“Box, Box, Box” - Applying Game Theory to the Singapore Grand Prix 2023

40.316 Game Theory | Singapore University of Technology and Design

Team 11 “Bwoah” Project Report 2024

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Executive Summary

This report examines the use of game theory in strategic planning at the 2023 Formula One Singapore Grand Prix, focusing on pre-race and adaptive in-race strategies under conditions like safety car deployments. It mainly explores two strategies: Medium-Hard (MH) with an ideal single pit stop at Lap 32, and Medium-Hard-Medium (MHM) with two stops at Laps 25 and 44 to utilize fresher medium tyres late in the race. A safety car at Lap 20 and a virtual safety car at Lap 44 required on-the-fly strategic adjustments. Mercedes, employing the MHM strategy, pitted during both interruptions, effectively using the periods to minimize time loss and maximize the benefit of new tyres. This move saw Mercedes finish 12 seconds ahead of their predicted time, showcasing the value of flexible, responsive strategy planning. In contrast, Ferrari's plan for a single stop was disrupted by the early safety car, compelling an unplanned pit stop that resulted in finishing 16 seconds slower than anticipated. This highlights the crucial role of adaptability and quick decision-making in Formula One, emphasizing that strategic plans should account for dynamic factors like safety car periods to improve competitive outcomes.

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Introduction

The world of Formula One, often perceived merely as cars racing around a track, transcends mere speed and driver skill. Rather, it is a realm where the prowess of engineers, coupled with strategic decisions dictates the line between victory and defeat. Diverse track configurations, from the narrow city streets of Singapore and Monaco to the expansive layouts of Belgium and Austria, imbue races with strategic complexity. Overtaking becomes a calculated maneuver, transforming each race into a battleground of wit and strategy. Our project delves into the intricate game theory underpinning these strategy-led contests.

Central to Formula One's strategic landscape is the array of tyre compounds provided by Pirelli, the sport's official tyre sponsor. Ranging from the swift soft compound to the resilient hard compound, teams face a strategic dilemma balancing speed and durability, particularly on demanding tracks such as Singapore. Each team receives 13 sets of tyres - comprising 8 softs, 3 mediums, and 2 hards, for the weekend's 5 sessions. Regulation dictates the utilization of at least two tyre compounds during the race, setting the stage for strategic tyre management.

A pivotal aspect of race strategy is the pit stop, a meticulously choreographed battle of speed and precision. Standard pit stops take roughly 30 seconds, yet the advent of a safety car or virtual safety car can truncate this duration to 20 seconds. While Pirelli gives a broad pit stop strategy, teams must fine-tune their approach to align with nuanced race conditions.

Within this strategic theater, our project spotlights the dynamics between two drivers of the sport: Carlos Sainz from Ferrari and Lewis Hamilton from Mercedes. In the recent race at Singapore in 2023, Ferrari found themselves with one set each of medium and hard tyres, while Mercedes boasted an additional set of mediums. While the conventional strategy dictated a single stop, transitioning from mediums to hards, Mercedes opted for a daring second pitstop. This strategic gambit aimed to re-equip the car with the faster medium tyres, facilitating overtaking maneuvers in pursuit of victory.

In essence, Formula One is a symphony of speed, strategy, and innovation, where the interplay of factors such as track design, tyre selection, and pit stop tactics can tilt the scales of victory. Our exploration into this intricate dynamic, particularly in the context of Singapore, sheds light on the strategic maneuver that defines success in this high-stakes sport. In the following segment, we delve deeper into the intricacies of the tyre selection and management, unraveling the strategic calculus that underpins Formula One's most compelling races.

Singapore Grand Prix: Tyre Strategy

The Singapore Grand Prix, part of the Formula 1 World Championship, is known for its demanding street circuit that puts a significant emphasis on strategy, especially when it comes to tyre choices. The race takes place at the Marina Bay Street Circuit, characterized by its high humidity, heat, and the fact that it's a night race, all of which add layers of complexity to the tyre strategy.

Typically, the ideal tyre strategy in Singapore varies significantly depending on a range of factors including the car's performance characteristics, the team's strategic philosophy, and, crucially, the race conditions (such as safety car periods which are quite common in Singapore due to the tight nature of the circuit).

1. Understanding tyre degradation

There are a total of five compounds of tyres in Formula One. For each Grand Prix, three compounds are chosen out of the five by considering factors such as the rate of tyre degradation on the particular track relative to other Grand Prix tracks, as well as sustainability and weather conditions. The three compounds of tyres will then be classified as either ‘Soft’, ‘Medium’ or ‘Hard’. The softest compound of the set usually degrades the fastest but provides the fastest speed advantage in the initial laps, and vice versa for the harder compounds.

Table 1: Typical advantage by tyre age and compound (Not adapted to Singapore)

Tyre compound	Time compared to baseline New Soft (sec per lap)
New Soft	0
New Medium	-0.4
New Hard	-0.7
Used Soft	-1
Used Medium	-1.3
Used Hard	-1.7

2. Results from Singapore Grand Prix 2023

The race began with Carlos Sainz Jr. of Ferrari on pole position while Lewis Hamilton of Mercedes started at 5th position. Logan Sargeant's crash led to a safety car period during Lap 20, during which both Mercedes and Ferrari drivers chose to pit to change out their Medium compound tyres to new sets of Hard compounds.

The race saw more incidents, including Esteban Ocon's retirement at Lap 44 that triggered a virtual safety car. Mercedes took advantage by pitting both drivers to change their Hard compounds for a set of new Medium compound tyres. As the race neared its end, Sainz strategically helped McLaren driver, Lando Norris maintain DRS, making it challenging for Russell and Hamilton to overtake. Russell's late crash helped Hamilton to third.

Making use of open-source public data via the fastf1 library on Python, we were able to extract the following information which were relevant to our analysis of both Mercedes' and Ferarri's tyre strategy.

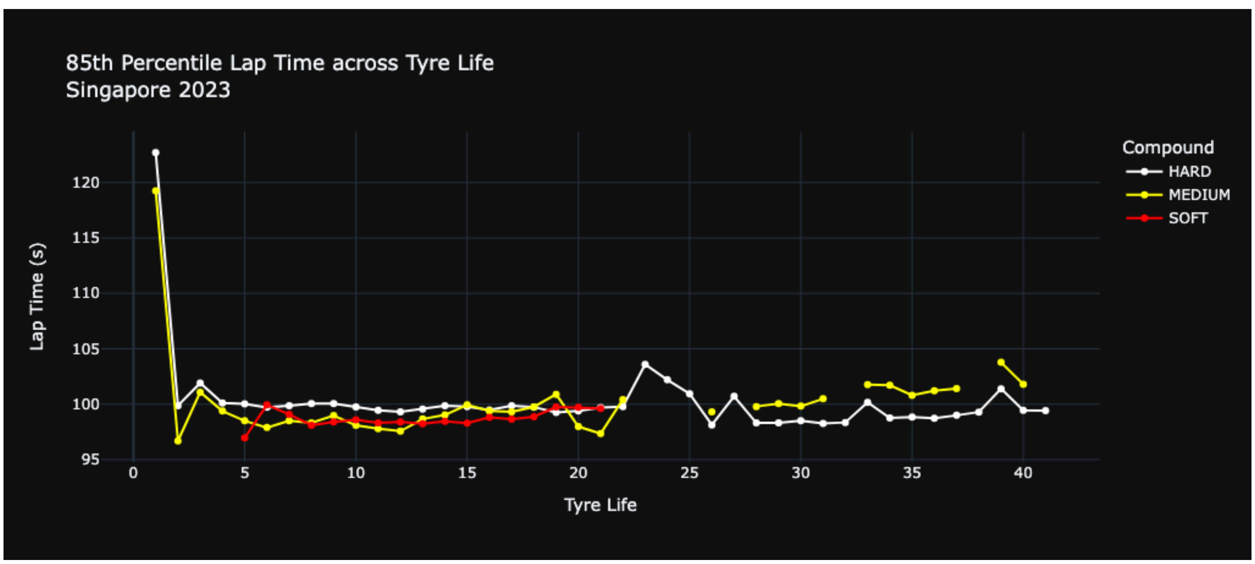

Figure 1*: Actual lap times of the 85th percentile using different tyre compounds*

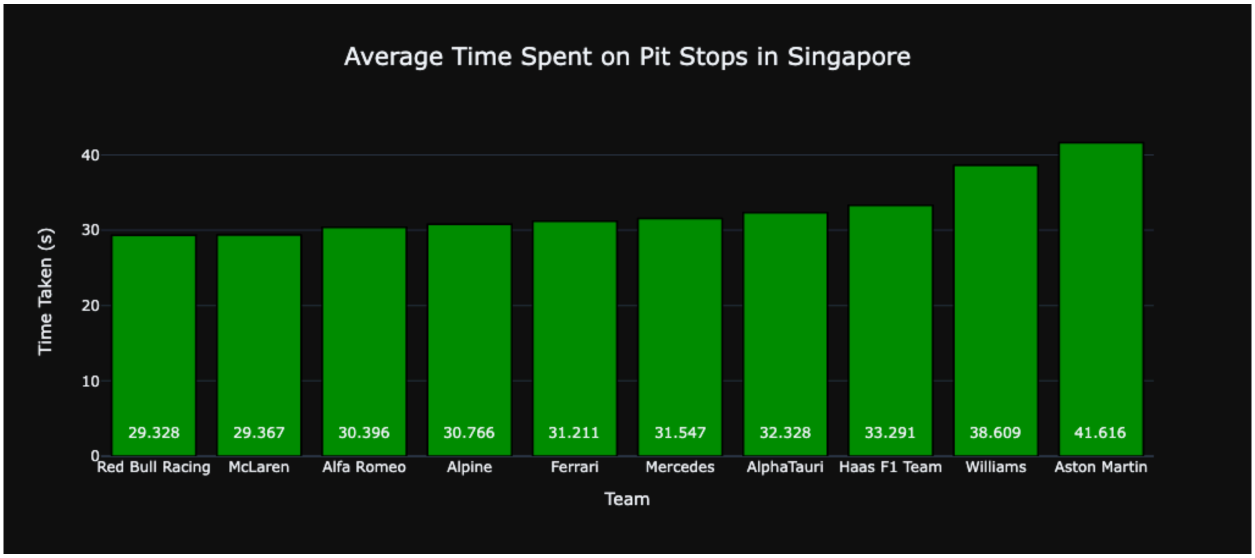

Figure 2*: Average pit stop time at Singapore 2023*

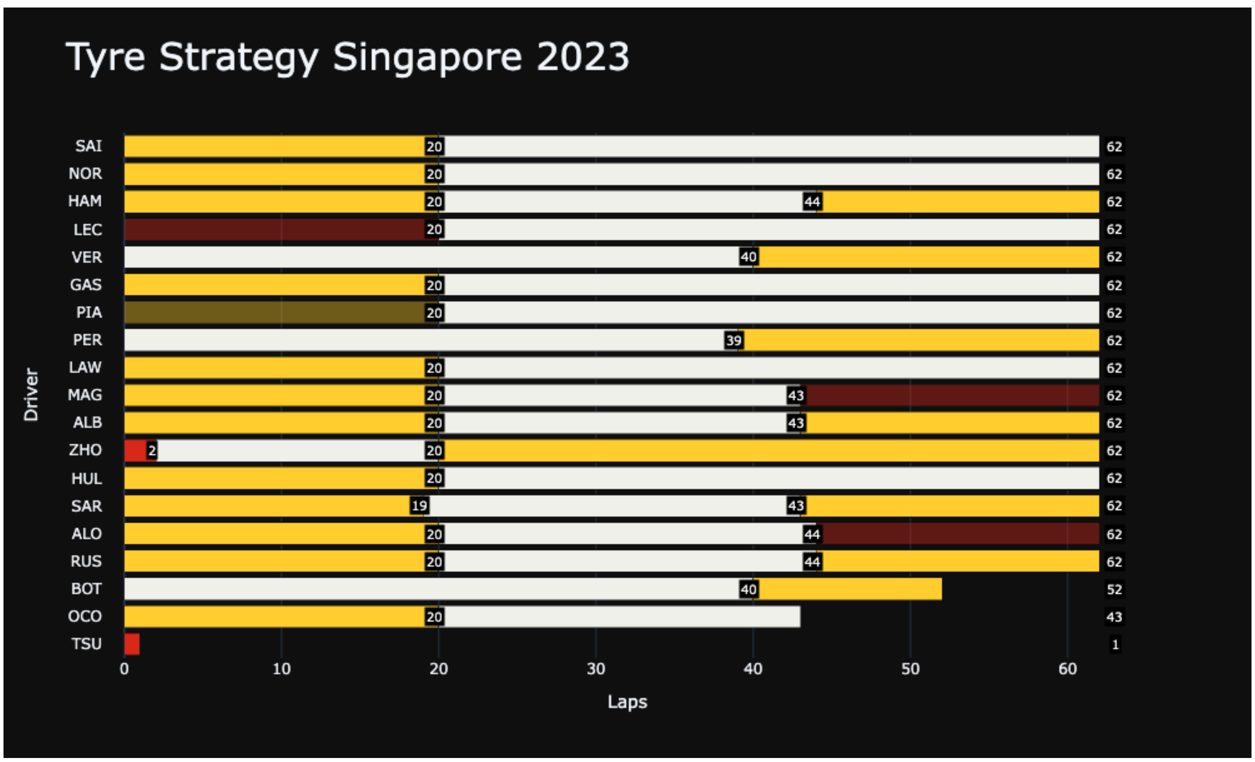

Figure 3*: Tyre strategy of all drivers in Singapore 2023*

3. Modeling Mercedes' Two-Stop Strategy using Game Theory

The following analysis will follow Race winner, Carlos Sainz from Ferrari and Podium second runner-up, Lewis Hamilton from Mercedes as our example players.

3.1 Game set-up

Formulating it as a two-player/two-team game, we assume that (1) the race is won by the driver who crosses the finish line the fastest, and (2) overtaking has no time cost. Additionally:

1. Normal pitstop takes 30s
2. Virtual Safety Car pit stop takes 20s
3. Safety Car pit stop takes 0s
4. Race Start tyres are cold, with tyre age 0 to account for a slow start
5. New H/M/S tyres at Pitstop have the following times (H<101.5s, M<100s, S<99s)
6. Used S Tyres have tyre age 20

Using data of the 85th percentile of lap times across all three tyre compounds, we attempt to first model the impact of tyre degradation on lap times and examine pit stop strategies, providing insights for optimal tyre selection and timing in racing. The following equations model the relationship between lap time and tyre age for each compound:

(New) Soft tyres: $T_{soft}(x) = 0.01063253x^2 - 0.2048096x + 99.321814$

(New) Medium tyres: $T_{medium}(x) = 0.0137390x^2 - 0.544534x + 103.666386$

(New) Hard tyres: $T_{hard}(x) = 0.00934275x^2 - 0.495842x + 105.265685$

For tyres placed during a pit stop, where tyre performance is influenced by reduced vehicle weight due to fuel consumption, which results in a fuel-corrected tyre age calculated by:

Soft Tyre Age after Pit: $T_{soft, Pit}(x) = T_{soft}(x + 2)$

Medium Tyre Age after Pit: $T_{medium, Pit}(x) = T_{medium}(x + 9)$

Hard Tyre Age after Pit: $T_{hard, Pit}(x) = T_{hard}(x + 11)$

3.2 Key findings

By modeling the game setup as per the above parameters, we were able to derive the following insights and key findings:

1. *General tyre performance*: From Figure 6 and Figure 7, we can observe that soft tyres deliver the quickest laps but degrade fastest. This makes them ideal for short bursts of speed, while hard tyres, though slower, last longer and are better for sustained periods with fewer pit stops.

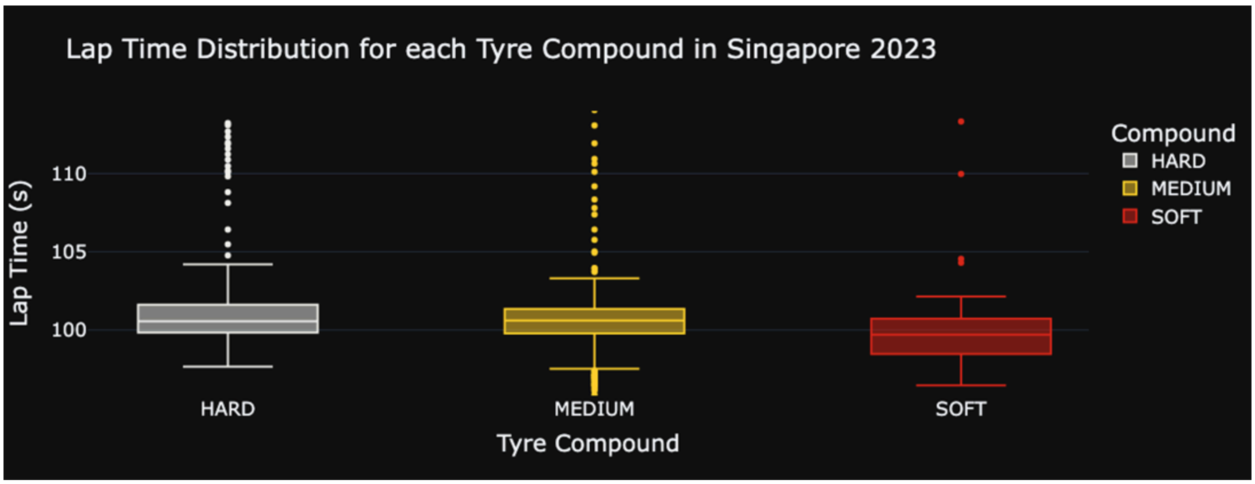

Figure 4: *Lap time distribution for each tyre compound in Singapore 2023*

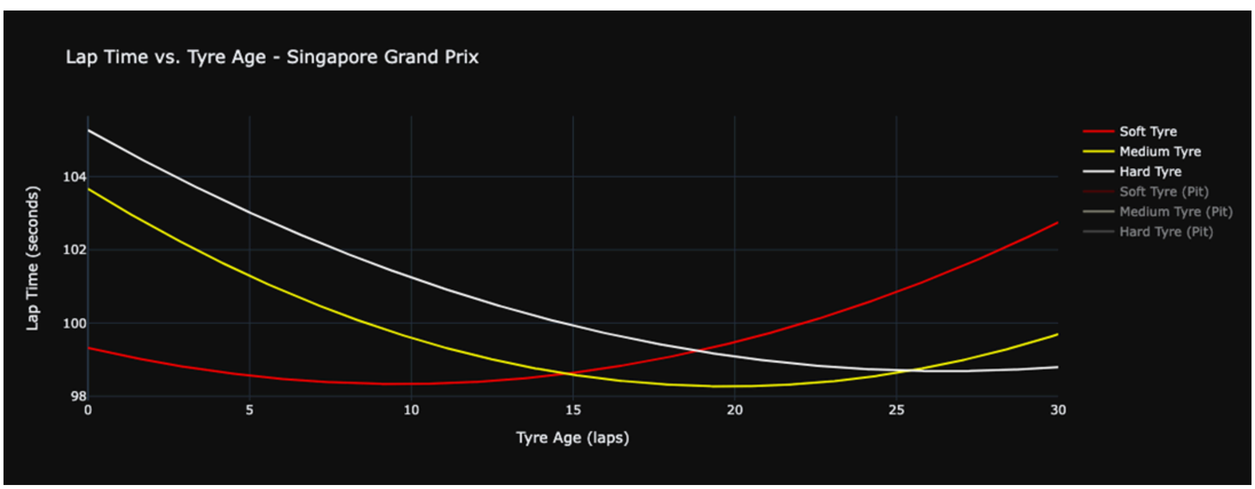

Figure 5: *Fuel-corrected tyre degradation and its effect on lap time*

2. *Impact of pitting*: Replacing worn tyres with new ones (regardless of the compound) generally results in faster lap times (Figure 8). This improvement is not solely due to the fresh tyre's performance but also the reduced weight of the car as fuel is consumed. Consequently, cars tend to perform better post-pit stop, highlighting the strategic importance of pit stop timing and tyre selection throughout the race.

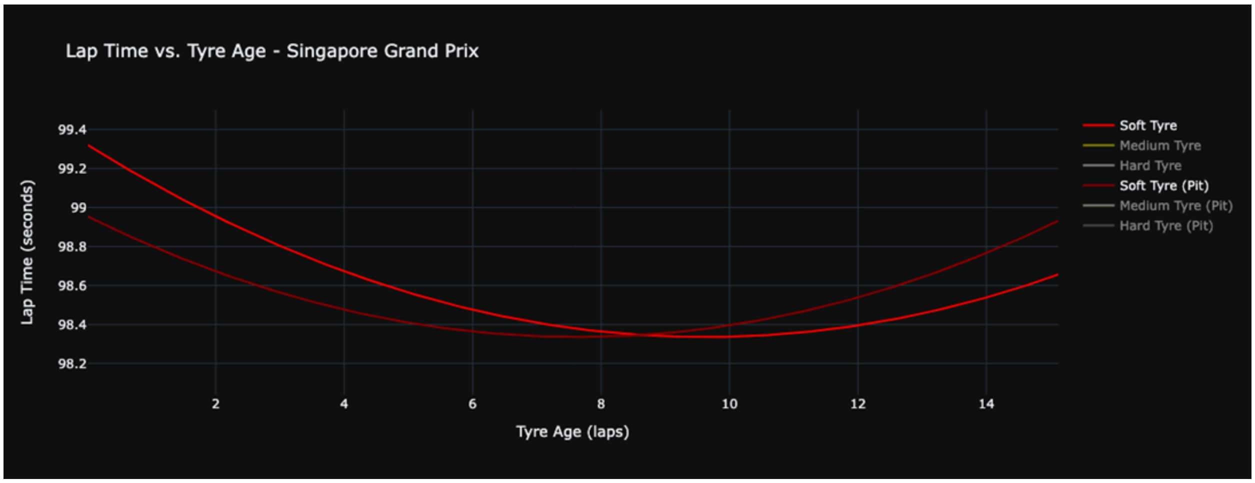

Figure 6*: Difference in lap time using new and used soft tyre compounds*

3.3 Pre-race strategy

The Tyre availability for both teams on the Race Day are given in Table 2. Given these allocations, both teams have meticulously outlined two potential strategies to navigate through the race's challenges effectively.

Table 2: Tyres Availability on Race Day during Singapore Grand prix 2023

	MERCEDES	**FERRARI**
New Medium	2	1
New Hard	1	1
Used Soft	4	4

3.3.1. Strategy 1: Medium-Hard Tyre Strategy

The first strategy under consideration is the Medium-Hard tyre strategy, where drivers would start on Medium tires before switching to Hard tires for the latter part of the race. This strategy hinges on an optimization function $R_1(L)$ aimed at minimizing total race time, with the crucial step of solving for L, the lap to switch from Medium to Hard tires, by setting the first derivative with respect to L to zero.

$$Minimize\ R_1(L) = \sum_{L=1}^{62} T_{Medium}(L) + T_{PitStop} + T_{Hard_{Pit}}(62 - L)$$

Let's start by summing the equations for $T_{Medium}(x)$ and $T_{Hard_{Pit}}(x)$ from 0 to L

$$\sum_{x=1}^{L} T_{Medium}(x) = 0.00457967492347389L^3 - 0.265397280795867L^2 + 103.396409260413L$$

$$\sum_{x=1}^{L} T_{Hard_{Pit}}(x) = 0.00311425018543565L^3 + 0.140479536208261L^2 + 100.798298456162L$$

Back to optimization function $R_1(L)$

$$R_1(L) = 0.00457967492347389L^3 - 0.265397280795867L^2 + 103.396409260413L + 20 + 0.00311425018543565(62 - L)^3 + 0.140479536208261(62 - L)^2 + 100.798298456162(62 - L)$$

Solve for $\frac{dR_1}{dL} = 0$ and get the Positive value of L

```python
def R1(L):
    term1 = 0.00457967492347389*L**3 - 0.265397280795867*L**2 + 103.396409260413*L
    pit1 = 30
    term2 = 0.00311425018543565*(62-L)**3 - 0.140479536208261*(62-L)**2 + 100.798298456162*(62-L)
    return term1 + pit1 + term2

# Initial guess
x0_single = np.array([0])

# Bounds for L
bounds_single = [(0, 62)]

# Minimize the new function with constraints
res_single = minimize(R1, x0_single, bounds=bounds_single, method='SLSQP')
```

```python
res_single.x # Optimal Lap to Pit
```

```
array([32.47342934])
```

```python
res_single.fun # Total Race Time
```

```
6198.516537328564
```

Figure 7: *Code to derive Strategy 1*

Result from analysis of Strategy 1:

- **Optimal Lap to Pit** $L = 32$
- **Total Race Time** $R_1(32) = 6198.5871$ **Seconds**

3.3.2. Strategy 2: Medium-Hard-Medium Tyre Strategy

The second strategy, Medium-Hard-Medium, proposes starting and finishing the race on Medium tyres, with a stint on Hard tyres in the middle. This approach requires solving for two variables, L_1 and L_2, representing the optimal laps to switch from Medium to Hard and then back to Medium, respectively. This involves setting partial derivatives with respect to L_1 and L_2 to zero and solving the equations to find the optimal points.

$$Minimize\ R_2(L) = \sum_{L=1}^{62} T_{Medium}(L1) + T_{PitStop} + T_{Hard_{Pit}}(L2) + T_{Medium}(L1) + T_{PitStop} + T_{Medium_{Pit}}(62 - L1 - L2)$$

Let's start by summing the equations for $T_{Medium_{Pit}}(x)$ from 0 to L

$$\sum_{x=1}^{L} T_{Medium_{Pit}}(x) = 0.004579674923473 89L^3 + 0.141746057862072L^2 + 99.7321192124913L$$

Back to optimization function $R_2(L)$

$$R_2(L) = 0.00457967492347389{L_1}^3 - 0.265397280795867{L_1}^2 + 103.396409260413L_1 + 20 + 0.00311425018543565(L_2)^3 + 0.140479536208261(L_2)^2 + 100.798298456162(L_2) + 20 + 0.00457967492347389(62 - L_1 - L_2)^3 + 0.141746057862072(62 - L_1 - L_2)^2 + 99.7321192124913(62 - L_1 - L_2)$$

Solve for $\frac{\partial R_2}{\partial L_1} = 0$ and $\frac{\partial R_2}{\partial L_2} = 0$; Get the Positive Value for L_1 and L_2

```
def R2(L1, L2):
    term1 = 0.00457967492347389*L1**3 - 0.265397280795867*L1**2 + 103.396409260413*L1
    pit1 = 30
    term2 = 0.00311425018543565*L2**3 - 0.140479536208261*L2**2 + 100.798298456162*L2
    pit2 = 30
    term3 = 0.00311425018543565*(62-L1-L2)**3 - 0.140479536208261*(62-L1-L2)**2 + 100.798298456162*(62-L1-L2)

    return term1 + pit1 + term2 + pit2 + term3

# Define a wrapper function for R_correct_separate that takes a single argument (array of L1 and L2)
# This is needed because minimize function expects the function to have a single argument
def R_wrapper(L):
    L1, L2 = L
    return R2(L1, L2)

# Initial guess
x0 = np.array([30, 30])

# Bounds for L1 and L2
bounds = [(0, 62), (0, 62)]

# Minimize the function with constraints
res = minimize(R_wrapper, x0, bounds=bounds, method='SLSQP')
```

```
res.x # Optimal Lap to Pit
```

```
array([25.43720776, 18.28182363])
```

```
res.fun # Total Race Time
```

```
6223.390994913271
```

***Figure 8**: Code to derive Strategy 2*

Result from analysis of Strategy 2:

- **Optimal Lap to Pit** $L_1 = 25$ **and** $L_1 + L_2 = 44$
- **Total Race Time** $R_1(32) = 6223.4259$ **Seconds**

3.3.3. Optimal pregame race Strategy

The preliminary analysis suggests that the Medium-Hard tyre strategy is superior. This finding aligns with the strategy's popularity in the Singapore Grand Prix, indicating a tactical advantage in the race's unique conditions.

Table 3: Ideal tyre strategy for Carlos Sainz and Hamilton

Driver	**Strategy**
Carlos Sainz (SAI/FER)	Sainz's use of one set of Medium tyres followed by a switch to Hard tyres is a classic example of a *one-stop* strategy. This approach suggests that his team aimed for track position and hoped that the tyre degradation would allow them to maintain a competitive pace till the end of the race.
Lewis Hamilton (HAM/MER)	Hamilton's strategy of using two sets of Medium tyres and one set of Hard tyres is indicative of a *two-stop* strategy. This approach suggests that his team might have been preparing for more aggressive tyre management, higher degradation levels, or aiming to exploit the car's pace in different race phases. A two-stop strategy in Singapore can be beneficial if a safety car situation arises or if the tyre degradation is higher than expected, allowing for fresher tyres towards the end of the race and potentially faster lap times.

3.4. Actual race strategy

By retrospection, the timing of the safety car and virtual safety car worked to Mercedes's advantage, as it fell within the ideal pit windows for Mercedes. The Safety Car on Lap 20 resulted in cars bunching up, and removed all the time advantage between the leading driver and the rest of the grid. This allowed Hamilton to close the gap to the race leader, Sainz. It also allowed for a free pitstop without a loss of race time. Thus by Lap 21, both Sainz and Hamilton were on similar race pace and tyre compounds.

The appearance of the Virtual Safety Car on Lap 44 left both Mercedes and Ferrari to make a decision as seen in Table 4. The decision making at Lap 44 by both drivers and teams can be modeled by the following strategy options and their respective payoffs in the Pure Nash Equilibrium.

Table 4: Available strategy options and reward functions to Mercedes and Ferrari

Mercedes Strategy options	**Ferrari strategy options**
One-stop M1 (MH): $R1(L1)$	One-stop F1 (MH): $R1(L1)$
Two-stops M2 (MHM): $R2(L1, L2)$	Two-stops F2 (MHS(Used)): $R3(L1, L2)$
Two-stops M3 (MHS(Used)): $R3(L1, L2)$	

3.4.1. Perfect Information Pure Nash application to decision making at Lap 44

From Table 5 and Table 6 below, both drivers seek to minimize their total race time (i.e., payoff). By opting to switch to a 2-stop M2 strategy (MHM) by pitting for new Medium tyres helps Mercedes save at least 28s of race time compared to a one-stop M1 strategy. However, opting to switch to a 2-stop strategy (F2) by pitting for used Soft tyres is a disadvantage for Ferrari, adding an additional 35 seconds of race time compared to a one-stop strategy. By plotting it as a one-shot perfect information Nash Equilibrium, we see that Mercedes going for a 2-stop M2 (MHM) strategy is strictly dominant over the one-stop M1 (MH) strategy and two-stop M3 (MHS(Used)) strategy, and this explains why Mercedes stuck with M2.

Likewise the two-stop F2 (MHS(Used)) strategy is strictly dominated by one-stop F1 (MH) strategy for Ferrari. While the one-stop F1 (MH) strategy is 28 s slower than Mercedes' two-stop M2 (MHM) strategy, the best response for Ferrari will therefore be to minimize the time gap, and therefore stick to the one-stop F1 (MH) strategy regardless (Table 5, 6).

This results in a one-shot pure Nash equilibrium at (M2, M1), with a payoff of (6187, 6215) seconds for Mercedes and Ferrari respectively. Purely based on race pace, Mercedes is set to win the race at Singapore 2023. When contrasted to the ideal one-stop MH pre-game strategy with a payoff of (6199, 6199), Mercedes has been dealt a faster timing, while Ferrari's total race time increased overall due to their respective decisions to pit at the respective laps.

Table 5: Pure Nash equilibrium payoff matrix

	F1	**F2**
M1	$(R1(20),\ R1(20))$	$(R1(20),\ R3(20, 24))$
M2	$(R2(20, 24),\ R1(20))$	$(R2(20, 24),\ R3(20, 24))$
M3	$(R3(20, 24),\ R1(20))$	$(R3(20, 24),\ R3(20, 24))$

Table 6: Pure Nash equilibrium payoff matrix in numbers

	F1	**F2**
M1	(6215, **6215**)	(6215, 6250)
M2	(**6187**, **6215**)	(**6187**, 6250)
M3	(6250, **6215**)	(6250, 6250)

However, using a pure Nash Equilibrium to model the decision making is limited for the following reasons. First, the driver order is sequential with leaders and followers. The decisions made by the leading team or driver can influence the sequential decisions made by followers, who rationally attempt to reduce any time advantage made by the leading driver. Second, the race is dynamic and uncertain circumstances such as the appearance of a Safety Car or unfavorable weather may probabilistically introduce different scenarios that changes all team's tyre strategy. Unfortunately, most existing game theory-related Formula 1 literature stops here (Filatov, 2022; Kaiser, 2021; Whittle, 2012). A recent paper by Bonomi et al (2023), explored the use of evolutionary genetic algorithms (GA) to optimize race strategies. However, it does not dive deep into the rationale of decision making at each lap of the race. Hence, we are seeking to expand upon the analysis with data visualization and dynamic sequential extensive form games.

In addition, the Pure Nash model does not account for why both Mercedes and Ferrari decided to pit as the Safety Car appeared at Lap 20, even though they would have achieved a faster total race time by pitting at Laps 25 and Laps 32 respectively.

3.4.2. Dynamic sequential extensive form game application to Lap 20

As discussed in Section 3.3 of pre-game strategy, Mercedes' one-stop M1 (MH) strategy and Ferrari's one-stop F1 (MH) strategy were both team's best-case strategy. In this section, we will consider the decision as to why both Mercedes decided to switch to a two-stop M2 (MHM) strategy at Lap 20 as well as why Ferrari decided to pit at Lap 20, even though Ferrari would have achieved a faster total race time by pitting at Lap 32.

In the case of both the two-stop M2 (MHM) and one-stop (MH) strategy (Figure FR), the introduction of a safety car will always reduce their respective total race times (Figure 9, 10).

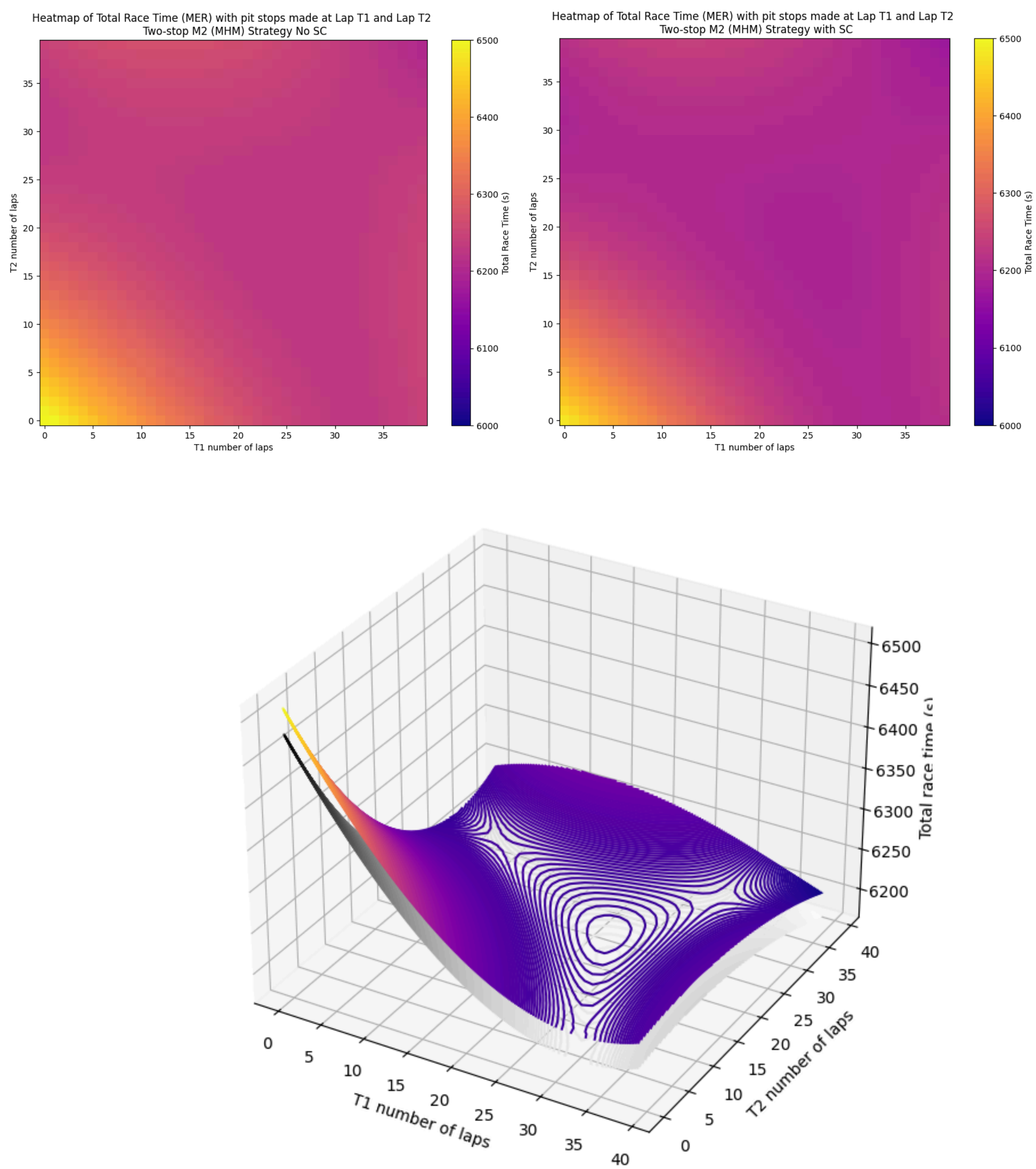

Figure 9*: Safety car will always result in a faster total race time across all laps (MHM)*

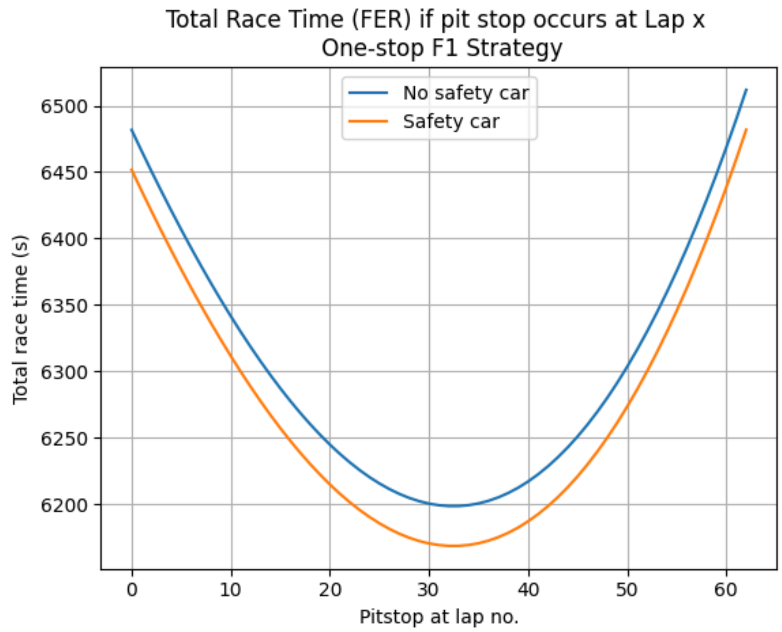

Figure 10: *Safety car will always result in a faster total race time across all laps (MH)*

By finding the total time difference between Ferrari and Mercedes Race time based on the deployment of the one-stop F1 and two-stop M2 strategy, we derive the following binary heatmap (Figure 11). Each sheet represents the lap at which Ferrari decides to pit to execute its one-stop F1 strategy. For each heatmap, the x-axis represents the lap at which Mercedes executes its first pitstop and the y-axis represents the lap that Mercedes executes its second pitstop in the two-stop M2 strategy. The black squares are outcomes in which Ferrari has a faster total race time, and non-black otherwise.

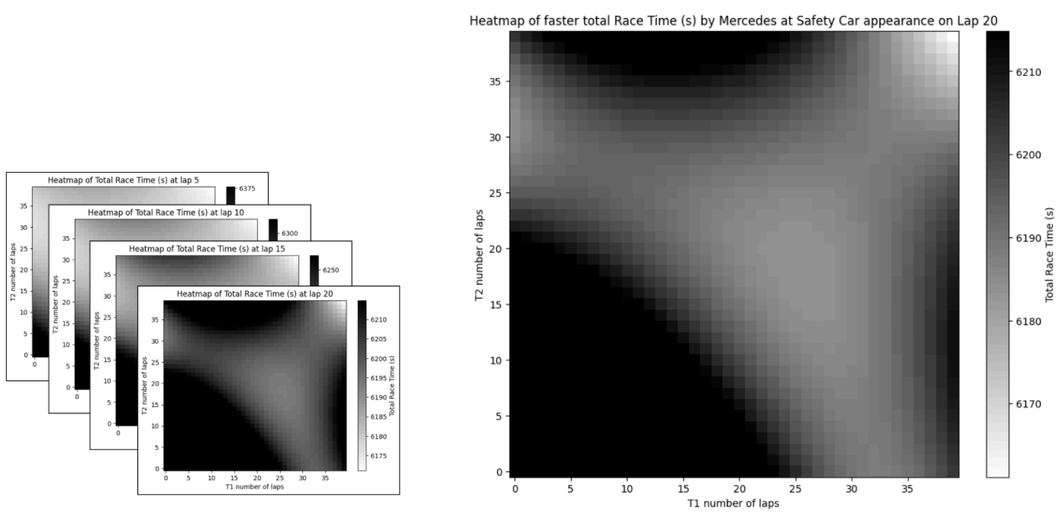

Figure 11*: Heatmap of total race time domination by Ferrari (black) and Mercedes (non-black)*

By plotting the total race time difference between Ferrari and Mercedes (Figure 12), we note the presence of a point of inflection at Lap 32 where Ferrari will achieve the fastest total race time over Mercedes. Ferrari will definitely want to pit at Lap 32 to minimize their total race time. In fact, if no safety car appears throughout the race, Ferrari can pit between Laps 23 and 43 and still achieve an overall faster race time regardless of when Mercedes chooses to pit for the two-stop strategy.

We can also observe that the introduction of the Safety Car will always provide Mercedes a larger second stop pit window with more winning opportunities deeper into the race. However, if there is a safety car appearance and Ferrari chooses to pit at Lap 32, Mercedes would have still lost the race despite having a more flexible tyre strategy.

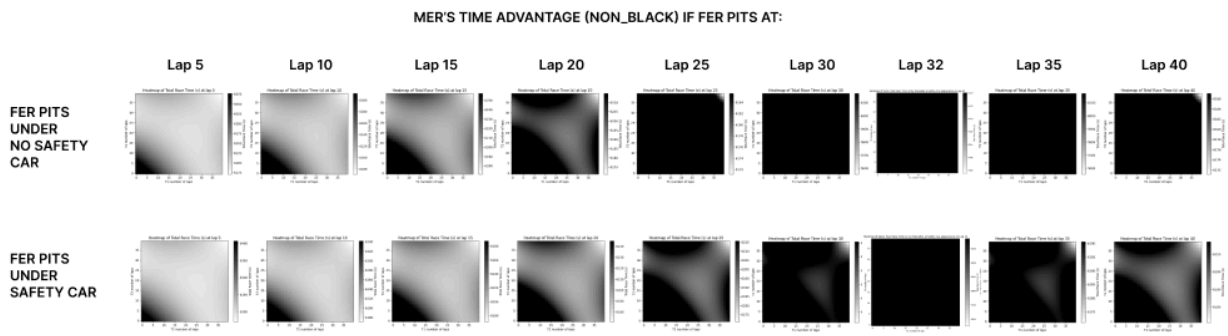

Figure 12: *Changes to MER's time advantage depending on when FER pits*

This means that as the race progresses without a safety car appearance, the optimal pit window that provides Mercedes an advantage with the two-stop M2 strategy is getting smaller. Should Mercedes have wanted to win the race with the two-stop M2 strategy, Mercedes will definitely want Ferrari to pit by Lap 23 should there be no safety car appearance throughout the race (Figure 13).

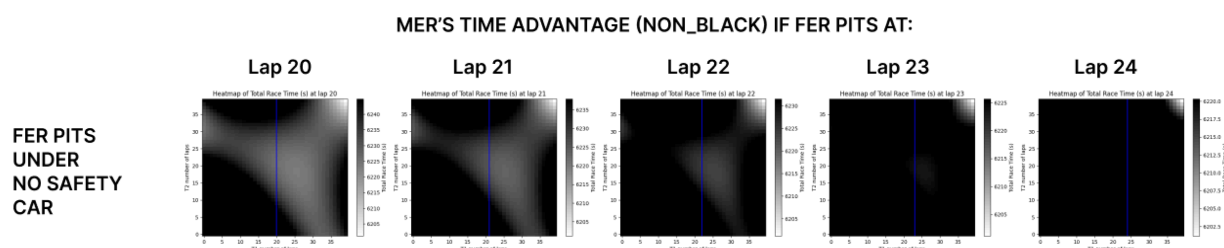

Figure 13: *Mercedes time advantage in the scenario where there is no safety car*

Alas, the safety car's appearance at Lap 20 played very well to Mercedes' advantage by opening up a new window of pit opportunities. This reduces the decision making down to the following finite 42-round sequentially rational extensive form game (Figure 14). The extensive form game at Lap 20 however, results in 77,658 different possible action spaces (42 + 42*42 + 42*42 + 42*42*42).

3.4.2.1. Finding subgame perfect equilibrium using backward induction

Right Branch

As discussed in the pre-game analysis, Ferrari's best opportunity to pit is at Lap 32 where they will dominate for the most part of the remaining race and minimize Mercedes' pit window the most. Thus working backwards from Lap 61 (final lap opportunity to pit) Ferrari will always choose to pit at an earlier lap until Lap 32. Likewise, Ferrari will always reject pitting between Laps 21 to 31 because they can achieve a better outcome at Lap 32.

This leaves the following two action spaces for Mercedes. If Mercedes pits at Lap 20, they open a very small pit window between next 17 to 25 laps (9-lap window) which they can still produce a time advantage. Or Mercedes could choose to not pit and be completely dominated at all remaining laps. Being sequentially rational, Mercedes will choose to Pit.

Left Branch

If Mercedes pits at Lap 20, they open a pit window between the next 8 to 33 laps (26-lap window) which they can still produce a time advantage. Or Mercedes could choose to not pit and be completely dominated at all remaining laps. Being sequentially rational, Mercedes will choose to Pit.

Root Node

Finally, knowing this, Ferrari will choose not to pit at Lap 20 in order to pit at Lap 32 instead. This will result in a subgame perfect Nash equilibrium where Ferrari picks {Dont Pit} and Mercedes picks {Pit, Pit}. The outcome of this SPE is Ferrari choosing to Dont Pit and Mercedes choosing to Pit.

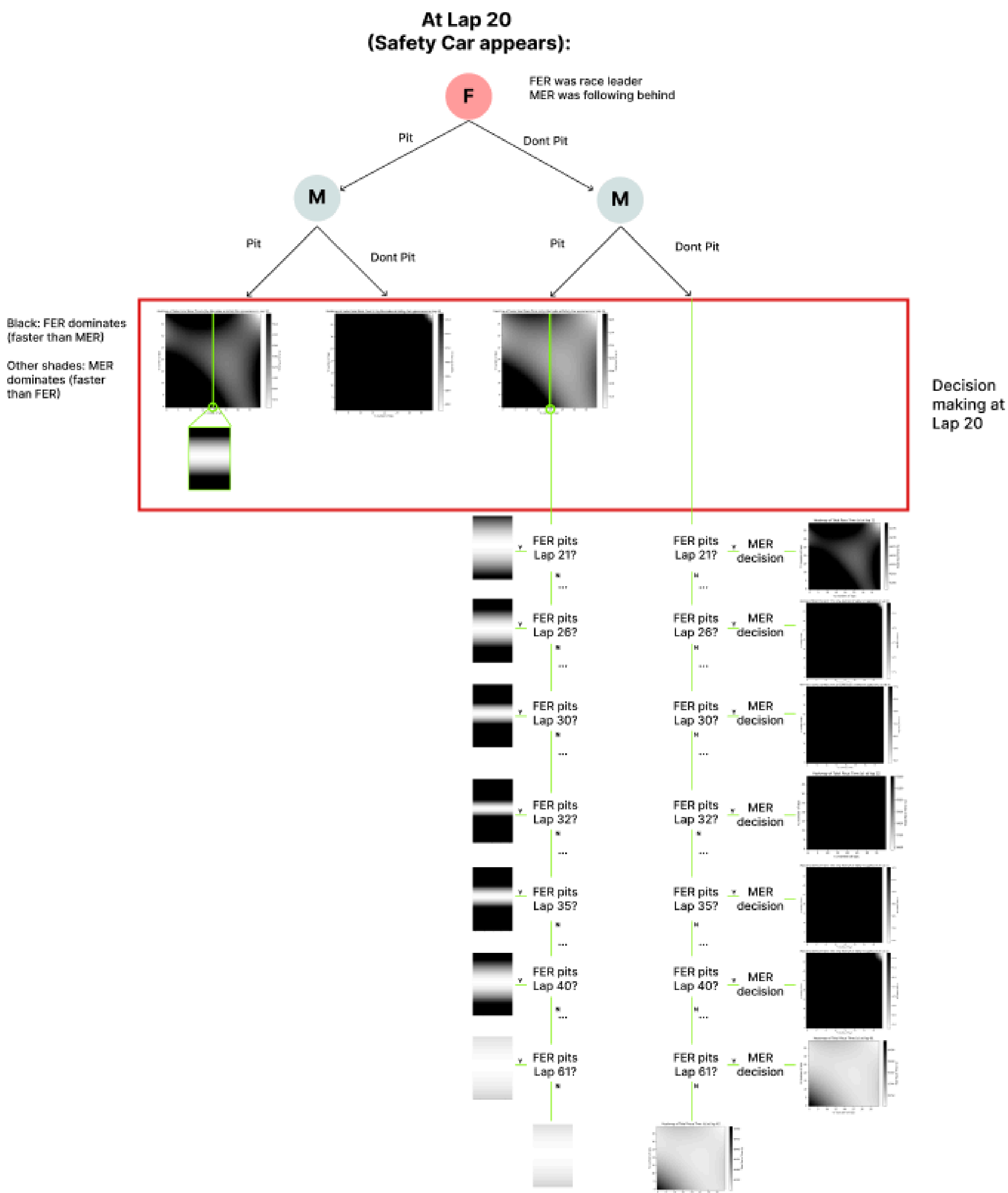

Figure 14(a): *Extensive form subgame for deciding whether to pit at Lap 20*

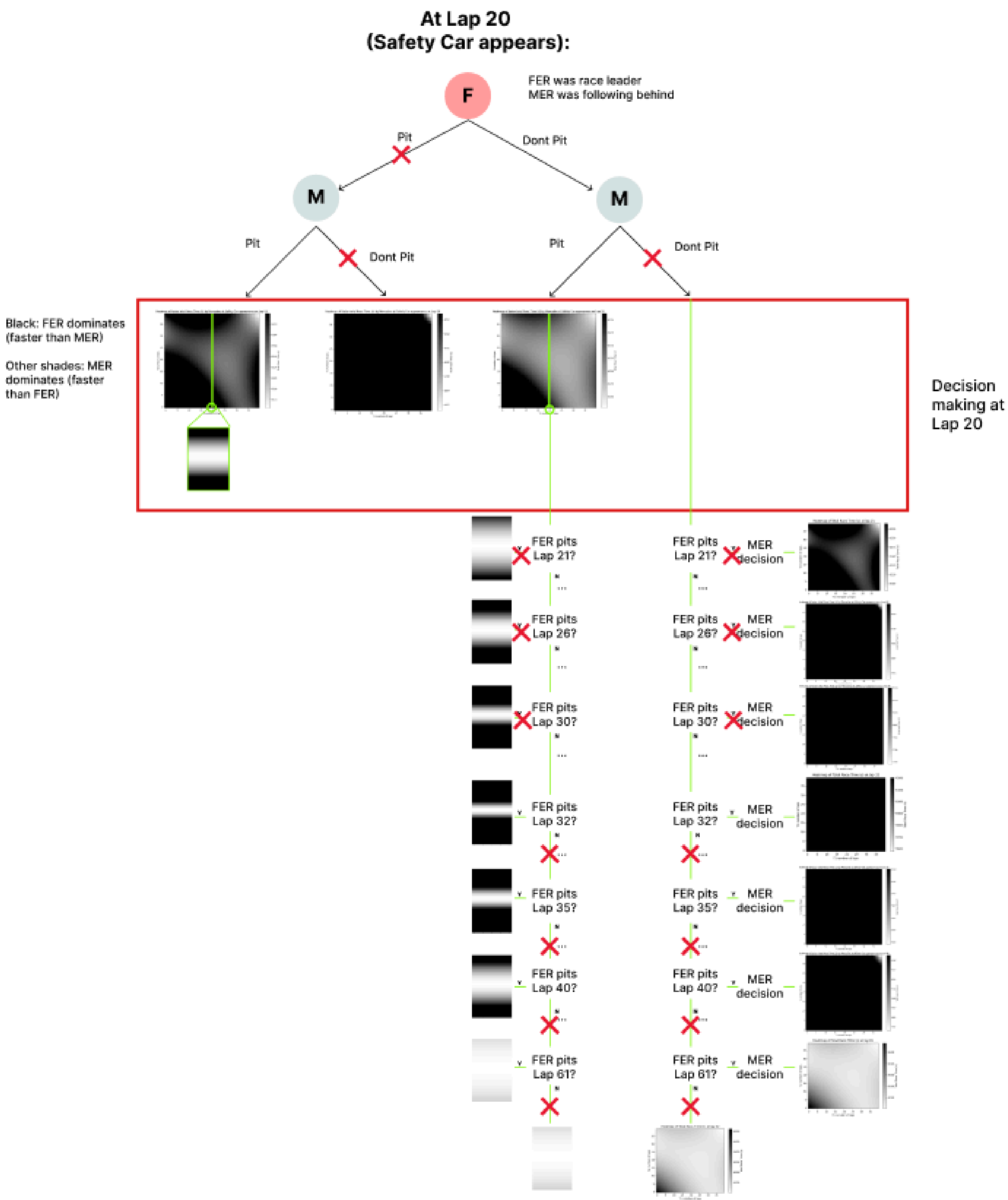

***Figure 14(b)**: Solved extensive form subgame for deciding whether to pit at Lap 20*

However, in the actual race, Ferrari chose to deviate from the subgame perfect equilibrium and pits instead at Lap 20! This decision does not make sense assuming that Ferrari is a sequentially rational player with perfect information. Indeed, another major deciding factor on choosing to pit at Lap 20 despite giving Mercedes a better time advantage would likely be the opportunities or difficulty of overtaking in the Singapore circuit (Figure 15).

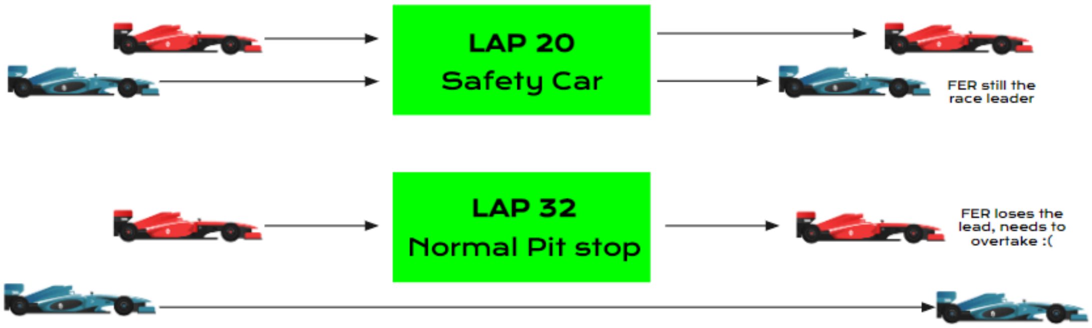

***Figure 15**: Position change if FER had pitted at Lap 20 versus Lap 32*

If Ferrari had pitted together with Mercedes at Lap 20 during the Safety Car, the first-mover's advantage as a leader will enable Ferrari to maintain its lead as it exits the pitstop. However, consider the alternate case where Ferrari had waited to pit at Lap 32 instead while Mercedes pitted at Lap 20. At Lap 32, Mercedes has a timing of 3203s while on 12-laps old Hard tyres. Meanwhile, Ferrari is on 32-laps old Medium tyres with a timing of 3190s. However, since Ferrari requires a normal pitstop at Lap 32, this adds an additional 30s and brings the total lap time of Ferrari to 3220s (Figure 16).

STRATEGY		Pitstop for Hard		
Mercedes	20 M	Pitstop Lap 20 (SC)	12 H	Total
Time	2003.80173	0	1199.23271	3203.034
Ferrari	32 M	Pitstop Lap 32 (Normal)	0 H	Total
Time	3190.34138	30	0	3220.341
				Time cost
				17.30694

***Figure 16**: Pitting under normal circumstances at Lap 32 will result in Ferrari losing pole*

This means that by pitting normally at Lap 32, even if Ferrari reduces Mercedes' pit window, Ferrari would have lost its position as the leader, and have to execute an overtaking maneuver to get back pole position. This is known as a classic undercut maneuver by Mercedes who pitted earlier. Given the difficulty in overtaking in Singapore, it is reasonable why Ferrari would have chosen to deviate from the more beneficial outcome of a faster race pace to pit at Lap 20 and retain its lead.

Singapore Grand Prix: Overtaking Strategy

4. Modeling the overtaking game

4.1. But Ferrari still won despite Mercedes' tyre advantage!

What was not accounted for by Mercedes is a combination of underestimating the level of strategic thinking/aggressive defensive employed by Sainz, the limitation of minimizing total race time as a metric of winning the race, and the overall difficulty in overtaking in the Singapore circuit. Viewed in retrospect and in the data, Mercedes had caught up with Ferrari by the 57th lap, with just six more laps to the end of the race. However, Mercedes was still unable to overtake Ferrari and take the lead despite having the pace advantage.

As drivers get into close proximity, a new form of sequentially rational subgame emerges with regards to overtaking and the risks associated with it. This is a shortfall of our previous two models that did not account for such interactions at close proximity, and does not account for the advantage of the drag reduction system (DRS) in speed gain.

Let's now assume the following constraints:

1. Overtaking can only happen when the follower's speed v_f is faster than the leader's speed v_l
2. Each driver is sequentially rational and seeks to maximize their payoff
3. Thus, a follower can choose to not overtake even if he has a faster speed than the leader. This leads to the two action space: {Cooperate/Not Overtake, Not Cooperate/Overtake}
4. The reward or payoff is embodied in the championship points gained (i.e., 25 points for 1st position, 18 points for 2nd position, and 15 points for 3rd position).

We get the following dynamic extensive form game tree:

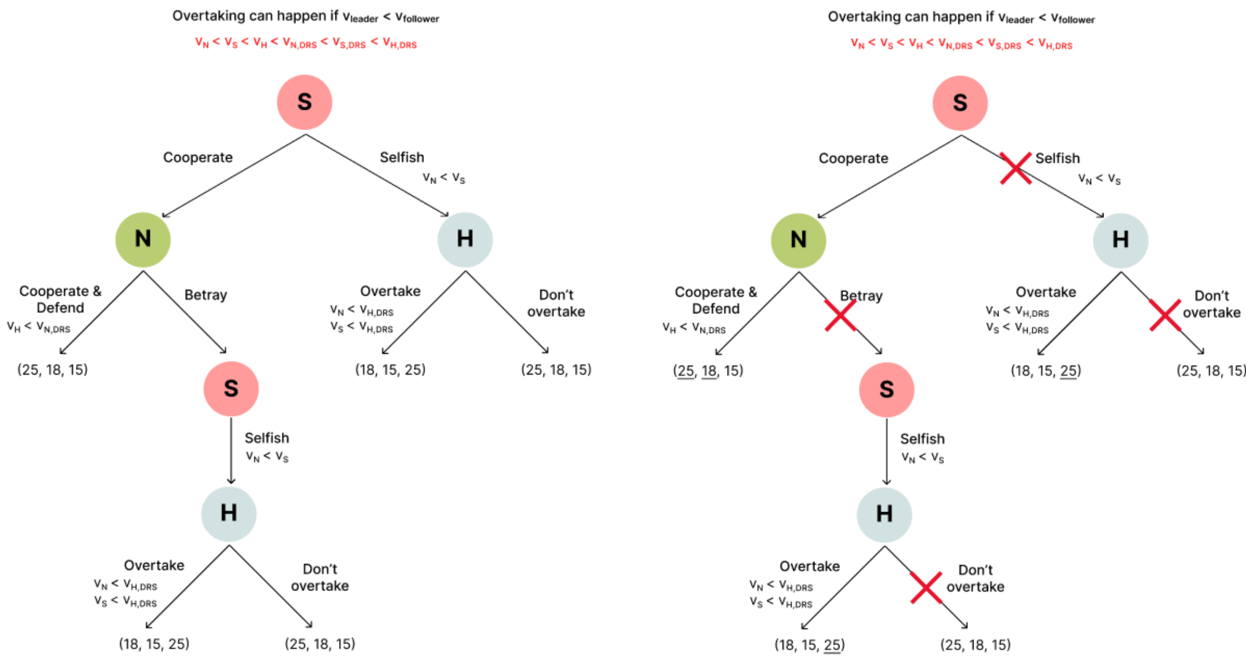

***Figure 17**: Unique overtaking decisions and its dynamic form games*

Between Sainz (Ferrari), Norris (McLaren) and Hamilton (Mercedes), Hamilton will prefer to overtake to gain pole position given his pace advantage with his fresh Medium tyres. Knowing this, Norris will choose to cooperate with Sainz and defend against Hamilton because he will end up with a faster DRS speed over Hamilton, which secures him a second-place finish over a third-place finish if he were to betray Sainz. Knowing this, Sainz will choose to cooperate and help Norris gain DRS speed because it will assure him of a first-place finish through Norris' defense instead of letting Hamilton overtake and cost him a second-place finish.

A subgame perfect Nash equilibrium exists at (25, 18, 15) with an outcome path of Sainz choosing to cooperate and Norris Choosing the Cooperate and Defend. As a result, the initial pace advantage gained by Hamilton from using fresh Medium tyres is negated as he is unable to catch up to Norris who now has a faster speed by being "towed" by Sainz using DRS. Beyond this particular scenario, overtaking decisions throughout the race can also be influenced by the driver's driving style and their associated crash probability.

4.2. Overtaking Game model

First, modifying the Formula 1 process to include three competitors is challenging, so our study focuses on two players, labeled Player 1 (P1, Ferrari) and Player 2 (P2, Mercedes), competing against each other.

We understand that P1 starts in a higher initial position compared to P2. The competition between P1 and P2 involves decisions about driving aggressively or conservatively, which influence their ultimate race positions and the associated risk of crashing. The payoff matrix designed for this scenario will display their anticipated scores, taking into account these elements.

Table 7: The Content of Game

Players	P1, P2
Position Payoff	**25** points for 1st place, **18** points for 2nd place
Players Strategies	**Aggressive (A)**: High risk of crashing but can improve position **Conservative (C):** Low risk of crashing, maintains position
Crash Probabilities (Supposing all players possess equivalent driving proficiency**)**	**P[A]**: **30%** chance of crashing **P[C]**: **10%** chance of crashing
Crash Impact	Crashing results in **0** points
Position Changes (if no crash)	If P2 is aggressive and P1 is conservative, P2 moves to 1st and P1 goes to 2nd. If both are aggressive or both are conservative, their positions remain the same as they started.
Expected Payoff (To arrive at the final payoff, we must sum up four different cases)	**Case 1**: P1 and P2 do not crash both **Case 2**: P1 and P2 crash both **Case 3**: P1 crashes and P2 does not crash **Case 4**: P1 does not crash and P2 crashes

Table 8: Final Payoff Matrix

P1\P2	**A**	**C**
A	(?, ?)	(?, ?)
C	(?, ?)	(?, ?)

Next, let's proceed to populate this table.

P1 and P2 choose to drive aggressively both, (A, A):

- **Case 1**: P1 and P2 do not crash both, the position does not change; the position payoff would be (25,18)
  And the probability of this case is : $P[Case\ 1]\ =\ 0.7\ *\ 0.7\ =.0.49$
  The payoff of P1 and P2: $U_1 =\ 0.49\ *\ 25\ =\ 12.25;\ U_2 =\ 0.49\ *\ 18\ =\ 8.82$
- **Case 2**: P1 and P2 both crash, they both get payoff of 0, (0,0)
  $P[Case\ 2]\ =\ 0.3\ *\ 0.3\ =\ 0.09$
  The payoff of P1 and P2: $U_1\ =\ U_2\ =\ 0$
- **Case 3**: P1 crashes and P2 does not crash, P1 goes to 1st, their payoff would be (0,25)
  $P[Case\ 3]\ =\ 0.3\ *\ 0.7\ =\ 0.21$
  The payoff of P1 and P2: $U_1 = 0.21\ *\ 0\ =\ 0;\ U_2 =\ 0.21\ *\ 25\ =\ 5.25$
- **Case 4**: P1 does not crash and P2 crashes, P1 remain 1st, their payoff would be (25,0)
  $P[Case\ 4]\ =\ 0.7\ *\ 0.3\ =\ 0.21$
  The payoff of P1 and P2: $U_1 = 0.21\ *\ 25\ =\ 5.25;\ U_2 = 0$
- **The Final payoff of** P1 and P2 when **(A, A)**:
  $U_{1,tot} = 12.25\ +\ 0\ +\ 0\ +\ 5.25\ =\ 17.5;\ U_{2,tot} =\ 8.82\ +\ 0\ +\ 5.25\ +\ 0\ =\ 14.07$

P1 chooses A, P2 chooses C, (A, C):

- **Case 1:** Position payoff: (25,18); $P[Case\ 1]\ =\ 0.7\ *\ 0.9\ =\ 0.63$
  The payoff of P1 and P2: $U_1 = 0.63\ *\ 25\ =\ 15.75;\ U_2 =\ 0.63\ *\ 18\ =\ 11.34$
- **Case 2**: Position payoff: (0,0); $P[Case\ 2]\ =\ 0.3\ *\ 0.1\ =\ 0.03$
  The payoff of P1 and P2: $U_1\ =\ U_2\ =\ 0$
- **Case 3**: Position payoff: (0,25); $P[Case\ 3]\ =\ 0.3\ *\ 0.9\ =\ 0.27$
  The payoff of P1 and P2: $U_1 = 0;\ U_2 =\ 0.27\ *\ 25\ =\ 6.75$
- **Case 4**:Position payoff: (25,0); $P[Case\ 4]\ =\ 0.7\ *\ 0.1\ =\ 0.07$
  The payoff of P1 and P2: $U_1 = 0.07\ *\ 25\ =\ 1.75;\ U_2 = 0$
- **The Final payoff of (A, C)** between P1 and P2:
  $U_{1,tot} = 15.75\ +\ 0\ +\ 0\ +\ 1.75\ =\ 17.5;\ U_{2,tot} =\ 11.34\ +\ 0\ +\ 6.75\ +\ 0\ =\ 18.09$

After Repeating the same process to **(C, A)** and **(C, C),** we finalize the payoff matrix below:

Table 9: Final Payoff Matrix

P1\P2	**A**	**C**
A	(17.5, 14.07)	(17.5,**18.09**)
C	(**18.09**,**17.5**)	(**22.5**,16.83)

Based on the presented payoff matrix, Player 2's strategy choice does not influence Player 1's decision when Player 1 opts for an aggressive approach to maintain their leading position. It's advisable for Player 2 to adopt a conservative strategy to mitigate the risk of crashing, which could result in lower payoffs. Opting for a conservative strategy is the optimal choice for Player 1 as well, ensuring both players achieve higher payoffs compared to aggressive maneuvers, regardless of Player 2's strategy. Interestingly, when trailing behind, Player 2 might take the risk of driving aggressively to potentially secure a higher payoff.

The **Nash Equilibrium (C, A)** occurs when Player 1 selects the Conservative Strategy and Player 2 chooses the Aggressive Strategy while Player 1 leads the race. However, consider the case where we incorporate the dynamic probability of crashing associated with both aggressive and conservative driving. This probability fluctuates due to various factors such as weather conditions and player status, introducing variability into the game dynamics.We'll delve further into the topic later, using p1and p2 to represent the likelihood of crashing while driving aggressively and conservatively, respectively: $P[A] = p1, P[C] = p2$, where $1 > p1 > p2 > 0$. The new payoff matrix table is attached below:

Table 10: New Payoff Matrix

P1\P2	**A**	**C**
A	25(1-p1), (1-p1)(18+7p1)	25(1-p1), **(1-p2)(18+7p1)**
C	(1-p2)(18+7p1), 25(1-p1)	**25(1-p2)**, (1-p2)(18+7p2)

For easier reference, we will redefine the payoffs for P1 and P2:

Table 11: Redefined Payoff Matrix

P1\P2	**A**	**C**
A	Q1, Q2	Q1, **Q3**
C	Q3, Q1	**Q4**, Q5

It is clear that Q3 > Q5, Q4> Q1, given by p1 > p2 and when:

- P1 chooses A: P2 would choose **Q3**
- P1 chooses C: P2 would choose **Max(Q1, Q5)**
- P2 chooses A: P1 would choose **Max(Q1, Q3)**
- P2 chooses C: P1 would choose **Q4**

Then the relation among **Q1, Q3, and Q5** would lead to different situations**:**

1. **Q1 > Q3 > Q5:**

Table 12

P1\P2	A	C
A	**Q1**, Q2	Q1, **Q3**
C	Q3, **Q1**	**Q4**, Q5

According to Table 12, there is no Nash Equilibrium at all. And any (p1,p2) in the domain from Figure 18 would satisfy this situation. When a Nash equilibrium does not exist, it prompts players to look for alternative approaches and solutions, emphasizing the complexity and dynamic nature of strategic decision-making. This can lead to richer analyses and more creative strategies, reflecting the adaptive and often unpredictable nature of real-world strategic interactions.

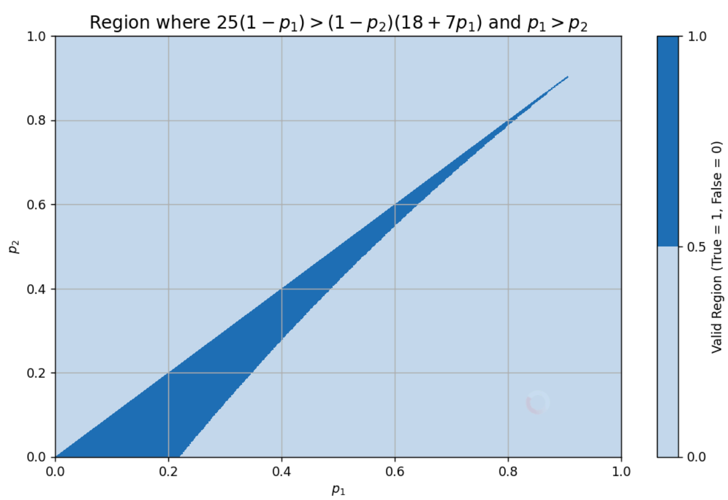

Figure 18*: Domain where Q1 >Q3 >Q5*

2. **Q3 > Q1 > Q5:**

Table 13

P1\P2	A	C
A	Q1, Q2	Q1, **Q3**

C	**Q3**, **Q1**	**Q4**, Q5

The **NE (C, A)** occurs when Player 1 selects the Conservative Strategy and Player 2 chooses the Aggressive Strategy which is consistent with the result shown in **Table 9.** In this scenario, if P1 adopts a conservative approach, P2 must opt for an aggressive strategy. Conversely, P2 should refrain from being aggressive when P1 chooses the same, as it could result in significantly reduced payoffs. The domain of p1 and p2 is shown in Figure 19.

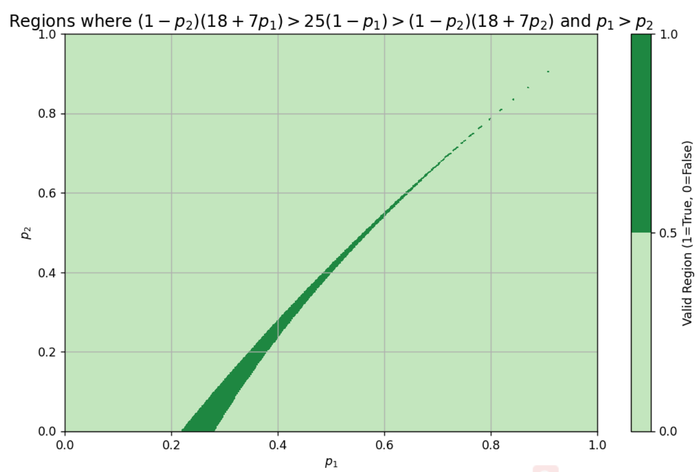

Figure 19: *Domains where Q3> Q1>Q5*

3. **Q3> Q5 > Q1:**

Table 14

P1\P2	**A**	**C**
A	Q1, Q2	Q1, **Q3**
C	**Q3**, Q1	**Q4**, **Q5**

In this scenario, the **Nash Equilibrium (C, C)** is observed, indicating that both P1 and P2 are advised to adopt a conservative approach to achieve the optimal outcome. This choice is particularly advantageous due to a slightly elevated probability of crashing, as depicted in Figure 20, attributed to certain factors.

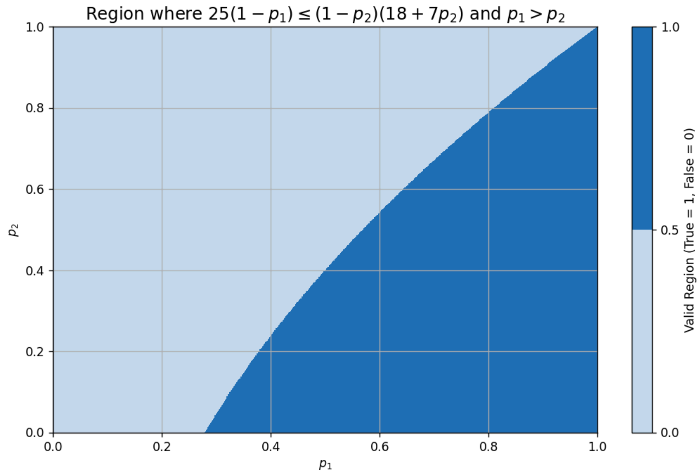

***Figure 20**: Domains where Q3 > Q5 > Q1*

4.2.1. Overtaking Analysis Conclusion

Based on normal and reasonable research, such as the example of determining crash probabilities beforehand, Player 2 (Mercedes), if lagging behind Player 1 (Ferrari), should take risks to overtake in this model to maximize payoff. However, in most cases, the third scenario is likely, where both parties opt for cautious strategies to maximize their own gains. Similarly, the variability of circumstances on the racetrack and changes in crash probabilities due to the environment and the drivers themselves can lead to different outcomes for overtaking strategies in the three cases, much like the absence of a Nash equilibrium in the first case, which may necessitate cooperation and various forms of gameplay between the players.

4.3. How could Mercedes have won then?

Let us consider an alternate situation. We have the base situation of the strategy that Mercedes performed in the race, the only other strategy they could have done is to not pit at all, and save the 30 seconds they lost by pitting.

Singapore being a track with low speed and a lot of corners, means that tyres degrade faster than other circuits. This means that Mercedes would not have been better off continuing instead of pitting, as the time lost from pitting would be 30 seconds, but they would have gained those 30 seconds back from the rest of the race and ended up further ahead than if they did not pit at all.

We believe Mercedes performed their most optimal strategy and received their most favorable outcome. The reason they could not win was because of the strategy performed by Ferrari and Sainz to give Norris the DRS to protect himself.

5. Conclusion

The analysis of tyre strategies at the Singapore Grand Prix reveals that the Medium-Hard (MH) strategy typically shaves off about 25 seconds compared to the Medium-Hard-Medium (MHM) strategy, projecting the fastest race time at 6199 seconds. Despite this, in the in-game analysis, Mercedes outperformed with an MHM strategy, finishing in 6187 seconds, ahead of Ferrari's MH strategy, underscoring the importance of adaptability in response to real-time conditions and competitor actions. Particularly, the timing of the safety car is crucial; if deployed before lap 32, MHM is preferable, otherwise MH is recommended. Despite missing out on a victory, the Mercedes F1 Team's bold execution under pressure highlights the critical role of strategic ingenuity and flexibility in Formula 1's high-stakes environment. This race not only demonstrated their resilience but also provided valuable lessons that will refine their strategies in the pursuit of excellence, emphasizing that success in Formula 1 requires continuous adaptation and learning from every race.

6. Limitations and future works

The game theory models (Pure Nash equilibrium, Dynamic extensive form games) used in this project are relatively simple in accounting for just race times based on tyre strategies, the effects of safety cars, aggressiveness and crash probabilities for a two-player game. In actual fact, there are other variables to be accounted for and we believe that it can be further developed to consider other race uncertainties such as the effect of unfavorable weather and the effect of downforce at different circuits. We can also consider extending it to consider N-players in the future.

The assumption of complete information also limits the model since a driver’s racing style may change depending on circumstances and many team’s race strategies are unknown until the very moment it is executed. The information asymmetries in an actual race could be considered for future iterations of the project but intentionally left out due to the broadness of the scope and time limitations for the project.

Nonetheless, there is promise in its application to other Formula 1 races in the future. Beyond Formula 1, the overtaking model can also be applied to analyze overtaking and aggressive driver behavior on the public roads which could potentially address road rage and minimize petty road accidents.

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