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Assignment-6

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Problem Statement:

For Parabola,

Find the area of the circle $4x^2 + 4y^2 = 9$ which is interior to the parabola $x^2 = 4y$.

SOLUTION:

Given:

Equation of circle is

$$4x^2 + 4y^2 = 9 \quad (1)$$

Equation of Parabola is

$$x^2 = 4y \quad (2)$$

From (2) we can say that Parabola is concave towards positive y axis.

From equation (1) radius of circle is,

$$r = \frac{3}{2} \quad (3)$$

To Find

To find the intersection points and area of shaded region shown in figure

STEP-1

The given circle and parabola can be expressed as conics with parameters,

For circle,

$$\mathbf{V}_1 = 4\mathbf{I} \quad (4)$$

So,

$$\mathbf{V}_1 = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \quad (5)$$

$$\mathbf{u}_1 = 0 \quad (6)$$

$$f_1 = -9 \quad (7)$$

$$\mathbf{V}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (8)$$

$$\mathbf{u}_2 = -\begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad (9)$$

$$f_2 = 0 \quad (10)$$

STEP-2

The intersection of the given conics is obtained as

$$\mathbf{x}^T (\mathbf{V}_1 + \mu \mathbf{V}_2) \mathbf{x} + 2 (\mathbf{u}_1 + \mu \mathbf{u}_2)^T \mathbf{x} \quad (11)$$

$$+ (f_1 + \mu f_2) = 0 \quad (12)$$

$$\mathbf{V}_1 + \mu \mathbf{V}_2 = \begin{pmatrix} \mu + 4 & 0 \\ 0 & 4 \end{pmatrix} \quad (13)$$

$$\mathbf{u}_1 + \mu \mathbf{u}_2 = -\begin{pmatrix} 0 \\ 2\mu \end{pmatrix} \quad (14)$$

$$f_1 + \mu f_2 = -9 \quad (15)$$

This conic is a single straight line if and only if,

$$\begin{vmatrix} \mathbf{V}_1 + \mu \mathbf{V}_2 & \mathbf{u}_1 + \mu \mathbf{u}_2 \\ (\mathbf{u}_1 + \mu \mathbf{u}_2)^T & f_1 + \mu f_2 \end{vmatrix} = 0 \quad (16)$$

And,

$$|\mathbf{V}_1 + \mu \mathbf{V}_2| = 0 \quad (17)$$

Substituting equation (13), (14) and (15) in equation (16)

We get,

$$\Rightarrow \begin{vmatrix} \mu + 4 & 0 & 0 \\ 0 & 4 & -2\mu \\ 0 & -2\mu & -9 \end{vmatrix} = 0 \quad (18)$$

Solving the above equation we get,

$$\mu^3 + 4\mu^2 + 9\mu + 36 = 0 \quad (19)$$

gives,

$$\mu = -4 \quad (20)$$

Thus, the parameters for a straight line can be expressed as

$$\mathbf{V} = \mathbf{V}_1 + \mu \mathbf{V}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}, \quad (21)$$

$$\mathbf{u} = \mathbf{u}_1 + \mu \mathbf{u}_2 = \begin{pmatrix} 0 \\ 8 \end{pmatrix} \quad (22)$$

$$f = -9, \quad (23)$$

$$\Rightarrow \mathbf{D} = \mathbf{V}, \mathbf{P} = \mathbf{I} \quad (24)$$

Thus, the desired pair of straight lines are

$$(\sqrt{|\lambda_1|} \pm \sqrt{|\lambda_2|}) \mathbf{P}^\top (\mathbf{x} - \mathbf{c}) = 0 \quad (25)$$

$$\Rightarrow (0 \pm 2) \mathbf{x} - \mathbf{c} = 0 \quad (26)$$

$$\text{or, } \mathbf{x} = \mathbf{c} + \kappa \begin{pmatrix} \pm 2 \\ 0 \end{pmatrix} \quad (27)$$

STEP-3

The points of intersection of the line is given by,

$$L: \mathbf{x} = \mathbf{q} + \kappa \mathbf{m} \quad \kappa \in \mathbb{R} \quad (28)$$

with the conic section,

$$\mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \quad (29)$$

are given by

$$\mathbf{x}_i = \mathbf{q} + \kappa_i \mathbf{m} \quad (30)$$

where,

$$\kappa_i = \frac{1}{\mathbf{m}^\top \mathbf{V} \mathbf{m}} \left(-\mathbf{m}^\top (\mathbf{V} \mathbf{q} + \mathbf{u}) \pm \sqrt{[\mathbf{m}^\top (\mathbf{V} \mathbf{q} + \mathbf{u})]^2 - (\mathbf{q}^\top \mathbf{V} \mathbf{q} + 2\mathbf{u}^\top \mathbf{q} + f) (\mathbf{m}^\top \mathbf{V} \mathbf{m})} \right) \quad (31)$$

On substituting

$$\mathbf{q} = \begin{pmatrix} 0 \\ 0.5 \end{pmatrix} \quad (32)$$

$$\mathbf{m} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad (33)$$

With the given Parabola,

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (34)$$

$$\mathbf{u} = -\begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad (35)$$

$$f = 0 \quad (36)$$

The value of κ ,

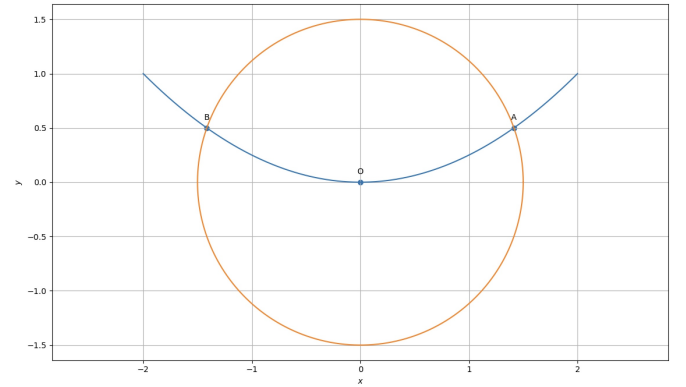
$$\kappa = \sqrt{2}, -\sqrt{2} \quad (37)$$

The points of intersection with Parabola along circle are

$$\mathbf{A} = \begin{pmatrix} \sqrt{2} \\ 0.5 \end{pmatrix} \quad (38)$$

$$\mathbf{B} = \begin{pmatrix} -\sqrt{2} \\ 0.5 \end{pmatrix} \quad (39)$$

Result



From the figure,

Total area of portion is given by,

$$A = \int_{-\sqrt{2}}^{\sqrt{2}} g(x) - f(x) dx \quad (40)$$

Where $g(x)$ is area of circle and $f(x)$ is the area of parabola around the points

$$A = \int_{-\sqrt{2}}^{\sqrt{2}} \frac{\sqrt{9-4x^2}}{2} - \frac{x^2}{4} dx \quad (41)$$

Area A is,

$$A = 3.0053609 m^2 \quad (42)$$

Construction

Points	coordinates
A	$\begin{pmatrix} \sqrt{2} \\ 0.5 \end{pmatrix}$
B	$\begin{pmatrix} -\sqrt{2} \\ 0.5 \end{pmatrix}$

Download the code
Github link: Assignment-6.