Poisson Cluster Process: Bridging the Gap Between PPP and 3GPP HetNet Models

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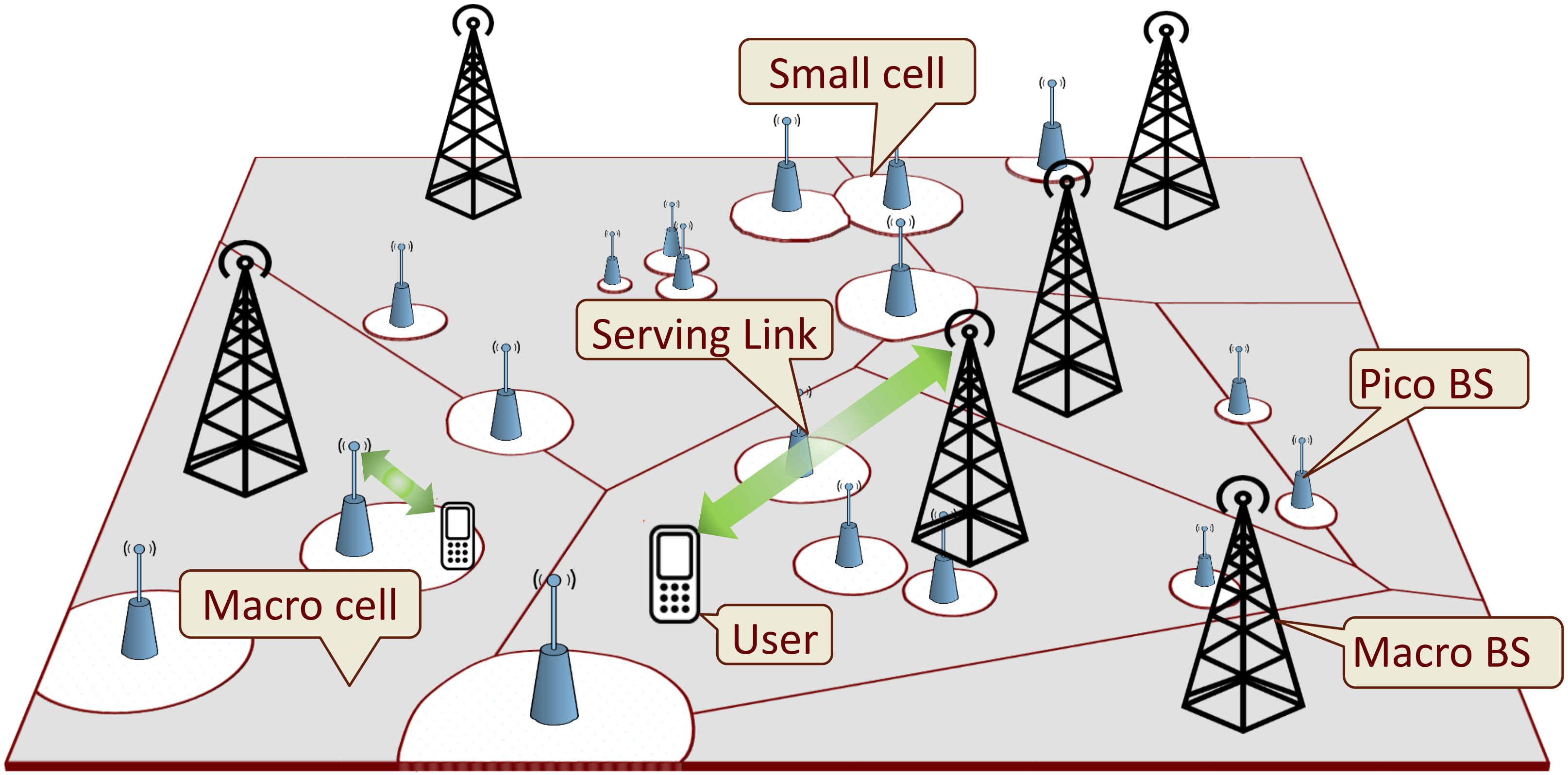
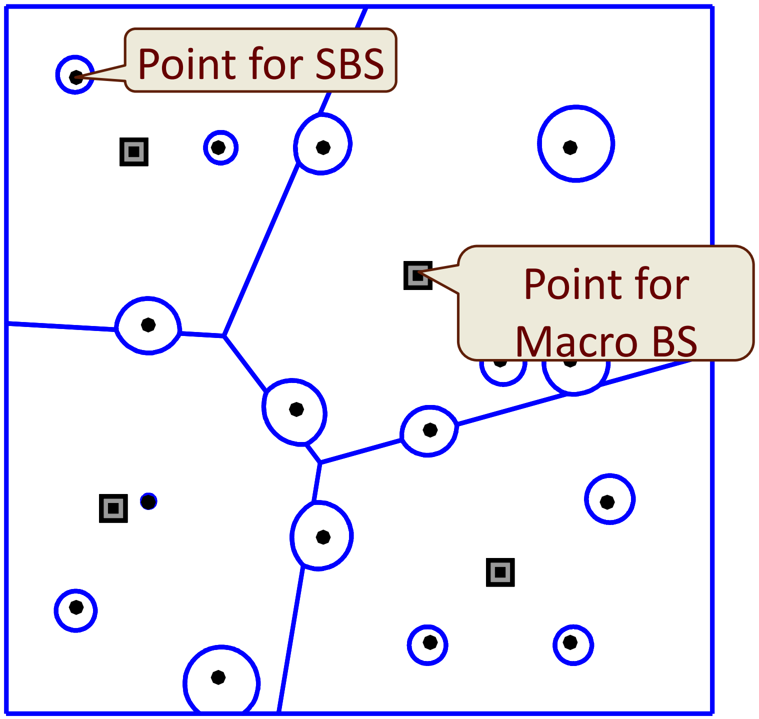
HetNet: Different Components



HetNet with different BSs and coverage regions

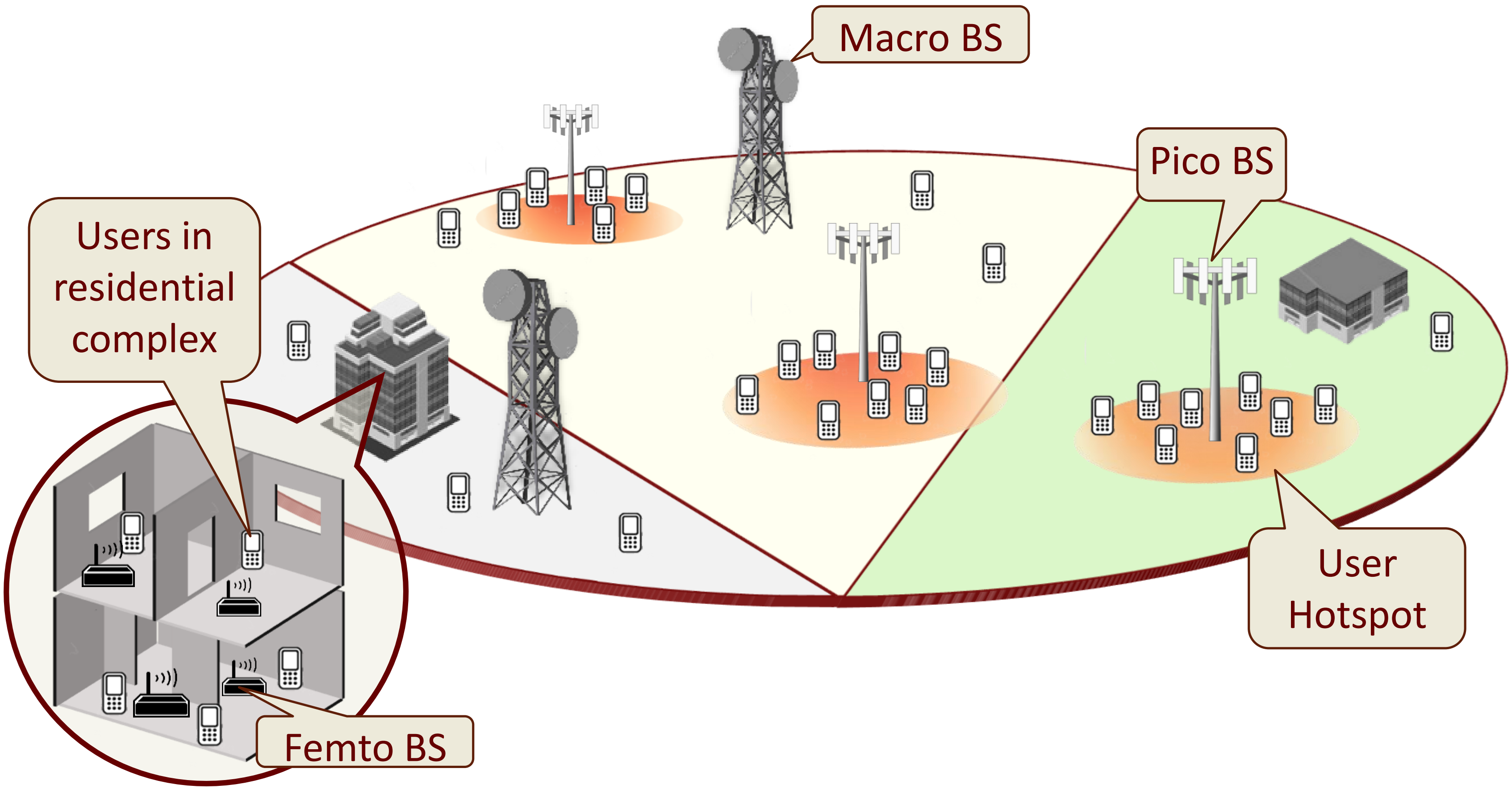
PPP Model of HetNets



A Two-tier Baseline HetNet Model

- BSs of different tiers (macro and small cells) and users are modeled as homogeneous Poisson point process (PPP) over $\mathbb{R}^2$.
- The statistics of $\mathtt{SINR}$ at the origin is computed.
- Due to stationarity, this result is used to characterize coverage probability of a typical user of the network.

PPP Model: How far from actual HetNet?



A Simple Illustration of HetNet

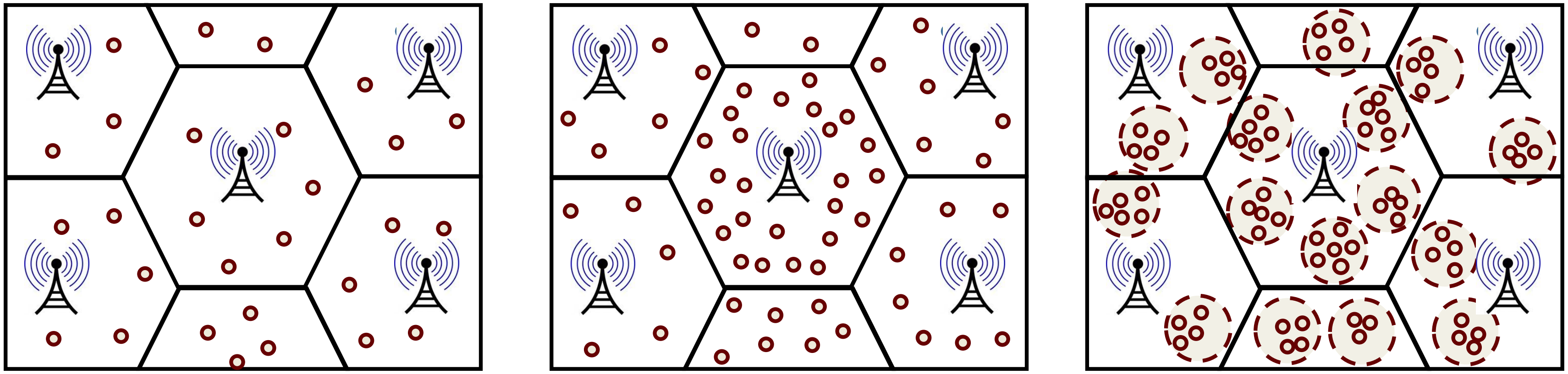
PPP Model: How far from actual HetNet?

- Key components missing in the baseline PP model:
  - Non-uniformity in user distribution:
    - Modeling all users as an independent PPP is not realistic.
    - Fraction of users form spatial clusters (*Hotspots*), e.g. users in public places and residential areas.
  - Correlation between small cell BS (SBS) and user location:
    - Operators deploy pico BSs at higher density at user hotspot.
    - Users install their own femto BSs.
  - Inter and Intra BS-tier Dependence: *
    - BS locations are not necessarily independent.
    - Site planning for deploying BSs introduces correlation in BS locations.

* N. Deng, W. Zhou, and M. Haenggi, "Heterogeneous cellular network models with dependence," IEEE JSAC, 2015.

3GPP Models: User Distribution

- 3GPP considers different configurations of SBSs and users in HetNet simulation model.



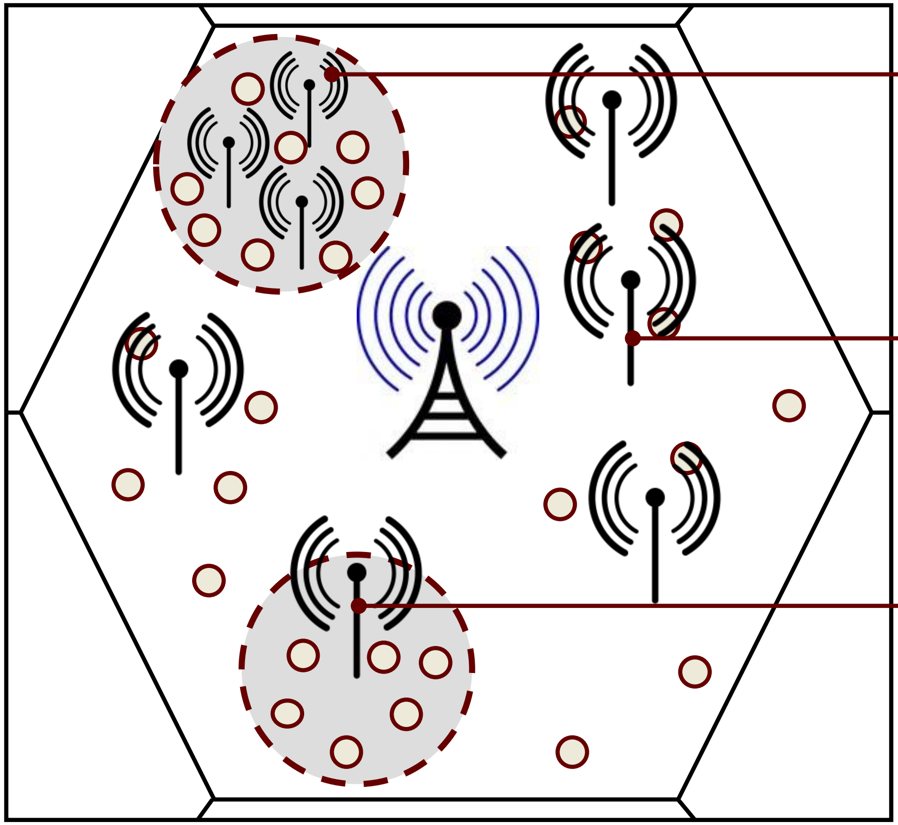
Users "uniform" across macro cells

Users "non-uniform" across macro cells

Users forming clusters within a disc

User Configurations in 3GPP HetNet Model

3GPP Model: SBS Distribution

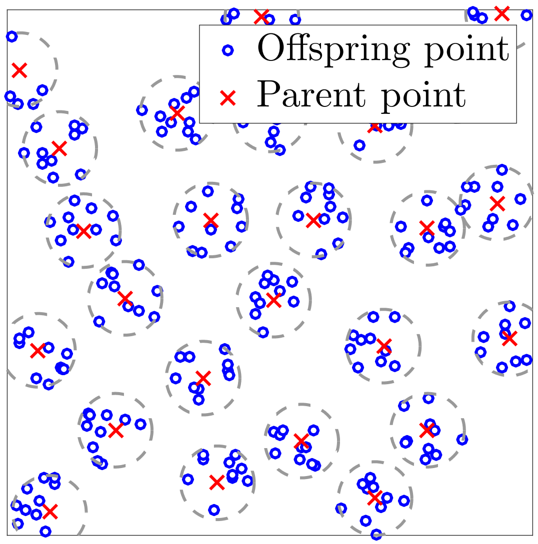


SBS Configurations in 3GPP HetNet Model

- SBSs deployed at higher density at certain areas (indoor models)
- SBSs deployed randomly or under some site planning.
- SBSs at the center of user hotspot.

How can PCP enhance HetNet Model?

- **Poisson Cluster Process** (PCP) is more appropriate abstraction for user and BS distributions considered by 3GPP.



Matern Cluster Process:
An example of PCP

Definition: Poisson Cluster Process (PCP)

A PCP is generated from a PPP Φ_{p} called the **Parent PPP**, by replacing each point $\mathbf{z}_i$ by a finite offspring point process $\mathcal{B}_i$ where each point is independently and identically distributed around origin.

$$\Phi = \bigcup_{\mathbf{z}_i \in \Phi_{\mathrm{p}}} \mathbf{z}_i + \mathcal{B}_i .$$

Definition: Matern Cluster Process

- $\#\mathcal{B}_i \sim \mathtt{Poisson}(\bar{m})$
- Each point in $\mathcal{B}_i$ is uniformly distributed inside disc of radius r_{d} centered at origin.

PCP Based HetNet Models

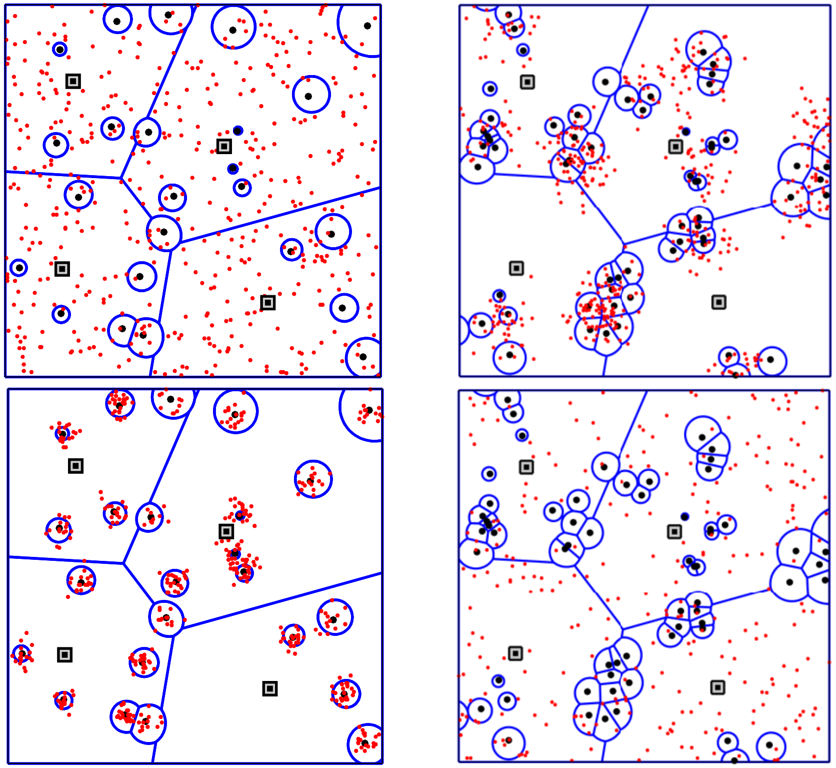
- New HetNet models with combination of PPP and PCP have closer resemblance to 3GPP HetNet models.

Model 1: Users PPP, BSs PPP (Baseline)

Model 3: Users PCP, BSs PCP

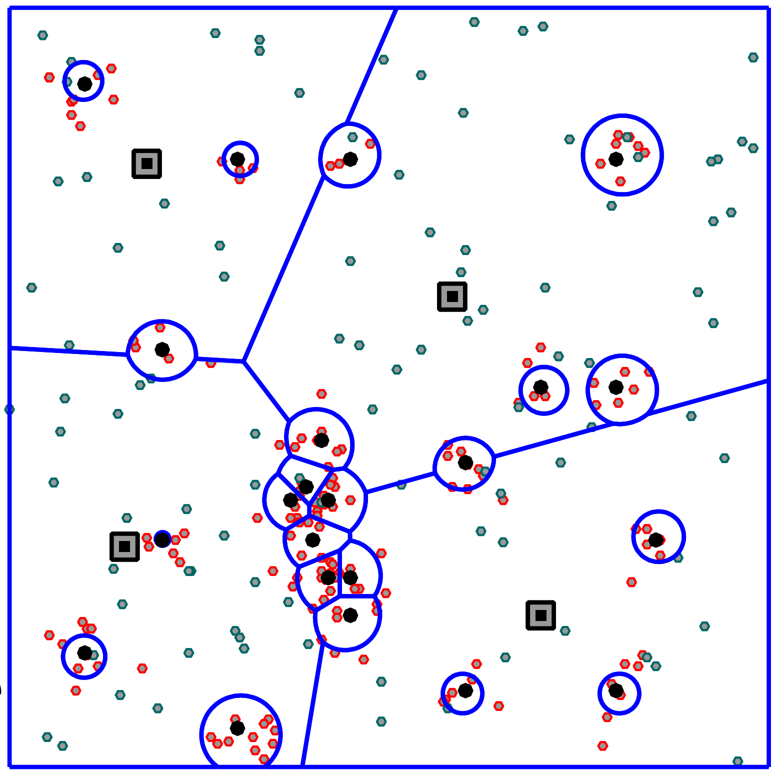
Model 2: Users PCP, BSs PPP

Model 4: Users PPP, BSs PCP



Unified HetNet Model: PCP meets PPP

- We consider K tier HetNet with two sets of BS PPs:
  - K_1 BS tiers are modeled as independent PPPs.
  - K_2 BS tiers are modeled as independent PCPs.
- User Distribution:
  - Users can be independent PPP.
  - Users can be a PCP correlated with the BS distribution.
- For this setup, we derive coverage probability.



Models 1-4 appear as special cases of this unified framework.

User Association: Max SIR and Max Power

- Two types of cell association are considered.
  - **Max Power**: Connect to the BS providing maximum average received power.
    - Contact and Nearest distance distributions of PCP and PPP are the key components of analysis*.
  - **Max SIR**: Connect to the BS providing maximum instantaneous SIR (assuming interference limited network).
    - The analysis is well-known for the Baseline model **.
    - We generalize this approach to enable coverage analysis for the proposed Unified Model.

*   M. Afshang, C. Saha, and H. S. Dhillon, "Nearest-Neighbor and Contact Distance Distributions for Thomas Cluster Process", IEEE Wireless Commun. Letters, 2016.

** H. S. Dhillon, R. K. Ganti, F. Baccelli, and J. G. Andrews, "Modeling and analysis of K-tier downlink heterogeneous cellular networks," IEEE Journal on Sel. Areas in Commun., 2012.

Construction of 0^{th} tier

- If the users are PCP, the user PP is correlated with one BS PPP Φ_k.
- Correlation can be handled by constructing the 0^{th} tier (Φ_0).
  - If users are clustered around Φ_k which is a PPP, Φ_0 is only the BS of Φ_k at cluster center (**z**) of the typical user.
  - If users same parent PPP of Φ_k which is a PCP, Φ_0 is the set of BSs in Φ_k belonging to the same cluster center (**z**) of the typical user.

Coverage Probability

- Coverage probability, or equivalently CCDF of SIR , is:

$$P_c = \mathbb{P}(\mathtt{SIR} \geq \beta) = \bigcup_{k\in\mathcal{K}} \mathbb{E}\left[\mathbf{1}\left(\bigcup_{\mathbf{x}\in\Phi_k} \frac{P_k h_{\mathbf{x}} \|\mathbf{x}\|^{-\alpha}}{\mathcal{I}(\Phi_k \setminus \{\mathbf{x}\}) + \sum_{j\in\mathcal{K}\setminus\{k\}} \mathcal{I}(\Phi_j)} > \beta\right)\right]$$

where $\mathcal{I}(\Phi_i) = \sum_{y\in\Phi_i} P_i h_{\mathbf{y}} \|\mathbf{y}\|^{-\alpha}$ is the interference from all BSs in Φ_i.

- For $\beta > 1$, P_c can be expressed as sum of probability that serving BS belongs to $\Phi_k (k \in \mathcal{K})$.

$$P_c = \sum_{k\in\mathcal{K}} P_{c(k)} = \sum_{k\in\mathcal{K}} \mathbb{E}\left[\mathbf{1}\left(\sum_{\mathbf{x}\in\Phi_k} \frac{P_k h_{\mathbf{x}} \|\mathbf{x}\|^{-\alpha}}{\mathcal{I}(\Phi_k \setminus \{\mathbf{x}\}) + \sum_{j\in\mathcal{K}\setminus\{k\}} \mathcal{I}(\Phi_j)} > \beta\right)\right]$$

Coverage Probability

- By Rayleigh fading assumption

$$\sum_{k\in\mathcal{K}}\mathbb{E}\Big[\sum_{\mathbf{x}\in\Phi_k}\exp\Big(-\frac{\beta}{P_k}(\mathcal{I}(\Phi_k\setminus\{\mathbf{x}\})\Big)\exp\Big(-\frac{\beta}{P_k}(\sum_{j\in\mathcal{K}\setminus\{k\}}\mathcal{I}(\Phi_j))\|\mathbf{x}\|^{\alpha}\Big)\Big]$$

- Taking expectation with respect to fading

$$\sum_{k\in\mathcal{K}}\underbrace{\mathbb{E}\Big[\sum_{\mathbf{x}\in\Phi_k}\prod_{\mathbf{y}\in\Phi_k\setminus\{\mathbf{x}\}}\frac{1}{1+\frac{\beta}{P_k}\left(\frac{\|\mathbf{x}\|}{\|\mathbf{y}\|}\right)^{-\alpha}}\Big]\mathbb{E}\Big[\prod_{j\in\mathcal{K}\setminus\{k\}}\exp\Big(-\frac{\beta}{P_k}(\mathcal{I}(\Phi_j))\|\mathbf{x}\|^{\alpha}\Big)\Big]}_{P_{c(k)}\text{: Per-tier Coverage}}$$

Coverage Probability

$$P_{\mathrm{c}(k)} = \underbrace{\mathbb{E}\left[\sum_{\mathbf{x}\in\Phi_k}\prod_{\mathbf{y}\in\Phi_k\setminus\{\mathbf{x}\}}\frac{1}{1+\frac{\beta}{P_k}\left(\frac{\|\mathbf{x}\|}{\|\mathbf{y}\|}\right)^{-\alpha}}\right]}\prod_{j\in\mathcal{K}\setminus\{k\}}\underbrace{\mathbb{E}\left[\exp\left(-\frac{\beta}{P_k}\mathcal{I}(\Phi_j)\|\mathbf{x}\|^{\alpha}\right)\right]}$$

This term belongs to a general family of functional over point processes called **sum-product functional**.

Can be expressed in terms of **probability generating functional (PGFL)** of Φ_j.

Sum-Product Functional

$$\mathbb{E}\left[\sum_{\mathbf{x}\in\Phi_k}\prod_{\mathbf{y}\in\Phi_k\setminus\{\mathbf{x}\}}\frac{1}{1+\frac{\beta}{P_k}\left(\frac{\|\mathbf{x}\|}{\|\mathbf{y}\|}\right)^{-\alpha}}\right] \equiv \mathbb{E}\left[\sum_{\mathbf{x}\in\Psi} g(\mathbf{x})\prod_{\mathbf{y}\in\Psi\setminus\{\mathbf{x}\}} v(\mathbf{x},\mathbf{y})\right],$$

Definition

We define Sum-product functional of a point process Ψ can be defined as:

$$\mathbb{E}\left[\sum_{\mathbf{x}\in\Psi} g(\mathbf{x})\prod_{\mathbf{y}\in\Psi\setminus\{\mathbf{x}\}} v(\mathbf{x},\mathbf{y})\right],$$

where $g(\mathbf{x}) : \mathbb{R}^2 \mapsto [0,1]$ and $v(\mathbf{x},\mathbf{y}) : [\mathbb{R}^2 \times \mathbb{R}^2] \mapsto [0,1]$ are measurable.

- There are more general versions of Sum-product functional *.
- There will be different expressions of the sum-product functional depending on $\Phi_k \ (k \in \mathcal{K})$.

* U. Schilcher, S. Toumpis, M. Haenggi, A. Crismani, G. Brandner and C. Bettstetter, "Interference Functionals in Poisson Networks," in *IEEE Trans. Inf. Theory*, Jan. 2016.

Sum product functional

- We require sum-product functional of Φ_k when:

Case 1: Φ_k is PPP

Case 2: Φ_k is PCP

Campbell-Mecke Theorem can be applied to evaluate sum-product functional.

Case 3: $\boxed{\Phi_k = \Phi_0}$

- The third case occurs when the users are modeled as PCP having same parent PPP as that of some PCP distributed BS tier.
- Has to be evaluated separately as Φ_0 is a **finite** PP.

Sum product functional: PPP

Lemma 1:

The sum product functional of Ψ when Ψ is a PPP can be expressed as follows:

$$\mathbb{E}\left[\sum_{\mathbf{x}\in\Psi} g(\mathbf{x}) \prod_{\mathbf{y}\in\Psi\setminus\{\mathbf{x}\}} v(\mathbf{x},\mathbf{y})\right] = \int_{\mathbb{R}^2} g(\mathbf{x}) G(v(\mathbf{x},\mathbf{y}))\Lambda(\mathrm{d}\mathbf{x}),$$

where $\Lambda(\cdot)$ is the mean measure of Ψ, $G(\cdot)$ denotes the PGFL of Ψ.

H. S. Dhillon, R. K. Ganti, F. Baccelli, and J. G. Andrews, "Modeling and analysis of K-tier downlink heterogeneous cellular networks," IEEE JSAC, 2012.

Sum product functional: PCP

- In PCP, selecting a point from **x** implies selecting a tuple **(x, z)**, where **z** is the cluster center of **x.**

Lemma 2:

The sum product functional of Ψ when Ψ is a PCP can be expressed as follows:

$$\mathbb{E}\left[\sum_{\mathbf{x}\in\Psi} g(\mathbf{x}) \prod_{\mathbf{y}\in\Psi\setminus\{\mathbf{x}\}} v(\mathbf{x},\mathbf{y})\right] = \iint_{\mathbb{R}^2\times\mathbb{R}^2} g(\mathbf{x})\widetilde{G}(v(\mathbf{x},\mathbf{y})|\mathbf{z})\Lambda(\mathrm{d}\mathbf{x},\mathrm{d}\mathbf{z})$$

with

$$\Lambda(\mathrm{d}\mathbf{x},\mathrm{d}\mathbf{z}) = \lambda_{\mathrm{p}_k}\bar{m}_k(\mathbf{x}-\mathbf{z})\mathrm{d}\mathbf{z}\,\mathrm{d}\mathbf{x}$$

$\widetilde{G}(\cdot)$ denotes PGFL of Ψ with respect to its reduced Palm distribution.

Sum product functional: A special Case

- Conditioning on the number of offspring points is the key to the analysis of sum product functional of **finite** PP Φ_0, i.e., BSs in Φ_k belonging to the same cluster center of the typical user.

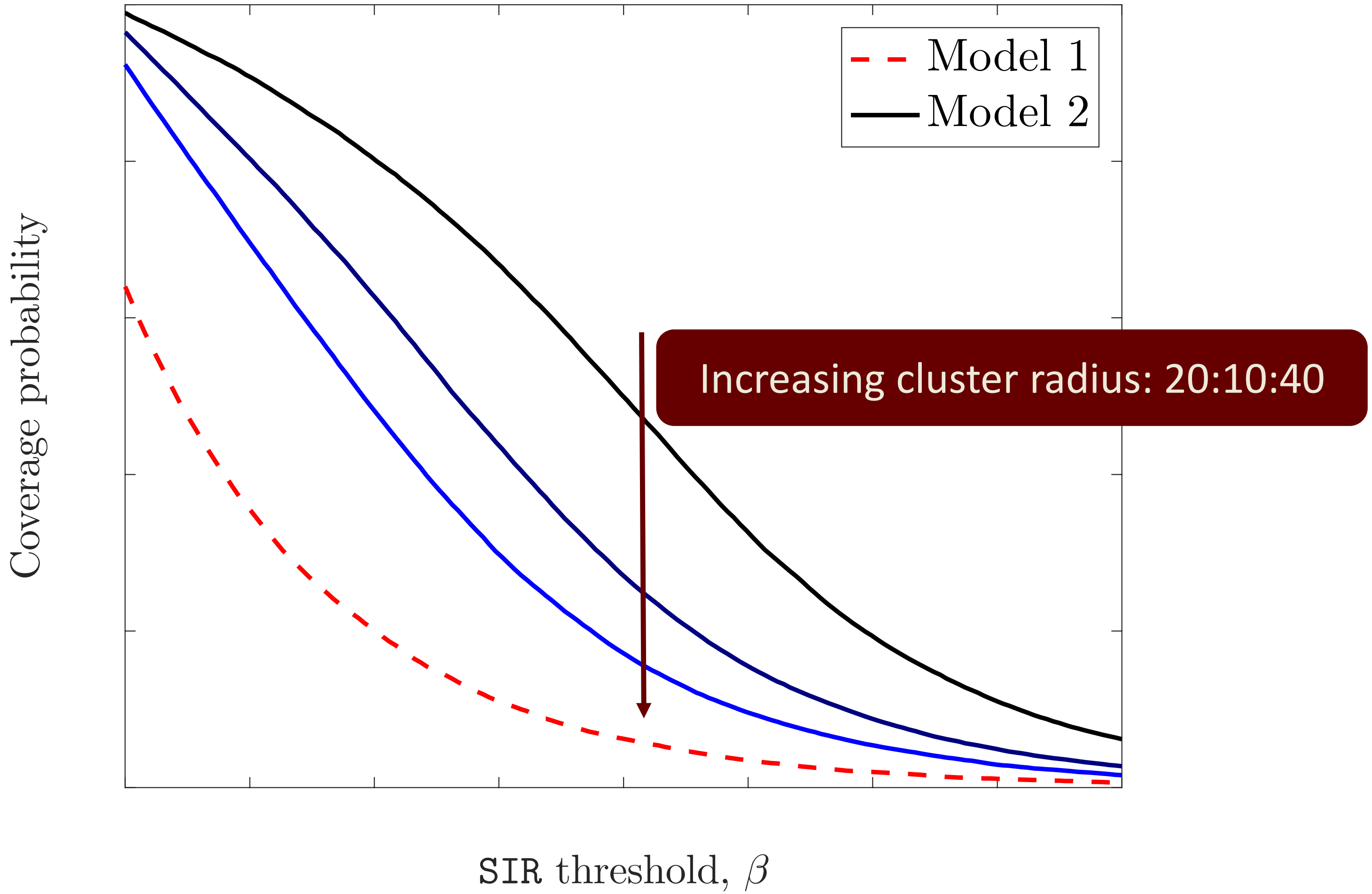
Lemma 3:

The sum product functional of Ψ when Ψ is a cluster of a randomly chosen point of Φ_k $(k \in \mathcal{K}_2)$ can be expressed as follows:

$$\mathbb{E}\left[\sum_{\mathbf{x}\in\Psi} g(\mathbf{x}) \prod_{\mathbf{y}\in\Psi\setminus\{\mathbf{x}\}} v(\mathbf{x},\mathbf{y})\right]$$

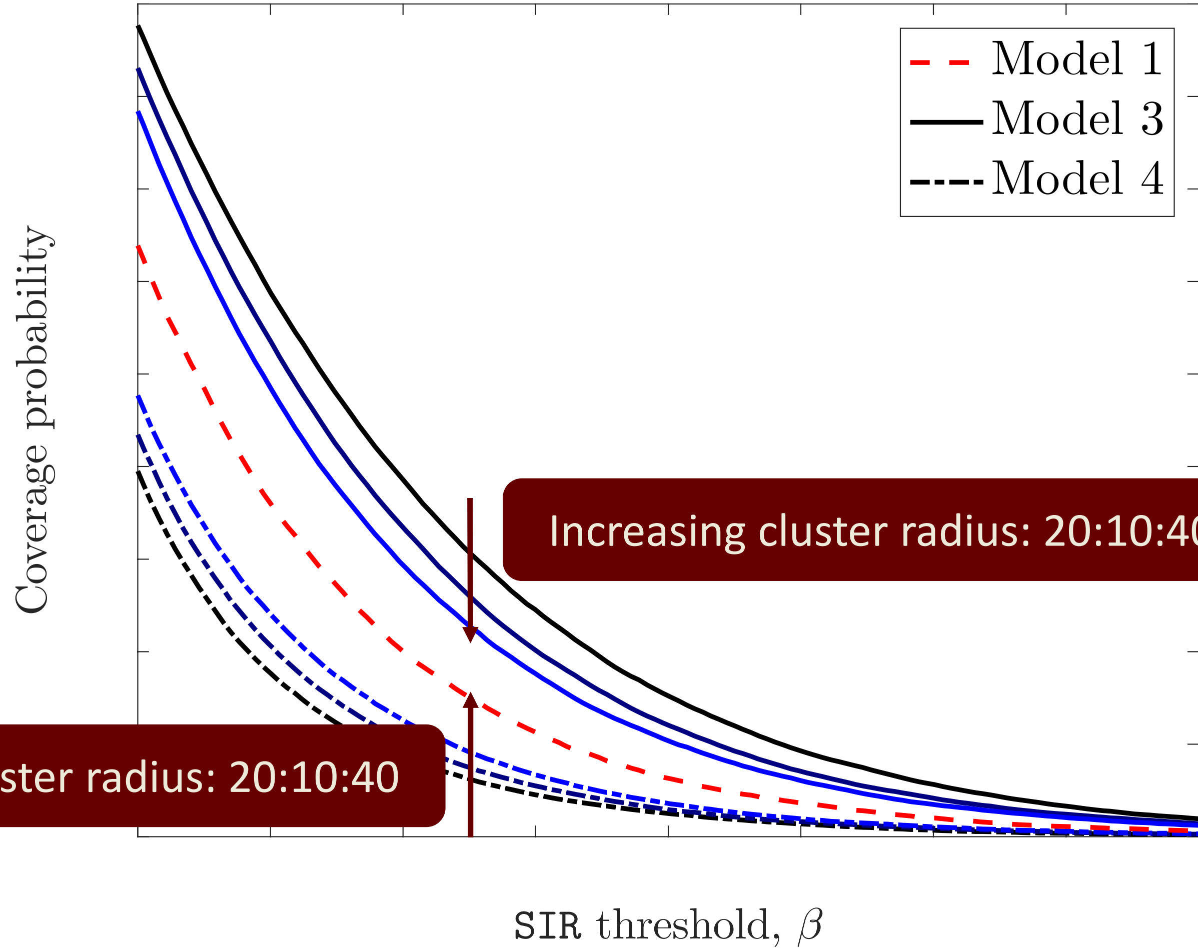
$$= \sum_{n=1}^{\infty} \int_{\mathbb{R}^2} g(\mathbf{x}) \left(\int_{\mathbb{R}^2} (v(\mathbf{x},\mathbf{y}) f_k(\mathbf{y}-\mathbf{z}) \mathrm{d}\mathbf{y} \right)^{n-1} f_k(\mathbf{x}-\mathbf{z}) \mathrm{d}\mathbf{x}\, n^2 \frac{p_{k(n)}}{\bar{m}_k}$$

Results: Model 1 Vs. Model 2



In Model 2, the coverage probability decreases as cluster radius increases and shifts towards that of Model 1.

Results: Models 1,3 and 4



Increasing cluster radius: 20:10:40

Increasing cluster radius: 20:10:40

Increasing cluster radius has a conflicting effect on coverage probability of Model 3 and 4: coverage probability of Model 4 increases whereas that of Model 3 decreases.

Conclusive Remarks

- We propose a unified framework for modeling K-tier HetNet.
- Derived coverage probability for this model under max-SIR connectivity.
  - Sum-product functionals for PCP and finite point processes were derived as key components of analysis.
- Special cases of the Model closely resemble BS and user configurations in 3GPP simulations of HetNet.
- Future Directions:
  - Rate analysis
  - Incorporation of shadowing and general channel models
  - Extension to uplink
  - Developing unified framework for Max-power association
  - Modeling spatial separation of BSs *

* M. Afshang and H. S. Dhillon, "Spatial modeling of device-to-device networks: Poisson cluster process meets Poisson hole process," in Proc., IEEE Asilomar, Nov. 2015,

Relevant Publications

- M. Afshang, H. S. Dhillon, "Poisson Cluster Process Based Analysis of HetNets with Correlated User and Base Station Locations", submitted, Available online: arxiv.org/abs/1612.07285
- C. Saha, M. Afshang, H. S. Dhillon, "Enriched K-Tier HetNet Model to Enable the Analysis of User-Centric Small Cell Deployments", IEEE Trans. on Wireless Commun., 2017.
- M. Afshang, C. Saha, and H. S. Dhillon, "Nearest-Neighbor and Contact Distance Distributions for Thomas Cluster Process", IEEE Wireless Commun. Letters, 2016.
- M. Afshang, H. S. Dhillon, and P. H. J. Chong , "Modeling and Analysis of Clustered Device-to-Device Networks" *IEEE Trans. on Wireless Commun., 2016*.
- M. Afshang, H. S. Dhillon, and P. H. J. Chong, "Fundamentals of Cluster-Centric Content Placement in Cache-Enabled Device-to-Device Networks " *IEEE Trans. on Commun.*,2016.

Thank You for Your Attention

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Coverage Probability

$$= \sum_{k \in \mathcal{K}} \mathbb{E}\left[\mathbf{1}\left(\sum_{\mathbf{x} \in \Phi_k} \frac{P_k h_{\mathbf{x}} \|\mathbf{x}\|^{-\alpha}}{\mathcal{I}(\Phi_k \setminus \{\mathbf{x}\}) + \sum_{j \in \mathcal{K} \setminus \{k\}} \mathcal{I}(\Phi_j)} > \beta\right)\right]$$

By $\beta > 1$ assumption

$$= \sum_{k \in \mathcal{K}} \mathbb{E}\left[\sum_{\mathbf{x} \in \Phi_k} \mathbb{P}\left(h_{\mathbf{x}} > \frac{\beta}{P_k}\left(\mathcal{I}(\Phi_k \setminus \{\mathbf{x}\}) + \sum_{j \in \mathcal{K} \setminus \{k\}} \mathcal{I}(\Phi_j)\right) \|\mathbf{x}\|^{\alpha}\right)\right]$$

$$= \sum_{k \in \mathcal{K}} \mathbb{E}\left[\sum_{\mathbf{x} \in \Phi_k} \exp\left(-\frac{\beta}{P_k}(\mathcal{I}(\Phi_k \setminus \{\mathbf{x}\})\right)\right.$$
$$\left. \times \exp\left(-\frac{\beta}{P_k}\left(\sum_{j \in \mathcal{K} \setminus \{k\}} \mathcal{I}(\Phi_j)\right) \|\mathbf{x}\|^{\alpha}\right)\right]$$

By Rayleigh assumption

$$= \sum_{k \in \mathcal{K}} \mathbb{E}\left[\sum_{\mathbf{x} \in \Phi_k} \prod_{\mathbf{y} \in \Phi_k \setminus \{\mathbf{x}\}} \frac{1}{1 + \frac{\beta}{P_k}\left(\frac{\|\mathbf{x}\|}{\|\mathbf{y}\|}\right)^{-\alpha}}\right] \mathbb{E}\left[\prod_{j \in \mathcal{K} \setminus \{k\}} \exp\left(-\frac{\beta}{P_k}(\mathcal{I}(\Phi_j)) \|\mathbf{x}\|^{\alpha}\right)\right]$$

Sum product functional is the key to the derivation of coverage

Sum product functional: A special Case

- In order to model the correlation between user and BS, the sum product functional of the cluster within which user is located should be handled separately.

Lemma 3:

The sum product functional of Ψ when Ψ is a cluster of a randomly chosen point of Φ_k $(k \in \mathcal{K}_2)$ can be expressed as follows:

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \exp\Bigg(-\bar{m}_k\Bigg(\int_{\mathbb{R}^2}\Big(1-v(\mathbf{x},\mathbf{y})\Big)f_k(\mathbf{y}-\mathbf{z})\mathrm{d}\mathbf{y}\Bigg)\Bigg)$$
$$\times\Big(\bar{m}_k\int_{\mathbb{R}^2} v(\mathbf{x},\mathbf{y})f_k(\mathbf{y}-\mathbf{z})\mathrm{d}\mathbf{y}+1\Big)f_k(\mathbf{x}-\mathbf{z})f_k(\mathbf{z})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{z}.$$