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Assignment-6

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Problem Statement:

Find the area of the circle $4x^2 + 4y^2 = 9$ which is interior to the parabola $x^2 = 4y$.

SOLUTION:

Given:

Equation of circle is

$$4x^2 + 4y^2 = 9 \tag{1}$$

Equation of Parabola is

$$x^2 = 4y \tag{2}$$

From (2) we can say that Parabola is concave towards positive y axis.

From equation (1) radius of circle is,

$$r = \frac{3}{2} \tag{3}$$

To Find

To find the intersection points and area of shaded region shown in figure

STEP-1

The given circle and parabola can be expressed as conics with parameters,
For circle,

$$\mathbf{V}_1 = 4\mathbf{I} \tag{4}$$

So,

$$\mathbf{V}_1 = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \tag{5}$$

$$\mathbf{u_1} = 0 \tag{6}$$

$$f_1 = -9 \tag{7}$$

For Parabola,

$$\mathbf{V}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{8}$$

$$\mathbf{u_2} = -\begin{pmatrix} 0 \\ 2 \end{pmatrix} \tag{9}$$

$$f_2 = 0 \tag{10}$$

STEP-2

The intersection of the given conics is obtained as

$$\mathbf{x}^\top (\mathbf{V}_1 + \mu\mathbf{V}_2)\,\mathbf{x} + 2\,(\mathbf{u}_1 + \mu\mathbf{u}_2)^\top \mathbf{x} \tag{11}$$
$$+ (f_1 + \mu f_2) = 0 \tag{12}$$

$$\mathbf{V}_1 + \mu\mathbf{V}_2 = \begin{pmatrix} \mu + 4 & 0 \\ 0 & 4 \end{pmatrix} \tag{13}$$

$$\mathbf{u}_1 + \mu\mathbf{u}_2 = -\begin{pmatrix} 0 \\ 2\mu \end{pmatrix} \tag{14}$$

$$f_1 + \mu f_2 = -9 \tag{15}$$

This conic is a single straight line if and only if,

$$\begin{vmatrix} \mathbf{V}_1 + \mu\mathbf{V}_2 & \mathbf{u}_1 + \mu\mathbf{u}_2 \\ (\mathbf{u}_1 + \mu\mathbf{u}_2)^\top & f_1 + \mu f_2 \end{vmatrix} = 0 \tag{16}$$

And,

$$\left|\mathbf{V}_1 + \mu\mathbf{V}_2\right| = 0 \tag{17}$$

Substituting equation (13),(14) and (15) in equation (16) We get,

$$\implies \begin{vmatrix} \mu + 4 & 0 & 0 \\ 0 & 4 & -2\mu \\ 0 & -2\mu & -9 \end{vmatrix} = 0 \tag{18}$$

Solving the above equation we get,

$$\mu^3 + 4\mu^2 + 9\mu + 36 = 0 \tag{19}$$

gives,

$$\mu = -4 \tag{20}$$

Thus, the parameters for a straight line can be expressed as

$$\mathbf{V} = \mathbf{V}_1 + \mu\mathbf{V}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}, \tag{21}$$

$$\mathbf{u} = \mathbf{u}_1 + \mu\mathbf{u}_2 = -\begin{pmatrix} 0 \\ 8 \end{pmatrix} \tag{22}$$

$$f = -9, \tag{23}$$

$$\implies \mathbf{D} = \mathbf{V}, \mathbf{P} = \mathbf{I} \tag{24}$$

Thus, the desired pair of straight lines are

$$\begin{pmatrix} \sqrt{|\lambda_1|} & \pm\sqrt{|\lambda_2|} \end{pmatrix} \mathbf{P}^\top (\mathbf{x} - \mathbf{c}) = 0 \tag{25}$$

$$\implies \begin{pmatrix} 0 & \pm 2 \end{pmatrix} \mathbf{x} - \mathbf{c} = 0 \tag{26}$$

$$\text{or, } \mathbf{x} = \mathbf{c} + \kappa \begin{pmatrix} \pm 2 \\ 0 \end{pmatrix} \tag{27}$$

STEP-3

The points of intersection of the line is given by,

$$L: \quad \mathbf{x} = \mathbf{q} + \kappa\mathbf{m} \quad \kappa \in \mathbb{R} \tag{28}$$

with the conic section,

$$\mathbf{x}^\top\mathbf{V}\mathbf{x} + 2\mathbf{u}^\top\mathbf{x} + f = 0 \tag{29}$$

are given by

$$\mathbf{x}_i = \mathbf{q} + \kappa_i\mathbf{m} \tag{30}$$

where,

$$\kappa_i = \frac{1}{\mathbf{m}^T\mathbf{V}\mathbf{m}} \left(-\mathbf{m}^T\left(\mathbf{V}\mathbf{q} + \mathbf{u}\right) \pm \sqrt{\left[\mathbf{m}^T\left(\mathbf{V}\mathbf{q} + \mathbf{u}\right)\right]^2 - \left(\mathbf{q}^T\mathbf{V}\mathbf{q} + 2\mathbf{u}^T\mathbf{q} + f\right)\left(\mathbf{m}^T\mathbf{V}\mathbf{m}\right)}\right) \tag{31}$$

On substituting

$$\mathbf{q} = \begin{pmatrix} 0 \\ 0.5 \end{pmatrix} \tag{32}$$

$$\mathbf{m} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \tag{33}$$

With the given Parabola,

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{34}$$

$$\mathbf{u} = -\begin{pmatrix} 2 \\ 0 \end{pmatrix} \tag{35}$$

$$f = 0 \tag{36}$$

The value of κ ,

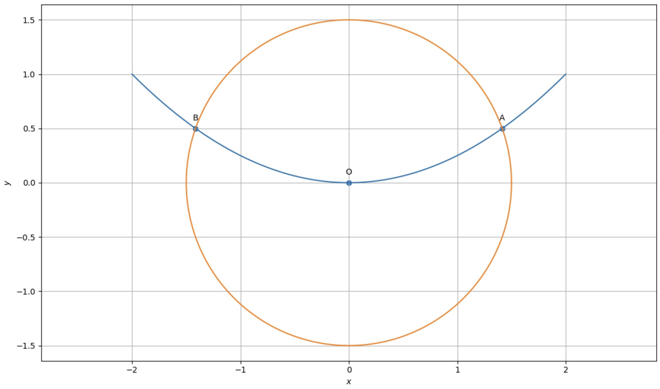
$$\kappa = \sqrt{2}, -\sqrt{2} \tag{37}$$

The points of intersection with Parabola along circle are

$$\mathbf{A} = \begin{pmatrix} \sqrt{2} \\ 0.5 \end{pmatrix} \tag{38}$$

$$\mathbf{B} = \begin{pmatrix} -\sqrt{2} \\ 0.5 \end{pmatrix} \tag{39}$$

Result



From the figure,
Total area of portion is given by,

$$A = \int_{-\sqrt{2}}^{\sqrt{2}} g(x) - f(x)\, dx \tag{40}$$

Where g(x) is area of circle and f(x) is the area of parabola around the points

$$A = \int_{-\sqrt{2}}^{\sqrt{2}} \frac{\sqrt{9 - 4x^2}}{2} - \frac{x^2}{4}\, dx \tag{41}$$

Area A is,

$$A = 3.0053609\, m^2 \tag{42}$$

Construction

Points	**coordinates**
A	$\begin{pmatrix} \sqrt{2} \\ 0.5 \end{pmatrix}$
B	$\begin{pmatrix} -\sqrt{2} \\ 0.5 \end{pmatrix}$

Download the code
Github link: Assignment-6.