Game Theory of 2023 F1 Singapore GP

Group 11

Presentation Outline

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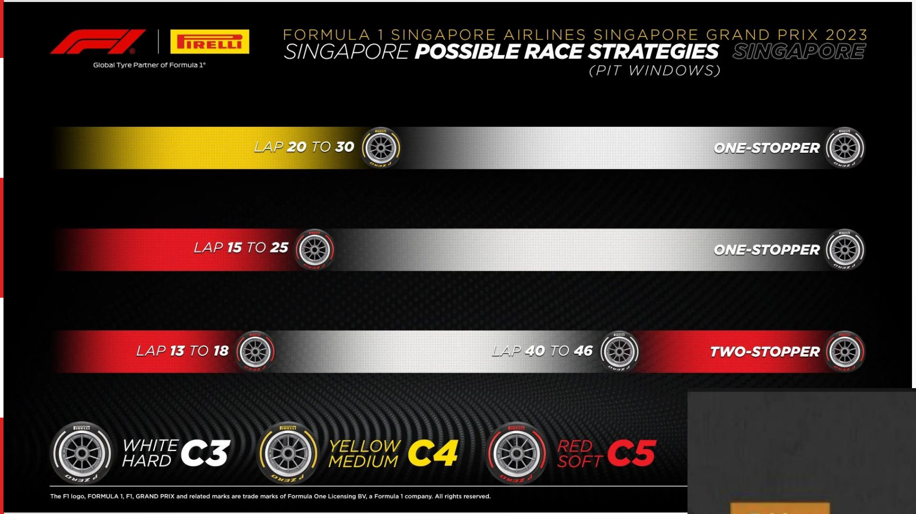
1 Introduction

Conventional vs Street Circuits

The Singapore Grand Prix



Meet the F1 Drivers

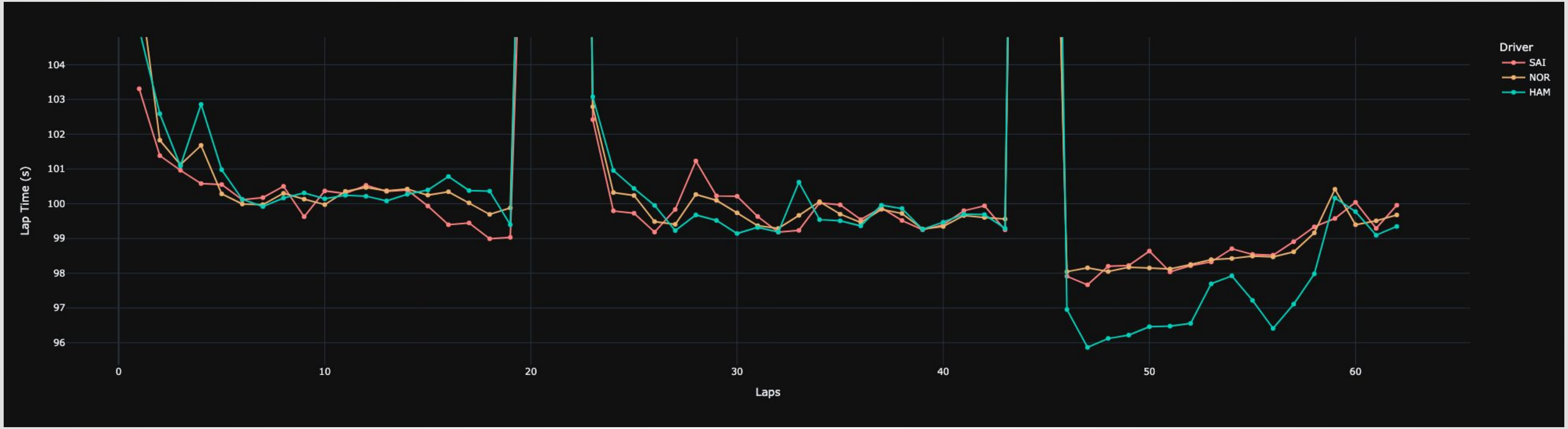
Carlos Sainz

Lewis Hamilton

2 Pregame Analysis

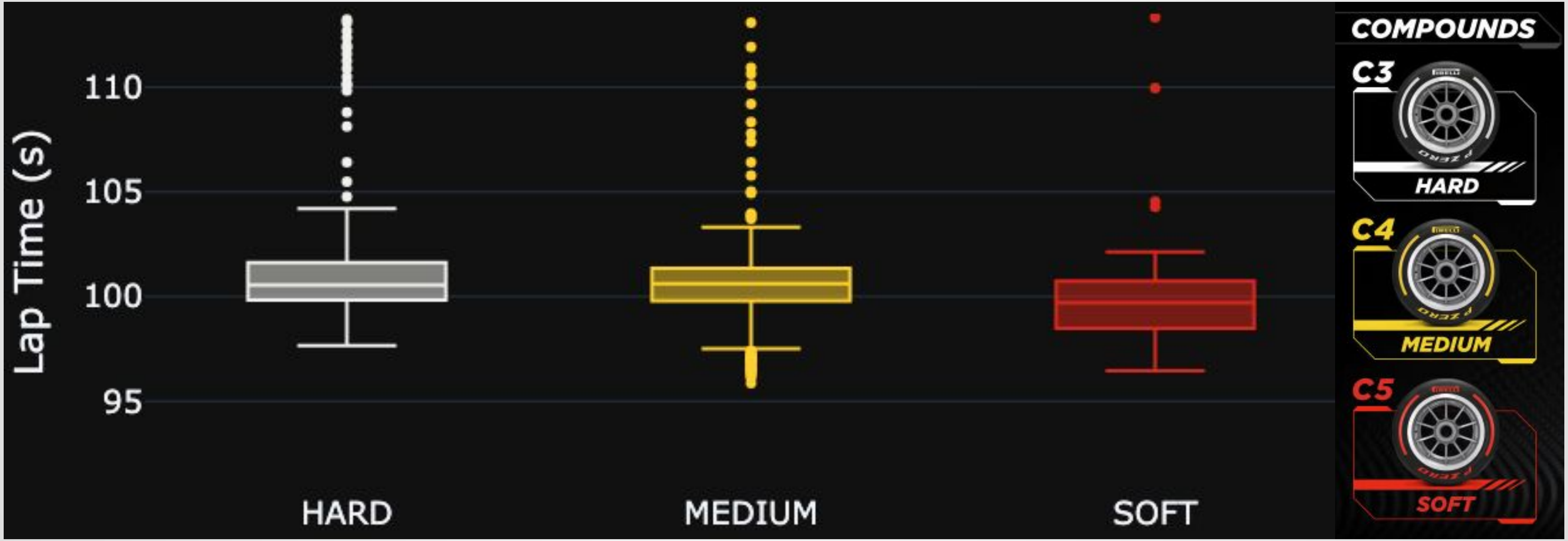
Insights from the 2023 Grand Prix

01 Lap Times and Tyre Compound of the Top 3 Drivers in the Singapore Grand Prix



Insights from the 2023 Grand Prix

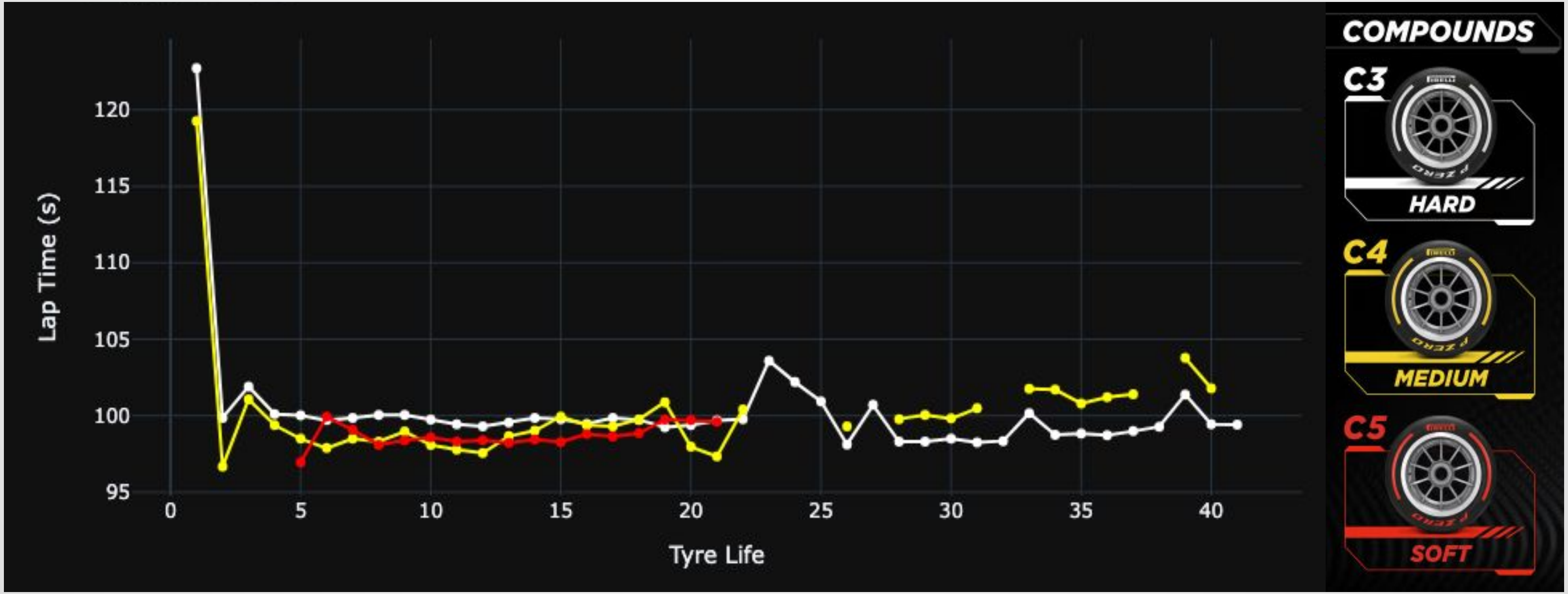
02 Lap Distribution for Each Tyre Compound (85th Percentile)



Insights from the 2023 Grand Prix

03

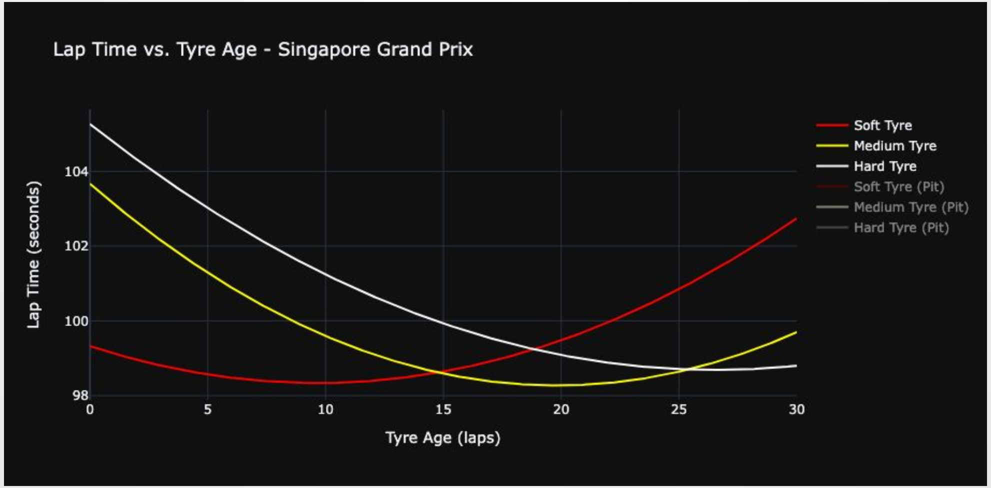
Lap Time across Tyre Life (85th Percentile)



Dynamics of Tyre Degradation

Tyre Degradation Equations

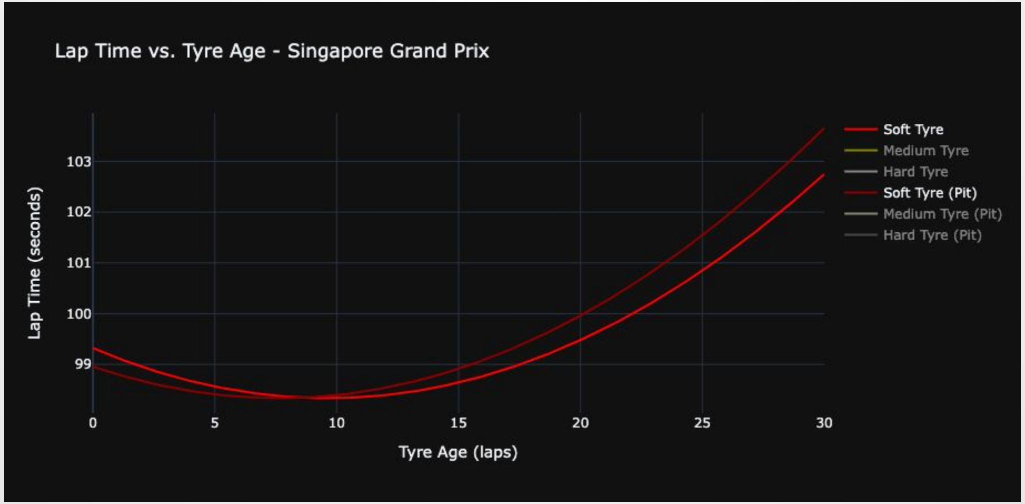
Soft Tyre Degradation	$T_{Soft}(x) = 0.010632527089783267x^2 - 0.20480955237358175x + 99.32181409958719$
Medium Tyre Degradation	$T_{Medium}(x) = 0.01373902477042166x^2 - 0.5445335863621549x + 103.66638621613211$
Hard Tyre Degradation	$T_{Hard}(x) = 0.009342750556306952x^2 - 0.49584233521158183x + 105.26568511257031$



Dynamics of Tyre Degradation

Performance boost post-pit due to reduced vehicle weight and new tyres.

Soft Tyre Pit Degradation	$T_{Soft_{Pit}}(x) = T_{Soft}(x + 2)$
Medium Tyre Pit Degradation	$T_{Medium_{Pit}}(x) = T_{Medium}(x + 9)$
Hard Tyre Pit Degradation	$T_{Hard_{Pit}}(x) = T_{Hard}(x + 11)$



Strategic Pregame Race Models

01

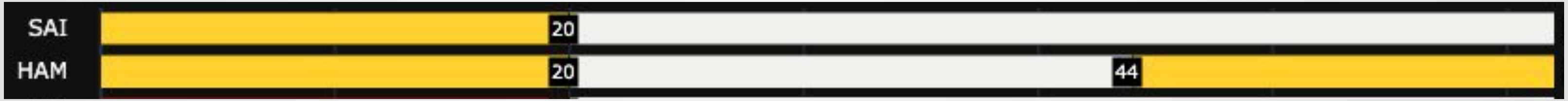
Medium-Hard

Start with Medium Tyres
Pit around Lap 20-35
End Race with Hard Tyres

02

Medium-Hard-Medium

Start with Medium Tyres
Pit around Lap 20-30
Use Hard Tyres in the Middle
Pit again around Lap 40-45
End with Medium Tyres



Medium-Hard Tyre Strategy

- Minimize Race Time R1(L)

$$Minimize\ R_1(L) = \sum_{L=1}^{62} T_{Medium}(L) + T_{\mathrm{PitStop}} + T_{\mathrm{Hard}_{Pit}}(62 - L)$$

- Summation of Equations from 0 to L

$$\sum_{x=1}^{L} T_{Medium}(x) = 0.00457967492347389L^3 - 0.265397280795867L^2 + 103.396409260413L$$

$$\sum_{x=1}^{L} T_{\mathrm{Hard}_{Pit}}(x) = 0.00311425018543565L^3 + 0.140479536208261L^2 + 100.798298456162L$$

- Optimization Equation R1(L)

$$\begin{aligned}R_1(L) = 0.00457967492347389L^3 - 0.265397280795867L^2 + 103.396409260413L + 20 \\ + 0.00311425018543565(62 - L)^3 + 0.140479536208261(62 - L)^2 \\ + 100.798298456162(62 - L)\end{aligned}$$

Medium-Hard Tyre Strategy

- Solve for the First Derivative = 0 and get the Positive Value of L
- Optimal Lap to Pit L = 32
- Total Race Time = 6198.52s

```python
def R1(L):
    term1 = 0.00457967492347389*L**3 - 0.265397280795867*L**2 + 103.396409260413*L
    pit1 = 30
    term2 = 0.00311425018543565*(62-L)**3 - 0.140479536208261*(62-L)**2 + 100.798298456162*(62-L)
    return term1 + pit1 + term2

# Initial guess
x0_single = np.array([0])

# Bounds for L
bounds_single = [(0, 62)]

# Minimize the new function with constraints
res_single = minimize(R1, x0_single, bounds=bounds_single, method='SLSQP')
```

✓ 0.0s

```python
res_single.x # Optimal Lap to Pit
```

✓ 0.0s

```
array([32.47342934])
```

```python
res_single.fun # Total Race Time
```

✓ 0.0s

```
6198.516537328564
```

Medium-Hard-Medium Tyre Strategy

- Minimize Race Time R2(L)

$$Minimize\ R_2(L) = \sum_{L=1}^{62} T_{Medium}(L1) + T_{\text{PitStop}} + T_{\text{Hard}_{Pit}}(L2) + T_{Medium}(L1) + T_{\text{PitStop}} + T_{\text{Medium}_{Pit}}(62 - L1 - L2)$$

- Summation of Equations from 0 to L

$$\sum_{x=1}^{L} T_{\text{Medium}_{Pit}}(x) = 0.00457967492347389L^3 + 0.141746057862072L^2 + 99.7321192124913L$$

- Optimization Equation R2(L)

$$\begin{aligned} R_2(L) = 0.00457967492347389{L_1}^3 - 0.265397280795867{L_1}^2 + 103.396409260413L_1 + 20 \\ + 0.00311425018543565(L_2)^3 + 0.140479536208261(L_2)^2 + 100.798298456162(L_2) \\ + 20 + 0.00457967492347389(62 - L_1 - L_2)^3 + 0.141746057862072(62 - L_1 - L_2)^2 \\ + 99.7321192124913(62 - L_1 - L_2) \end{aligned}$$

Medium-Hard-Medium Tyre Strategy

- Solve for the Partial First Derivative = 0 and get the Positive Value of L1 and L2
- Optimal Lap to Pit
  - L1 = 25
  - L2 = 44
- Total Race Time = 6223.39s

```python
def R2(L1, L2):
    term1 = 0.00457967492347389*L1**3 - 0.265397280795867*L1**2 + 103.396409260413*L1
    pit1 = 30
    term2 = 0.00311425018543565*L2**3 - 0.140479536208261*L2**2 + 100.798298456162*L2
    pit2 = 30
    term3 = 0.00311425018543565*(62-L1-L2)**3 - 0.140479536208261*(62-L1-L2)**2 + 100.798298456162*(62-L1-L2)

    return term1 + pit1 + term2 + pit2 + term3

# Define a wrapper function for R_correct_separate that takes a single argument (array of L1 and L2)
# This is needed because minimize function expects the function to have a single argument
def R_wrapper(L):
    L1, L2 = L
    return R2(L1, L2)

# Initial guess
x0 = np.array([30, 30])

# Bounds for L1 and L2
bounds = [(0, 62), (0, 62)]

# Minimize the function with constraints
res = minimize(R_wrapper, x0, bounds=bounds, method='SLSQP')
```

✓ 0.0s

```python
res.x # Optimal Lap to Pit
```

✓ 0.0s

```
array([25.43720776, 18.28182363])
```

```python
res.fun # Total Race Time
```

✓ 0.0s

```
6223.390994913271
```

Pregame Analysis Conclusion

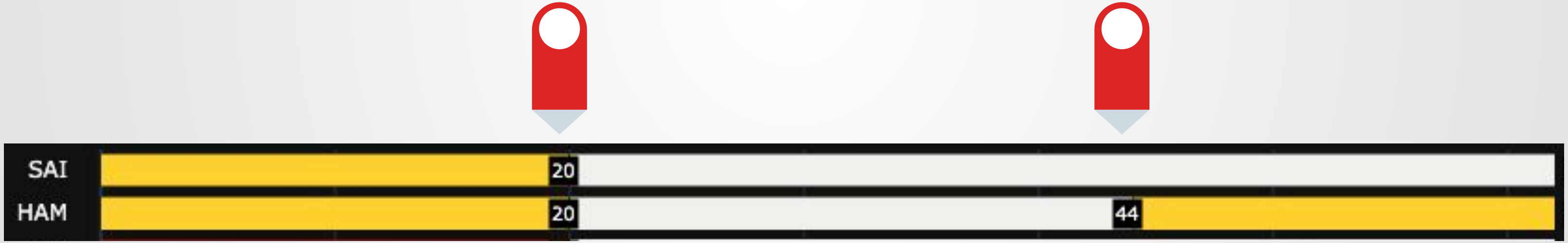
Medium-Hard Tyre Strategy is the Ideal Strategy for F1 Teams in the Singapore Grand Prix

3 Ingame Analysis (Race Time)

What actually happened.

LAP 20
Safety Car

LAP 44
Virtual Safety Car



Available Strategy options

Based on tyre options available, a combination of the following strategies can be chosen:

	MERCEDES	FERRARI
New Medium	2	1
New Hard	1	1
Used Soft	4	4

Mercedes Strategy options	Ferrari strategy options
One-stop M1 (MH): R1(L1)	One-stop F1 (MH): R1(L1)
Two-stops M2 (MHM): R2(L1,L2)	Two-stops F2 (MHS(Used)): R3(L1,L2)
Two-stops M3 (MHS(Used)): R3(L1,L2)	

Why Mercedes pit at Lap 44?

At Lap 44, MER opting for a two-stop M2 strategy is the most ideal choice for fastest race time!
FER opting for a one-stop F1 strategy is most ideal at minimizing time gap!

	F1	**F2**
M1	(R1(20), R1(20))	(R1(20), R3(20,24))
M2	(R2(20,24), R1(20))	(R2(20,24), R3(20,24))
M3	(R3(20,24), R1(20))	(R3(20,24), R3(20,24))

Nash Equilibrium exists at when MER chooses M2 and FER chooses F1

	F1	**F2**
M1	(6215, **6215**)	(6215, 6250)
M2	(**6187**, **6215**)	(**6187**, 6250)
M3	(6250, **6215**)	(6250, 6250)

Payoff of (6187, 6215)

Actual vs pre-game results

As a result of the SC at Lap 20 and VSC at Lap 44:

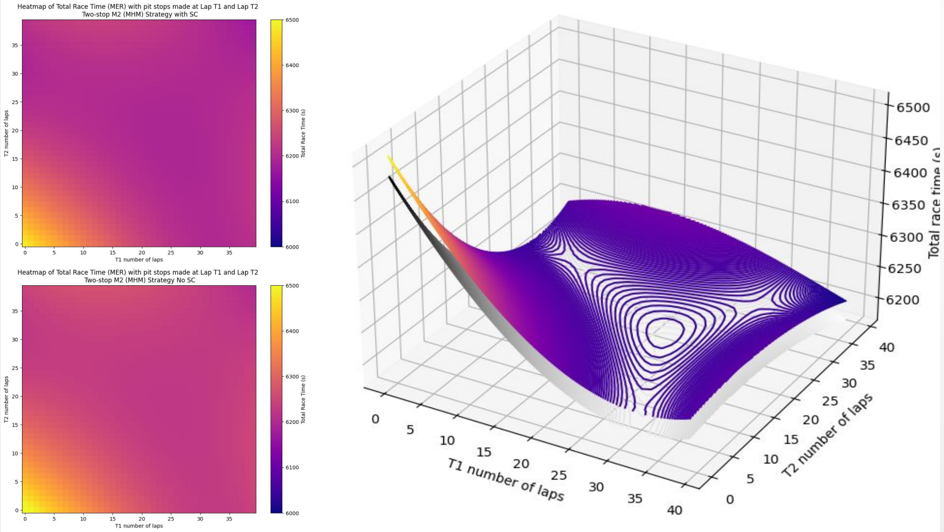
- MER performed 12s better than its pre-game strategy
- FER performed 16s worse than its pre-game strategy

	MER	**FER**
Pre-game	6199	6199
Actual game	**6187 (-12s)**	**6215 (+16s)**

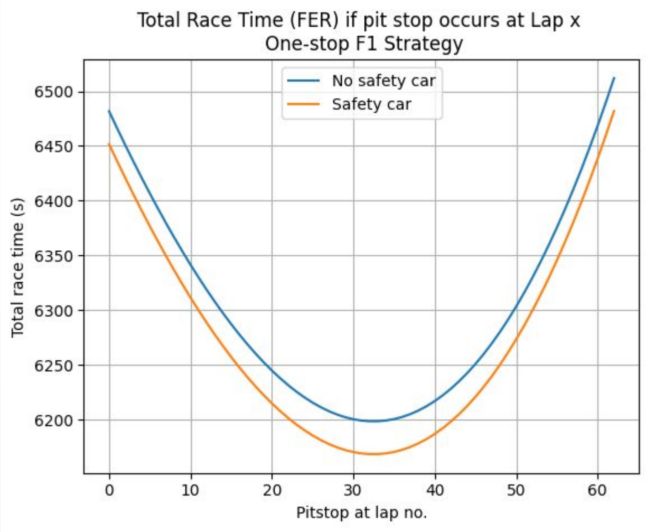
Question: Why is this the case?

Total race time is always dominant under (virtual) safety car

MER's total race time

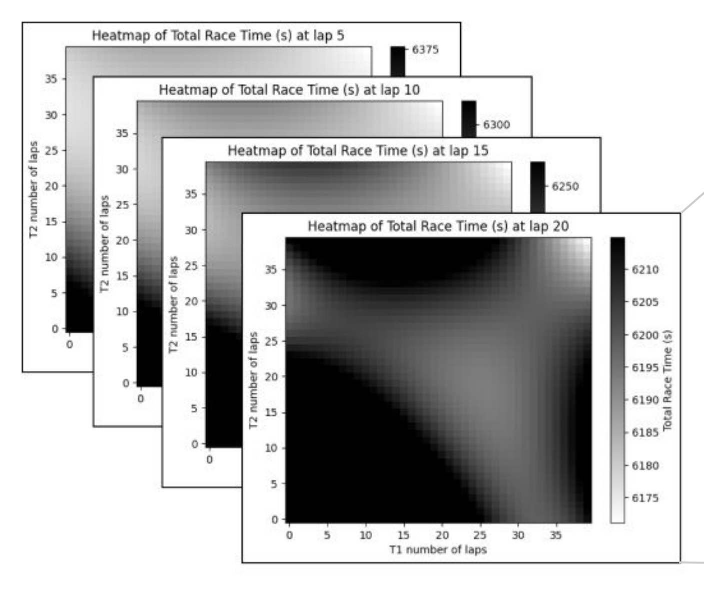


FER's total race time



Strategic Pitting Decisions

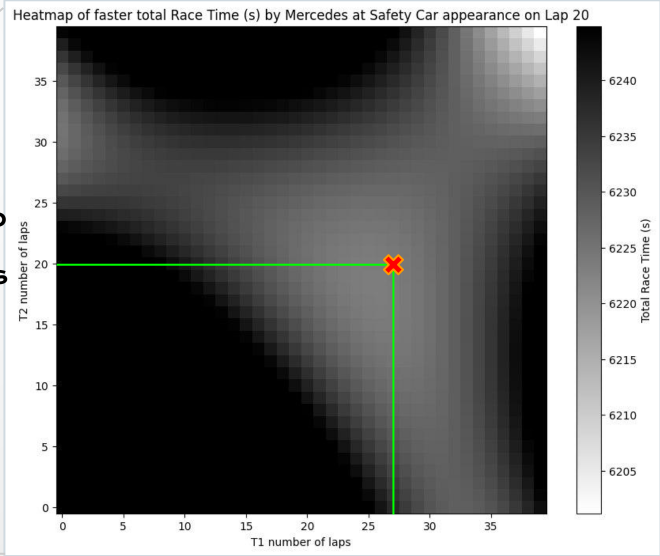
Which lap should Ferrari pit?



If Ferrari Pits at Lap 20:

Black: FER faster than MER
Non-black: otherwise

Which lap should Mercedes do their second pit stop?



Which lap should Mercedes do their first pit stop?

Strategic Pitting Decisions

Modelling Approach

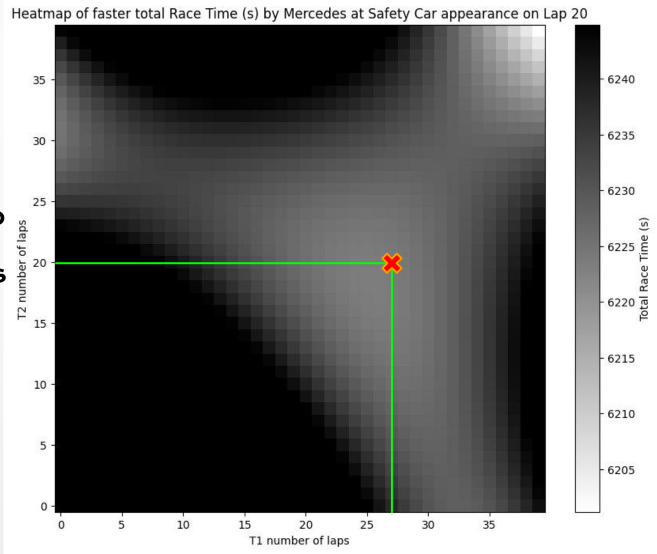
- Use of estimated race timings to predict opponents decisions and optimize own strategy

Predicted Outcomes

- Analysis of how strategic pitting affects overall race time and positioning relative to competitors.

If Ferrari Pits at Lap 20:

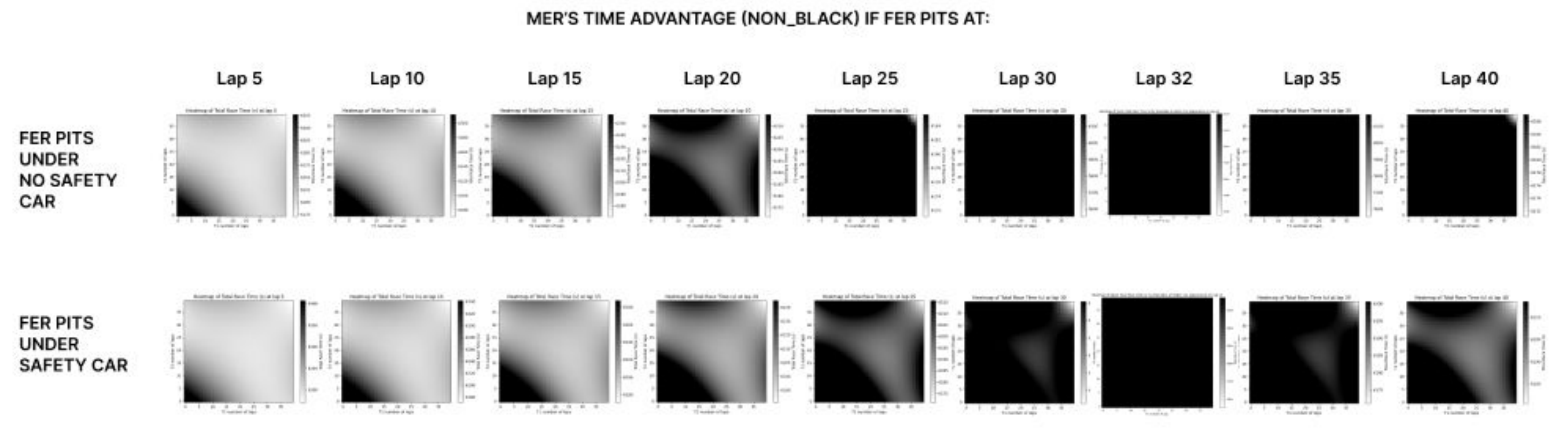
Which lap should Mercedes do their second pit stop?



Which lap should Mercedes do their first pit stop?

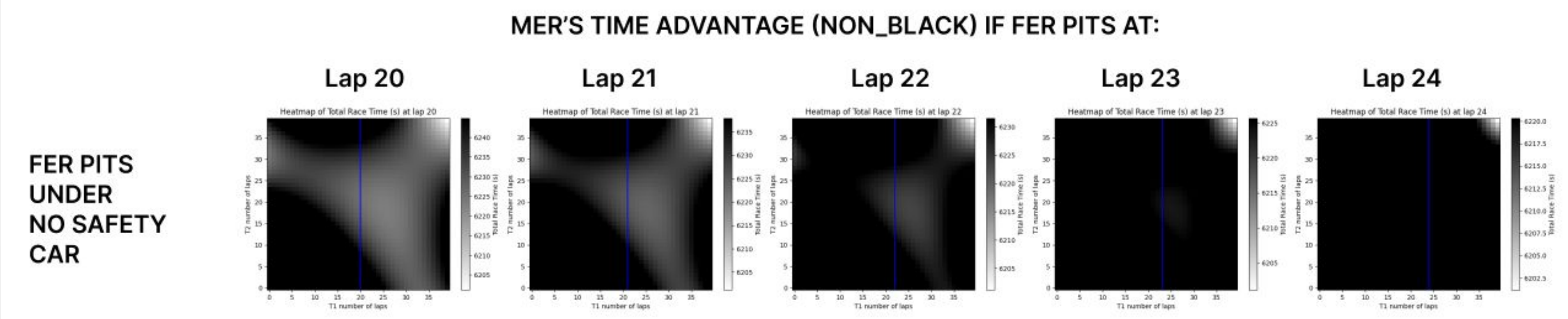
1.Why Mercedes pit at Lap 20?

Regardless of safety car or not, Mercedes' pit window for implementing a two-stop M2 strategy gets shorter if Ferrari pits later into the race, and will definitely lose if Ferrari pits at Lap 32.



1.Why Mercedes pit at Lap 20?

If there was no safety car by Lap 23, Mercedes needs Ferrari to pit at that lap if they wanted a time advantage from a two-stop M2 strategy.



A saving grace!

LAP 20

Safety Car

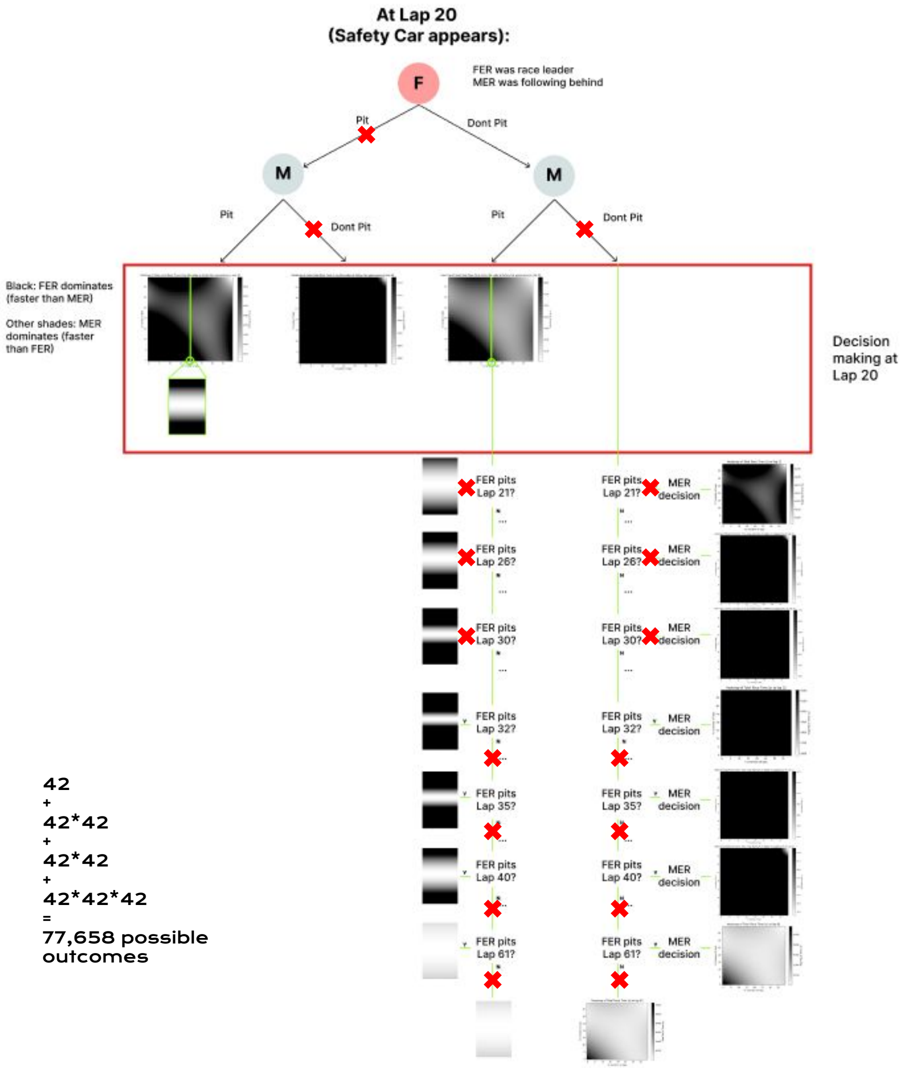
This results in a 42-round subgame!

FER will choose to pit at Lap 32 in order to minimize their total race time, and minimize MER pit window.

Knowing this, MER will always choose to pit in order to maximize its own pit window

Therefore, as race leader FER will always choose to Not Pit.

Question: Why did Ferrari pit then?



42
+
42*42
+
42*42
+
42*42*42
=
77,658 possible outcomes

Is Ferrari irrational??

	Pitstop lap	FER
Pre-game	32	6199
Actual game	20	**6215 (+16s)**

Overtaking is difficult!

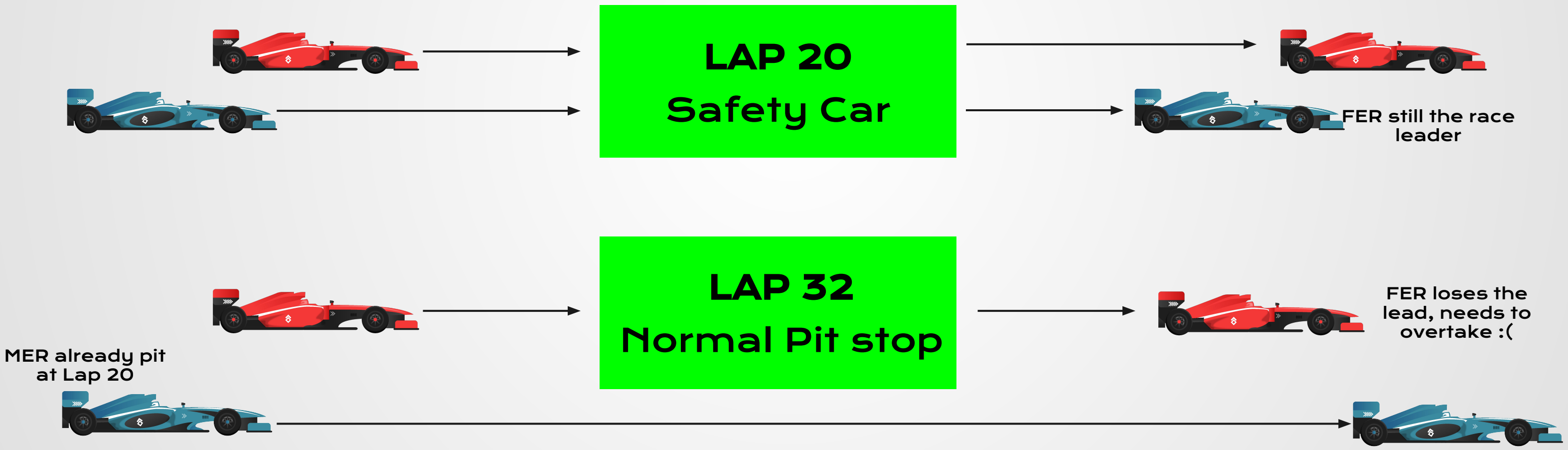
Time taken by Mercedes and Ferrari at Lap 32:

STRATEGY		Pitstop for Hard		
Mercedes	20 M	Pitstop Lap 20 (SC)	12 H	Total
Time	2003.80173	0	1199.23271	3203.034
Ferrari	32 M	Pitstop Lap 32 (Normal)	0 H	Total
Time	3190.34138	30	0	3220.341
				Time cost
				17.30694

This would result in an undercut from Mercedes!

Overtaking is difficult!

FER chose to pit at Lap 20 to maintain their race lead!



Ferrari's decision to pit at Lap 20 almost costed them in the last 15 laps of the race, with MER chasing down FER with a faster pace. But FER held on.

4 Overtaking Analysis

Overtaking Game model

Players	P1, P2
Position Payoff	1st: **25** points; 2nd: **18** points; We assume that the initial position of P1 is 1st and P2 is 2nd
Players Strategies	**Aggressive(A)** : High risk of crashing but can improve the position **Conservative(C):** Low risk of crashing but maintains position
Crash Prob.	**P[A] = 0.3, P[C] = 0.1**
Crash Impact	Crashing results in **0** points
Position Changes(If no crash)	If P2 is A and P1 is C, P2 move to 1st and P2 move to 2nd; if they are both C or A, their position would maintain the same
Expected Payoff (Consider 4 cases)	**Case 1:** P1 and P2 do not crash ; **Case 2:** P1 and P2 crash both **Case 3:** P1 does crash, P2 not ; **Case 4:** P2 does crash, P1 not

Let's begin by examining **(A, A)**, where both P1 and P2 opt for aggressive driving, U1 and U2 represent the payoff of P1 and P2 in each case, P represent the probability of the case

Case 1 :
P[Case 1] = 0.7*0.7 = 0.49; U1 = 0.49*25 = 12.25, U2 = 0.49*18 = 8.82

Case 2:
P[Case 2] = 0.3*0.3 = 0.09; U1 = 0, U2 = 0

Case 3:
P[Case 1] = 0.3*0.7 = 0.21; U1 = 0, U2 = 0.21*25 = 5.25

Case 4:
P[Case 1] = 0.7*0.3 = 0.21; U1 = 0.21*25 = 5.25, U2 = 0

The Final Payoff of P1 and P2 when **(A, A):**

U1,tot = 12.25+0+0+5.25 = 17.5, U2,tot = 8.82 + 0 + 5.25 + 0 = 14.07

We complete the payoff matrix by repeating the process for (C, A), (A, C), and (C, C), determining that the **Nash Equilibrium (NE)** is **(C, A)** which P1 chooses to be Conservative and P2 choose to be Aggressive

P1\P2	A	C
A	17.5, 14.07	17.5, 18.09
C	18.09, 17.5	22.5, 16.83

Next, we attempt to standardize the payoff matrix by setting P[A] = p1 and P[C] = p2 to observe its behavior, where 1 > p1 > p2 > 0

P1\P2	A	C
A	25(1-p1), (1-p1)(18+7p1)	25(1-p1), (1-p2)(18+7p1)
C	(1-p2)(18+7p1), 25(1-p1)	25(1-p2), (1-p2)(18+7p2)

For easier reference, we will redefine the payoff for P1 and P2

P1\P2	A	C
A	Q1, Q2	Q1, Q3
C	Q3, Q1	Q4, Q5

It is clear that Q3 > Q5, Q4> Q1, given by p1 > p2 and when:

P1 chooses A: P2 would choose Q3
P1 chooses C: P2 would choose Max(Q1, Q5)
P2 chooses A: P1 would choose Max(Q1, Q3)
P2 chooses C: P1 would choose Q4

And there would be 3 situations among Q1, Q3,Q5:

1.Q1 > Q3 > Q5

P1\P2	A	C
A	Q1, Q2	Q1, Q3
C	Q3, Q1	Q4, Q5

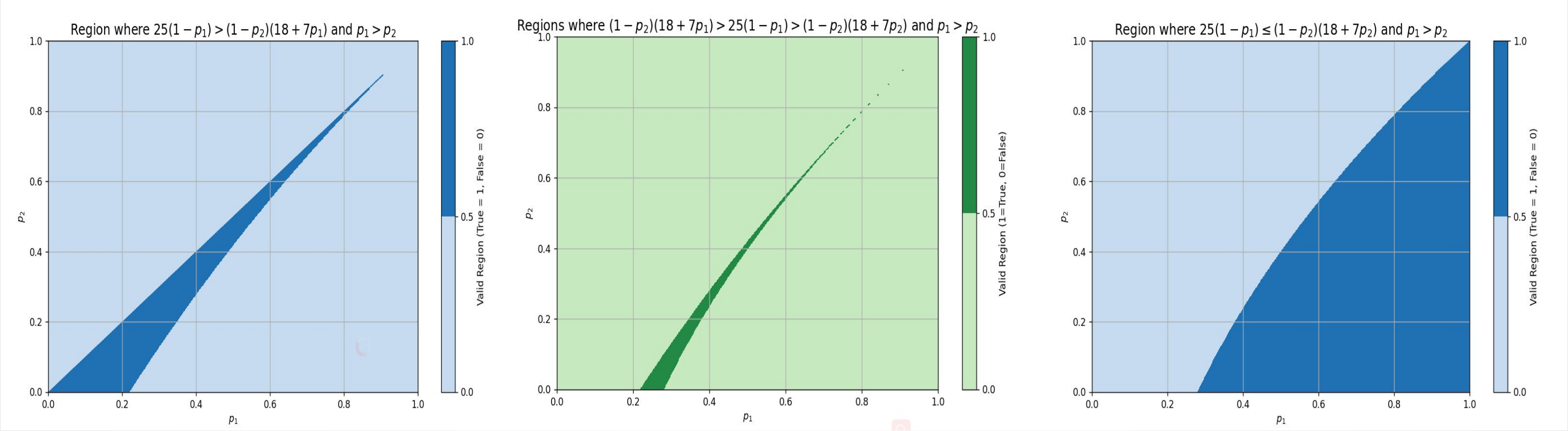
2.Q3 > Q1 > Q5

P1\P2	A	C
A	Q1, Q2	Q1, Q3
C	Q3, Q1	Q4, Q5

3.Q3 > Q5 > Q1

P1\P2	A	C
A	Q1, Q2	Q1, Q3
C	Q3, Q1	Q4, Q5

And the domain of each situation below:



1. Q1 > Q3 > Q5

2. Q3 > Q1 > Q5

3. Q3 > Q5 > Q1

5 Conclusion

Use Tyre Strategies Wisely

- As a pre-game strategy, the MH tyre strategy is the most optimal for Singapore GP
    - Total Race Time (MH) = 6198.52s Total Race Time (MHM) = 6223.39s
- Ingame strategy consists of unpredictable and dynamic events that require reactive decision making.
    - Total Race Time FER (MH) = 6215s > Total Race Time MER (MHM) = 6187s

Use Tyre Strategies Wisely

01

Medium-Hard

If No Safety Car throughout the race, and if Ferrari (race leader) does not pit before Lap 23, MH Performs better than MHM

02

Medium-Hard-Medium

If a Safety Car comes before Lap 32, MHM may perform better than MH (subject to Mercedes' 2nd pit stop)

Thank You