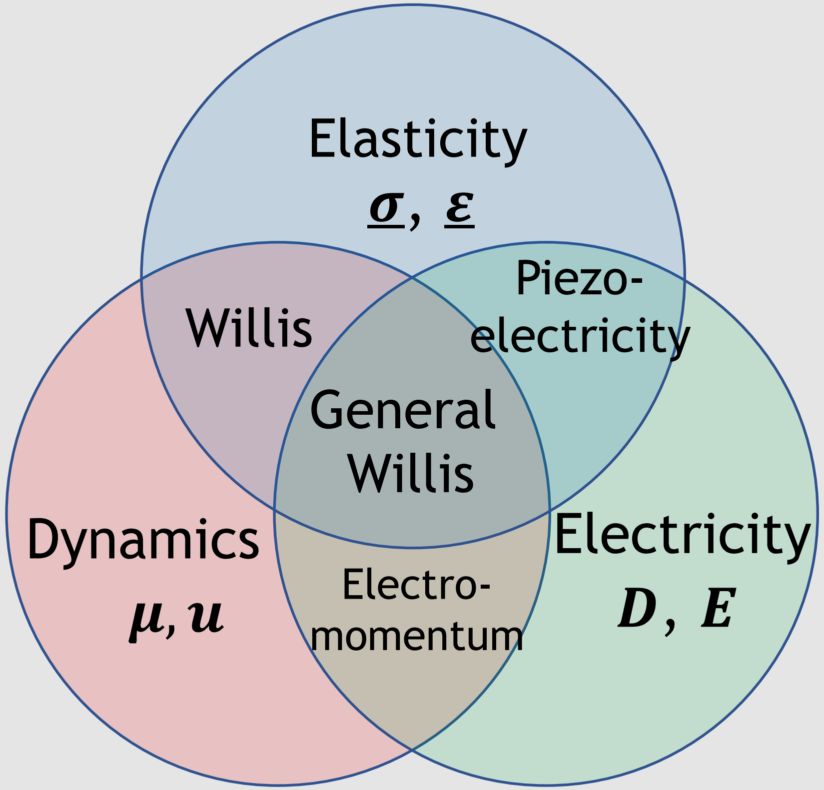


Pernas-Salomón & Shmuel, 2020

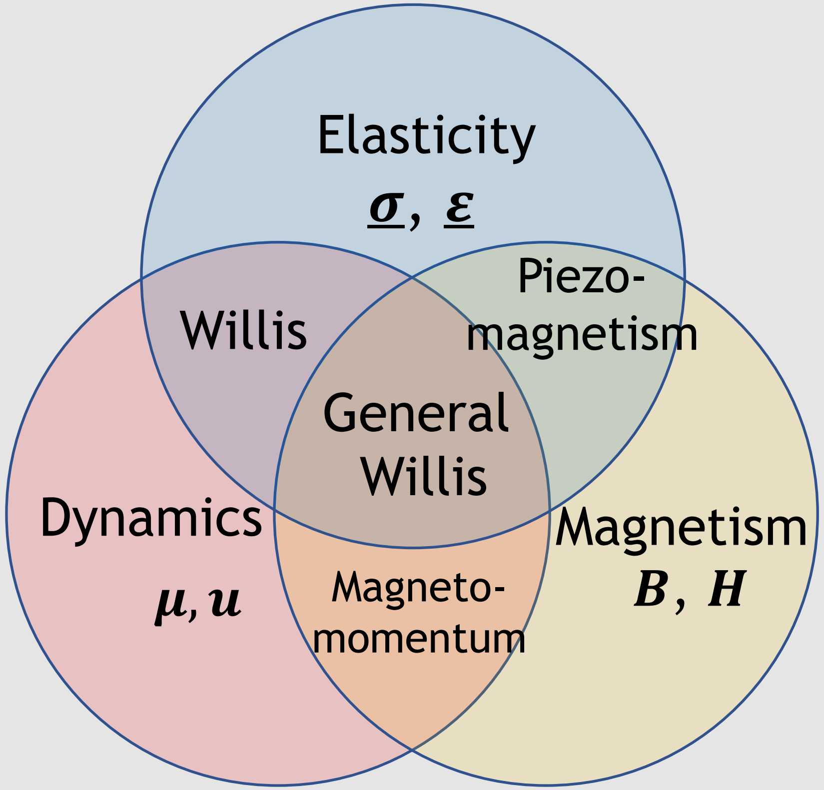


Gokani et al., 2023

Electromomentum coupling usually does not consider electrodynamics



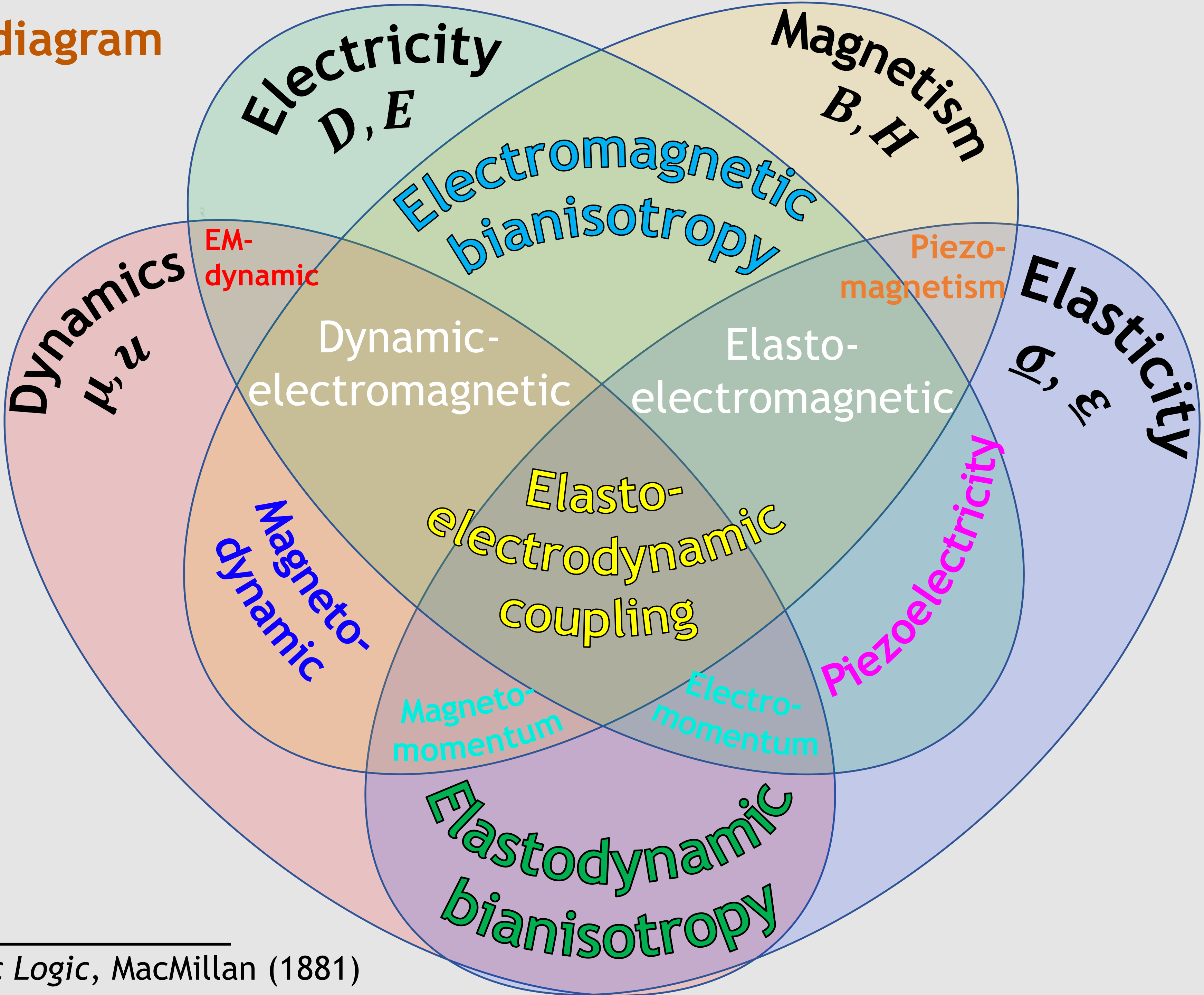
quasielectrostatic



quasimagnetostatic

Adapted from Fig. 1 of Pernas-Salomón and Shmuel, PRA **14** (2020)

4-way Venn diagram



J. Venn, *Symbolic Logic*, MacMillan (1881)

Reciprocity

$$\cancel{\int_A (p_1 \boldsymbol{u}_2 - p_2 \boldsymbol{u}_1 + \boldsymbol{E}_1\times\boldsymbol{H}_2 - \boldsymbol{E}_2\times\boldsymbol{H}_1)\cdot\hat{n}\,dA} = 0 \text{ by the Sommerfeld radiation condition}$$

$$= \cancel{\int_V (\boldsymbol{u}_2\cdot\boldsymbol{f}_1 - \boldsymbol{u}_1\cdot\boldsymbol{f}_2 + p_1 q_2 - p_2 q_1 + \boldsymbol{E}_2\cdot\boldsymbol{J}_1 - \boldsymbol{E}_1\cdot\boldsymbol{J}_2 + \boldsymbol{H}_1\cdot\boldsymbol{K}_2 - \boldsymbol{H}_2\cdot\boldsymbol{K}_1)\,dV} = 0 \text{ by reciprocity}$$

$$-i\omega\int_V \underbrace{(\boldsymbol{u}_1\cdot\boldsymbol{\mu}_2 - \boldsymbol{u}_2\cdot\boldsymbol{\mu}_1 + p_1\varepsilon_2 - p_2\varepsilon_1 + \boldsymbol{H}_1\cdot\boldsymbol{B}_2 - \boldsymbol{H}_2\cdot\boldsymbol{B}_1 - \boldsymbol{E}_1\cdot\boldsymbol{D}_2 + \boldsymbol{E}_2\cdot\boldsymbol{D}_1)}_{=0}\,dV$$

Acousto-electrodynamic constitutive relations

$$\begin{pmatrix} \varepsilon \\ \boldsymbol{\mu} \\ \boldsymbol{D} \\ \boldsymbol{B} \end{pmatrix} = \begin{pmatrix} -\beta & \boldsymbol{\gamma}^{\mathsf{T}} & \boldsymbol{b}^{\mathsf{T}} & \boldsymbol{d}^{\mathsf{T}} \\ -\boldsymbol{\eta} & \underline{\boldsymbol{\rho}} & \underline{\boldsymbol{v}} & \underline{\boldsymbol{m}} \\ \boldsymbol{c} & \underline{\boldsymbol{w}} & \underline{\boldsymbol{\epsilon}} & \underline{\boldsymbol{\xi}} \\ \boldsymbol{e} & \underline{\boldsymbol{n}} & \underline{\boldsymbol{\zeta}} & \underline{\boldsymbol{\mu}} \end{pmatrix} \begin{pmatrix} p \\ \boldsymbol{u} \\ \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}$$

$$\boldsymbol{\gamma}_1 = \boldsymbol{\gamma}^{\mathrm{e}} + \boldsymbol{\gamma}^{\mathrm{o}}$$
$$\boldsymbol{\gamma}_2 = \boldsymbol{\gamma}^{\mathrm{e}} - \boldsymbol{\gamma}^{\mathrm{o}}$$
$$\boldsymbol{\eta}_1 = \boldsymbol{\eta}^{\mathrm{e}} + \boldsymbol{\eta}^{\mathrm{o}}$$
$$\boldsymbol{\eta}_2 = \boldsymbol{\eta}^{\mathrm{e}} - \boldsymbol{\eta}^{\mathrm{o}}$$
$$\underline{\boldsymbol{\zeta}}_1 = \underline{\boldsymbol{\zeta}}^{\mathrm{e}} + \underline{\boldsymbol{\zeta}}^{\mathrm{o}}$$
$$\underline{\boldsymbol{\zeta}}_2 = \underline{\boldsymbol{\zeta}}^{\mathrm{e}} - \underline{\boldsymbol{\zeta}}^{\mathrm{o}}$$

etc....

$$\begin{aligned}
&\boldsymbol{u}_1\cdot[(\underline{\boldsymbol{\rho}}-\underline{\boldsymbol{\rho}}^{\mathsf{T}})\boldsymbol{u}_2] + \boldsymbol{H}_1\cdot[(\underline{\boldsymbol{\mu}}-\underline{\boldsymbol{\mu}}^{\mathsf{T}})\boldsymbol{H}_2] + \boldsymbol{E}_2\cdot[(\underline{\boldsymbol{\epsilon}}-\underline{\boldsymbol{\epsilon}}^{\mathsf{T}})\boldsymbol{E}_1 \\
&+(\boldsymbol{\eta}^{\mathrm{e}}+\boldsymbol{\gamma}^{\mathrm{e}})\cdot(p_1\boldsymbol{u}_2 - p_2\boldsymbol{u}_1) + (\boldsymbol{\eta}^{\mathrm{o}}-\boldsymbol{\gamma}^{\mathrm{o}})\cdot(p_1\boldsymbol{u}_2 + p_2\boldsymbol{u}_1) \\
&-\boldsymbol{E}_1\cdot[(\underline{\boldsymbol{\zeta}}^{\mathrm{e}})^{\mathsf{T}} + \underline{\boldsymbol{\xi}}^{\mathrm{e}}]\boldsymbol{H}_2 - \boldsymbol{E}_1\cdot[(\underline{\boldsymbol{\zeta}}^{\mathrm{o}})^{\mathsf{T}} - \underline{\boldsymbol{\xi}}^{\mathrm{o}}]\boldsymbol{H}_2 \\
&+\boldsymbol{E}_2\cdot[(\underline{\boldsymbol{\zeta}}^{\mathrm{e}})^{\mathsf{T}} + \underline{\boldsymbol{\xi}}^{\mathrm{e}}]\boldsymbol{H}_1 - \boldsymbol{E}_2\cdot[(\underline{\boldsymbol{\zeta}}^{\mathrm{o}})^{\mathsf{T}} - \underline{\boldsymbol{\xi}}^{\mathrm{o}}]\boldsymbol{H}_1 \\
&+\boldsymbol{u}_1\cdot[\underline{\boldsymbol{v}}^{\mathrm{e}} + (\underline{\boldsymbol{w}}^{\mathrm{e}})^{\mathsf{T}}]\boldsymbol{E}_2 - \boldsymbol{u}_1\cdot[\underline{\boldsymbol{v}}^{\mathrm{o}} - (\underline{\boldsymbol{w}}^{\mathrm{o}})^{\mathsf{T}}]\boldsymbol{E}_2 \\
&-\boldsymbol{u}_2\cdot[(\underline{\boldsymbol{w}}^{\mathrm{e}})^{\mathsf{T}} + \underline{\boldsymbol{v}}^{\mathrm{e}}]\boldsymbol{E}_1 + \boldsymbol{u}_2\cdot[(\underline{\boldsymbol{w}}^{\mathrm{o}})^{\mathsf{T}} - \underline{\boldsymbol{v}}^{\mathrm{o}}]\boldsymbol{E}_1 \\
&+\boldsymbol{u}_1\cdot[\underline{\boldsymbol{m}}^{\mathrm{e}} - (\underline{\boldsymbol{n}}^{\mathrm{e}})^{\mathsf{T}}]\boldsymbol{H}_2 - \boldsymbol{u}_1\cdot[\underline{\boldsymbol{m}}^{\mathrm{o}} + (\underline{\boldsymbol{n}}^{\mathrm{o}})^{\mathsf{T}}]\boldsymbol{H}_2 \\
&+\boldsymbol{u}_2\cdot[(\underline{\boldsymbol{n}}^{\mathrm{e}})^{\mathsf{T}} - \underline{\boldsymbol{m}}^{\mathrm{e}}]\boldsymbol{H}_1 - \boldsymbol{u}_2\cdot[(\underline{\boldsymbol{n}}^{\mathrm{o}})^{\mathsf{T}} + \underline{\boldsymbol{m}}^{\mathrm{o}}]\boldsymbol{H}_1 \\
&+p_1(\boldsymbol{b}^{\mathrm{e}}+\boldsymbol{c}^{\mathrm{e}})\cdot\boldsymbol{E}_2 - p_1(\boldsymbol{b}^{\mathrm{o}}-\boldsymbol{c}^{\mathrm{o}})\cdot\boldsymbol{E}_2 \\
&-p_2(\boldsymbol{c}^{\mathrm{e}}+\boldsymbol{b}^{\mathrm{e}})\cdot\boldsymbol{E}_1 + p_2(\boldsymbol{c}^{\mathrm{o}}-\boldsymbol{b}^{\mathrm{o}})\cdot\boldsymbol{E}_1 \\
&+p_2(\boldsymbol{e}^{\mathrm{e}}-\boldsymbol{d}^{\mathrm{e}})\cdot\boldsymbol{H}_1 - p_2(\boldsymbol{e}^{\mathrm{o}}+\boldsymbol{d}^{\mathrm{o}})\cdot\boldsymbol{H}_1 \\
&+p_1(\boldsymbol{d}^{\mathrm{e}}-\boldsymbol{e}^{\mathrm{e}})\cdot\boldsymbol{H}_2 - p_1(\boldsymbol{d}^{\mathrm{o}}+\boldsymbol{e}^{\mathrm{o}})\cdot\boldsymbol{H}_2 = 0
\end{aligned}$$

$$\underline{\boldsymbol{\rho}} = \underline{\boldsymbol{\rho}}^{\mathsf{T}}, \quad \underline{\boldsymbol{\mu}} = \underline{\boldsymbol{\mu}}^{\mathsf{T}}, \quad \underline{\boldsymbol{\epsilon}} = \underline{\boldsymbol{\epsilon}}^{\mathsf{T}}$$

$$\boldsymbol{\gamma} = \boldsymbol{\eta}^{\mathrm{o}} - \boldsymbol{\eta}^{\mathrm{e}}$$

$$\left.\begin{aligned}
\underline{\boldsymbol{\xi}} &= (\underline{\boldsymbol{\zeta}}^{\mathrm{o}})^{\mathsf{T}} - (\underline{\boldsymbol{\zeta}}^{\mathrm{e}})^{\mathsf{T}} \\
\underline{\boldsymbol{v}} &= (\underline{\boldsymbol{w}}^{\mathrm{o}})^{\mathsf{T}} - (\underline{\boldsymbol{w}}^{\mathrm{e}})^{\mathsf{T}} \\
\underline{\boldsymbol{m}} &= (\underline{\boldsymbol{n}}^{\mathrm{e}})^{\mathsf{T}} - (\underline{\boldsymbol{n}}^{\mathrm{o}})^{\mathsf{T}} \\
\boldsymbol{b} &= \boldsymbol{c}^{\mathrm{o}} - \boldsymbol{c}^{\mathrm{e}} \\
\boldsymbol{d} &= \boldsymbol{e}^{\mathrm{e}} - \boldsymbol{e}^{\mathrm{o}}
\end{aligned}\right\} \text{new results}$$

Sieck et al., PRB **96**, 104303 (2017)
Kong, Proc. IEEE **60**, 9 (1972)

Passivity

Lossless, passive condition

$$\Re(\nabla\cdot\boldsymbol{I}+\nabla\cdot\boldsymbol{S})=0$$

Time-avg. Poynting vectors

$$\boldsymbol{I}=p\boldsymbol{u}^*/2,\ \boldsymbol{S}=\boldsymbol{E}\times\boldsymbol{H}^*/2$$

Time-harmonic governing eqs.

$$\nabla p=i\omega\boldsymbol{\mu} \longleftarrow \text{momentum}$$
$$\nabla\cdot\boldsymbol{u}=-i\omega\varepsilon \longleftarrow \text{continuity}$$
$$\nabla\times\boldsymbol{E}=i\omega\boldsymbol{B} \longleftarrow \text{Faraday}$$
$$\nabla\times\boldsymbol{H}=-i\omega\boldsymbol{D} \longleftarrow \text{Ampere}$$

Acousto-electrodynamic constitutive relations

$$\begin{pmatrix}\varepsilon\\ \boldsymbol{\mu}\\ \boldsymbol{D}\\ \boldsymbol{B}\end{pmatrix}=\begin{pmatrix}-\beta & \boldsymbol{\gamma}^\top & \boldsymbol{b}^\top & \boldsymbol{d}^\top\\ -\boldsymbol{\eta} & \underline{\boldsymbol{\rho}} & \underline{\boldsymbol{v}} & \underline{\boldsymbol{m}}\\ \boldsymbol{c} & \underline{\boldsymbol{w}} & \underline{\boldsymbol{\epsilon}} & \underline{\boldsymbol{\xi}}\\ \boldsymbol{e} & \underline{\boldsymbol{n}} & \underline{\boldsymbol{\zeta}} & \underline{\boldsymbol{\mu}}\end{pmatrix}\begin{pmatrix}p\\ \boldsymbol{u}\\ \boldsymbol{E}\\ \boldsymbol{H}\end{pmatrix}$$

$$\frac{i\omega}{4}(\boldsymbol{\mu}\cdot\boldsymbol{u}^*-\boldsymbol{\mu}^*\cdot\boldsymbol{u}+\varepsilon^*p-\varepsilon p^*+\boldsymbol{B}\cdot\boldsymbol{H}^*-\boldsymbol{B}^*\cdot\boldsymbol{H}-\boldsymbol{D}^*\cdot\boldsymbol{E}+\boldsymbol{D}\cdot\boldsymbol{E}^*)=0$$

$$\begin{aligned}&\frac{i\omega}{4}[-\boldsymbol{u}^*\cdot(\underline{\boldsymbol{\rho}}^\dagger-\underline{\boldsymbol{\rho}})\boldsymbol{u}+2i\Im(\beta)|p|^2-\\ &(\boldsymbol{\eta}-\boldsymbol{\gamma}^*)\cdot p\boldsymbol{u}^*+(\boldsymbol{\eta}^*-\boldsymbol{\gamma})\cdot p^*\boldsymbol{u}\\ &+\boldsymbol{E}^*\cdot(\underline{\boldsymbol{\epsilon}}-\underline{\boldsymbol{\epsilon}}^\dagger)\boldsymbol{E}+\boldsymbol{H}^*\cdot(\underline{\boldsymbol{\mu}}-\underline{\boldsymbol{\mu}}^\dagger)\boldsymbol{H}\\ &-\boldsymbol{E}^*\cdot(\underline{\boldsymbol{\zeta}}^\dagger-\underline{\boldsymbol{\xi}})\boldsymbol{H}+\boldsymbol{H}^*\cdot(\underline{\boldsymbol{\zeta}}-\underline{\boldsymbol{\xi}}^\dagger)\boldsymbol{E}+\boldsymbol{u}^*\cdot(\underline{\boldsymbol{v}}-\underline{\boldsymbol{w}}^\dagger)\boldsymbol{E}\\ &+\boldsymbol{u}^*\cdot(\underline{\boldsymbol{m}}-\underline{\boldsymbol{n}}^\dagger)\boldsymbol{H}+\boldsymbol{E}^*\cdot(\underline{\boldsymbol{w}}-\underline{\boldsymbol{v}}^\dagger)\boldsymbol{u}+\boldsymbol{H}^*\cdot(\underline{\boldsymbol{n}}-\underline{\boldsymbol{m}}^\dagger)\boldsymbol{u}\\ &+p(\boldsymbol{b}^*+\boldsymbol{c})\cdot\boldsymbol{E}^*+p(\boldsymbol{d}^*+\boldsymbol{e})\cdot\boldsymbol{H}^*\\ &-p^*(\boldsymbol{b}+\boldsymbol{c}^*)\cdot\boldsymbol{E}-p^*(\boldsymbol{d}+\boldsymbol{e}^*)\cdot\boldsymbol{H}]=0\end{aligned}$$

$$\Im\beta=0$$
$$\underline{\boldsymbol{\rho}}^\dagger=\underline{\boldsymbol{\rho}}$$
$$\boldsymbol{\eta}=\boldsymbol{\gamma}^*$$
$$\underline{\boldsymbol{\mu}}=\underline{\boldsymbol{\mu}}^\dagger$$
$$\underline{\boldsymbol{\epsilon}}=\underline{\boldsymbol{\epsilon}}^\dagger$$
$$\underline{\boldsymbol{\zeta}}=\underline{\boldsymbol{\xi}}^\dagger$$

$$\left.\begin{aligned}\underline{\boldsymbol{w}}^\dagger&=\underline{\boldsymbol{v}}\\ \underline{\boldsymbol{n}}^\dagger&=\underline{\boldsymbol{m}}\\ \boldsymbol{c}&=-\boldsymbol{b}^*\\ \boldsymbol{d}&=-\boldsymbol{e}^*\end{aligned}\right\}\ \text{new results}$$

*What about a reciprocal **and** passive acousto-electrodynamic medium?*

Sieck et al., PRB **96**, 104303 (2017)
Kong, Proc. IEEE **60**, 9 (1972)

Reciprocity *and* passivity

TEXAS
The University of Texas at Austin

Results from reciprocity

$$\underline{\rho} = \underline{\rho}^{\mathsf{T}}$$
$$\underline{\mu} = \underline{\mu}^{\mathsf{T}}$$
$$\underline{\epsilon} = \underline{\epsilon}^{\mathsf{T}}$$
$$\gamma = \eta^{\mathrm{o}} - \eta^{\mathrm{e}}$$
$$\underline{\xi} = (\underline{\zeta}^{\mathrm{o}})^{\mathsf{T}} - (\underline{\zeta}^{\mathrm{e}})^{\mathsf{T}}$$
$$\underline{v} = (\underline{w}^{\mathrm{o}})^{\mathsf{T}} - (\underline{w}^{\mathrm{e}})^{\mathsf{T}}$$
$$\underline{m} = (\underline{n}^{\mathrm{e}})^{\mathsf{T}} - (\underline{n}^{\mathrm{o}})^{\mathsf{T}}$$
$$\boldsymbol{b} = \boldsymbol{c}^{\mathrm{o}} - \boldsymbol{c}^{\mathrm{e}}$$
$$\boldsymbol{d} = \boldsymbol{e}^{\mathrm{e}} - \boldsymbol{e}^{\mathrm{o}}$$

Equation for $\mathfrak{Re}\,(\nabla\cdot\boldsymbol{I} + \nabla\cdot\boldsymbol{S}) \geq 0$ for reciprocity *and* passivity:

$$\omega\Big[\frac{1}{2}\boldsymbol{u}^*\cdot\mathfrak{Im}(\underline{\rho})\boldsymbol{u} + \frac{1}{2}\mathfrak{Im}(\beta)|p|^2 - \mathfrak{Im}(\chi^{\mathrm{o}}_{pu})\cdot\mathfrak{Re}(p\boldsymbol{u}^*) + \mathfrak{Im}(\chi^{\mathrm{e}}_{pu})\cdot\mathfrak{Im}(p\boldsymbol{u}^*)$$
$$+\frac{1}{2}\boldsymbol{E}^*\cdot\mathfrak{Im}(\underline{\epsilon})\boldsymbol{H} + \frac{1}{2}\boldsymbol{H}^*\cdot\mathfrak{Im}(\underline{\mu})\boldsymbol{H} - \mathfrak{Im}(\underline{\chi}^{\mathrm{o}}_{EH}):\mathfrak{Re}(\boldsymbol{H}^*\otimes\boldsymbol{E}) - \mathfrak{Im}(\underline{\chi}^{\mathrm{e}}_{EH}):\mathfrak{Im}(\boldsymbol{H}\otimes\boldsymbol{E}^*)$$
$$+\mathfrak{Im}(\underline{\chi}^{\mathrm{o}}_{uE}):\mathfrak{Re}(\boldsymbol{E}\otimes\boldsymbol{u}^*) + \mathfrak{Im}(\underline{\chi}^{\mathrm{e}}_{uE}):\mathfrak{Im}(\boldsymbol{E}\otimes\boldsymbol{u}^*) - \mathfrak{Im}(\underline{\chi}^{\mathrm{o}}_{uH}):\mathfrak{Im}(\boldsymbol{H}^*\otimes\boldsymbol{u}) + \mathfrak{Im}(\underline{\chi}^{\mathrm{e}}_{uH}):\mathfrak{Re}(\boldsymbol{H}\otimes\boldsymbol{u})$$
$$-\mathfrak{Im}(\chi^{\mathrm{o}}_{pE})\cdot\mathfrak{Im}(p\boldsymbol{E}^*) + \mathfrak{Im}(\chi^{\mathrm{e}}_{pE})\cdot\mathfrak{Re}(p\boldsymbol{E}^*) + \mathfrak{Im}(\chi^{\mathrm{o}}_{pH})\cdot\mathfrak{Re}(p\boldsymbol{H}^*) - \mathfrak{Im}(\chi^{\mathrm{e}}_{pH})\cdot\mathfrak{Im}(p\boldsymbol{H}^*)\Big] \geqslant 0.$$

$\mathfrak{Re}\,(\nabla\cdot\boldsymbol{I} + \nabla\cdot\boldsymbol{S}) \geq 0$

$$-\frac{i\omega}{4}[-\boldsymbol{u}^*\cdot(\underline{\rho}^\dagger - \underline{\rho})\boldsymbol{u} + 2i\mathfrak{Im}(\beta)|p|^2 - (\eta - \gamma^*)\cdot p\boldsymbol{u}^* + (\eta^* - \gamma)\cdot p^*\boldsymbol{u}$$
$$+\boldsymbol{E}^*\cdot(\underline{\epsilon} - \underline{\epsilon}^\dagger)\boldsymbol{E} + \boldsymbol{H}^*\cdot(\underline{\mu} - \underline{\mu}^\dagger)\boldsymbol{H} - \boldsymbol{E}^*\cdot(\underline{\zeta}^\dagger - \underline{\xi})\boldsymbol{H} + \boldsymbol{H}^*\cdot(\underline{\zeta} - \underline{\xi}^\dagger)\boldsymbol{E}$$
$$+\boldsymbol{u}^*\cdot(\underline{v} - \underline{w}^\dagger)\boldsymbol{E} + \boldsymbol{u}^*\cdot(\underline{m} - \underline{n}^\dagger)\boldsymbol{H} + \boldsymbol{E}^*\cdot(\underline{w} - \underline{v}^\dagger)\boldsymbol{u} + \boldsymbol{H}^*\cdot(\underline{n} - \underline{m}^\dagger)\boldsymbol{u}$$
$$+p(\boldsymbol{b}^* + \boldsymbol{c})\cdot\boldsymbol{E}^* + p(\boldsymbol{d}^* + \boldsymbol{e})\cdot\boldsymbol{H}^* - p^*(\boldsymbol{b} + \boldsymbol{c}^*)\cdot\boldsymbol{E} - p^*(\boldsymbol{d} + \boldsymbol{e}^*)\cdot\boldsymbol{H}] \geqslant 0.$$

Definitions following Sieck[1]

$$\eta^{\mathrm{o}} = \chi^{\mathrm{o}}_{pu} \qquad \eta^{\mathrm{e}} = i\chi^{\mathrm{e}}_{pu}$$
$$\underline{\zeta}^{\mathrm{o}} = \underline{\chi}^{\mathrm{o}}_{EH} \qquad \underline{\zeta}^{\mathrm{e}} = i\underline{\chi}^{\mathrm{e}}_{EH}$$
$$\underline{n}^{\mathrm{o}} = i\underline{\chi}^{\mathrm{o}}_{uH}, \qquad \underline{n}^{\mathrm{e}} = \underline{\chi}^{\mathrm{e}}_{uH}$$
$$\boldsymbol{c}^{\mathrm{o}} = i\chi^{\mathrm{o}}_{pE}, \qquad \boldsymbol{c}^{\mathrm{e}} = \chi^{\mathrm{e}}_{pE}$$
$$\underline{w}^{\mathrm{o}} = \underline{\chi}^{\mathrm{o}}_{uE}, \qquad \underline{w}^{\mathrm{e}} = i\underline{\chi}^{\mathrm{e}}_{uE}$$
$$\boldsymbol{e}^{\mathrm{o}} = \chi^{\mathrm{o}}_{pH}, \qquad \boldsymbol{e}^{\mathrm{e}} = i\chi^{\mathrm{e}}_{pH}$$

Lossless case

$$\mathfrak{Im}(\underline{\rho}) = 0$$
$$\mathfrak{Im}(\beta) = 0$$
$$\mathfrak{Im}(\underline{\epsilon}) = 0$$
$$\mathfrak{Im}(\underline{\mu}) = 0$$

and

$$\mathfrak{Im}(\text{all }\chi) = 0$$

[1]Sieck et al., PRB **96**, 104303 (2017)

Reciprocity and passivity

Reciprocity

$$\underline{\rho} = \underline{\rho}^{\mathsf{T}}$$
$$\underline{\mu} = \underline{\mu}^{\mathsf{T}}$$
$$\underline{\epsilon} = \underline{\epsilon}^{\mathsf{T}}$$
$$\gamma = \eta^{\mathrm{o}} - \eta^{\mathrm{e}}$$
$$\underline{\xi} = (\underline{\zeta}^{\mathrm{o}})^{\mathsf{T}} - (\underline{\zeta}^{\mathrm{e}})^{\mathsf{T}}$$
$$\underline{v} = (\underline{w}^{\mathrm{o}})^{\mathsf{T}} - (\underline{w}^{\mathrm{e}})^{\mathsf{T}}$$
$$\underline{m} = (\underline{n}^{\mathrm{e}})^{\mathsf{T}} - (\underline{n}^{\mathrm{o}})^{\mathsf{T}}$$
$$\boldsymbol{b} = \boldsymbol{c}^{\mathrm{o}} - \boldsymbol{c}^{\mathrm{e}}$$
$$\boldsymbol{d} = \boldsymbol{e}^{\mathrm{e}} - \boldsymbol{e}^{\mathrm{o}}$$

+

Passivity

$$\begin{aligned} -\mathfrak{Re}(\nabla\cdot \boldsymbol{I} + \nabla\cdot \boldsymbol{S}) = -\frac{i\omega}{4}\big[&-\boldsymbol{u}^*\cdot(\underline{\rho}^\dagger - \underline{\rho})\boldsymbol{u} + 2i\,\mathfrak{Im}(\beta)|p|^2 - (\boldsymbol{\eta}-\boldsymbol{\gamma}^*)\cdot p\boldsymbol{u}^* + (\boldsymbol{\eta}^*-\boldsymbol{\gamma})\cdot p^*\boldsymbol{u} \\ &+ \boldsymbol{E}^*\cdot(\underline{\epsilon}-\underline{\epsilon}^\dagger)\boldsymbol{E} + \boldsymbol{H}^*\cdot(\underline{\mu}-\underline{\mu}^\dagger)\boldsymbol{H} - \boldsymbol{E}^*\cdot(\underline{\zeta}^\dagger-\underline{\xi})\boldsymbol{H} + \boldsymbol{H}^*\cdot(\underline{\zeta}-\underline{\xi}^\dagger)\boldsymbol{E} \\ &+ \boldsymbol{u}^*\cdot(\underline{v}-\underline{w}^\dagger)\boldsymbol{E} + \boldsymbol{u}^*\cdot(\underline{m}-\underline{n}^\dagger)\boldsymbol{H} + \boldsymbol{E}^*\cdot(\underline{w}-\underline{v}^\dagger)\boldsymbol{u} + \boldsymbol{H}^*\cdot(\underline{n}-\underline{m}^\dagger)\boldsymbol{u} \\ &+ p(\boldsymbol{b}^*+\boldsymbol{c})\cdot\boldsymbol{E}^* + p(\boldsymbol{d}^*+\boldsymbol{e})\cdot\boldsymbol{H}^* - p^*(\boldsymbol{b}+\boldsymbol{c}^*)\cdot\boldsymbol{E} - p^*(\boldsymbol{d}+\boldsymbol{e}^*)\cdot\boldsymbol{H}\big] \geqslant 0 \end{aligned}$$

=

All $\mathfrak{Im}$ parts vanish for lossless case

$$\begin{aligned} -\mathfrak{Re}(\nabla\cdot \boldsymbol{I} + \nabla\cdot \boldsymbol{S}) = \omega\Big[&\frac{1}{2}\boldsymbol{u}^*\cdot\mathfrak{Im}(\underline{\rho})\boldsymbol{u} + \frac{1}{2}\mathfrak{Im}(\beta)|p|^2 - \mathfrak{Im}(\boldsymbol{\chi}^{\mathrm{o}}_{pu})\cdot\mathfrak{Re}(p\boldsymbol{u}^*) + \mathfrak{Im}(\boldsymbol{\chi}^{\mathrm{e}}_{pu})\cdot\mathfrak{Im}(p\boldsymbol{u}^*) \\ &+ \frac{1}{2}\boldsymbol{E}^*\cdot\mathfrak{Im}(\underline{\epsilon})\boldsymbol{H} + \frac{1}{2}\boldsymbol{H}^*\cdot\mathfrak{Im}(\underline{\mu})\boldsymbol{H} - \mathfrak{Im}(\underline{\chi}^{\mathrm{o}}_{EH}) : \mathfrak{Re}(\boldsymbol{H}^*\otimes\boldsymbol{E}) - \mathfrak{Im}(\underline{\chi}^{\mathrm{e}}_{EH}) : \mathfrak{Im}(\boldsymbol{H}\otimes\boldsymbol{E}^*) \\ &+ \mathfrak{Im}(\underline{\chi}^{\mathrm{o}}_{uE}) : \mathfrak{Re}(\boldsymbol{E}\otimes\boldsymbol{u}^*) + \mathfrak{Im}(\underline{\chi}^{\mathrm{e}}_{uE}) : \mathfrak{Im}(\boldsymbol{E}\otimes\boldsymbol{u}^*) - \mathfrak{Im}(\underline{\chi}^{\mathrm{o}}_{uH}) : \mathfrak{Im}(\boldsymbol{H}^*\otimes\boldsymbol{u}) + \mathfrak{Im}(\underline{\chi}^{\mathrm{e}}_{uH}) : \mathfrak{Re}(\boldsymbol{H}\otimes\boldsymbol{u}) \\ &- \mathfrak{Im}(\boldsymbol{\chi}^{\mathrm{o}}_{pE})\cdot\mathfrak{Im}(p\boldsymbol{E}^*) + \mathfrak{Im}(\boldsymbol{\chi}^{\mathrm{e}}_{pE})\cdot\mathfrak{Re}(p\boldsymbol{E}^*) + \mathfrak{Im}(\boldsymbol{\chi}^{\mathrm{o}}_{pH})\cdot\mathfrak{Re}(p\boldsymbol{H}^*) - \mathfrak{Im}(\boldsymbol{\chi}^{\mathrm{e}}_{pH})\cdot\mathfrak{Im}(p\boldsymbol{H}^*)\Big] \geqslant 0 \end{aligned}$$

Sieck et al., PRB **96**, 104303 (2017)
Kong, Proc. IEEE **60**, 9 (1972)

Verification of source-driven constitutive relations

Acousto-electrodynamic constitutive relations

$$\begin{pmatrix} \varepsilon \\ \boldsymbol{\mu} \\ \boldsymbol{D} \\ \boldsymbol{B} \end{pmatrix} = \begin{pmatrix} -\beta & \boldsymbol{\gamma}^{\mathsf{T}} & \boldsymbol{b}^{\mathsf{T}} & \boldsymbol{d}^{\mathsf{T}} \\ -\boldsymbol{\eta} & \underline{\boldsymbol{\rho}} & \underline{\boldsymbol{v}} & \underline{\boldsymbol{m}} \\ \boldsymbol{c} & \underline{\boldsymbol{w}} & \underline{\boldsymbol{\epsilon}} & \underline{\boldsymbol{\xi}} \\ \boldsymbol{e} & \underline{\boldsymbol{n}} & \underline{\boldsymbol{\zeta}} & \underline{\boldsymbol{\mu}} \end{pmatrix} \begin{pmatrix} p \\ \boldsymbol{u} \\ \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}$$

$$\operatorname{Im}\beta = 0$$
$$\underline{\boldsymbol{\rho}}^{\dagger} = \underline{\boldsymbol{\rho}}$$
$$\boldsymbol{\eta} = \boldsymbol{\gamma}^{*}$$
$$\underline{\boldsymbol{\mu}} = \underline{\boldsymbol{\mu}}^{\dagger}$$
$$\underline{\boldsymbol{\epsilon}} = \underline{\boldsymbol{\epsilon}}^{\dagger}$$
$$\underline{\boldsymbol{\zeta}} = \underline{\boldsymbol{\xi}}^{\dagger}$$

$$\underline{\boldsymbol{w}}^{\dagger} = \underline{\boldsymbol{v}}$$
$$\underline{\boldsymbol{n}}^{\dagger} = \underline{\boldsymbol{m}}$$
$$\boldsymbol{c} = -\boldsymbol{b}^{*}$$
$$\boldsymbol{d} = -\boldsymbol{e}^{*}$$

Matches result of passivity: theory conserves energy!

Box 1. Source-driven theory[1]

$$\boldsymbol{\eta} = \boldsymbol{\eta}^{\mathrm{o}} + \boldsymbol{\eta}^{\mathrm{e}} = \chi^{\mathrm{o}}_{pu} + i\chi^{\mathrm{e}}_{pu}$$
$$\underline{\boldsymbol{\zeta}} = \underline{\boldsymbol{\zeta}}^{\mathrm{o}} + \underline{\boldsymbol{\zeta}}^{\mathrm{e}} = \underline{\chi}^{\mathrm{o}}_{EH} + i\underline{\chi}^{\mathrm{e}}_{EH}$$
$$\underline{\boldsymbol{w}} = \underline{\boldsymbol{w}}^{\mathrm{o}} + \underline{\boldsymbol{w}}^{\mathrm{e}} = \underline{\chi}^{\mathrm{o}}_{uE} + i\underline{\chi}^{\mathrm{e}}_{uE}$$
$$\underline{\boldsymbol{n}} = \underline{\boldsymbol{n}}^{\mathrm{o}} + \underline{\boldsymbol{n}}^{\mathrm{e}} = i\underline{\chi}^{\mathrm{o}}_{uH} + \underline{\chi}^{\mathrm{e}}_{uH}$$
$$\boldsymbol{c} = \boldsymbol{c}^{\mathrm{o}} + \boldsymbol{c}^{\mathrm{e}} = i\chi^{\mathrm{o}}_{pE} + \chi^{\mathrm{e}}_{pE}$$
$$\boldsymbol{e} = \boldsymbol{e}^{\mathrm{o}} + \boldsymbol{e}^{\mathrm{e}} = \chi^{\mathrm{o}}_{pH} + i\chi^{\mathrm{e}}_{pH}$$

Box 2. By reciprocity

$$\boldsymbol{\gamma} = \boldsymbol{\eta}^{\mathrm{o}} - \boldsymbol{\eta}^{\mathrm{e}} = \chi^{\mathrm{o}}_{pu} - i\chi^{\mathrm{e}}_{pu}$$
$$\underline{\boldsymbol{\xi}} = (\underline{\boldsymbol{\zeta}}^{\mathrm{o}} - \underline{\boldsymbol{\zeta}}^{\mathrm{e}})^{\mathsf{T}} = (\underline{\chi}^{\mathrm{o}}_{EH} - i\underline{\chi}^{\mathrm{e}}_{EH})^{\mathsf{T}}$$
$$\underline{\boldsymbol{v}} = (\underline{\boldsymbol{w}}^{\mathrm{o}})^{\mathsf{T}} - (\underline{\boldsymbol{w}}^{\mathrm{e}})^{\mathsf{T}} = (\underline{\chi}^{\mathrm{o}}_{uE} - i\underline{\chi}^{\mathrm{e}}_{uE})^{\mathsf{T}}$$
$$\underline{\boldsymbol{m}} = -(\underline{\boldsymbol{n}}^{\mathrm{o}})^{\mathsf{T}} + (\underline{\boldsymbol{n}}^{\mathrm{e}})^{\mathsf{T}} = (-i\underline{\chi}^{\mathrm{o}}_{uH} + \underline{\chi}^{\mathrm{e}}_{uH})^{\mathsf{T}}$$
$$\boldsymbol{b} = \boldsymbol{c}^{\mathrm{o}} - \boldsymbol{c}^{\mathrm{e}} = i\chi^{\mathrm{o}}_{pE} - \chi^{\mathrm{e}}_{pE}$$
$$\boldsymbol{d} = -\boldsymbol{e}^{\mathrm{o}} + \boldsymbol{e}^{\mathrm{e}} = -(\chi^{\mathrm{o}}_{pH} - i\chi^{\mathrm{e}}_{pH})$$

Compare Box 1, 2

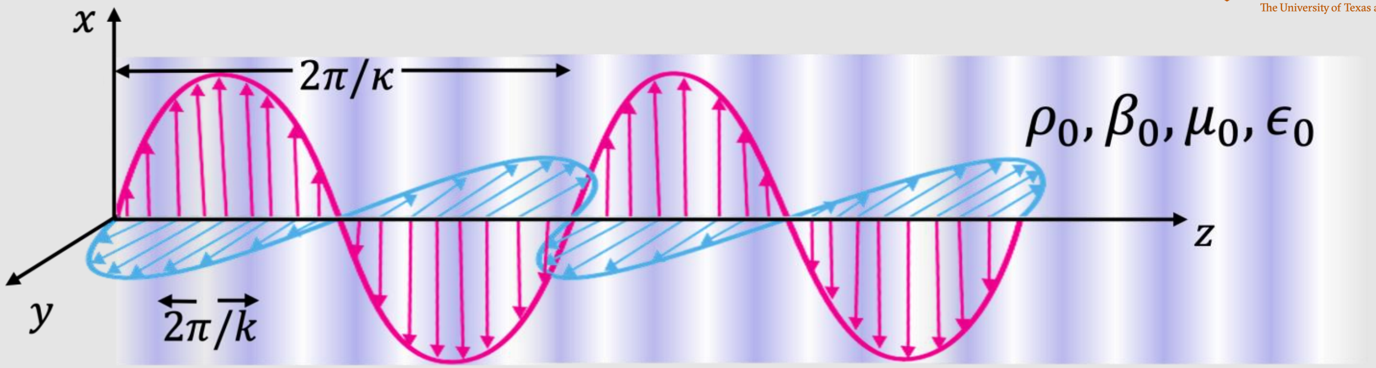
$$\boldsymbol{\eta} = \boldsymbol{\gamma}^{*}$$
$$\underline{\boldsymbol{\zeta}} = \underline{\boldsymbol{\xi}}^{\dagger}$$
$$\underline{\boldsymbol{w}}^{\dagger} = \underline{\boldsymbol{v}}$$
$$\boldsymbol{d} = -\boldsymbol{e}^{*}$$
$$\underline{\boldsymbol{n}}^{\dagger} = \underline{\boldsymbol{m}}$$
$$\boldsymbol{c} = -\boldsymbol{b}^{*}$$

---

[1] Gokani et al., *J. Acoust. Soc. Am.* **153**, A120 (2023).

Even & odd couplings

$$
\begin{aligned}
ik p_{\text{ext}} &= i\omega\rho_0 u_{\text{ext}} + f_{\text{ext}}\\
ik u_{\text{ext}} &= i\omega\beta_0 p_{\text{ext}} + q_{\text{ext}}\\
i\kappa E_{\text{ext}} &= i\omega\mu_0 H_{\text{ext}} - K_{\text{ext}}\\
i\kappa H_{\text{ext}} &= i\omega\epsilon_0 E_{\text{ext}} - J_{\text{ext}}
\end{aligned}
$$



$$
p_{\text{eff}} = p_{\text{ext}} + \frac{1}{1-k_0^2/k^2}\left(-\frac{k_0^2}{k^2}\beta_0^{-1}\bar{m}_{\text{eff}} + \frac{k_0}{k}c_0\bar{d}_{\text{eff}}\right)
$$

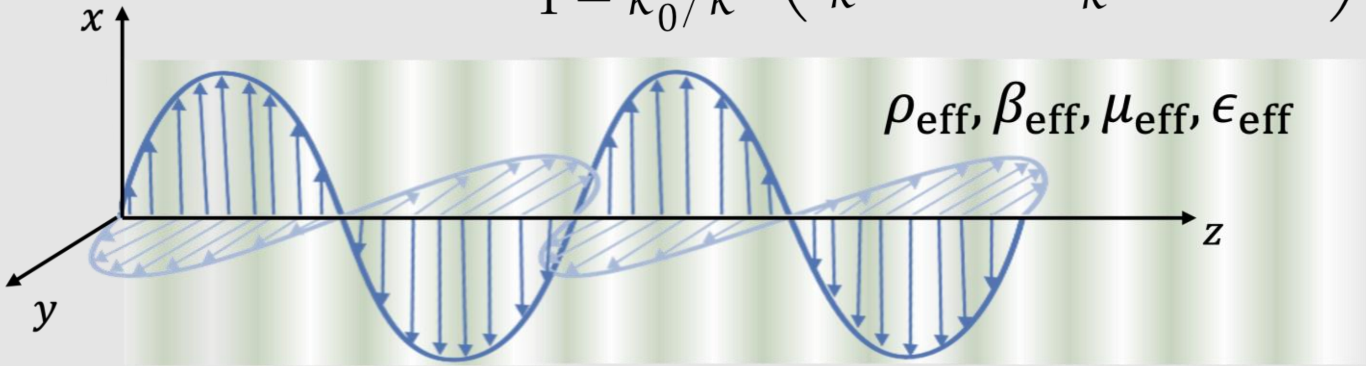
$$
u_{\text{eff}} = u_{\text{ext}} + \frac{1}{1-k_0^2/k^2}\left(-\frac{k_0}{k}c_0\bar{m}_{\text{eff}} + \frac{k_0^2}{k^2}\rho_0^{-1}\bar{d}_{\text{eff}}\right)
$$

$$
E_{\text{eff}} = E_{\text{ext}} + \frac{1}{1-\kappa_0^2/\kappa^2}\left(\frac{\kappa_0^2}{\kappa^2}\epsilon_0^{-1}\bar{P}_{\text{eff}} + \frac{\kappa_0}{\kappa}C_0\bar{M}_{\text{eff}}\right)
$$

$$
H_{\text{eff}} = H_{\text{ext}} + \frac{1}{1-\kappa_0^2/\kappa^2}\left(\frac{\kappa_0}{\kappa}C_0\bar{P}_{\text{eff}} + \frac{\kappa_0^2}{\kappa^2}\mu_0^{-1}\bar{M}_{\text{eff}}\right)
$$

$$
\begin{aligned}
p_{\text{ext}} &= V[-\beta_0^{-1}(\tilde{\alpha}_{pp} - S_{\text{ac}})\bar{m}_{\text{eff}} + c_0(-i\tilde{\alpha}_{pu} - S_{\text{ac}})\bar{d}_{\text{eff}} - i(\beta_0\epsilon_0)^{-1/2}\tilde{\alpha}_{pE}\bar{P}_{\text{eff}}]\\
u_{\text{ext}} &= V[-c_0(i\tilde{\alpha}_{pu} - S_{\text{ac}})\bar{m}_{\text{eff}} + \rho_0^{-1}(\tilde{\alpha}_{uu} - S_{\text{ac}})\bar{d}_{\text{eff}} - i(\rho_0\epsilon_0)^{-1/2}\tilde{\alpha}_{uE}\bar{P}_{\text{eff}}]\\
E_{\text{ext}} &= V[-i(\beta_0\epsilon_0)^{-1/2}\tilde{\alpha}_{pE}\bar{m}_{\text{eff}} - i(\rho_0\epsilon_0)^{-1/2}\tilde{\alpha}_{uE}\bar{d}_{\text{eff}} + \epsilon_0^{-1}(\tilde{\alpha}_{EE} - S_{\text{em}})\bar{P}_{\text{eff}} - C_0 S_{\text{em}}\bar{M}_{\text{eff}}]\\
H_{\text{ext}} &= V[-C_0 S_{\text{em}}\bar{P}_{\text{eff}} - \mu_0^{-1}(S_{\text{em}} - \tilde{\alpha}_{HH})\bar{M}_{\text{eff}}]
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon_{\text{eff}} &= -\beta_0 p_{\text{eff}} + \bar{m}_{\text{eff}}\\
\mu_{v,\text{eff}} &= \rho_0 u_{\text{eff}} + \bar{d}_{\text{eff}}\\
D_{\text{eff}} &= \epsilon_0 E_{\text{eff}} + \bar{P}_{\text{eff}}\\
B_{\text{eff}} &= \mu_0 H_{\text{eff}} + \bar{M}_{\text{eff}}
\end{aligned}
$$



Interpretation of form of constitutive relations

$$\begin{pmatrix} \varepsilon_{\text{eff}} \\ \mu_{v,\text{eff}} \\ D_{\text{eff}} \\ B_{\text{eff}} \end{pmatrix} = \begin{pmatrix} -\beta_{\text{eff}} & \chi^{\text{o}}_{pu} - i\chi^{\text{e}}_{pu} & i\chi^{\text{o}}_{pE} - \chi^{\text{e}}_{pE} & -(\chi^{\text{o}}_{pH} - i\chi^{\text{e}}_{pH}) \\ -(\chi^{\text{o}}_{pu} + i\chi^{\text{e}}_{pu}) & \rho_{\text{eff}} & \chi^{\text{o}}_{uE} - i\chi^{\text{e}}_{uE} & -i\chi^{\text{o}}_{uH} + \chi^{\text{e}}_{uH} \\ i\chi^{\text{o}}_{pE} + \chi^{\text{e}}_{pE} & \chi^{\text{o}}_{uE} + i\chi^{\text{e}}_{uE} & \epsilon_{\text{eff}} & \chi^{\text{o}}_{EH} \\ \chi^{\text{o}}_{pH} + i\chi^{\text{e}}_{pH} & i\chi^{\text{o}}_{uH} + \chi^{\text{e}}_{uH} & \chi^{\text{o}}_{EH} & \mu_{\text{eff}} \end{pmatrix} \begin{pmatrix} p_{\text{eff}} \\ u_{\text{eff}} \\ E_{\text{eff}} \\ H_{\text{eff}} \end{pmatrix}$$

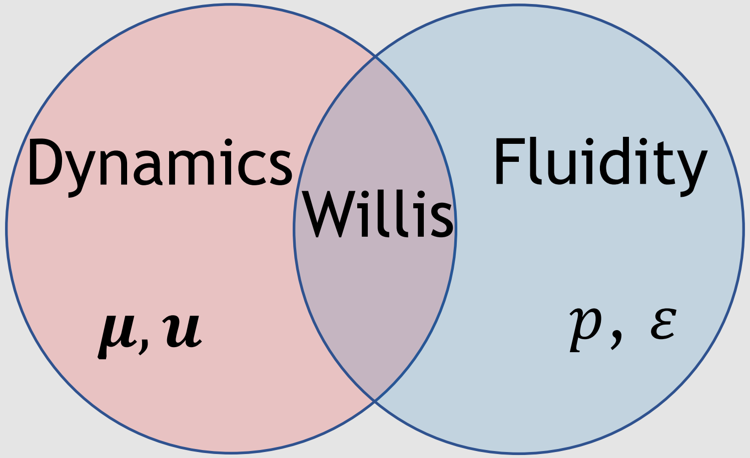
Phase relations

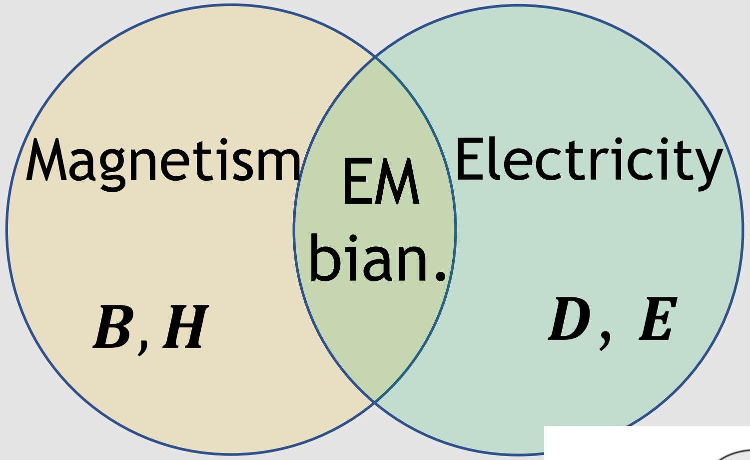
- ε_{eff} in phase with E_{eff}; D_{eff} in phase with p_{eff}
- $\mu_{v,\text{eff}}$ in phase with H_{eff}; B_{eff} in phase with u_{eff}
- ε_{eff} and H_{eff} related by time derivative
- $\mu_{v,\text{eff}}$ and E_{eff} related by time derivative

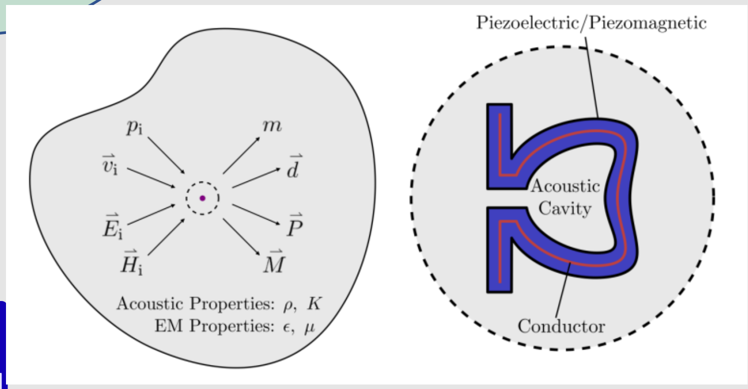
Insights

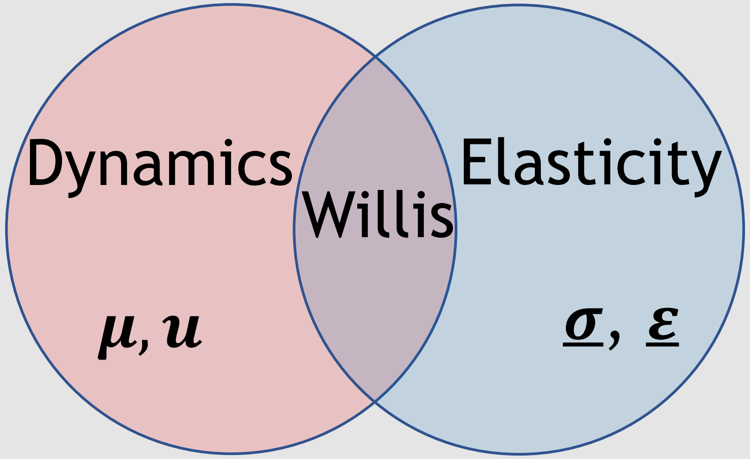
Gokani et al., *J. Acoust. Soc. Am.* **153**, A120 (2023).

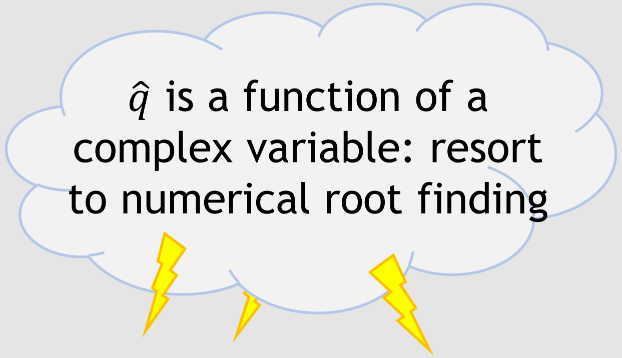
Figure board











Fubini coefficients (--)

$$B_1 = 1 - \frac{1}{8}\sigma^2 + \mathcal{O}(\sigma^4)$$

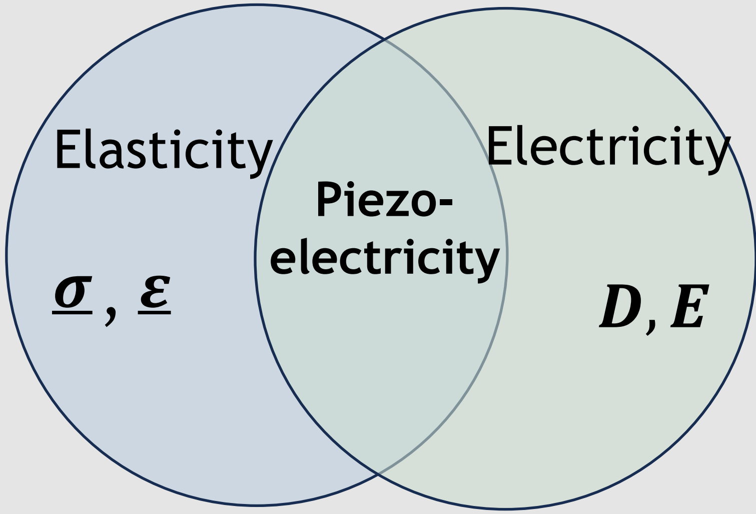
$$B_2 = \frac{1}{2}\sigma + \mathcal{O}(\sigma^3)$$

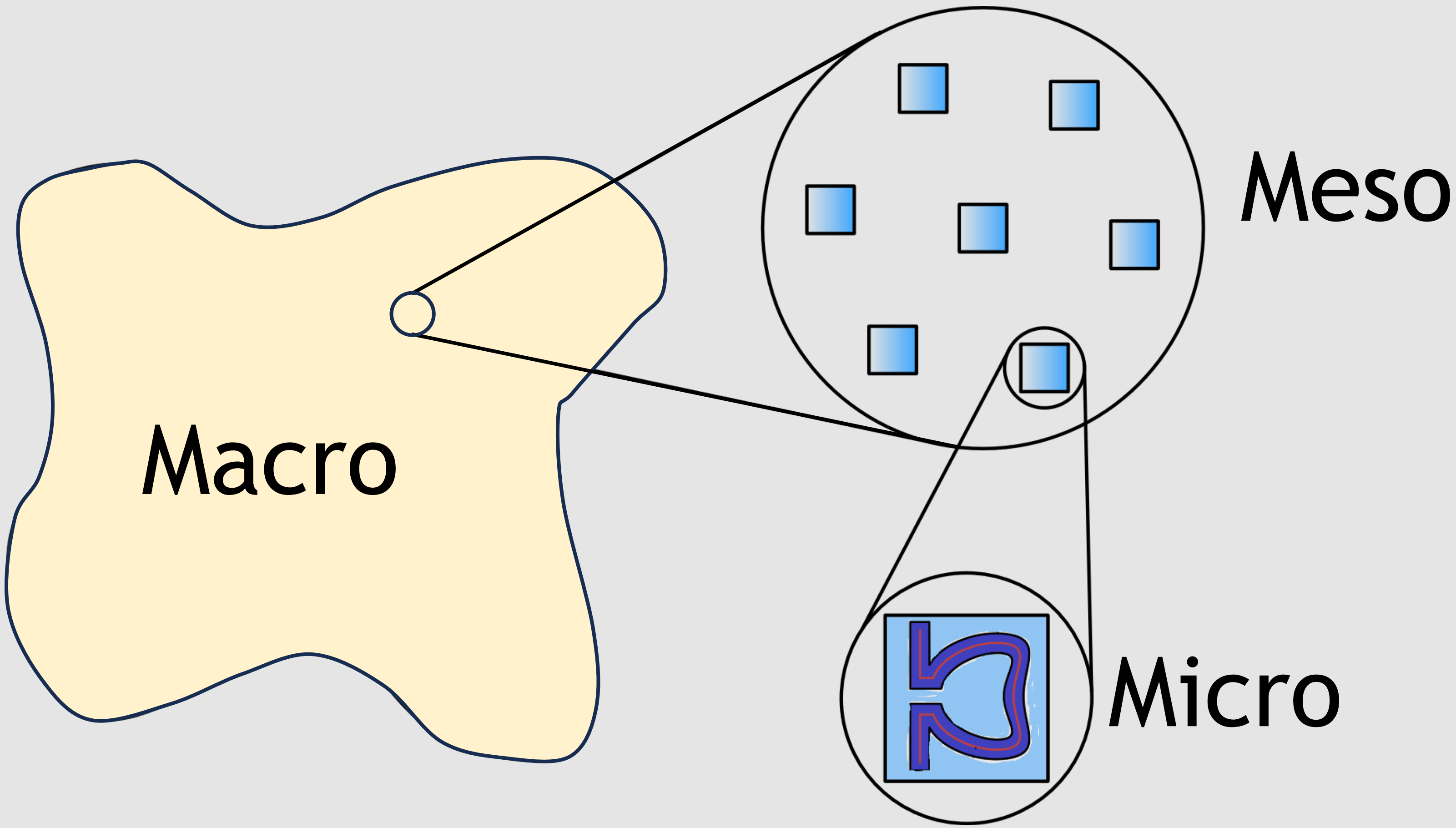
$$B_3 = \frac{3}{8}\sigma^2 + \mathcal{O}(\sigma^4)$$

$$B_4 = \frac{1}{3}\sigma^3 + \mathcal{O}(\sigma^5)$$

$$B_5 = 0.3255\,\sigma^4 + \mathcal{O}(\sigma^6)$$

$$B_n = \frac{(n\sigma)^{n-1}}{2^{n-1}n!} + \mathcal{O}(\sigma^{n+1}).$$

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