Applied Linear Algebra for Signal Processing, Data Analytics and Machine Learning

Basics:

We consider vectors for representing data

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \leftarrow \text{ if } v_1, v_2 \cdots v_n \in R$$

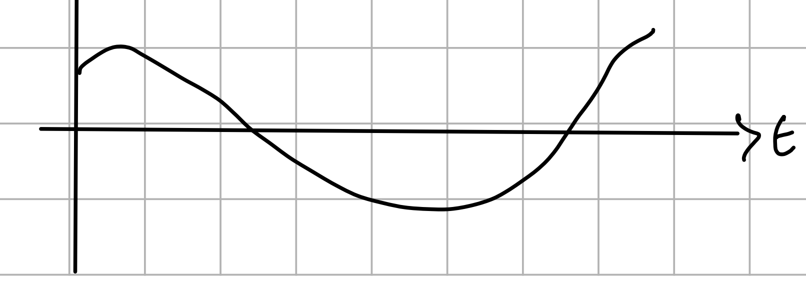
then $\vec{v} \in R^n$

else if $v_1, v_2 \cdots v_n \in \mathbb{C}$ (complex no.)

then $\vec{v} \in \mathbb{C}^n$

We can represent many types of data as vector, for example

Signals:



Location : $\begin{bmatrix} - \\ - \\ - \\ - \end{bmatrix}$

and so on

Basics operation on vectors

Addition $u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ $\quad v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

$$u + v = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}$$

Scalar Multiplication $\quad k\vec{v} = k \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} kv_1 \\ kv_2 \\ \vdots \\ kv_n \end{bmatrix}$

Linear Combination :

$\vec{U_1}, \vec{U_2}, \vec{U_3}, \dots \vec{U_m}$: vectors

$k_1, k_2, k_3 \dots k_n$: Scalars

$k_1 v_1 + k_2 v_2 + k_3 v_3 + k_4 \dots k_n v_n \leftarrow$ Linear combination of vectors

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Inner product

Consider arbitary vectors , $\bar{U}, \bar{V} \in \mathbb{R}^n$

$$\bar{U} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \qquad \bar{V} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$\bar{U}^T = \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix}$$

Inner product of $\bar{U}$ and $\bar{V}$ is: $\bar{U}^T V = [u_1 u_2 \dots u_n] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

$\langle \bar{U}, \bar{V} \rangle$

= a number (Scalar)

for complex vectors

Hermatian

$$\langle \bar{U}, \bar{V} \rangle = \bar{u}^H \bar{v}$$

$$= \begin{bmatrix} u_1^* & u_2^* & \cdots & u_n^* \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Conjugate of u_1

$$= \sum_{i=1}^{n} u_i^* v_i$$

Orthogonality:

Two vectors are orthogonal if their inner product is zero

for

$\mathbb{R}^n$: $\bar{u}^T \bar{v} = 0$

$\mathbb{C}^n$: $\bar{u}^H \bar{v} = 0$

Ex:

$$\bar{u} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \qquad \bar{v} = \begin{pmatrix} 2 \\ 2 \\ -2 \\ -2 \end{pmatrix}$$

$$\bar{u}^T \bar{v} = 2 - 2 - 2 + 2$$

$$= 0$$

Complex Sinusoids

$$\bar{u} = \begin{bmatrix} 1 \\ e^{j 2\pi/n} \\ e^{j 4\pi/n} \\ \vdots \\ e^{j 2\pi (n-1)/n} \end{bmatrix} \rightarrow \text{freq of } \bar{u} = \frac{1}{n}$$

$$\bar{v} = \begin{bmatrix} 1 \\ e^{j 4\pi/n} \\ e^{j 6\pi/n} \\ \vdots \\ e^{j 4\pi (n-1)/n} \end{bmatrix} \xrightarrow{\text{freq}} \frac{2}{n}$$

$\bar{v}$ and $\bar{u}$ are two complex sinusoids

- Complex sinusoids at the frequency $0, \frac{1}{n}, \frac{2}{n}, \ldots \frac{n-1}{n}$ are orthogonal

- This property is very important in Fourier analysis

Norm of a vector

denoted by $\|v\|$

$$\text{norm of } u = \sqrt{\langle \bar{u}, \bar{u} \rangle}$$

$$= \sqrt{u_1^2 + u_2^2 + u_3^2 + u_4^2 \cdots u_n^2}$$

$\|\bar{u}\|_2 \leftarrow$ length of the vector

for complex vector:

$$\|\bar{u}\| = \sqrt{|u_1|^2 + |u_2|^2 \ldots + |u_n|^2}$$

mod

Easy to see: $\|\bar{u}\| \geq 0$

Video-2

$\|\bar{u}\| \leftarrow$ norm of a vector

$\|\bar{u}\| = 1$ means $\bar{u}$ is a unit norm vector

$\frac{\bar{u}}{\|\bar{u}\|} = \tilde{u} \leftarrow$ unit norm vector for any vector $\bar{u}$

Norm has interesting applications for signals

Ex: for a signal vector $\bar{u}$

$$\|\bar{u}\|^2 = u_1^2 + u_2^2 + u_3^2 + \dots u_n^2$$

$$= \text{Energy of a signal}$$

$$\bar{u} = \begin{bmatrix} -1 \\ 3 \\ -2 \\ 1 \end{bmatrix}$$

$$\|\bar{u}\| = \sqrt{1 + 9 + 4 + 1}$$

$$= \sqrt{15}$$

$$\frac{\bar{u}}{\|\bar{u}\|} = \frac{1}{\sqrt{15}} \begin{bmatrix} 1 \\ 3 \\ -2 \\ 1 \end{bmatrix}$$

unit norm

Distance

$$\bar{u}, \bar{v}$$

$$\|\bar{u} - \bar{v}\|$$

$$= \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \dots (u_n - v_n)^2}$$

Cauchy-Schwarz inequality

or CS inequality

for real vectors:

$$|\langle \bar{u}, \bar{v} \rangle| \leq \|\bar{u}\| \, \|\bar{v}\|$$

magnitude

or

$$|\langle \bar{u}, \bar{v} \rangle|^2 \leq \|\bar{u}\|^2 \, \|\bar{v}\|^2$$

for complex vectors

$$|\bar{u}^H \bar{v}|^2 \leq \|\bar{u}\|^2 \, \|\bar{v}\|^2$$

Simple proof:

Consider real vectors:

$$\bar{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \bar{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\bar{u}^T \bar{v} = u_1 v_1 + u_2 v_2$$

$$(\bar{u}^T \bar{v})^2 = (u_1 v_1 + u_2 v_2)^2$$

$$= u_1^2 v_1^2 + u_2^2 v_2^2 + 2 u_1 v_1 u_2 v_2$$

now if we use the theorem

Geometric mean $\leq$ arthematic mean

(This is nothing but:

$$\sqrt{ab} \leq \frac{a+b}{2}$$
)

using that it becomes

$$2 u_1 u_2 v_1 v_2 \leq u_1^2 v_2^2 + u_2^2 v_1^2$$

$$(\bar{u}^T \bar{v})^2 \leq u_1^2 v_1^2 + u_2^2 v_2^2 + u_1^2 v_2^2 + u_2^2 v_1^2$$

$$= (u_1^2 + u_2^2)(v_1^2 + v_2^2)$$

$$= \|\bar{u}\|^2 \|\bar{v}\|^2$$

Simple example:

$$\bar{u} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \qquad \bar{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$\bar{u}^T \bar{v} = 1 - 2 + 3 - 4$$
$$= -2$$

$$(\bar{u}^T \bar{v})^2 = (-2)^2 = 4$$

$$\|\bar{u}\|^2 = 1 + 1 + 1 + 1 = 4$$

$$\|\bar{v}\|^2 = 1 + 4 + 9 + 16 = 30$$

So it holds

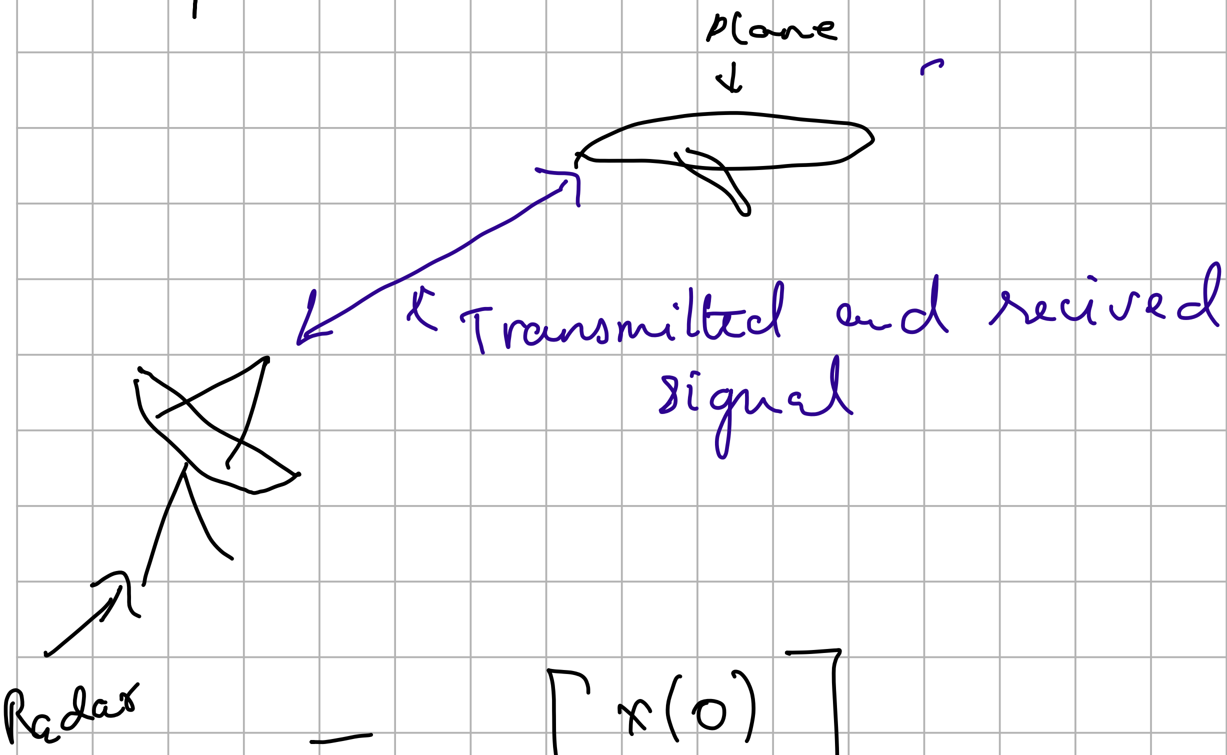
$(\bar{u}^T \bar{v}) \leftarrow$ Correlation between 2 signals $\bar{u}, \bar{v}$

Has many interesting application

Example: Radar signal processing

Radar application

Setup



$$\underset{\text{Transmitted signal}}{\bar{x}} = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(n) \end{bmatrix}$$

$$\bar{y} = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(n \end{bmatrix}$$

recived signal (measured after certain delay)

If object present :

$$\bar{y} = \bar{x} + \bar{w}$$

recived Transmitted noice (additive white Gaussian noice)

(dont worry about it!)

So if object is present, then $\bar{y}$ is highly co-related with $\bar{x}$

So $|\bar{x}^T \bar{y}|$ is high

Inner product

if no object (Target)

$\bar{y} = \bar{w}$ (no reflected signal)

so $\bar{y}$ is just noice

So to determine presence of target we corelate recived signal with transmitted signal

So if $|\bar{x}^T \bar{y}|^2 \geq \gamma$

threshold gamma

Target present

else

Target absent

This radar is called detection problem or also called binary hypothesis testing ✓

Inner product can also be used for determining the similarity b/w signals

Ex: $\bar{x}_1$, $\bar{x}_2$

$$\text{if } |\bar{x}_1^H \bar{x}_2|^2 \geq \gamma$$

similar

else not similar