Digital Communication Through Simulation

Chirag Shah

CONTENTS

Syntax for executing C program in Linux terminal

```
gcc -o file_name file_name.c -I../include -
    ↪ lm
```

Note

include file has coeffs.f file which has to included with the execution command for C

Syntax for executing python program in Linux terminal

```
python3 file_name.py
```

Chapter 1 Basics of Probability

1.1 Important Properties of the Probability Measure

$$P(E^c) = 1 - P(E) \tag{1.1.0.1}$$

where E^c denotes the complement of E This property implies that ,

$$P(E^c) + P(E) = 1 \tag{1.1.0.2}$$

$$P(\varnothing) = 0 \tag{1.1.0.3}$$

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) \tag{1.1.0.4}$$

If two events E_1 and E_2 are mutually exclusive then ,

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) \tag{1.1.0.5}$$

If

$$E_1 \subseteq E_2 \tag{1.1.0.6}$$

Then,

$$P(E_1) \le P(E_2) \tag{1.1.0.7}$$

This says that if event E_1 is contained in E_2 then occurrence of E_1 means E_2 has occurred but the converse is not true.

1.2 Conditional Probability

Conditional probability is known as the possibility of an event or outcome happening, based on the existence of a previous event or outcome. It is calculated by multiplying the probability of the preceding event by the renewed probability of the succeeding, or conditional, event.

Let event E_1 has occurred but are actually interested in event E_2. Knowing that of E_1 has occurred changes the probability of E_2 occurring.

The probability of E_2 occurring given that event E_1 has occurred.

$$P(E_2/E_1) = \begin{cases} \frac{P(E_2 \bigcup E_1}{P(E_1)} & if P(E_1) \neq 0 \\ 0 & otherwise \end{cases} \tag{1.2.0.1}$$

1.3 Uniform and Non-Uniform distribution

The probability density function (PDF) of the uniform distribution is given by,

$$P(x) = \frac{1}{b-a} \tag{1.3.0.1}$$

For all the values

$$f(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & x \le a / x \ge b \end{cases} \tag{1.3.0.2}$$

The Cumulative distribution function (CDF) of the uniform distribution is given by,

$$F(x) = \frac{x-a}{b-a} \tag{1.3.0.3}$$

For all the values

$$F(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \le x \le b \\ 1 & x > b \end{cases} \tag{1.3.0.4}$$

The Gaussian distribution function of the Non uniform distribution is given by,

$$f(x,\mu,\sigma)=\frac{1}{\sigma\sqrt{2\pi}}\exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right) \tag{1.3.0.5}$$

Chapter 2 Two Dice

2.1 Sum of Independant Random Variables

Introduction

Let X and Y be random variables.

The distribution of their sum is given by,

$$T=X+Y \tag{2.1.0.1}$$

To determine the distribution of T we need to calculate

$$f_T(t)=P(t=t)=P(X+Y=t) \tag{2.1.0.2}$$

Which we can do by summing the joint p.m.f. over the appropriate values:

$$\sum_{(x,y):x+y=t} f(x,y) \tag{2.1.0.3}$$

Example:

To calculate the total number of bets that Xavier and Yolanda win,

By calculating $P(X+Y=T)$ for t=0,1,2,3,....,8.

The probabilities that we would need to sum for t=4 are highlighted in the joint p.m.f. table below

$$\begin{array}{cc|cccc}
 & 5 & 0 & 0 & 0 & .0238\\
 & 4 & \boxed{0} & 0 & .0795 & .0530\\
y & 3 & 0 & \boxed{.0883} & .1766 & .0294\\
 & 2 & .0327 & .1963 & \boxed{.0981} & 0\\
 & 1 & .0726 & .1090 & 0 & \boxed{0}\\
 & 0 & .0404 & 0 & 0 & 0\\
\hline
 & & 0 & 1 & 2 & 3\\
 & & & & x &
\end{array}.$$

Fig. 2.1.0.1: Joint p.m.f fot t=4

For a fixed value of t, x determines the value of y and and vice versa,

In General,

$$y=t-x \tag{2.1.0.4}$$
$$\tag{2.1.0.5}$$

Using the above equation

$$f_T(t)=\sum_x f(x,t-x) \tag{2.1.0.6}$$

This is the general equation for the p.m.f. of the sum T.

For Independent random variables

Let X and Y be independent random variables.

Then, the p.m.f. of T

$$T=X+Y \tag{2.1.0.7}$$

Its convolution of the p.m.f.s of X AND Y is

$$f_T=f_X*f_Y \tag{2.1.0.8}$$

The convolution operator (*) is defined as

$$f_T(t)=\sum_x f(x)\cdot f_Y(t-x) \tag{2.1.0.9}$$

The joint distribution is the product of the marginal distributions We have,

$$f_T(t)=f_X(x)\cdot f_Y(t-x) \tag{2.1.0.10}$$

Two dice, one blue and one grey, are thrown at the same time. The event defined by the sum of the two numbers appearing on the top of the dice can have 11 possible outcomes 2, 3, 4, 5, 6, 6, 8, 9, 10, 11 and 12. A student argues that each of these outcomes has a probability $\frac{1}{11}$. Do you agree with this argument? Justify your answer.

2.1.1 *The Uniform Distribution:* Let $X_i\in\{1,2,3,4,5,6\}, i=1,2,$ be the random variables representing the outcome for each die. Assuming the dice to be fair, the probability mass function (pmf) is expressed as

$$p_{X_i}(n)=\Pr(X_i=n)=\begin{cases}\frac{1}{6} & 1\le n\le 6\\ 0 & otherwise\end{cases} \tag{2.1.1.1}$$

The desired outcome is

$$X=X_1+X_2, \tag{2.1.1.2}$$
$$\implies X\in\{1,2,\ldots,12\} \tag{2.1.1.3}$$

The objective is to show that

$$p_X(n)\neq\frac{1}{11} \tag{2.1.1.4}$$

2.1.2 *Convolution:*

From (2.1.1.2),

$$p_X(n)=\Pr(X_1+X_2=n)=\Pr(X_1=n-X_2) \tag{2.1.2.1}$$

$$=\sum_k \Pr(X_1=n-k|X_2=k)\,p_{X_2}(k) \tag{2.1.2.2}$$

after unconditioning. $\because$ X_1 and X_2 are independent,

$$\Pr(X_1=n-k|X_2=k)$$
$$=\Pr(X_1=n-k)=p_{X_1}(n-k) \tag{2.1.2.3}$$

From (2.1.2.2) and (2.1.2.3),

$$p_X(n)=\sum_k p_{X_1}(n-k)p_{X_2}(k)=p_{X_1}(n)*p_{X_2}(n) \tag{2.1.2.4}$$

where $*$ denotes the convolution operation. Substituting from (2.1.1.1) in (2.1.2.4),

$$p_X(n)=\frac{1}{6}\sum_{k=1}^{6}p_{X_1}(n-k)=\frac{1}{6}\sum_{k=n-6}^{n-1}p_{X_1}(k) \tag{2.1.2.5}$$

$$\because p_{X_1}(k) = 0, \quad k \le 1, k \ge 6. \tag{2.1.2.6}$$

From (2.1.2.5),

$$p_X(n) = \begin{cases} 0 & n < 1 \\ \frac{1}{6}\sum_{k=1}^{n-1} p_{X_1}(k) & 1 \le n-1 \le 6 \\ \frac{1}{6}\sum_{k=n-6}^{6} p_{X_1}(k) & 1 < n-6 \le 6 \\ 0 & n > 12 \end{cases} \tag{2.1.2.7}$$

Substituting from (2.1.1.1) in (2.1.2.7),

$$p_X(n) = \begin{cases} 0 & n < 1 \\ \frac{n-1}{36} & 2 \le n \le 7 \\ \frac{13-n}{36} & 7 < n \le 12 \\ 0 & n > 12 \end{cases} \tag{2.1.2.8}$$

satisfying (2.1.1.4).

2.1.3 *The Z-transform:* The Z-transform of $p_X(n)$ is defined as

$$P_X(z) = \sum_{n=-\infty}^{\infty} p_X(n) z^{-n}, \quad z \in \mathbb{C} \tag{2.1.3.1}$$

From (2.1.1.1) and (2.1.3.1),

$$P_{X_1}(z) = P_{X_2}(z) = \frac{1}{6}\sum_{n=1}^{6} z^{-n} \tag{2.1.3.2}$$

$$= \frac{z^{-1}\left(1 - z^{-6}\right)}{6\left(1 - z^{-1}\right)}, \quad |z| > 1 \tag{2.1.3.3}$$

upon summing up the geometric progression.

$$\because p_X(n) = p_{X_1}(n) * p_{X_2}(n), \tag{2.1.3.4}$$

$$P_X(z) = P_{X_1}(z) P_{X_2}(z) \tag{2.1.3.5}$$

The above property follows from Fourier analysis and is fundamental to signal processing. From (2.1.3.3) and (2.1.3.5),

$$P_X(z) = \left\{\frac{z^{-1}\left(1 - z^{-6}\right)}{6\left(1 - z^{-1}\right)}\right\}^2 \tag{2.1.3.6}$$

$$= \frac{1}{36}\frac{z^{-2}\left(1 - 2z^{-6} + z^{-12}\right)}{\left(1 - z^{-1}\right)^2} \tag{2.1.3.7}$$

Using the fact that

$$p_X(n-k) \overset{\mathcal{H}}{\longleftrightarrow} Z P_X(z) z^{-k}, \tag{2.1.3.8}$$

$$n u(n) \overset{\mathcal{H}}{\longleftrightarrow} Z \frac{z^{-1}}{\left(1 - z^{-1}\right)^2} \tag{2.1.3.9}$$

after some algebra, it can be shown that

$$\frac{1}{36}\left[(n-1)\,u(n-1) - 2\,(n-7)\,u(n-7) + (n-13)\,u(n-13)\right] \overset{\mathcal{H}}{\longleftrightarrow} Z \frac{1}{36}\frac{z^{-2}\left(1 - 2z^{-6} + z^{-12}\right)}{\left(1 - z^{-1}\right)^2} \tag{2.1.3.10}$$

where

$$u(n) = \begin{cases} 1 & n \ge 0 \\ 0 & n < 0 \end{cases} \tag{2.1.3.11}$$

From (2.1.3.1), (2.1.3.7) and (2.1.3.10)

$$p_X(n) = \frac{1}{36}\left[(n-1)\,u(n-1) - 2\,(n-7)\,u(n-7) + (n-13)\,u(n-13)\right] \tag{2.1.3.12}$$

which is the same as (2.1.2.8). Note that (2.1.2.8) can be obtained from (2.1.3.10) using contour integration as well.

2.1.4 The experiment of rolling the dice was simulated using Python for 10000 samples. These were generated using Python libraries for uniform distribution. The frequencies for each outcome were then used to compute the resulting pmf, which is plotted in Figure 2.1.4.1. The theoretical pmf obtained in (2.1.2.8) is plotted for comparison.

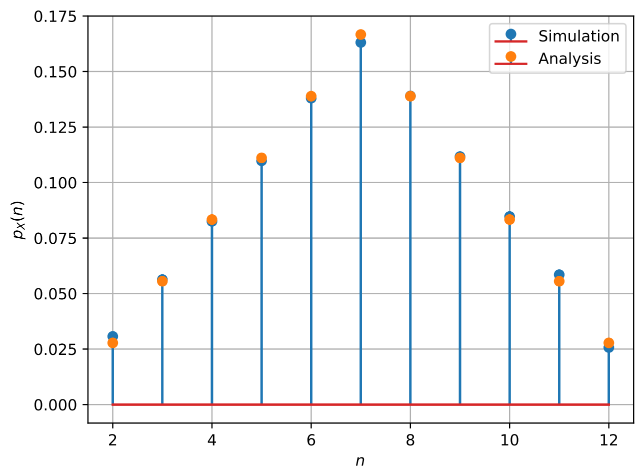

Fig. 2.1.4.1: Plot of $p_X(n)$. Simulations are close to the analysis.

2.1.5 Code of above solution

```
/codes/chapter2/two_dice.py
```

2.1.6 For executing python program in linux terminal

```
python3 two_dice.py
```

Chapter 3 Random Numbers

3.1 Uniform Random Numbers

It's just a random number where each possible number is just as likely as any other possible number.
Example : A fair die is a uniform random number generator for numbers between 1 and 6 inclusive.
The probability density function of the uniform distribution is given by,

$$P(x) = \frac{1}{b-a} \tag{3.1.0.1}$$

$$f(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & x \le a / x \ge b \end{cases} \tag{3.1.0.2}$$

The Cumulative distribution function of the uniform distribution is given by,

$$F(x) = \frac{x-a}{b-a} \tag{3.1.0.3}$$

$$F(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \le x \le b \\ 1 & x > b \end{cases} \tag{3.1.0.4}$$

Let U be a uniform random variable between 0 and 1.

3.1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files and execute the C program.

```
codes/chapter3/gen_samp.c
```

3.1.2 For executing C program in linux terminal

```
gcc -o gen_samp gen_sam.c -I../include -
    ↪ lm
```

3.1.3 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \le x) \tag{3.1.3.1}$$

Solution: The following code plots Fig. 3.1.3.1

```
codes/chapter3/cdf_plot.py
```

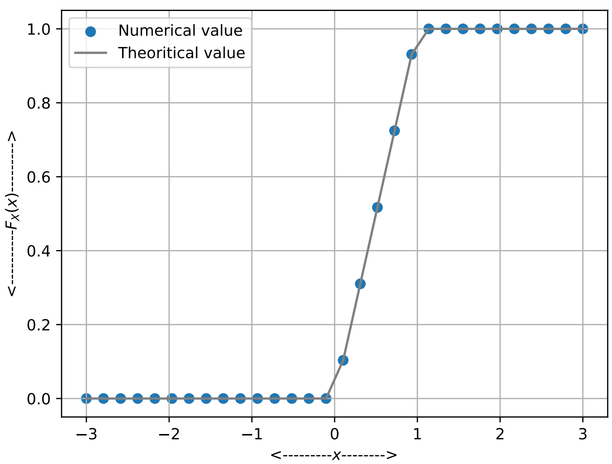

Fig. 3.1.3.1: The CDF of U

3.1.4 Find a theoretical expression for $F_U(x)$.
Solution:
$F_U(x)$ is defined as $\int_{-\infty}^{x} f_U(x)\,dx$

$$F_U(x) = \int_{-\infty}^{x} f_U(x)\,dx \tag{3.1.4.1}$$

For the random variable which is uniform throughout U, $f_U(x)$ is

$$f_U(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & elsewhere \end{cases} \tag{3.1.4.2}$$

Substituting it, we get

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 0 \end{cases} \tag{3.1.4.3}$$

3.1.5 To find Mean,
The mean of U is given by,

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{3.1.5.1}$$

3.1.6 To find variance,
The variance of U is given by,

$$\text{var}[U] = E[U - E[U]]^2 \tag{3.1.6.1}$$

Write a C program to find the mean and variance of U.
Solution: Source code is given below

```
codes/chapter3/gen_samp.c
```

Output

```
Mean is: 0.500031
Variance is: 0.083247
```

3.1.7 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{3.1.7.1}$$

Solution: For a random variable X,
Mean μ_X is given by,

$$\mu_X = \int_{-\infty}^{\infty} x dF_U(x) = E[X] \tag{3.1.7.2}$$

Substituting from (3.1.4.3) $F_U(x)$ in the above equations, we get

$$E[X] = \mu_U = \frac{1}{2} \tag{3.1.7.3}$$

Variance σ_X^2 is given by,

$$\sigma_X^2 = \int_{-\infty}^{\infty} x^2 dF_U(x) - \mu_X^2 = E\left[X^2\right] - \mu_X^2 \tag{3.1.7.4}$$

Substituting from (3.1.4.3) $F_U(x)$ in the above equations, we get

$$E\left[X^2\right] - \mu_X^2 = \sigma_U^2 = \frac{1}{12} \tag{3.1.7.5}$$

3.2 Central Limit Theorem

The central limit theorem states that whenever a random sample of size n is taken from any distribution with mean and variance, then the sample mean will be approximately normally distributed with mean and variance. The larger the value of the sample size, the better the approximation to the normal.
Formula

$$Z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} \tag{3.2.0.1}$$

3.2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{3.2.1.1}$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Solution: Source code

```
codes/chapter3/gau_cdf_pdf.c
```

Output

```
Mean is : 0.000630
Variance is : 1.000150
```

3.2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The properties of a CDF are

$$\frac{dF_X(x)}{dx} \geq 0 \tag{3.2.2.1}$$
$$F_X(\infty) = 0 \tag{3.2.2.2}$$
$$F_X(-\infty) = 1 \tag{3.2.2.3}$$

Below code for generating CDF of X

```
codes/chapter3/gau_cdf.py
```

The CDF of X is shown in fig below

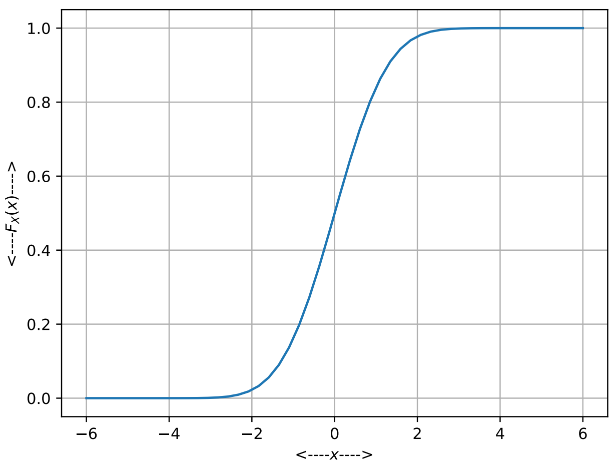

Fig. 3.2.2.1: The CDF of X

3.2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat.
The PDF of X is defined as,

$$p_X(x) = \frac{d}{dx}F_X(x) \tag{3.2.3.1}$$

What properties does the PDF have?
Solution: The properties of PDF are

$$f_X(x) \geq 0 \tag{3.2.3.2}$$
$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1 \tag{3.2.3.3}$$

Below code for generating PDF of X

```
codes/chapter3/gau_pdf.py
```

The PDF of X is shown in fig below

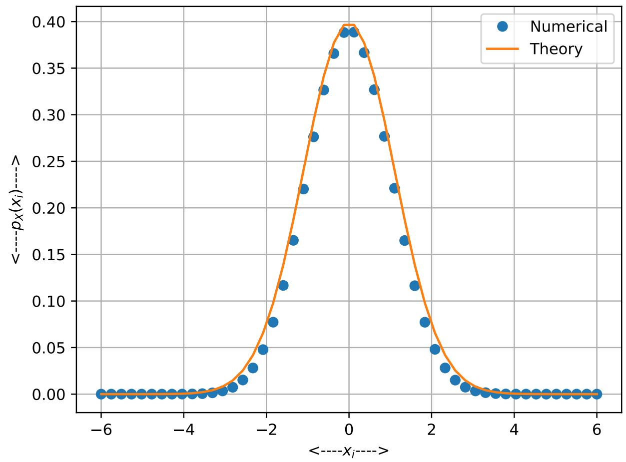

Fig. 3.2.3.1: The PDF of X

3.2.4 Find the mean and variance of X by writing a C program.
Solution: Below code gives you mean and variance of X

```
codes/chapter3/gau_cdf_pdf.c
```

Output

```
Mean is : 0.000630
Variance is : 1.000150
```

3.2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{3.2.5.1}$$

repeat the above exercise theoretically.
Solution: Substituting the below equation in (3.2.5.2),

$$E[X] = \mu_X = \int_{-\infty}^{\infty} x dF_U(x) \tag{3.2.5.2}$$

We get,

$$E[X] = \mu_X = \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)dx \tag{3.2.5.3}$$
$$\mu_X = \frac{1}{\sqrt{2\pi}}\left[-\exp\left(-\frac{x^2}{2}\right)\right]_{-\infty}^{\infty} \tag{3.2.5.4}$$
$$E[X] = \mu_X = 0 \tag{3.2.5.5}$$

Substituting the below equation in (3.2.5.2),

$$\sigma_X^2 = \int_{-\infty}^{\infty} x^2 dF_U(x) - \mu_X^2 = -\mu_X^2 \tag{3.2.5.6}$$

$$\sigma_X^2 = \int_{-\infty}^{\infty} x^2\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)dx - 0 \tag{3.2.5.7}$$
$$\sigma_X^2 = \frac{2}{\sqrt{\pi}}\int_0^{\infty} x^2\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)dx$$

Let a = $\frac{x^2}{2}$ and using gama function, We have,

$$\sigma_X^2 = \frac{2}{\sqrt{\pi}} \int_0^\infty t^{\frac{3}{2}-1} \exp\left(-t\right)\, dt \tag{3.2.5.8}$$

$$\Gamma(x) = \int_0^\infty e^{-z} z^{x-1}\, \mathrm{d}x \tag{3.2.5.9}$$

$$\sigma_X^2 = 2\frac{1}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2} = 1 \tag{3.2.5.10}$$

$$E\left[X^2\right] = \sigma_X^2 = 1$$

3.3 From Uniform to Other

3.3.1 Generate samples of

$$V = -2\ln\left(1-U\right) \tag{3.3.1.1}$$

and plot its CDF.
Solution:
Using uni.dat the CDF of V is plotted in Fig. 3.3.1.1
Code for loading and plotting graph from uni.dat

```
codes/chapter3/cdf_v.py
```

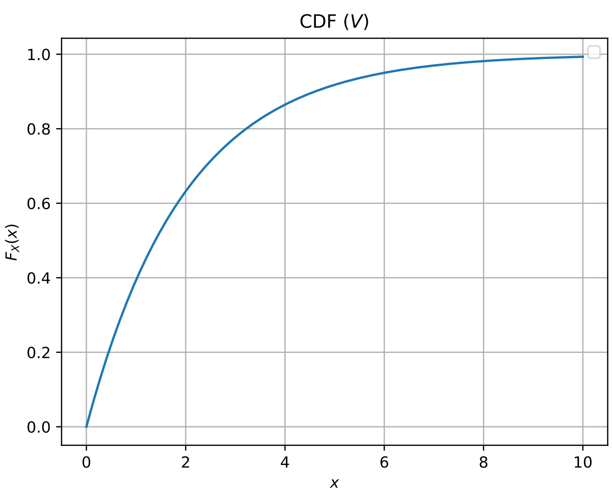

Fig. 3.3.1.1: The CDF of V

3.3.2 Find a theoretical expression for $F_V(x)$.

$$F_V(w) = P(V < w) \tag{3.3.2.1}$$

We know that ,

$$V = -2\ln\left(1-U\right) \tag{3.3.2.2}$$

Substituting the above eq in $F_V(w)$

We get,

$$F_V(w) = P(-2\ln\left(1-U\right)) < w) \tag{3.3.2.3}$$

$$= P(\ln(1-U) \geq -\frac{w}{2}) \tag{3.3.2.4}$$

Taking log to exp

$$= P(1-U \geq \exp\left(-\frac{w}{2}\right)) \tag{3.3.2.5}$$

$$= P(U < 1-\exp\left(\frac{-w}{2}\right)) \tag{3.3.2.6}$$

Now ,

$$F_V(w) = F_U(1-e^{\frac{-w}{2}}) \tag{3.3.2.7}$$

$F_U(x)$ is dedined as,

$$F_U(x) = \begin{cases} 1 & x > 0 \\ x & 0 \leq x \leq 1 \\ 0 & x < 0 \end{cases} \tag{3.3.2.8}$$

Using the above equation in (2.3.2.7)
$F_V(w)$ is defined as

$$F_V(w) = \begin{cases} 1-e^{\frac{-w}{2}} & w \geq 0 \\ 0 & w < 0 \end{cases} \tag{3.3.2.9}$$

3.4 Triangular Distribution

3.4.1 Find the theoretical expressions for the PDF and CDF of T.
Solution: The CDF and PDF of T is given by,
For PDF
Using ,

$$f_U(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & elsewhere \end{cases} \tag{3.4.1.1}$$

We have,

$$f_T(x) = \begin{cases} x & 0 \leq x \leq 1 \\ 2-x & 1 \leq x \leq 2 \\ 0 & \text{other wise} \end{cases} \tag{3.4.1.2}$$

For CDF of T
Using,

$$F_U(x) = \int_{-\infty}^{x} f_U(x)\, dx \tag{3.4.1.3}$$

We have,

$$F_T(x) = \begin{cases} 1 & x > 2 \\ 2x - \frac{x^2}{2} - 1 & 1 \leq x \leq 2 \\ \frac{x^2}{2} & 0 \leq x \leq 1 \\ 0 & \text{other wise} \end{cases} \tag{3.4.1.4}$$

3.4.2 Generate

$$T = U_1 + U_2 \tag{3.4.2.1}$$

Solution: C code is given below

```
codes/chapter3/two_uni.c
```

3.4.3 Find the CDF of T.
Solution: After running above C code push the u1.dat and u2.dat in python code given below.

```
codes/chapter3/two_uni_cdf.py
```

Below fig shows the CDF of T.

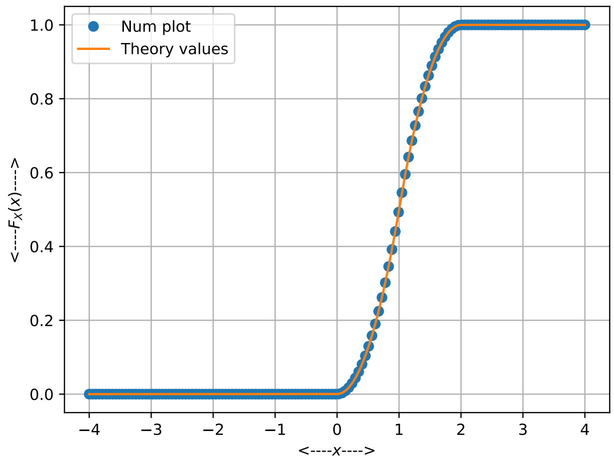

Fig. 3.4.3.1: The CDF of T

3.4.4 Find the PDF of T.
Solution: The PDF of T is plotted in Fig. 3.4.4.1 using the code below

```
codes/chapter3/two_uni_pdf.py
```

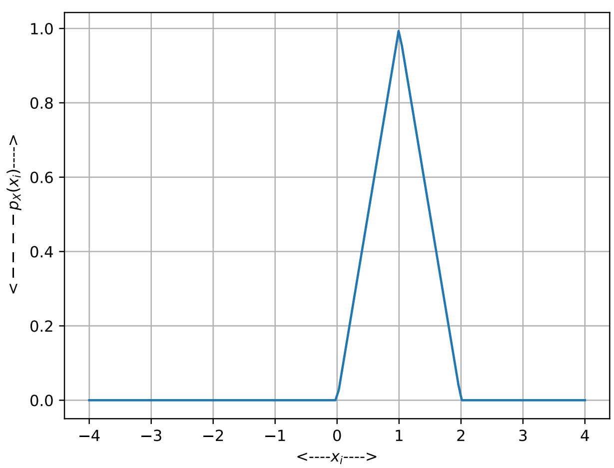

Fig. 3.4.4.1: The PDF of T

3.4.5 Verify your results through a plot.
Solution: Verified in the above figures.

Chapter 4 Maximum Likelihood Detection: BPSK

4.1 Maximum Likelihood

4.1.1 Generate equiprobable $X \in \{1, -1\}$.
Solution:
Equiprobable of X is generated using the below code
Code

```
from numpy import random
import matplotlib.pyplot as plt
import seaborn as sns
b=random.binomial(n=1, p=0.5, size=1000)
print(b)
sns.distplot(b, hist=False, label='
    binomial')
plt.show()
```

```
codes/chapter4/eq_pb.py
```

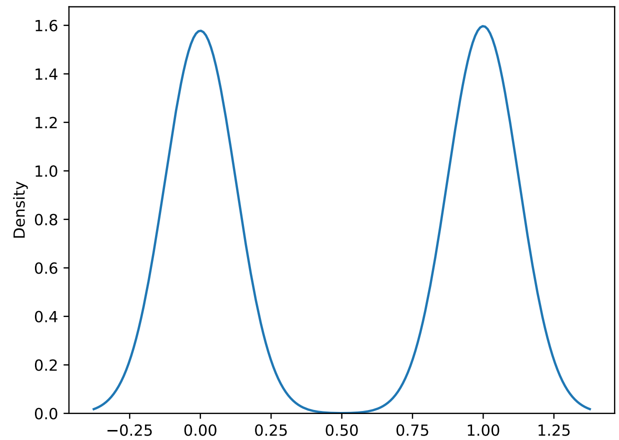

Fig. 4.1.1.1: equiprobable of X

4.1.2 Generate

$$Y = AX + N, \tag{4.1.2.1}$$

where $A = 5$ dB, and $N \sim \mathcal{N}(0, 1)$.
Solution: Using the below code y can be generated.

```
import numpy as np
from numpy import random
import matplotlib.pyplot as plt
import seaborn as sns

n = 100
X = np.random.binomial(1, 0.5,n)*2-1
N = np.random.normal(0, 1,n)

a = 5
A = (0.1*5)**10
Y = A*X + N

sns.distplot(random.normal(loc=50, scale
    =5, size=1000), hist=False, label=
    'normal')
sns.distplot(random.binomial(n=100, p
    =0.5, size=1000), hist=False,
    label='binomial')

print(Y)
plt.show()
```

Code is given below

```
codes/chapter4/value_y.py
```

4.1.3 Plot Y using a scatter plot.
Solution: Using the below python code Y can be plotted.

```
codes/chapter4/scatter_plot_y.py
```

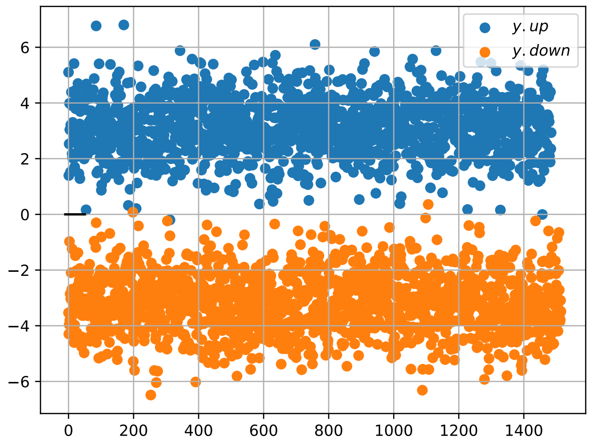

Fig. 4.1.3.1: Scatter plot of Y

4.1.4 Guess how to estimate X from Y.
Solution:

$$y \underset{-1}{\overset{1}{\gtrless}} 0 \tag{4.1.4.1}$$

By using the above equation we can estimate X from Y

4.1.5 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{4.1.5.1}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1 | X = -1\right) \tag{4.1.5.2}$$

Solution: Based on the decision rule :

$$y \underset{-1}{\overset{1}{\gtrless}} 0 \tag{4.1.5.3}$$

By using the above equation
1. To find

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{4.1.5.4}$$

$$\Pr\left(\hat{X} = -1 | X = 1\right) \tag{4.1.5.5}$$

$Y < 0$ When $\hat{X} = -1$

$$\Pr\left(Y < 0 | X = 1\right) \tag{4.1.5.6}$$

We know that $Y = AX + N$

$$\Pr\left(AX + N < 0 | X = 1\right) \tag{4.1.5.7}$$

substitute $X = 1$ we get

$$\Pr\left(A + N < 0\right) \tag{4.1.5.8}$$

Putting A to RHS we get

$$\Pr\left(\hat{X} = -1 | X = 1\right) = \Pr\left(N < -A\right) \tag{4.1.5.9}$$

Similarly,

2. To find

$$P_{e|1} = \Pr\left(\hat{X} = 1 | X = -1\right) \tag{4.1.5.10}$$

$$\Pr\left(\hat{X} = 1 | X = -1\right) \tag{4.1.5.11}$$

$Y > 0$ When $\hat{X} = 1$

$$\Pr\left(Y > 0 | X = -1\right) \tag{4.1.5.12}$$

We know that $Y = AX + N$

$$\Pr\left(AX + N > 0 | X = -1\right) \tag{4.1.5.13}$$

substitute $X = -1$ we get

$$\Pr\left(-A + N > 0\right) \tag{4.1.5.14}$$

Putting A to RHS we get

$$\Pr\left(\hat{X} = 1 | X = -1\right) = \Pr\left(N > A\right) \tag{4.1.5.15}$$

Now Since , $N \sim \mathcal{N}(0, 1)$,

$$P_{e|0} = P_{e|1} = \Pr\left(N > A\right) \tag{4.1.5.16}$$

4.1.6 Find P_e assuming that X has equiprobable symbols.
Solution:
By using the Q-function

$$Q(x) = \frac{1}{\sqrt{2\pi}} . \frac{1}{2} \int_x^\infty e^-u^2 \ du. \tag{4.1.6.1}$$

and Using,

$$P_e = \left(\Pr\left(X = 1\right) P_{e|1}\right) + \left(\Pr\left(X = -1\right) P_{e|0}\right) \tag{4.1.6.2}$$

Here by question X is equiprobable So,

$$P_r(X = \pm 1) = \frac{1}{2} \tag{4.1.6.3}$$

Substituting here,

$$P_e = \frac{1}{2}(P_{e|0} + P_{e|1}) \tag{4.1.6.4}$$

Substituting it we have,

$$P_e = \Pr\left(A < N\right) \tag{4.1.6.5}$$

The below equation can be written using Q-Function.

$$Q(A) = P_e \tag{4.1.6.6}$$
$$P_e = Q(A) \tag{4.1.6.7}$$

4.1.7 Verify by plotting the theoretical P_e with respect to A from 0 to 10 dB.
Solution: The theoretical P_e is plotted in Fig. 4.1.7.1, along with numerical estimations from generated samples of Y. The below code is used for the plot,

```
codes/chapter4/theo_pe.py
```

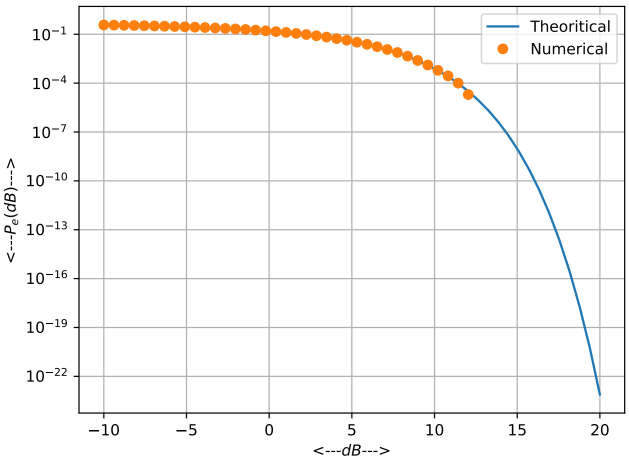

Fig. 4.1.7.1: $P_e(db)$ versus db plot

4.1.8 Now, consider a threshold δ while estimating X from Y. Find the value of δ that maximizes the theoretical P_e.

Solution:

Using decision rule,

$$y \underset{-1}{\overset{1}{\gtrless}} \delta \tag{4.1.8.1}$$

We have,

$$P_{e|0} = \Pr\left(\hat{X} = -1|X = 1\right) \tag{4.1.8.2}$$

$$\Pr\left(\hat{X} = -1|X = 1\right) \tag{4.1.8.3}$$

$Y < 0$ When $\hat{X} = -1$

$$\Pr\left(Y < 0|X = 1\right) \tag{4.1.8.4}$$

We know that $Y = AX + N$

$$\Pr\left(AX + N < 0|X = 1\right) \tag{4.1.8.5}$$

substitute $X = 1$ we get

$$\Pr\left(A + N < 0\right) \tag{4.1.8.6}$$

Putting A to RHS we get

$$\Pr\left(\hat{X} = -1|X = 1\right) = \Pr\left(N < -A\right) \tag{4.1.8.7}$$

Now replacing zero as δ We get,

$$\Pr\left(Y < \delta|X = 1\right) \tag{4.1.8.8}$$
$$\Pr\left(A + N < \delta\right) \tag{4.1.8.9}$$
$$\Pr\left(N < -A + \delta\right) \tag{4.1.8.10}$$
$$Q(A - \delta) \tag{4.1.8.11}$$

Therefore

$$P_{e|1} = Q(A - \delta) \tag{4.1.8.12}$$

Similarly,

$$P_{e|1} = \Pr\left(\hat{X} = 1|X = -1\right) \tag{4.1.8.13}$$

$$\Pr\left(\hat{X} = 1|X = -1\right) \tag{4.1.8.14}$$

$Y > 0$ When $\hat{X} = 1$

$$\Pr\left(Y > 0|X = -1\right) \tag{4.1.8.15}$$

We know that $Y = AX + N$

$$\Pr\left(AX + N > 0|X = -1\right) \tag{4.1.8.16}$$

substitute $X = -1$ we get

$$\Pr\left(-A + N > 0\right) \tag{4.1.8.17}$$

Putting A to RHS we get

$$\Pr\left(\hat{X} = 1|X = -1\right) = \Pr\left(N > A\right) \tag{4.1.8.18}$$

Similarly, replacing zero as δ
We have,

$$P_{e|1} \Pr\left(\hat{X} = 1|X = -1\right) \tag{4.1.8.19}$$
$$\Pr\left(Y > \delta|X = -1\right) \tag{4.1.8.20}$$
$$\Pr\left(N > A + \delta\right) \tag{4.1.8.21}$$
$$Q(A + \delta) \tag{4.1.8.22}$$

Therefore

$$P_{e|0} = Q(A + \delta) \tag{4.1.8.23}$$

Using the below equation

$$P_e = \frac{1}{2}(P_{e|0} + P_{e|1}) \tag{4.1.8.24}$$

Using the above $P_{e|0}$ and $P_{e|1}$ values P_e is written as,

$$P_e = \frac{1}{2}Q(A + \delta) + \frac{1}{2}Q(A - \delta) \tag{4.1.8.25}$$

Using the integral for Q-function from (4.1.6.1),

$$Q(x) = \frac{1}{\sqrt{2\pi}}.\frac{1}{2}\int_x^\infty e^-u^2\ du. \tag{4.1.8.26}$$

P_e can be written as,

$$P_e = k(\int_{A+\delta}^\infty \exp\left(-\frac{u^2}{2}\right) du + \int_{A-\delta}^\infty \exp\left(-\frac{u^2}{2}\right) du) \tag{4.1.8.27}$$

where k is a constant
By Differentiating the above equation (4.1.9.10) wrt δ , we get

$$\frac{dP_e}{d\delta} = \exp\left(-\frac{(A+\delta)^2}{2}\right) - \exp\left(-\frac{(A-\delta)^2}{2}\right) d\delta. \tag{4.1.8.28}$$

By equating it to zero we get,

$$\exp\left(-\frac{(A+\delta)^2}{2}\right) - \exp\left(-\frac{(A-\delta)^2}{2}\right) = 0$$
$$-\exp\left(\frac{(A+\delta)^2 - (A-\delta)^2}{2}\right) = 1$$
$$\exp\left(-2A\delta\right) = 1$$

Therefore

$$\delta = 0$$

For $\delta = 0$ P_e is maximum.

4.1.9 Repeat the above exercise when

$$p_X(0) = p \tag{4.1.9.1}$$

Solution:
We know that

$$P_e = (\Pr(X = 1) P_{e|1}) + (\Pr(X = -1) P_{e|0}) \tag{4.1.9.2}$$

Now $P_{e|1}$ and $P_{e|0}$ is defined as,

$$P_{e|1} = Q(A + \delta) \tag{4.1.9.3}$$

$$P_{e|0} = Q(A - \delta) \tag{4.1.9.4}$$

Since non equiprobabl X is defined as

$$\Pr(X = 1) = (1 - p) \tag{4.1.9.5}$$

$$\Pr(X = -1) = p \tag{4.1.9.6}$$

From the above equation and Since X is not equiprobable, P_e is given by,

$$P_e = (1 - p)P_{e|1} + pP_{e|0} \tag{4.1.9.7}$$

$$P_e = (1 - p)Q(A + \delta) + pQ(A - \delta) \tag{4.1.9.8}$$

Using the integral for Q-function

$$Q(x) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{2} \int_x^\infty e^-u^2 \, du. \tag{4.1.9.9}$$

From (4.1.9.10) P_e can be defined as P_e can be written as,

$$P_e = k(\int_{A+\delta}^\infty \exp\left(-\frac{u^2}{2}\right) du + \int_{A-\delta}^\infty \exp\left(-\frac{u^2}{2}\right) du) \tag{4.1.9.10}$$

Using the above equation we can write for $p_X(0) = p$

$$P_e = \left(k((1 - p) \int_{A+\delta}^\infty \exp\left(-\frac{u^2}{2}\right) du\right) + \left(p \int_{A-\delta}^\infty \exp\left(-\frac{u^2}{2}\right) du\right) \tag{4.1.9.11}$$

k is a constant.
Differentiate P_e wrt δ and equate to zero
We have

$$\frac{dP_e}{d\delta} = (1 - p) \exp\left(-\frac{(A + \delta)^2}{2}\right) - p \exp\left(-\frac{(A - \delta)^2}{2}\right) d\delta. \tag{4.1.9.12}$$

Following the same steps as in problem 4.1.8, δ for maximum P_e evaluates to,

$$\delta = \frac{1}{2A} \ln\left(\frac{1}{p} - 1\right) \tag{4.1.9.13}$$

Equating the above equation to zero

$$(1 - p) \exp\left(-\frac{(A + \delta)^2}{2}\right) - (p) \exp\left(-\frac{(A - \delta)^2}{2}\right) = 0$$

$$\exp\left(-\frac{(A + \delta)^2 - (A - \delta)^2}{2}\right) = \frac{p}{(1 - p)}$$

$$\exp(-2A\delta) = \frac{p}{(1 - p)}$$

Therefore

$$\delta = \frac{1}{2A} \log\left(\frac{1}{p} - 1\right)$$

Chapter 5 Transformation of Random Variables

5.1 Gaussian to Other

5.1.1 Let $X_1 \sim \mathcal{N}(0, 1)$ and $X_2 \sim \mathcal{N}(0, 1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{5.1.1.1}$$

Solution:
Below code gives CDF of V.

```
codes/chapter5/sum_cdf
```

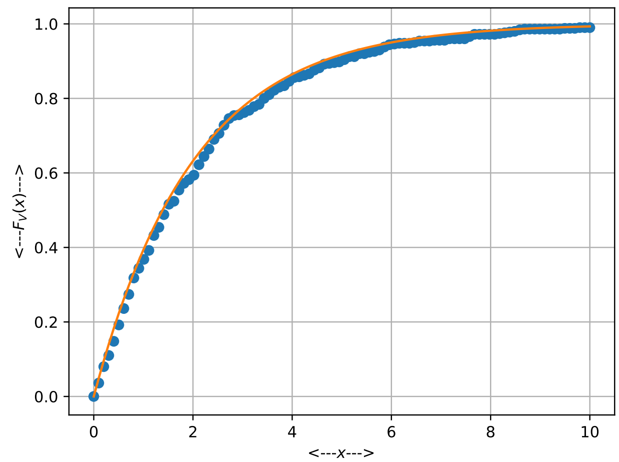

Fig. 5.1.1.1: CDF of V

Below code gives PDF of V.

```
codes/chapter5/sum_pdf
```

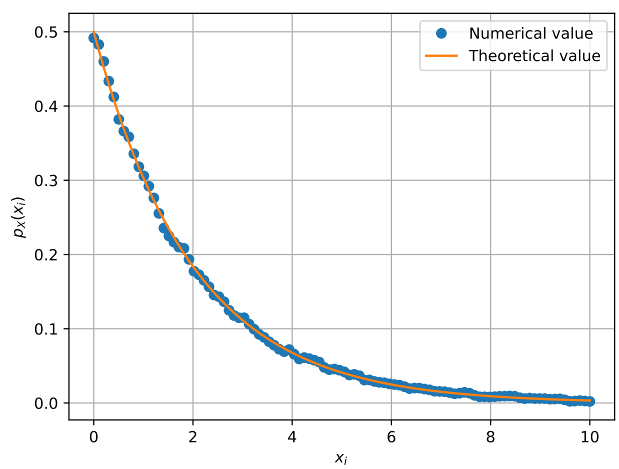

Fig. 5.1.1.2: PDF of V

5.1.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \tag{5.1.2.1}$$

find α.
Solution:
Below code gives CDF of V with different alfa values.

```
codes/chapter5/value_alfa.py
```

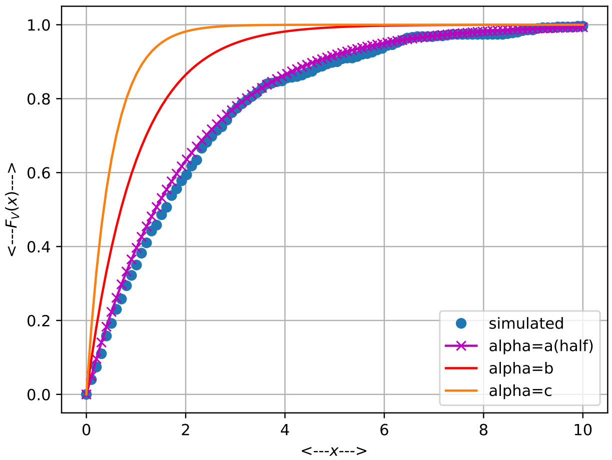

Fig. 5.1.2.1: PDF of V

From the above figure $\alpha = 0.5$.

5.1.3 Plot the CDF and PDF of

$$A = \sqrt{V} \tag{5.1.3.1}$$

Solution:
Below code gives you CDF of A.

```
codes/chapter5/sq_root_cdf.py
```

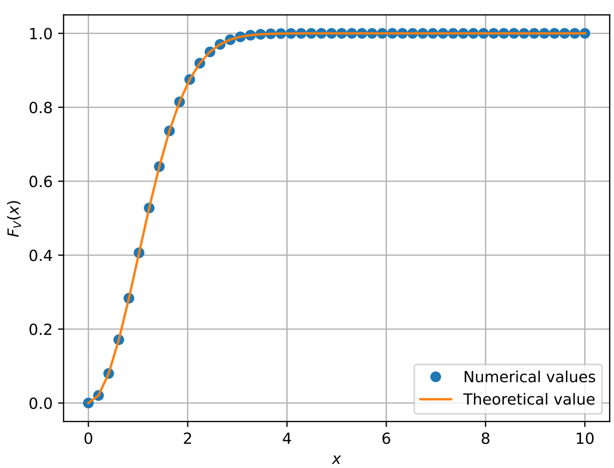

Fig. 5.1.3.1: CDF of A

Below code gives you CDF of A.

```
codes/chapter5/sq_root_cdf.py
```

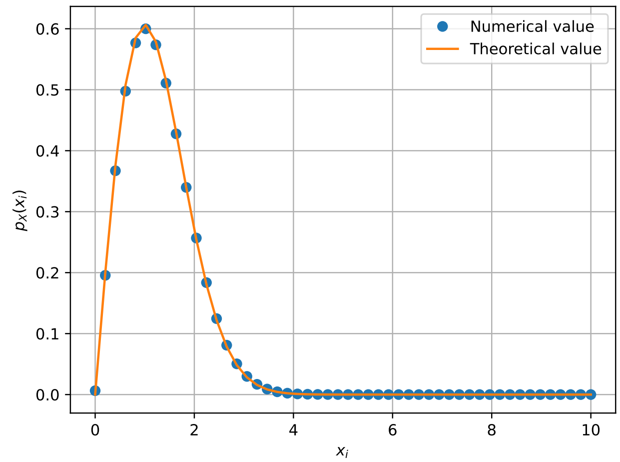

Fig. 5.1.3.2: PDF of A

5.2 Conditional Probability

5.2.1 Plot

$$P_e = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{5.2.1.1}$$

for

$$Y = AX + N, \tag{5.2.1.2}$$

where A is Raleigh with $E\left[A^2\right] = \gamma, N \sim \mathcal{N}(0,1), X \in (-1,1)$ for $0 \le \gamma \le 10$ dB.
Solution: In below figure the dots are required values
Below code gives the plot of P_e

```
codes/chapter5/value_pe.py
```

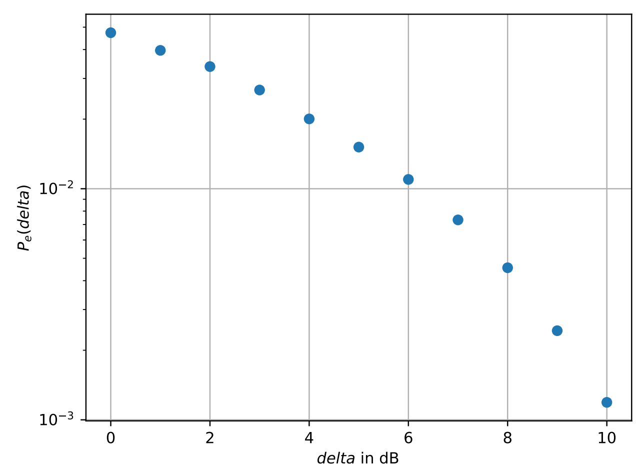

Fig. 5.2.1.1: Value Of P_e

5.2.2 Assuming that N is a constant, find an expression for P_e. Call this $P_e(N)$
Solution:
The value of P_e is given by,

$$P_e = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{5.2.2.1}$$

$$\hat{X} = \begin{cases} -1 & N < -A \\ 1 & N > A \end{cases} \tag{5.2.2.2}$$

For $\hat{X} = -1$ $Y < 0$ always and $Y = AX + N$

$$P_e = \Pr\left(AX + N < 0 | X = 1\right) \tag{5.2.2.3}$$

$$P_e = \Pr\left(A + N < 0\right) \quad \text{for X=1} \tag{5.2.2.4}$$

$$P_e = \Pr\left(A < -N\right) \tag{5.2.2.5}$$

The P_e is defind wrt constant N

$$P_e(N) = \begin{cases} F_X(N) & N \geq 0 \\ 0 & N < 0 \end{cases} \tag{5.2.2.6}$$

For random variable X with $E\left[X^2\right] = \gamma$, CDF is given by

$$F_X(x) = 1 - \exp\left(-\frac{X^2}{\gamma}\right) \quad \text{for } x \geq 0 \tag{5.2.2.7}$$

Substituting (5.2.2.7) in (5.2.2.6),

$$P_e(N) = \begin{cases} 1 - \exp\left(-\frac{N^2}{\gamma}\right) & N \geq 0 \\ 0 & N < 0 \end{cases} \tag{5.2.2.8}$$

or

When $N < 0$

$$P_e(N) = \int_{-\infty}^{N} F_X(x)\,dx \int_{-\infty}^{0} dx + \int_{0}^{N} F_X(x)\,dx \tag{5.2.2.9}$$

$$P_e(N) = \int_{0}^{N} \frac{x}{\sigma^2} \exp\left(\frac{-x^2}{2\sigma^2}\right) dx \tag{5.2.2.10}$$

Now P_e(N) is defined as,

$$P_e(N) = \begin{cases} 1 - \exp\left(-\frac{N^2}{2\sigma^2}\right) & N \geq 0 \\ 0 & N < 0 \end{cases} \tag{5.2.2.11}$$

5.2.3 For a function g,

$$E\left[g(X)\right] = \int_{-\infty}^{\infty} g(x) p_X(x)\,dx \tag{5.2.3.1}$$

Find $P_e = E\left[P_e(N)\right]$.

Solution:

Using $P_e(N)$ from (5.2.2.6)

$$P_e(N) = \begin{cases} 1 - \exp\left(-\frac{N^2}{\gamma}\right) & N \geq 0 \\ 0 & N < 0 \end{cases} \tag{5.2.3.2}$$

and We know that,

$$P_N(x) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right) \tag{5.2.3.3}$$

$$\begin{aligned} P_e = E\left[P_e(N)\right] &= \int_{-\infty}^{\infty} P_e(x) p_N(x)\,dx \\ &= \int_{0}^{\infty} \left(1 - e^{-\frac{x^2}{\gamma}}\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx \end{aligned}$$

$$P_e = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\frac{x^2}{2}}\,dx \tag{5.2.3.4}$$

By integrating the above equation.
We get

$$P_e = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\gamma}{2+\gamma}}$$

5.2.4 Plot P_e in problems 5.2.1 and 5.2.3 on the same graph w.r.t γ. Comment.

Solution:

Below code gives you CDF of A.

```
codes/chapter5/value_pe_del.py
```

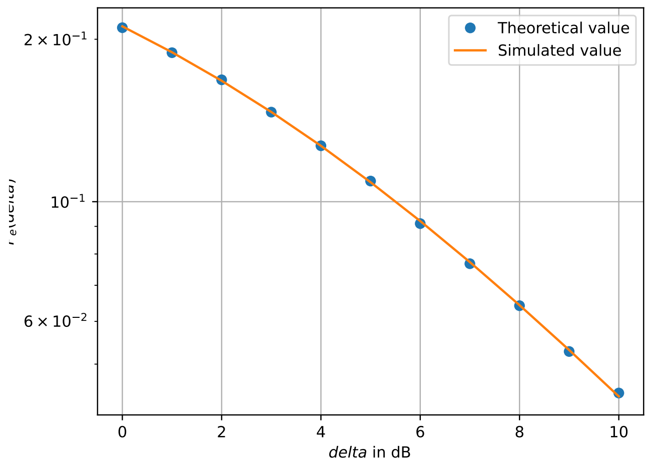

Fig. 5.2.4.1: P_e versus γ