WAVE PHENOMENA

ME/EE 384N-8—Spring 2024

Homework Problems

Fourier Transforms and Dirac Delta Functions

1-1. If $\mathcal{F}\{f(x,y)\} = F(k_x,k_y)$ and $\mathcal{F}\{g(x,y)\} = G(k_x,k_y)$, show that

(a) $\mathcal{F}\{f(x-a,y-b)\} = F(k_x,k_y)e^{-ik_xa-ik_yb}$

(b) $\mathcal{F}\{f(x,y)**g(x,y)\} = F(k_x,k_y)G(k_x,k_y)$

Here and below, make use of any other identities to complete your proofs of the identities in the problem statements. For example, you may use the relation in part (a), if relevant, in your proof of the identity in part (b).

1-2. [After Williams, *Fourier Acoustics*.] If $\mathcal{F}_x\{f(x)\} = F(k_x)$, $\mathcal{F}_x\{g(x)\} = G(k_x)$, and $\mathcal{F}_x\{h(x)\} = H(k_x)$, derive the convolution theorem for

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(k_x)G(k_x)H(k_x)e^{ik_xx}\,dk_x$$

Are the resulting operations associative?

1-3. Let $f(x) = A(x/L)\mathrm{rect}(x/2L)$, which describes an N wave of length $2L$ and peak amplitude A. The N wave is a reasonably good approximation of a sonic boom waveform. Calculate its Fourier transform by using known transforms of related elementary functions. Specifically, sketch $f(x)$ and its first derivative $f'(x)$, identify the Fourier transforms of the elementary functions in $f'(x)$, and then use an appropriate theorem to convert the result to $F(k_x) = \mathcal{F}_x\{f(x)\}$. Show that algebraic manipulation of the final result yields

$$F(k_x) = -i2AL\left[\frac{\sin k_xL}{(k_xL)^2} - \frac{\cos k_xL}{k_xL}\right]$$

The expression inside the brackets may be recognized as $j_1(k_xL)$, the spherical Bessel function of the first kind, of order 1. Thus $F(0) = 0$ as required because $\int_{-\infty}^{\infty} f(x)\,dx = 0$ due to $f(x)$ being an odd function.

1-4. In class we obtained the Fourier transform $F(k_x)$ of the triangle function $f(x) = A\Lambda(x/L)$ from the known transform of the rectangle function by making use of the shifting and differentiation theorems. Obtain the transform of the same triangle function, again from that of the rectangle function, but now by representing the former as the convolution of the latter with itself (noting that to do so the amplitude of the rectangle function must be adjusted appropriately), and then making use of the convolution theorem.

1-5. If $\mathcal{H}\{f(\rho)\} = F_H(\kappa)$, show that

(a) $\mathcal{H}\{f(\alpha\rho)\} = \alpha^{-2} F_H(\kappa/\alpha)$ for $\alpha > 0$

(b) $\mathcal{H}^{-1}\{\kappa^2 F_H(\kappa)\} = -f''(\rho) - \rho^{-1} f'(\rho)$

where primes indicate derivatives with respect to the argument. One way to obtain the relation in part (b) is by making use of Bessel's equation:

$$\left(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} + \kappa^2 - \frac{n^2}{\rho^2}\right) J_n(\kappa\rho) = 0$$

1-6. Show that $\mathcal{H}\{J_0(\alpha\rho)\} = \alpha^{-1}\delta(\kappa - \alpha)$.

1-7. [After Goodman, *Introduction to Fourier Optics* (3rd edition).] Polar coordinates prove most useful for calculating Fourier transforms when the function is separable in this coordinate system, i.e., $f(\rho, \phi) = f_\rho(\rho) f_\phi(\phi)$. The special case in which the function is independent of ϕ yields $\mathcal{F}\{f(\rho)\} = 2\pi\mathcal{H}\{f(\rho)\}$.

(a) Let $f_\phi(\phi) = e^{in\phi}$ and show that

$$F(\kappa, \psi) = (-i)^n e^{in\psi} 2\pi F_H^n(\kappa)$$

where $F(\kappa, \psi) = \mathcal{F}\{f_\rho(\rho)e^{in\phi}\}$ is the desired Fourier transform and $F_H^n(\kappa) = \mathcal{H}_n\{f_\rho(\rho)\}$ is the Hankel transform of order n. Note that the result $F(\kappa) = 2\pi F_H(\kappa)$ is recovered for $n = 0$. Hint: Make use of the identities

$$e^{i\alpha\sin\beta} = \sum_{m=-\infty}^{\infty} J_m(\alpha) e^{im\beta}$$

and $J_{-n}(\alpha) = (-1)^n J_n(\alpha)$.

(b) Now let $f_\phi(\phi)$ be an arbitrary function of ϕ and show that

$$F(\kappa, \psi) = \sum_{m=-\infty}^{\infty} c_m (-i)^m e^{im\psi} 2\pi F_H^m(\kappa)$$

where

$$c_m = \frac{1}{2\pi}\int_0^{2\pi} f_\phi(\phi) e^{-im\phi}\, d\phi$$

are the coefficients in the Fourier series expansion of $f_\phi(\phi)$.

1-8. If the 3D spatial Fourier transform is defined by

$$\mathcal{F}_{\mathrm{3D}}\{f(x, y, z)\} = \iiint_{-\infty}^{\infty} f(x, y, z) e^{-i(k_x x + k_y y + k_z z)}\, dxdydz$$

show that for a spherically symmetric function $f(r)$

$$\mathcal{F}_{\mathrm{3D}}\{f(r)\} = 4\pi \int_0^{\infty} f(r) j_0(kr)\, r^2 dr$$

where $j_0(kr) = (\sin kr)/kr$ is the spherical Bessel function of the first kind, of order 0. One way to proceed is as follows. Since the field is spherically symmetric choose $\mathbf{k} = k\,\mathbf{e}_z$ such that $\mathbf{k}\cdot\mathbf{r} = kr\cos\theta$, where θ is the polar angle. Then make the variable substitution $u = kr\cos\theta$ to evaluate the integral over θ.

1-9. With the delta function represented as $\delta(x) = \lim_{\epsilon\to 0}\mu(x,\epsilon)$, express $\mu(x,\epsilon)$ in terms of the following distributions:

(a) $J_n(x)$

(b) $j_n(x)$

where J_n is the Bessel function of the first kind, and j_n the spherical Bessel function of the first kind, each of order n, with n being an integer. Do your results impose any restriction on n? You may find it useful to consult a handbook of mathematical functions, such as the NIST Digital Library of Mathematical Functions, which is available online.

1-10. Show that $x\delta'(x) = -\delta(x)$. Prove this relation by showing that $-x\delta'(x)$ satisfies the sifting property.

1-11. Evaluate the following integrals:

(a) $\int_{-\infty}^{\infty} f(x)\delta(x^2 - a^2)\,dx$

(b) $\int_{-\infty}^{\infty} f(x)\delta(\sin x)\,dx$

1-12. The 3D Fourier transform $F(k) = \mathcal{F}_{3\mathrm{D}}\{f(r)\}$ of a spherically symmetric function $f(r)$ is given in Prob. 1-8. Show that the Fourier integral theorem $\mathcal{F}_{3\mathrm{D}}^{-1}\{\mathcal{F}_{3\mathrm{D}}[f(r)]\} = f(r)$ is satisfied if the inverse transform is given by

$$\mathcal{F}_{3\mathrm{D}}^{-1}\{F(k)\} = \frac{1}{2\pi^2}\int_0^{\infty} F(k)j_0(kr)\,k^2dk$$

Hint: Employ the integral representation of the delta function in terms of spherical Bessel functions.

1-13. Using the coordinate transform relation for $\delta(x)\delta(y)$ (see posted handout), show that

$$\delta(x - x_0)\delta(y - y_0) = \frac{\delta(\rho - \rho_0)}{\rho}\delta(\phi - \phi_0)$$

where (x_0, y_0) and (ρ_0, ϕ_0) represent the same point in cartesian and polar coordinates, respectively. Specifically, show that

$$f(\rho_0, \phi_0) = \int_0^{2\pi}\int_0^{\infty} f(\rho,\phi)\frac{\delta(\rho - \rho_0)}{\rho}\delta(\phi - \phi_0)\,J(\rho,\phi)d\rho d\phi$$

where $J(\rho, \phi)$ is the Jacobian of the transformation from (x, y) to (ρ, ϕ) coordinates.

1-14. The second Rayleigh integral contains the expression

$$\mu(x, y, z) = ikz\left(\frac{1}{ikr} - 1\right)\frac{e^{ikr}}{2\pi r^2}$$

where $r = (x^2 + y^2 + z^2)^{1/2}$. Show that in the limit $z \to 0$ this expression satisfies the sifting property, i.e., that $\lim_{z\to 0}\iint_{-\infty}^{\infty} f(x, y)\mu(x, y, z)\,dxdy = f(0, 0)$, thus demonstrating that a 2D delta function is obtained in this limit: $\lim_{z\to 0}\mu(x, y, z) = \delta(x)\delta(y)$. Hint: Make the variable substitutions $u = x/z$ and $v = y/z$ before taking the limit $z \to 0$, after which it may be helpful to evaluate the resulting integrals in polar coordinates.

Fourier Acoustics

2-1. Download the posted Matlab code FourierAcousticProp.m, and read the posted comments on this code.

(a) Check how well the computed axial propagation curve for a circular piston compares with the exact solution shown in Fig. 13.8(a) of Blackstock's book. Specifically, compare your computation using the Fourier acoustics code with Eq. (C-4) on p. 455 of Blackstock on the same graph.

(b) Create a plot of the angular spectrum of the source pressure in which the radiation circle defined by $K_x^2 + K_y^2 = K^2$ is superimposed, inside which are modes that propagate and outside which are evanescent modes. The dimensionless angular spectrum of the source pressure is obtained from the velocity spectrum by multiplying the latter by K/K_z. It is probably best to create a color plot of the angular spectrum (e.g., using `imagesc`), rather than a contour plot, and superimpose a ring with a sharply contrasting color. Such a plot will indicate the relative amounts of radiated and reactive power in the near field. Observe how radiation patterns change as ka is varied for several different source geometries, and how this relates to the fraction of energy in the angular spectrum that lies within the radiation circle. Note also the singularity along the radiation circle that results from K_z going to zero at those points. See p. 31 in *Fourier Acoustics* by Williams for further discussion.

(c) The mean (i.e., time-averaged) intensity vector field is of considerable practical interest. Among other things, a calculation or measurement of this field can reveal the presence of energy sinks on radiating surfaces. See the posted excerpt from *Sound Intensity* by Fahy, and note that whereas there is no energy sink in a dipole source [Fig. 4.10(a), showing all energy flowing away from the dipole], when the antiphase monopoles constituting the dipole have different strengths the weaker of the two becomes an energy sink [Fig. 4.11(c)]. Illustrate energy flow from source to sink using the angular spectrum code. To get you started, the following script will generate plots of the intensity field at a prescribed distance Z (i.e., `Zmax`) along the X and Y axes, making use of the vector components `Ix`, `Iy`, and `Iz` that are commented out of the posted version of the code:

```
Ix_xz=Ix(N/2+1,:); Iz_xz=Iz(N/2+1,:); y0=zeros(1,N);
quiver(x,y0,Ix_xz,Iz_xz)
xlim([-2 2]), xlabel('x/a');
title(['Intensity Vectors in x-z Plane (y=0)'])

Iy_yz=Iy(:,N/2+1)'; Iz_yz=Iz(:,N/2+1)'; x0=zeros(1,N);
quiver(y,x0,Iy_yz,Iz_yz)
xlim([-2 2]), xlabel('y/a');
title(['Intensity Vectors in y-z Plane (x=0)'])
```

Try using the "shaded dipole" source function and plot the intensity vectors along the x axis close to the source plane (`Zmax` $\simeq 0$) for small ka. Why do you think the observed energy flow from "source" to "sink" disappears for large ka?

2-2. A point source with volume velocity Q is mounted in a rigid baffle, and the normal velocity in the source plane is given by $Q\delta(x)\delta(y)e^{-i\omega t}$. Convert the source condition to polar coordinates and use Hankel transforms to show that the radiated pressure field is

$$p = -i\omega Q \frac{\rho_0 e^{i(kr-\omega t)}}{2\pi r}$$

where $r = (x^2 + y^2 + z^2)^{1/2}$. To evaluate the inverse Hankel transform you will need the relation

$$\mathcal{H}\left\{ \frac{e^{ik\sqrt{z^2+\rho^2}}}{\sqrt{z^2+\rho^2}} \right\} = \frac{ie^{iz\sqrt{k^2-\kappa^2}}}{\sqrt{k^2-\kappa^2}}$$

which follows from entry 13 in the posted table of Hankel transforms by Debnath and Lokenath.

2-3. A baffled planar ring source of radius a and infinitesimal width w vibrates with normal velocity $u_0 w\delta(\rho - a)e^{-i\omega t}$.

(a) What is the volume velocity of this source?

(b) Use Hankel transforms to show that the pressure field along the z axis is

$$p_\omega(\rho = 0, z) = -ikaw\rho_0 c_0 u_0 \frac{e^{ik\sqrt{z^2+a^2}}}{\sqrt{z^2+a^2}}$$

This solution, derived by other means, appears on page 445 in the textbook by Blackstock. To evaluate the inverse Hankel transform you will need the relation provided in the previous problem.

(c) Along the axis of a circular piston of radius a the transition from near to far field occurs at the Rayleigh distance, $z \simeq \frac{1}{2}ka^2$; recall Fig. 13.8 in the textbook by Blackstock. Along the axis of a circular ring of radius a the above solution shows that the transition from near to far field occurs at $z \simeq a$. Provide a physical explanation for why the former depends on frequency whereas the latter does not.

2-4. An infinite plate in the plane $z = 0$ supports a circularly symmetric standing wave with normal velocity $u_0 J_0(k_b\rho)e^{-i\omega t}$, where $k_b = \omega/c_b$ is the bending wave number and c_b the bending wave speed.

(a) Use Hankel transforms to calculate the complex pressure field $p_\omega(\rho, z)$ radiated into the fluid. Note that the radiation forms a collimated sound beam.

(b) Calculate the corresponding particle velocity vector field $\mathbf{u}_\omega(\rho, z)$ in the fluid. Verify that (i) along the axis of symmetry ($\rho = 0$) the ρ component of the particle velocity vanishes, and (ii) in the source plane your result recovers the boundary condition on the z component of the particle velocity, $u_0 J_0(k_b\rho)$.

(c) Calculate the intensity vector field $\mathbf{I}_\omega(\rho, z)$ in the fluid for both $c_b > c_0$ (supersonic bending waves) and $c_b < c_0$ (subsonic bending waves).

You should have found in part (a) that the transverse distribution of the pressure field is always $J_0(k_b\rho)$, independent of z, and that the amplitude along the axis of the beam never decreases. When these solutions of the wave equation were "discovered" in optics they were labeled "nondiffracting beams" and generated considerable excitement at the time due to the prospect of creating extraordinarily large depths of field [J. Durnin, "Exact solutions for nondiffracting beams," J. Opt. Soc. Am. A **4**, 651 (1987)]. Practical aspects of this application are examined in a later homework problem.

2-5. A plane wave propagating in direction $(\theta, \phi) = (\theta_0, 0)$ possesses a phase term of the form $e^{i(kx \sin\theta_0 + kz\cos\theta_0 - \omega t)}$. Therefore, one method by which a sound beam radiated by a source in the plane $z = 0$ with normal velocity $u_0(x, y)$ $(\times e^{-i\omega t})$ may be steered in that direction is by applying a linear phase shading such that the source condition is replaced by $u_0(x, y)e^{ikx\sin\theta_0}$.

(a) If the angular spectrum of $u_0(x, y)$ is $U_0(k_x, k_y)$, show that

$$\mathcal{F}\{u_0(x, y)e^{ikx\sin\theta_0}\} = U_0(k_x - k\sin\theta_0, k_y)$$

If most of the energy in the angular spectrum $U_0(k_x, k_y)$ is concentrated at $(k_x, k_y) \simeq (0, 0)$, i.e., in the z direction, the phase shading thus shifts this energy to the direction determined by $(k_x, k_y) \simeq (k\sin\theta_0, 0)$.

(b) In spherical coordinates $(k_x, k_y) = (k\sin\theta\cos\phi, k\sin\theta\sin\phi)$, and thus the direction determined in part (a) is $(\theta, \phi) \simeq (\theta_0, 0)$. For at least two nonzero values of θ_0, use the propagation code for a circular piston of radius a to verify that the angular spectrum is translated, and the sound beam is rotated, by the amounts predicted by theory.

(c) Repeat part (b) by plotting the angular spectrum and pressure field for a uniform strip source of width $2a$, obtained by replacing, for example, $\exp[-(\rho/a)^{30}]$ with $\exp[-(x/a)^{30}]$ in the propagation code. Making the wave field 2D in this way often emphasizes physical phenomena of interest.

(d) Use the propagation code to show that you can simulate the rotation of two circular pistons, one centered at $(x, y) = (2a, 0)$ and the other at $(x, y) = (-2a, 0)$, such that their beam axes intersect at $(x, y, z) = (0, 0, \frac{1}{2}z_0)$, where $z_0 = \frac{1}{2}ka^2$ is the Rayleigh distance.

2-6. Transmission of a sound beam through a thin layer in the x-y plane that modulates the phase speed sinusoidally in the x direction can be modeled by multiplying the field in that plane by the phase term $e^{im\sin\kappa_0 x}$, where m is the amplitude of the modulation and κ_0 its spatial frequency. A model of this sort is referred to as a phase screen. The same modeling approach can be applied to reflection of a sound beam from a sinusoidally corrugated surface.

(a) If the angular spectrum of $p_0(x, y)$ is $P_0(k_x, k_y)$, show that

$$\mathcal{F}\{p_0(x, y)e^{im\sin\kappa_0 x}\} = \sum_{n=-\infty}^{\infty} J_n(m)P_0(k_x - n\kappa_0, k_y)$$

Hint: First show that $e^{iz\sin\theta} = \sum_n J_n(z)e^{in\theta}$ by setting $t = e^{i\theta}$ in Eq. (10.12.1) of the NIST Digital Library of Mathematical Functions.

(b) The effect of the sinusoidal phase modulation is thus to take an angular spectrum originally centered at $(k_x, k_y) = (0, 0)$ and recenter it at all angles for which $(k_x, k_y) = (n\kappa_0, 0)$, where the integer values $n = 0, \pm1, \pm2\ldots$ are referred to as diffraction orders. Representing the wave vectors in spherical coordinates as in part (b) of the previous problem, one sees that the angles are those for which $\sin\theta_n = n\kappa_0/k$ and $\phi_n = 0$. Use the propagation code to verify that diffraction does indeed produce pressure maxima at the predicted angles. Specifically, plot the pressure field in the x-z plane produced by a uniform strip source of width $2a$ by letting $p_0(x, y) = p_0 \exp[-(x/a)^{30}]$ with $ka = 50$, and for the phase screen set $m = 1$ and $\kappa_0/k = 0$, 0.1, 0.2, and 0.3. To assess the theoretical prediction, superimpose lines on the plots (either by hand or by computer) that extend outward from $(x, z) = (0, 0)$ at angles θ_n with respect to the z axis for the first several diffraction orders. Note: When using the propagation code to calculate radiation from a pressure source, remove the factor of k/k_z that appears in the inverse Fourier transform (even though for $ka \gg 1$ the difference this makes is usually insignificant).

(c) For a plane wave incident on the phase screen the transmitted field can be calculated analytically. Specifically, for $p_0(x, y) = p_0 = \text{const}$, show that with the angular spectrum in the source plane expressed as in part (a), the transmitted field is

$$p_\omega = p_0 \sum_{n=-\infty}^{\infty} J_n(m) \exp\Big\{in\kappa_0 x + i[k^2 - (n\kappa_0)^2]^{1/2} z\Big\}$$

Note that this expression reduces to the plane-wave result $p_\omega = p_0 e^{ikz}$ in the absence of the phase screen ($m = 0$), as required.

Discussion: The expression above is the same as that for a light wave transmitted through a sound wave propagating perpendicular to the path of the light wave. The density perturbation in the sound wave presents a sinusoidally varying index of refraction (phase screen) to the light wave. The expression above is the same as the second of Eqs. (13.1.21) in Morse and Ingard's *Theoretical Acoustics* if in the expression above one replaces p_0 by $\mathbf{E}_0$, m by $\eta K z$, κ_0 by k_s, and k by K, where K and k_s are the wave numbers of the light and the sound, respectively, and η characterizes the perturbation of the optical index of refraction. Then assume $K^2 \gg (nk_s)^2$, and ignore the time dependence in (13.1.21). The resulting expression corresponds to the Raman-Nath theory for the diffraction of light by sound. Raman was awarded the 1930 Nobel Prize in Physics "for his work on the scattering of light and for the discovery of the effect named after him", i.e., Raman scattering.

2-7. The pressure reflection coefficient for a plane wave in a fluid that is incident on an elastic half-space occupying the region $z \le 0$ may be expressed in terms of its wave vector component parallel to the interface, $\kappa = (k_x^2 + k_y^2)^{1/2} = k \sin\theta$, as

$$R(\kappa) = \frac{(k_t^2 - 2\kappa^2)^2\sqrt{k^2-\kappa^2} + 4\kappa^2\sqrt{(k^2-\kappa^2)(k_t^2-\kappa^2)(k_l^2-\kappa^2)} - (\rho_0/\rho_s)k_t^4\sqrt{k_l^2-\kappa^2}}{(k_t^2 - 2\kappa^2)^2\sqrt{k^2-\kappa^2} + 4\kappa^2\sqrt{(k^2-\kappa^2)(k_t^2-\kappa^2)(k_l^2-\kappa^2)} + (\rho_0/\rho_s)k_t^4\sqrt{k_l^2-\kappa^2}}$$

where θ is the angle of incidence with respect to the normal to the interface, $k = \omega/c_0$ is the wave number in the fluid, $k_l = \omega/c_l$ and $k_t = \omega/c_t$ are the wave numbers for the longitudinal and transverse (shear) waves propagating in the solid at speeds c_l and c_t, respectively, ρ_0 is the density of the fluid, and ρ_s is the density of the solid. The angular spectrum of the reflected pressure field at the interface is thus $R(\kappa)$ times the angular spectrum of the incident pressure field at the interface.

(a) Show that when the elastic medium is replaced by a second fluid the reflection coefficient reduces to the familiar result

$$R(\theta) = \frac{\rho_1 c_1 \cos\theta - \rho_0 c_0 \cos\theta_1}{\rho_1 c_1 \cos\theta + \rho_0 c_0 \cos\theta_1}$$

where θ_1 is the transmission angle with respect to the normal, and where $\rho_1 = \rho_s$, $c_1 = c_l$. Hint: Note that because the shear modulus is zero in a fluid, $c_t = 0$ and therefore $k_t = \infty$. Thus divide both the numerator and denominator of $R(\kappa)$ by k_t^4, take the limit $k_t \to \infty$, and then make use of Snell's law to obtain the desired result.

(b) Plot the magnitude and phase of $R(\theta)$ in the range $0 \le \theta \le 90°$ for a water-aluminum interface using the material parameters in Table I of Landsberger and Hamilton [J. Acoust. Soc. Am **109**, 488–500 (2001)]. Although you may choose a specific frequency to make these calculations, note that the reflection coefficient is independent of frequency (each term in the numerator and demoninator is proportional to ω^5, which can therefore be factored out and cancelled). When a critical angle is exceeded, take the positive square root of the resulting negative quantity, e.g., replace $(k_t^2 - \kappa^2)^{1/2}$ by $i(\kappa^2 - k_t^2)^{1/2}$ for $\kappa > k_t$. Matlab may automatically choose this root, but it is something to watch out for. Check your results against Fig. 4 of L&H.

(c) Modify the Matlab code FourierAcousticProp.m to model reflection of a sound beam from an elastic half-space. For simplicity, as well as to emphasize the physical phenomena of interest, consider 2D propagation such that $p_\omega = p_\omega(x,z)$, $k_y = 0$, and in the reflection coefficient $\kappa = k_x$. Use the same material properties as in part (b) together with the source parameters reported by L&H. Specifically, let the interface be the plane $z = 0$ and approximate a 1 MHz beam incident on the interface by a Gaussian pressure amplitude e^{-x^2/a^2} with $a = 1.2$ cm. Include linear phase shading to account for incidence at angle θ_0 in the x-z plane. Note that you will need to convert all wave numbers in the reflection coefficient to the dimensionless form used in the code by multiplying each wave number by source radius a, e.g., replace k_t by $K_t = k_t a = (c_0/c_t)K$.

Compare the incident and reflected pressure amplitudes in the plane $z = 0$ for $\theta_0 = 20°$, $\theta_0 = 30.4°$, and $\theta_0 = 40°$. Specifically, for each angle plot $|p_\omega(x, z = 0)|$ for both the incident and reflected fields on the same graph. Strong beam displacement should be apparent near the Rayleigh angle $\theta_R = 30.4°$, defined by $\sin\theta_R = c_0/c_R$, where c_R is the propagation speed of the Rayleigh wave. You should observe a minimum in the reflected beam profile where the radiation from the leaky Rayleigh wave interferes with the specularly reflected field (i.e., the reflected field predicted by geometrical acoustics).

(d) Now make color plots of the amplitudes of the incident and reflected beams in the x-z plane for same same incidence angles as in part (c). Both the incident and reflected beam should be shown on each plot. The reflected beam is calculated in a straightforward way using the propagation code. For the incident beam, it is sufficient to also calculate it using the propagation code as though it were a beam radiated *away* from the interface at angle $-\theta_0$. This is done merely as a visual aid to provide better perspective on displacement of the reflected beam.

The following features should be observed. (1) For $\theta_0 = 40°$, the incident and reflected beams should be symmetric about the z axis but otherwise identical due to total internal reflection. (2) For $\theta_0 = 20°$, the reflected beam should have lower intensity than the incident beam due to penetration of the incident beam into the solid below the critical angle. (3) At the Rayleigh angle $\theta_0 = 30.4°$, in addition to displacement and broadening of the reflected beam, an interference minimum should be observed that follows a path that is parallel to the axis of the reflected beam, visual demonstration of which has been provided by Schlieren photographs.

(e) Repeat parts (c) and (d) for an incident pressure amplitude on the interface that is uniform in the region $|x| \leq a = 1.2$ cm and zero outside. You may want to use a super-Gaussian amplitude distribution to smooth out the edges of the beam. Identify any qualitative differences with the results in parts (c) and (d).

Far Field and Focal Plane

3-1. If $p_\omega(r, \theta, \phi)$ represents the Fraunhofer approximation of the Rayleigh integral for the field radiated by a velocity source in the plane $z = 0$, show that $\partial p_\omega/\partial\theta = 0$ at $\theta = 90°$. What is the physical explanation for this result?

3-2. The second Rayleigh integral, which applies when the source condition in the plane $z = 0$ is expressed in terms of pressure as $p = p_0(x, y)e^{-i\omega t}$, is

$$p_\omega(x, y, z) = -\frac{ikz}{2\pi}\iint_{-\infty}^{\infty} p_0(x_0, y_0)\left(1 - \frac{1}{ikR}\right)\frac{e^{ikR}}{R^2}\, dx_0 dy_0$$

where $R = [(x - x_0)^2 + (y - y_0)^2 + z^2]^{1/2}$.

(a) Obtain the Fraunhofer approximation for $p_\omega(r, \theta, \phi)$. Note that this far-field limit requires the r dependence to be e^{ikr}/r. In addition to assuming $r \gg z_0$, where z_0 is the Rayleigh distance, you should find it necessary to assume $ka > 1$, where a is the characteristic source dimension, to achieve the result for the far field. Explain the need for this additional condition.

(b) As a result of the Fraunhofer approximation you should find that $p_\omega = 0$ at $\theta = 90°$. Show that this is so, and provide a physical explanation for this result.

3-3. A circular membrane of radius a is mounted in a rigid baffle in the plane $z = 0$. A loudspeaker in the region $z < 0$ radiates sound near a natural frequency of the membrane such that the normal velocity of the membrane is $u_0 J_0(\alpha_{0n}\rho/a)e^{-i\omega t}$, where α_{0n} are the roots of $J_0(x) = 0$.

(a) Show that when the membrane is driven in its nth mode, the angular dependence of the pressure in the far field is proportional to

$$\frac{J_0(ka\sin\theta)}{\alpha_{0n}^2 - (ka\sin\theta)^2}$$

The required integral may be solved using the Papoulis table of Hankel transforms.

(b) Now construct a directivity function $D_n(\theta)$ that is normalized in such a way that $D_n(\theta_{0n}) = 1$, where $\theta_{0n} = \arcsin(\alpha_{0n}/ka)$. Obtain the normalization factor by calculating analytically the value of the expression in part (a) for $ka\sin\theta = \alpha_{0n}$.

(c) In rectangular format, plot $|D_n(\theta)|$ for the lowest five natural modes using a few different values of ka, with ka held constant for each plot. Indicate with a dot the point on each curve at which $\theta = \theta_{0n}$. You should find that there is a prominent radiation lobe with a maximum in the neighborhood of $\theta = \theta_{0n}$ that is shifted farther off axis as n is increased.

(d) Using wave vector relations, provide a physical explanation for the prominent radiation lobe near the angle θ_{0n}. Hint: What is κ_n, the wave number associated with the motion of the membrane?

3-4. The directivity function for a circular piston of radius a is

$$D(\theta) = \frac{2J_1(ka\sin\theta)}{ka\sin\theta}$$

(a) Compare the far-field beam patterns in the x-z plane resulting from steering the beam in the direction $(\theta, \phi) = (\theta_0, 0)$ by pure rotation and by linear phase shading. Make the comparison for $\theta_0 = 30°$ and $\theta_0 = 60°$, first with $ka = 10$ and then with $ka = 50$. Therefore make four plots, one for each combination of θ_0 and ka, and with a curve for both the rotated and phase shaded source on each plot.

(b) You should find that the beam pattern for the phase shaded source broadens as θ_0 is increased. Provide a physical explanation for this effect.

3-5. A rectangular piston is mounted in a rigid baffle in such a way that it rocks in rigid body motion about the y axis with velocity distribution $u_0(x/a)\mathrm{rect}(x/2a)\mathrm{rect}(y/2b)e^{-i\omega t}$, like a seesaw.

(a) Show that the far field is described by the product of the two spherical Bessel functions j_0 and j_1, one a function of ka and the other a function of kb. Hint: Recall Prob. 1-3.

(b) Show that a dipole directivity is obtained for $ka \ll 1$ and $kb \ll 1$.

3-6. A baffled rectangular membrane occupies the region $0 \le x \le a$ and $0 \le y \le b$ in the plane $z = 0$. A loudspeaker in the region $z < 0$ radiates sound near a natural frequency of the membrane, such that the normal velocity of the membrane is $u_0 \sin(m\pi x/a)\sin(n\pi y/b)e^{-i\omega t}$. The angular spectrum of the velocity distribution is

$$
\begin{aligned}
U_0(k\alpha, k\beta) &= \tfrac{1}{4}u_0 ab e^{-i[ka\alpha + kb\beta + (m+n-2)\pi/2]} \\
&\quad\times \left[\frac{\sin[(ka\alpha + m\pi)/2]}{(ka\alpha + m\pi)/2} + (-1)^{m+1}\frac{\sin[(ka\alpha - m\pi)/2]}{(ka\alpha - m\pi)/2}\right] \\
&\quad\times \left[\frac{\sin[(kb\beta + n\pi)/2]}{(kb\beta + n\pi)/2} + (-1)^{n+1}\frac{\sin[(kb\beta - n\pi)/2]}{(kb\beta - n\pi)/2}\right]
\end{aligned}
$$

where $\alpha = \sin\theta\cos\phi$ and $\beta = \sin\theta\sin\phi$ are direction cosines. The same mode shape and angular spectrum are obtained if the membrane is replaced by a simply supported thin plate.

(a) Plot $|U_0(k\alpha, k\beta)|/u_0 ab$ versus $(ka\alpha, kb\beta)$ for $b = 2a$ and $(m, n) = (1, 1)$, (1,2), (2,1), (3,1), and (10,10). Interpret the locations of the peaks and zeros. For example, you should find that the volume velocity $U_0(0, 0)$ vanishes when either m or n is an even integer. Why should this be so?

(b) Plot the angular spectrum of the pressure in the plane $z = 0$, in the dimensionless form $|P_0(k\alpha, k\beta)|/\rho_0 c_0 u_0 ab$, for the same parameters as in part (b), first with $ka = 100$ and then with $ka = 10$. Interpret the results.

3-7. A nondiffracting sound beam is produced by source pressure $p_0 J_0(\alpha\rho)e^{-i\omega t}$ for $\alpha < k$. However, this Bessel beam requires a source with infinite surface area and infinite power, unlike a Gaussian beam, the source for which, while requiring infinite surface area, does not require infinite power. This problem is motivated by the first paper to investigate the effect of finite source aperture on the radiation of a Bessel beam [Durnin et al., "Diffraction-free beams," Phys. Rev. Lett. **58**, 1499–1501 (1987)].

(a) Obtain the Fraunhofer approximation of the sound field radiated by a Bessel source truncated by a circular aperture of radius a, i.e., with pressure given by $p_0 J_0(\alpha\rho)e^{-i\omega t}$ for $\rho \le a$ and zero for $\rho > a$. The field integral may be solved using the Papoulis table of Hankel transforms. You should find that the axial amplitude in the far field is proportional to $J_1(\alpha a)$, which passes through zero repeatedly as αa is increased. How does this behavior relate to the volume velocity of the source?

(b) The Rayleigh distance is where the transition from near field to far field occurs. One definition of Rayleigh distance is where the Fraunhofer approximation of the amplitude on axis equals the source amplitude. Verify that this definition yields the familiar result $\frac{1}{2}ka^2$ for a baffled circular piston of radius a. You may use the known far-field solution for a circular piston; you need not rederive it.

(c) Following the approach in part (b), determine the Rayleigh distance z_B for the beam in part (a). Check that in the limit $\alpha \to 0$ the result $\frac{1}{2}ka^2$ is obtained as in part (b). Explain why you should expect this limiting value of z_B.

(d) Like in part (a), you should find that z_B as defined in part (b) is proportional to $|J_1(\alpha a)|$. For large αa, z_B will thus be an oscillatory function with peak amplitude proportional to $1/\sqrt{\alpha a}$. Use the peak amplitude of the asymptotic expression for $|J_1(\alpha a)|$ to estimate z_B for $\alpha a \gg 1$.

(e) Following the approach in part (b), determine the Rayleigh distance z_G for a Gaussian beam radiated by the source distribution $p_0 e^{-\rho^2/w^2} e^{-i\omega t}$ (without truncation).

(f) Now consider the results presented by Durnin et al. Curve (a) in their Fig. 1 shows a Gaussian source distribution matched to a Bessel source distribution by equating their FWHM ("full width half maximum", defined in terms of intensity). Use this matching procedure to determine the numerical value of the dimensionless quantity αw.

(g) In Fig. 3(a) the authors compare calculations of the axial fields radiated by the two sources when the Bessel source is truncated by a circular aperture of radius a. Use the results in parts (d)–(f), together with data provided by Durnin et al., to calculate the numerical value of the ratio z_B/z_G for their simulations. Does your result appear to be consistent with Fig. 3(a)? Explain.

(h) Letting W_B be the power radiated by the truncated Bessel source and W_G be the power radiated by the infinite Gaussian source, calculate the numerical value of the ratio W_B/W_G. This power ratio is the same for both acoustics and optics provided the values of αa and αw are the same.

3-8. Diagnostic medical ultrasound employs rectangular sources that are both focused and steered. This problem investigates the use of a phased array to perform these functions.

(a) For a source in the plane $z = 0$, combine the function of focusing at point $(x, y, z) = (0, 0, d)$ with that of steering in direction $(\theta, \phi) = (\theta_0, 0)$ to obtain the pressure in the focal plane given the velocity source $u = u_0(x, y)e^{-i\omega t}$ in the absence of both focusing and steering. Express your result in terms of the Fourier transform of $u_0(x, y)$. At the order of approximation used in class to investigate focusing and steering individually, when combined are the two effects independent or interdependent? In particular, does changing the steering angle alter the focusing gain?

(b) The ideal rectangular source we wish to focus and steer has a 2:1 aspect ratio, with uniform velocity u_0 in the region where $|x| \leq 2w$ and $|y| \leq w$, and zero velocity elsewhere. When focused at distance d, show that the pressure distribution produced by this source in the focal plane is described by

$$\frac{\sin(2kwx/d)}{2kwx/d}\frac{\sin(kwy/d)}{kwy/d}$$

What is the focusing gain for this source?

(c) In practice, focusing and steering are accomplished using phased arrays. Consider a rectangular array with identical elements, each of which is defined by the velocity distribution $u = u_1(x, y)e^{-i\omega t}$ when centered at the origin, $(x, y) = (0, 0)$. From the product theorem for arrays, if $u_0(x, y)$ describes an array of elements each having velocity amplitude $u_1(x, y)$, the Fourier transform of an array of these elements is $U_0(k_x, k_y) = U_1(k_x, k_y)A(k_x, k_y)$, where $A(k_x, k_y)$ is the Fourier transform of an array of points at the same locations as the elements. Use this result to write an expression for the pressure in the focal plane if an unfocused transducer array represented by $u_0(x, y)$ is focused at distance d. Beam steering is no longer considered.

(d) We now wish to approximate the ideal rectangular source in part (b) with a phased array of identical circular elements, each having the same uniform velocity amplitude u_0. Start with 2 circular elements, each having radius $a = w$, with one centered at $(x, y) = (-w, 0)$ and the other at $(x, y) = (w, 0)$. How does the focusing gain for this two-element array compare with that in part (b). Define focusing gain the same way as for an unsteered source with uniform velocity, $G = |p_\omega(0, 0, d)|/\rho_0 c_0 u_0$. Now halve the element radius to $a = \frac{1}{2}w$, such that 8 elements fit into the same rectangular space when organized as two rows of four. What is the new focusing gain? If the element size is halved again, such that $a = \frac{1}{4}w$ and 32 elements fill the rectangular space as four rows of eight, and then the size is halved again, etc., what is the general result for the focusing gain?

(e) Now consider the field in the focal plane. For arrays with increasing element density as described in part (d), compare with the result in part (b) the beam patterns in the focal plane along the x and y axes, plotted versus the dimensionless coordinates kwx/d and kwy/d, respectively. Include the first three sidelobes in your plot along each axis. What number of elements is required to obtain reasonable agreement with the result in part (b)? Is this number different for agreement along the x axis versus along the y axis? If so, why? Label your curves according to the total number of elements, i.e., $N = N_x N_y = 2$, 8, 32, 128, etc.

3-9. The focusing gain achieved with a cylindrical lens differs qualitatively from that achieved with a spherical lens. To assess this difference let the normal velocity of an asymmetric Gaussian source in the plane $z = 0$ be $u_0 e^{-x^2/a^2} e^{-y^2/b^2} e^{-i\omega t}$, where ultimately b will be taken to be large relative to a such that the source will be wide in the y direction and narrow in the x direction. Focusing by a cylindrical lens is approximated by multiplying the velocity distribution in the source plane by $e^{-ikx^2/2d}$, where d is the distance between the source and focal plane. As $\lambda \to 0$ the focus becomes a line parallel to the y axis, rather than the focal point produced by a spherical lens.

(a) Show that the sound pressure $p_\omega(x, y, z)$ at the point in the middle of the focal plane may be expressed as

$$p_\omega(0,0,d) = -\frac{ik\rho_0 c_0}{2\pi d} u_0 e^{ikd} \mathcal{F}\Big\{e^{-x^2/a^2} e^{-y^2/\tilde{b}^2}\Big\}_{k_x=k_y=0}$$

where $\tilde{b}^2 = b^2/(1 - ikb^2/2d)$. To arrive at this result follow the same procedure that was used in class, beginning with the Rayleigh integral, to obtain the field produced in the focal plane by a spherical lens.

(b) Evaluate the Fourier transform in part (a), and use the result to show that

$$\lim_{b\to\infty} \frac{|p_\omega(0,0,d)|}{\rho_0 c_0 u_0} = \sqrt{\frac{ka^2}{2d}}$$

Notice the weaker focusing gain in a cylindrical wave field in comparison with that in a spherical wave field, e.g., the gain here increases by 3 dB per octave versus 6 dB per octave for a spherical wave.