Section 0

$\sqrt{4} = \pm 2$

$\sqrt{1} = i \in$ complex

$\sqrt{-64} = \pm 8i$

Complex no. $\underbrace{a}_{\text{real}} + \underbrace{ib}_{\text{imaginary}}$ form

adding complex no.

$$(1+9i)+(-3-8i) = -2+i$$

Multiply:

$$(2+3i)(4-8i) = 8-16i+12i-24i^2$$

$$= 8-4i+24$$

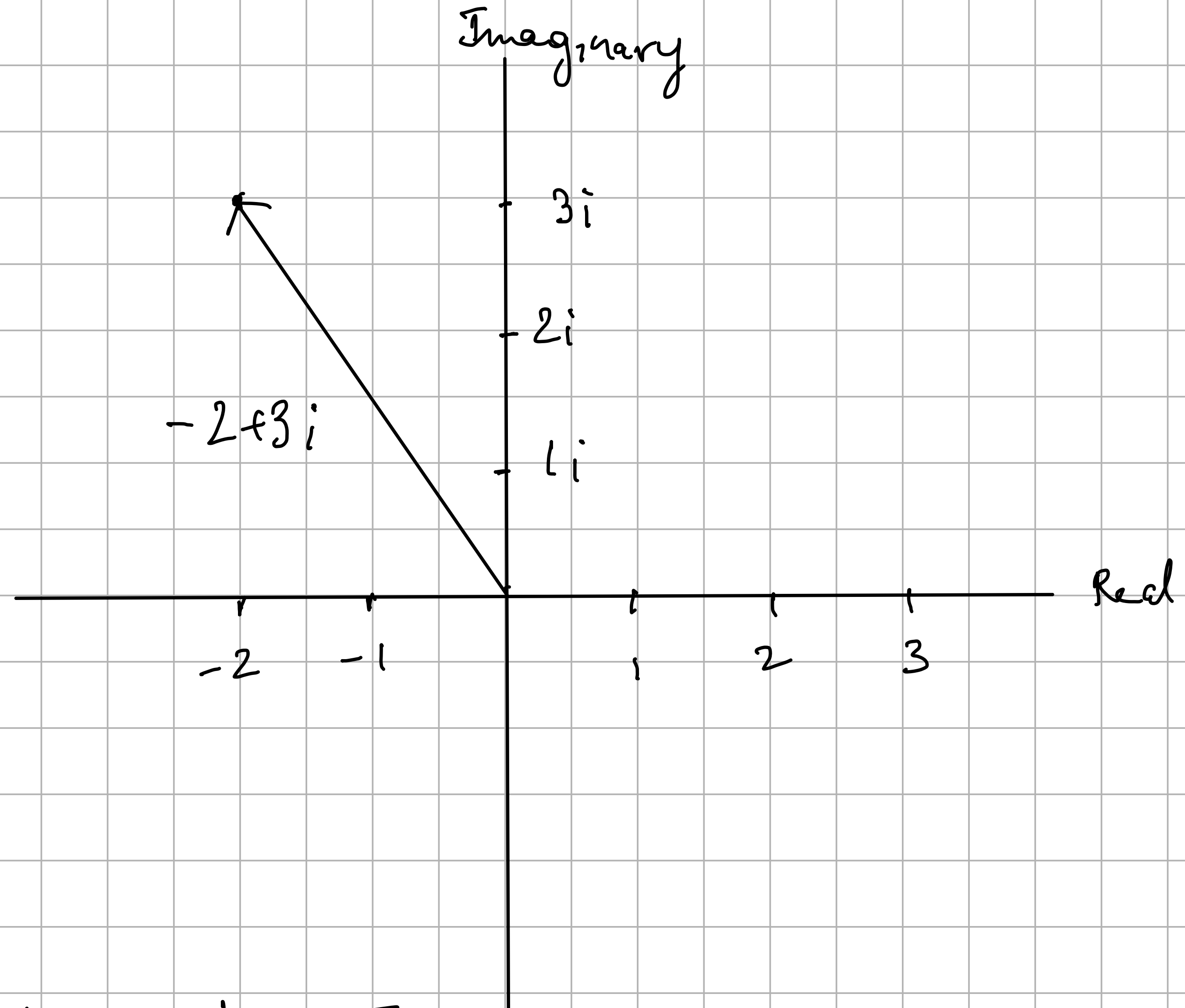
$$= 32-4i$$

Complex Conjugate $(a - ib)$

note $(a+ib)(a-ib) = a^2 + b^2 \in$ Real

Complex no. as vector [in imaginary plane]

$-2 + 3i$



$|-2+3i| = \sqrt{2^2+3^2} = \sqrt{13}$

magnitude

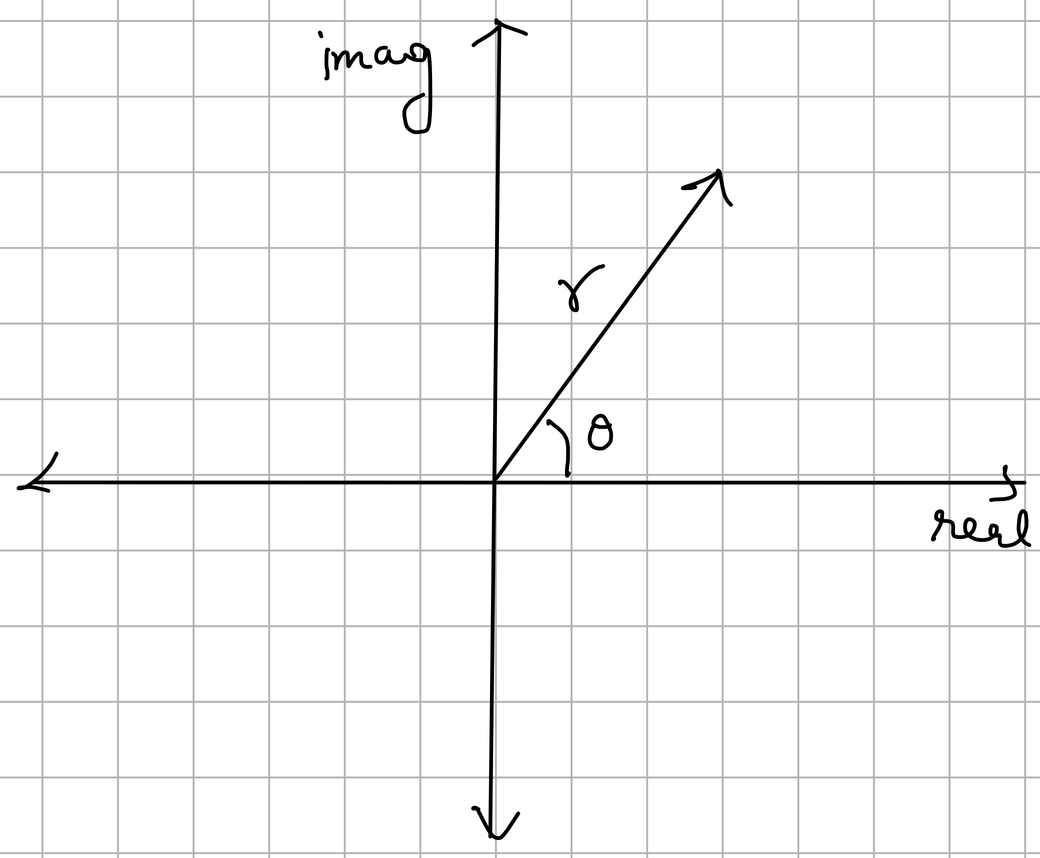
2nd form of representation (Polar form)

$$a + ib = r\cos\theta + i\sin\theta$$

magnitude angle with x axis

$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

Exponential form $\Rightarrow r e^{i\theta}$



normal form $1+i$

polar form: $\theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$

$r = \sqrt{1^2+1^2} = \sqrt{2}$

polar form: $\sqrt{2}\,\sin\frac{\pi}{4} + i\cos\left(\frac{\pi}{4}\right)$

$= 1 + \frac{i}{\sqrt{2}}$

exp form: $\sqrt{2}\,e^{\pi i/4}$

<u>Exp form is most used</u>

Complex conjugate of matrix

$$A = \begin{bmatrix} 2+3i & 0 \\ 5 & 3-i \end{bmatrix}$$

$$A^* = \begin{bmatrix} 2-3i & 0 \\ 5 & 3+i \end{bmatrix}$$

complex conjugate

Q)
$$B = \begin{bmatrix} 3+7i & 2e^{i\pi/3} \\ 5 & 3-i \end{bmatrix}$$

$$B^* = \begin{bmatrix} 3-7i & 2e^{-i\pi/3} \\ 5 & 3+i \end{bmatrix}$$

for a matrix A

$$(A^*)^T = (A^T)^* = \underbrace{A^\dagger}$$

← A dagger

Unitary Matrix U

a matrix U is unitary if

$U U^\dagger = I$ or $U^\dagger$ is inevrse of U

Hermitian Matrix

$$H = H^{\dagger}$$

Qbits

classic Bits $\rightarrow$ 0 or 1

Qbits $\rightarrow$ $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Superposition

∟ both 0 and 1 at the same time

Mathematically:

Lets take qbit ψ

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

α represents how much the qbit is of 0 state

β represents how much the qbit is in 1 state

Qbits are 0 or 1 based on
what we measured.
α and β are just representation
of the probibality of the states qbit is in

probiblity is measured by:

α state $= |\alpha|^2$

β state $= |\beta|^2$

So in $|0\rangle = (1)^2$ so 1 probiblity of
being 0 state

and as they are probiblity

$$|\alpha|^2 + |\beta|^2 = 1$$

Dirac Notation

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$= \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \alpha |0\rangle + \beta |1\rangle$$

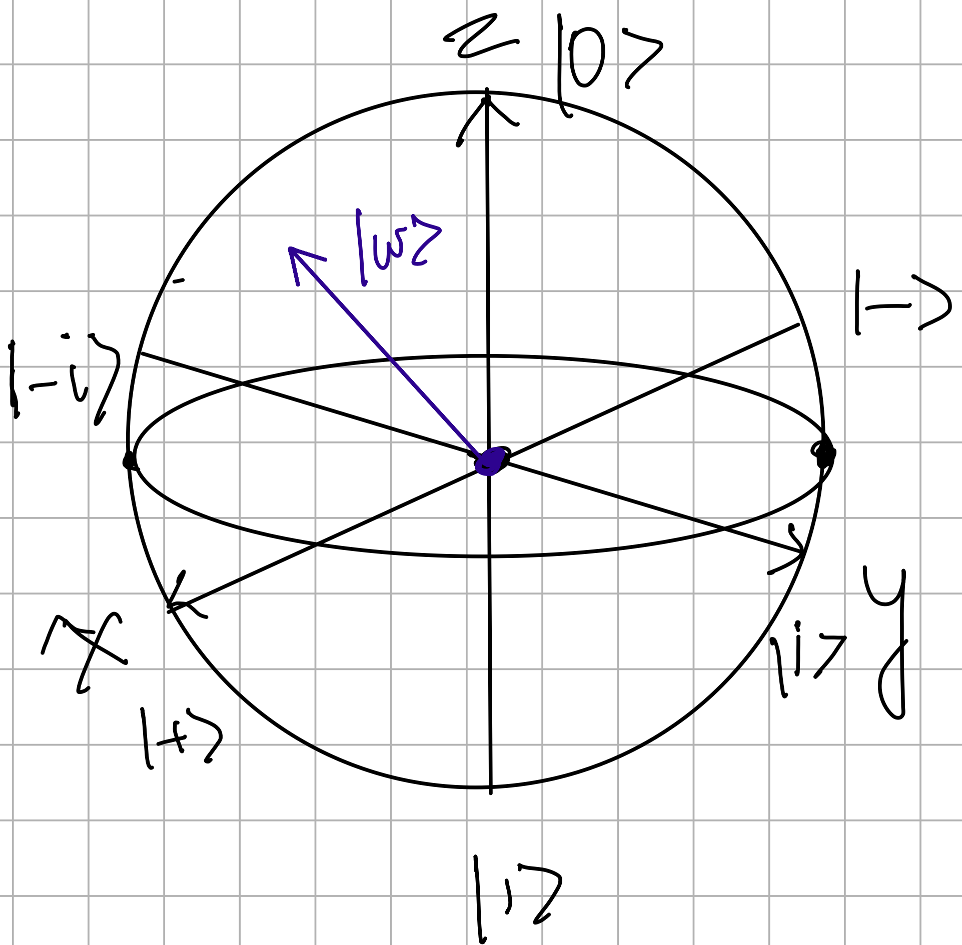
a)

$$|\psi\rangle = \begin{pmatrix} \frac{1}{2} \\ \frac{2\sqrt{3}}{4} \end{pmatrix}$$

$$= \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$$

Representating using Bloch sphere

representation of qbits:



Manuplating a Qbit - X, Y, Z gates

X-gate

↳ Rotates the qbit around the X axis by 180°

Y gate

↳ Rotates the qbit around the y axis by 180°

Z-gate

↳ rotates by 180° around Z axis

for applying the gates

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We can multiply these matrices to perform the gate operation

Let $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an arbitrary gate

$U|0\rangle = \begin{pmatrix} a \\ c \end{pmatrix}$ ← The first column shows the probablity of 0 state

$U|1\rangle = \begin{pmatrix} b \\ d \end{pmatrix}$ ← Second column shows the probablity of 1 state for a gate U

$$U|0\rangle = \begin{pmatrix} a \\ c \end{pmatrix}$$

$$= a|0\rangle + c|1\rangle$$

and

$$U|1\rangle = \begin{pmatrix} b \\ d \end{pmatrix} = b|0\rangle + d|1\rangle$$

Quantum gates are linear.

Let $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

then if we apply a gate U

$$U|\psi\rangle = U(\alpha|0\rangle + \beta|1\rangle)$$

$$= U\alpha|0\rangle + U\beta|1\rangle$$

$$\alpha U|0\rangle + \beta U|1\rangle$$

Example

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$Y|\psi\rangle = Y\left(\frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle\right)$$

$$= \frac{\sqrt{3}}{2} \; Y|0\rangle + \frac{1}{2} \; Y|1\rangle$$

$$= \frac{\sqrt{3}}{2} \begin{pmatrix} 0 \\ i \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -i \\ 0 \end{pmatrix}$$

$$= \frac{\sqrt{3}}{2} \, i \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{2}(-i) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{i\sqrt{3}}{2} \, |1\rangle - \frac{i}{2} \; |0\rangle$$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$Z\left(\alpha|0\rangle + \beta|1\rangle\right) = \alpha|0\rangle - \beta|1\rangle$$

prove it

$$Z(\alpha|0\rangle + \beta|1\rangle)$$

$$= \alpha \; Z|0\rangle + \beta \; Z|1\rangle$$

$$= \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$= \alpha |0\rangle + \beta(-1) |1\rangle$$

$$= \alpha |0\rangle - \beta |1\rangle$$

1.5 Intro to Global and Relative phase

$$\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \xrightarrow{\text{Zgate}} \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

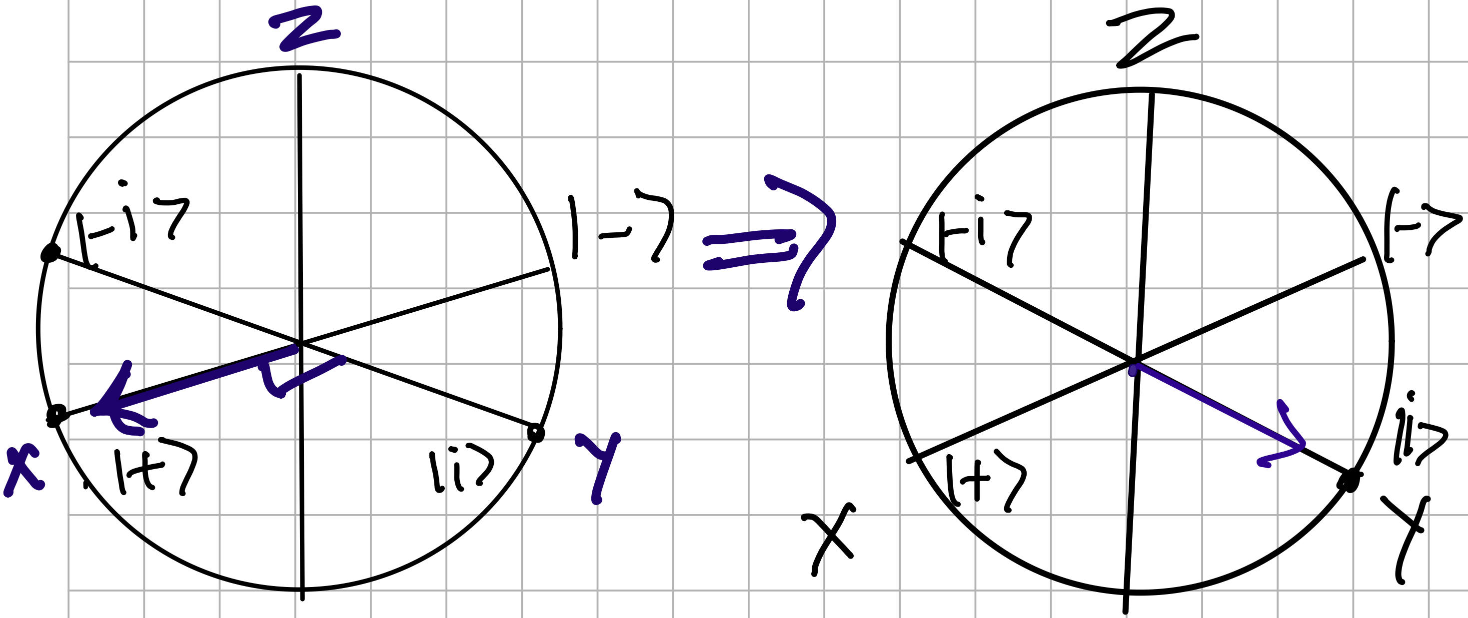
note that $-1 = e^{i\pi}$

and we rotated π radians

so we can write the results as

$$\xrightarrow{\text{Zgate}} \frac{1}{\sqrt{2}}|0\rangle + e^{i\pi} \frac{1}{\sqrt{2}}|1\rangle$$

another example:



$$\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

same
$$\begin{cases} \frac{1}{\sqrt{2}}|0\rangle + i\frac{1}{\sqrt{2}}|1\rangle \\ \frac{1}{\sqrt{2}}|0\rangle + e^{i\pi/2}\frac{1}{\sqrt{2}}|1\rangle \end{cases}$$

(i in exp form is $e^{i\pi/2}$)

So we use complex no. so in $e^{i\varphi}$, we can change φ value and easily rotate around a circle.

∴ so for any qbit can be represented as

$$|\psi\rangle = \alpha|0\rangle + e^{i\varphi}\beta|1\rangle$$

but why only for $|1\rangle$ and not $|0\rangle$

There are 2 types of Phase

Global Phase (entire qbit is multiplied by $e^{i\varphi}$)

$$e^{i\varphi}\left(\alpha|0\rangle + \beta|1\rangle\right)$$

$$e^{i\varphi}\alpha|0\rangle + \beta e^{i\varphi}|1\rangle$$

but we discard this because

$$e^{i\varphi}\alpha|0\rangle + \beta e^{i\varphi}|1\rangle$$

$$= \alpha|0\rangle + \beta|1\rangle \text{ (no change overall)}$$

Relative Phase (only one state is phased or multiplied by $e^{i\varphi}$)

$$\alpha|0\rangle + e^{i\varphi}\beta|1\rangle$$

But if we have a qbit that has amplitude in both 0 and 1 phase?

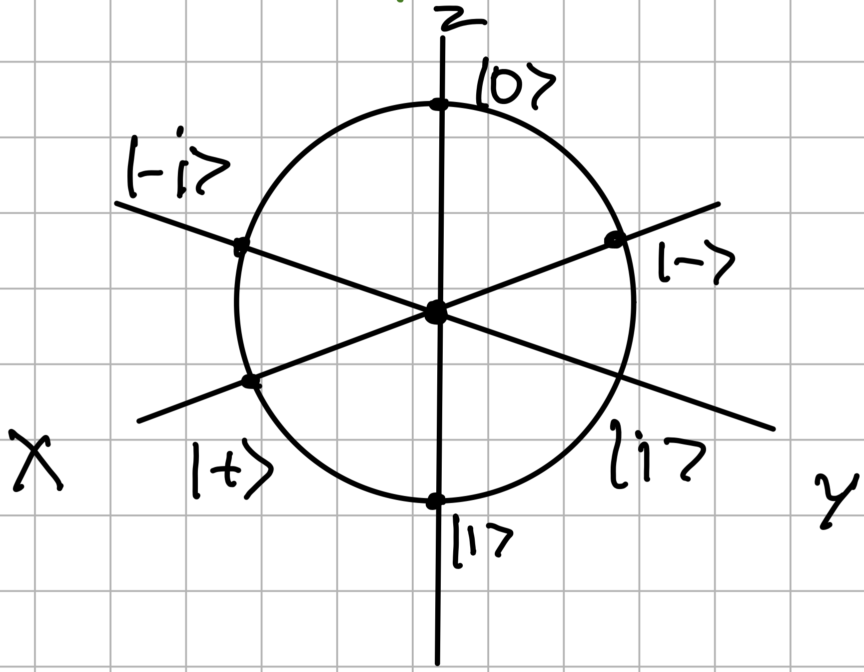
$$e^{i\theta}\alpha|0\rangle + e^{i\varphi}\beta|1\rangle$$

$$e^{i\theta}\left(\alpha|0\rangle + (e^{i\theta})^{-1} e^{i\varphi}\beta|1\rangle\right)$$

global so discard

$$\alpha|0\rangle + e^{i(\varphi-\theta)}\beta|1\rangle$$

1.6) The Hadamard Gate and the +, −, i and −i states



states meanings

$|+\rangle \Rightarrow \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

$|-\rangle \Rightarrow \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$

$|i\rangle \Rightarrow \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle$

$$|-i\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle$$

Hadamard Gate

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$|0\rangle \xrightarrow{H} |+\rangle$ $\quad$ $|+\rangle \xrightarrow{H} |0\rangle$

$|1\rangle \xrightarrow{H} |-\rangle$ $\quad$ $|-\rangle \xrightarrow{H} |1\rangle$

this means Hadamard Gate is its own inverse

So if we apply Hadamard gate to any qbit

$$H\left(\alpha|0\rangle + e^{i\varphi}\beta|1\rangle\right)$$

$$= \alpha H|0\rangle + e^{i\varphi}\beta H|1\rangle$$

$$= \alpha|+\rangle + e^{i\varphi}\beta|-\rangle$$

$$= \alpha\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) + e^{i\varphi}\beta\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)$$

$$= \left(\frac{\alpha + e^{i\varphi}\beta}{\sqrt{2}}\right)|0\rangle + \left(\frac{\alpha - e^{i\varphi}\beta}{\sqrt{2}}\right)|1\rangle$$

Note

$$|+\rangle \xrightarrow{H} |0\rangle$$

so

$$\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \xrightarrow{H} |0\rangle$$

even though $|+\rangle$ has 50-50 %

probiblity of being 0 or 1, when

Hadamar gate is applied, the answer

is always $|0\rangle$ state

Similarly for $|-\rangle$

$$|-\rangle \xrightarrow{H} |1\rangle$$

1.7) The Phase Gate (S and T gates)

$$S \text{ gate} \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/2} \end{pmatrix}$$

$$|0\rangle \xrightarrow{S} |0\rangle$$

$$|1\rangle \xrightarrow{S} e^{i\pi/2} |1\rangle$$

$$\alpha|0\rangle + \beta|1\rangle \xrightarrow{S} \alpha|0\rangle + e^{i\pi/2} |\beta\rangle$$

So it adds relative phase of $\pi/2$ radian

$$T \text{ gate} \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$$

$$|0\rangle \xrightarrow{T} |0\rangle$$

$$|1\rangle \xrightarrow{T} e^{i\pi/4} |1\rangle$$

So it adds $e^{i\pi/4}$ relative phase

for a $|\psi\rangle$ qubit

$$S^{\dagger}\left(S\left(|\psi\rangle \right)\right) = |\psi\rangle$$

hence S and $S^{\dagger}$ are inverse of each other

Similarly

$$T^{\dagger}\left(T\left(|\psi\rangle \right)\right) = |\psi\rangle$$

2.1 Representing Multiple Qbits mathematically

$\otimes$ ← Tensor product (like normal multiplication but with matrix form)

Two qbits in 0 state

$$|0\rangle \otimes |0\rangle = |00\rangle$$

Two qbits in superposition

$$(\alpha|0\rangle + \beta|1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle)$$

$$\alpha|0\rangle \otimes \gamma|0\rangle + \alpha|0\rangle \otimes \delta|1\rangle + \beta|1\rangle \otimes \gamma|0\rangle + \beta|1\rangle \otimes \delta|1\rangle$$

$$\alpha\gamma|00\rangle + \alpha\delta|01\rangle + \beta\gamma|10\rangle + \beta\delta|11\rangle$$

prob of measuring 00 $= |\alpha\gamma|^2$

" " 01 $= |\alpha\delta|^2$

10 $= |\beta\gamma|^2$

11 $= |\beta\delta|^2$

Example with 2 qbits

$$\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) \otimes \left(\frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle\right)$$

$$\frac{\sqrt{3}}{2\sqrt{2}}|00\rangle + \frac{1}{2\sqrt{2}}|01\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|10\rangle + \frac{1}{2\sqrt{2}}|11\rangle$$

If we want to add another qbit

then tensor it again

So for example

$$\left(\frac{\sqrt{3}}{2\sqrt{2}}|00\rangle + \frac{1}{2\sqrt{2}}|01\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|10\rangle + \frac{1}{2\sqrt{2}}|11\rangle\right) \otimes$$

$$\left(\frac{1}{\sqrt{3}}|0\rangle + e^{i\pi/4}\frac{\sqrt{2}}{\sqrt{3}}|1\rangle\right)$$

$$= \frac{1}{2\sqrt{2}}|000\rangle + \frac{1}{2\sqrt{6}}|001\rangle + \frac{1}{2\sqrt{2}}|100\rangle$$

$$+ \frac{1}{2\sqrt{6}}|110\rangle + e^{i\pi/4}\frac{1}{2}|001\rangle$$

$$+ \frac{1}{2\sqrt{3}}|011\rangle + \frac{1}{2}|101\rangle + \frac{1}{2\sqrt{3}}|110\rangle$$

Notations

$$\underbrace{|00\ldots0\rangle}_{n \text{ zeros}} \Rightarrow |0\rangle^{\otimes n}$$

So

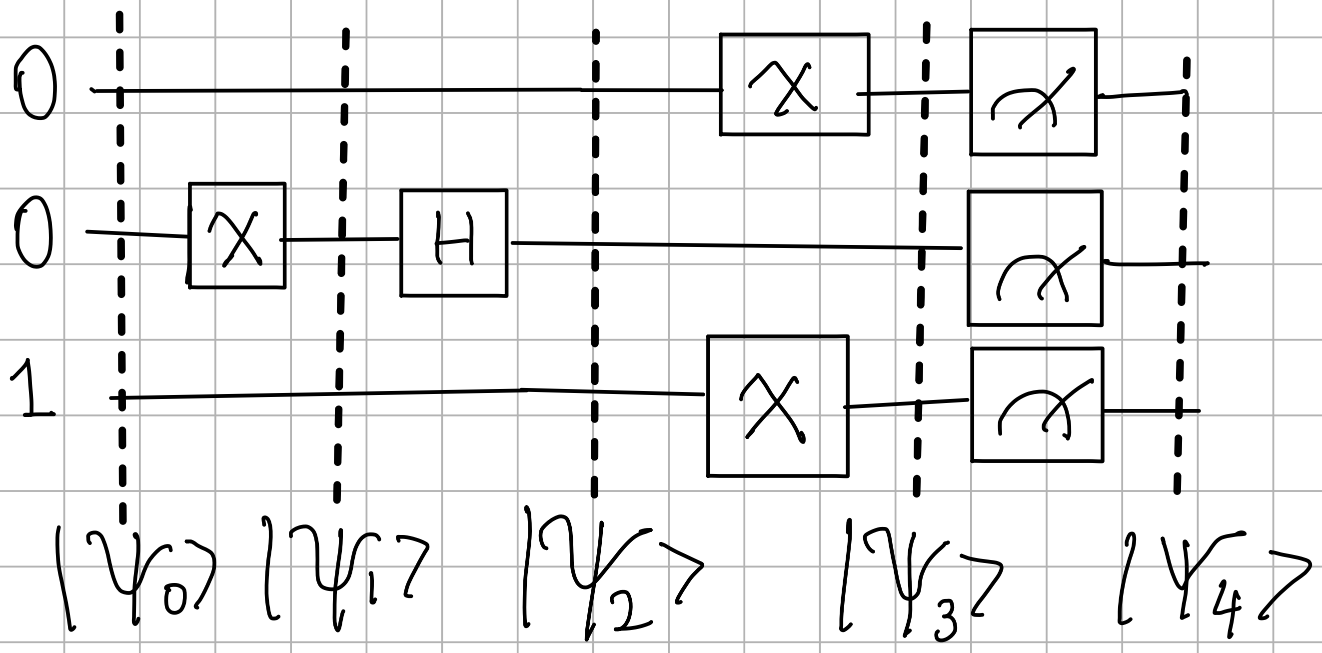
$$|1\rangle^{\otimes 5} \Rightarrow |11111\rangle$$

So how to apply a gate for a certain gate?

$$\frac{\sqrt{3}}{2}|000\rangle + \frac{1}{2\sqrt{2}}|101\rangle - \frac{1}{2\sqrt{2}}|110\rangle$$

Lets say we want to apply a X gate for 2nd qbit (highlated)

$\boxed{\text{We use quantum circuits.}}$



$|\psi_0\rangle = |001\rangle$

$|\psi_1\rangle = |011\rangle$ [X gate for 2nd qbit]

now we apply a H gate for second qbit for getting $|\psi_2\rangle$

so

$$|1\rangle \xrightarrow{H} |-\rangle$$

so

$|\psi_2\rangle = |0-1\rangle$

$$= \underbrace{|0\rangle \otimes \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)}_{\text{This first}} \otimes |1\rangle$$

$$= \left(\frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|01\rangle\right) \otimes |1\rangle$$

$$= \frac{1}{\sqrt{2}}|001\rangle - \frac{1}{\sqrt{2}}|011\rangle$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}\left(|001\rangle - |011\rangle\right)$$

for ψ_3 apply X gate to 1^{st} and 3^{rd} qbit

$$|\psi_3\rangle = \frac{1}{\sqrt{2}}\left(|100\rangle - |110\rangle\right)$$

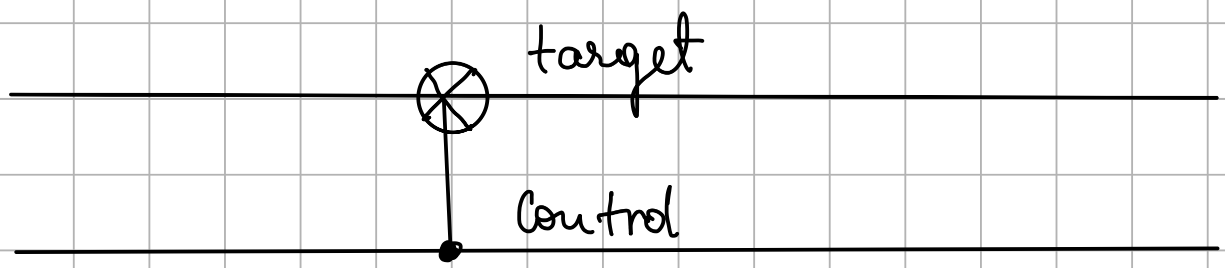
$$\underbrace{|\psi_4\rangle}_{\text{we just measure the qbit}} = \frac{1}{\sqrt{2}}\left(|100\rangle - |110\rangle\right)$$

So we get $|100\rangle$ state with prob $\frac{1}{2}$

and $|110\rangle$ with prob $\frac{1}{2}$

2.3 Multi qbit gates : CNOT, Toffoli and Controlled gates

CNOT / Controlled X gate



Cnot gate applies an X gate on target if control qbit is 1 else does nothing

Example:

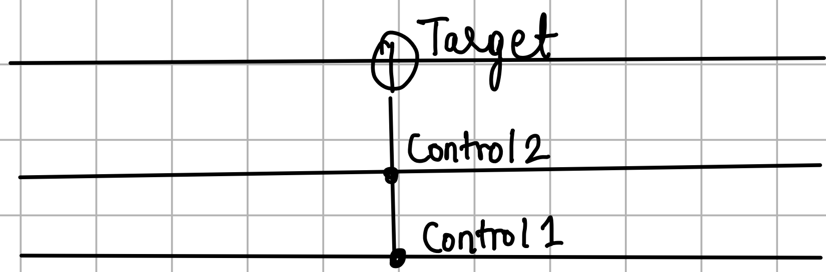
$$CNOT\left(\frac{\sqrt{3}}{4}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle + \frac{1}{4}|11\rangle\right)$$

$$\Rightarrow \frac{\sqrt{3}}{4} CNOT(|00\rangle) + \frac{1}{2} cnot(|01\rangle + \frac{1}{\sqrt{2}} cnot|10\rangle$$
$$+ \frac{1}{4} cnot|11\rangle$$

first qbit is control and second qbit is target (will be given)

$$\frac{\sqrt{3}}{4}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{\sqrt{2}}|11\rangle + \frac{1}{4}|10\rangle$$

Toffoli gate



Same as Cnot but 2 control qbits

if control 1 and control 2 are 1 then X gate target
else nothing

Example : (Take 2nd and 3rd as control and 4th as target)

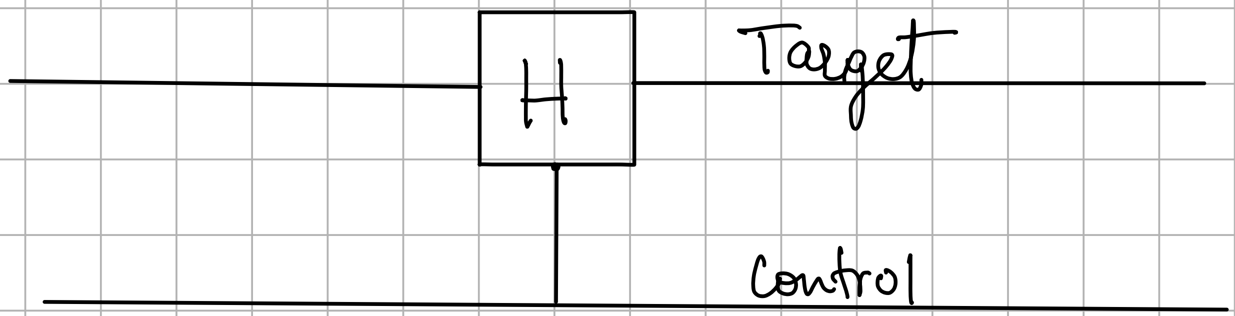
$$\text{TOFFOLI}\left(\frac{1}{\sqrt{2}}|0011\rangle + \frac{1}{\sqrt{2}}|0110\rangle\right)$$

$$= \frac{1}{\sqrt{2}}|0001\rangle + \frac{1}{\sqrt{2}}|0111\rangle$$

There are the same controlled gates

CY, CZ, CS, CT and CH ...

They do the operation only if the control qbit is in 1 else does nothing.
They are represented as follows



2.4) Measuring Singular Qbits.

how do we measure a single qbit
lets say 2nd qbit in the following

$$|\psi\rangle = \frac{1}{2}|00\rangle + \frac{1}{4}|01\rangle + \frac{e^{i\pi/2}}{\sqrt{2}}|10\rangle + \frac{\sqrt{3}}{4}|11\rangle$$

Prob(second qbit as 1) = Prob($|01\rangle$) + Prob($|11\rangle$)

$$= \left|\frac{1}{4}\right|^2 + \left|\frac{\sqrt{3}}{4}\right|^2$$

$$= \frac{1}{4}$$

Lets take another example

$$|\psi\rangle = \frac{1}{2}|00\rangle + \frac{1}{4}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle + \frac{\sqrt{3}}{4}|11\rangle$$

Say we measure the 1st qbit and get the reading of 1, so let $|\psi_1\rangle$ be the state after the measurement

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}|10\rangle + \frac{\sqrt{3}}{4}|11\rangle$$

we only take where the first state is 1 which was measured

But the probablily does not add up to become 1

So we normalize it to become 1

$$|\psi_1\rangle = A\left(\frac{1}{\sqrt{2}}|10\rangle + \frac{\sqrt{3}}{4}|11\rangle\right)$$

we find the value of A

$$\frac{1}{A} = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{\sqrt{3}}{4}\right)^2$$

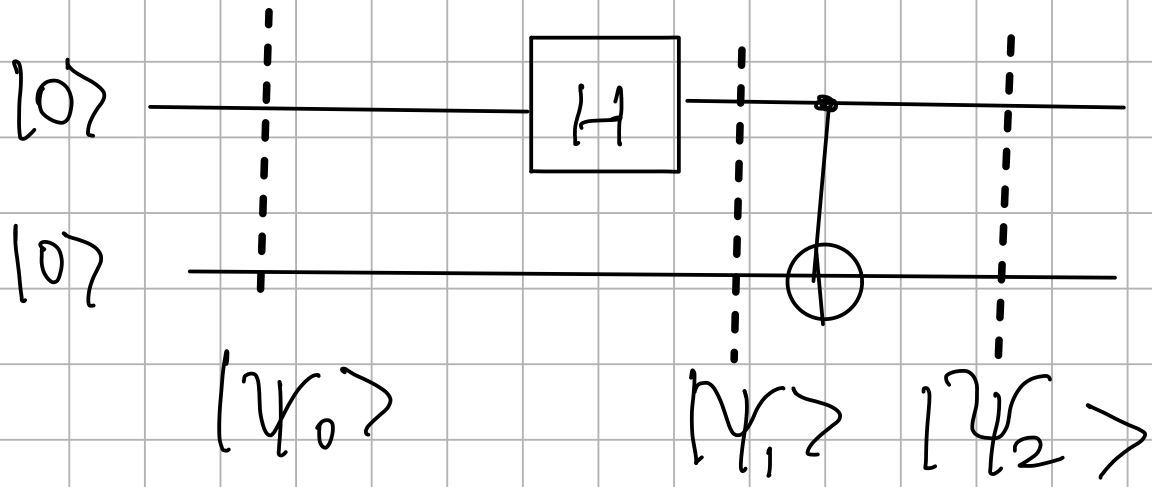
$$A = \frac{4}{\sqrt{11}}$$

So

$$|\psi_1\rangle = \frac{4}{\sqrt{22}}|10\rangle + \frac{\sqrt{3}}{\sqrt{11}}|11\rangle$$

We can use this for any no. of qbits.

2.5) Entanglement and Bell State



$$|\psi_0\rangle = |00\rangle$$

$$|\psi_1\rangle = |+0\rangle$$

$$= \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) \otimes |0\rangle$$

$$= \frac{1}{\sqrt{2}}\left(|00\rangle + |10\rangle\right)$$

$$|\psi_2\rangle = \mathrm{CNOT}\left(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

in this $|\psi_2\rangle$ note that

- if we measure one of the qbits (1^{st} or 2^{nd}) as 0, then the other collapse to 0

  then $|\psi_2\rangle \longrightarrow |00\rangle$

- if we measure one of the qbits as 1 the other will also collapse to 1

  $$|\psi_2\rangle \rightarrow |11\rangle$$

$\boxed{\text{This is called entanglement}}$

A state is entangled if it cannot be factored into tensor product of individual qbits.

Ex:

$\Rightarrow$ not entangled qbits:

$$\frac{\sqrt{3}}{2\sqrt{5}}|00\rangle + \frac{1}{2\sqrt{5}}|01\rangle + \frac{\sqrt{3}}{\sqrt{5}}|10\rangle + \frac{1}{\sqrt{5}}|11\rangle$$

that can be factoried into tensor product as

$$\left(\frac{1}{\sqrt{5}}|0\rangle + \frac{2}{\sqrt{5}}|1\rangle\right) \otimes \left(\frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle\right)$$

$\Rightarrow$ Entangled qbits

$$\frac{1}{\sqrt{2}}\left(|000\rangle + |011\rangle\right)$$

Types of entangled States

Maximally entangled

Qbits are maximally entangled if by measuring one of the qbits, we can surely tell the other qbit's measured state as 0 or 1

Ex: previous one

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

Sai

These maximally entangled states are called Bell states

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

$$|\phi^-\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right)$$

$$|\psi^+\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right)$$

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$$

Partially Entangled

Qbits are partially entangled if measuring one qbit affects the other qbits amplitude or phase.

Example:

$$|\psi\rangle = \sqrt{\frac{3}{5}}\,|00\rangle + \frac{1}{\sqrt{5}}\,|01\rangle + \frac{1}{2\sqrt{5}}\,|10\rangle + \frac{\sqrt{3}}{2\sqrt{5}}\,|11\rangle$$

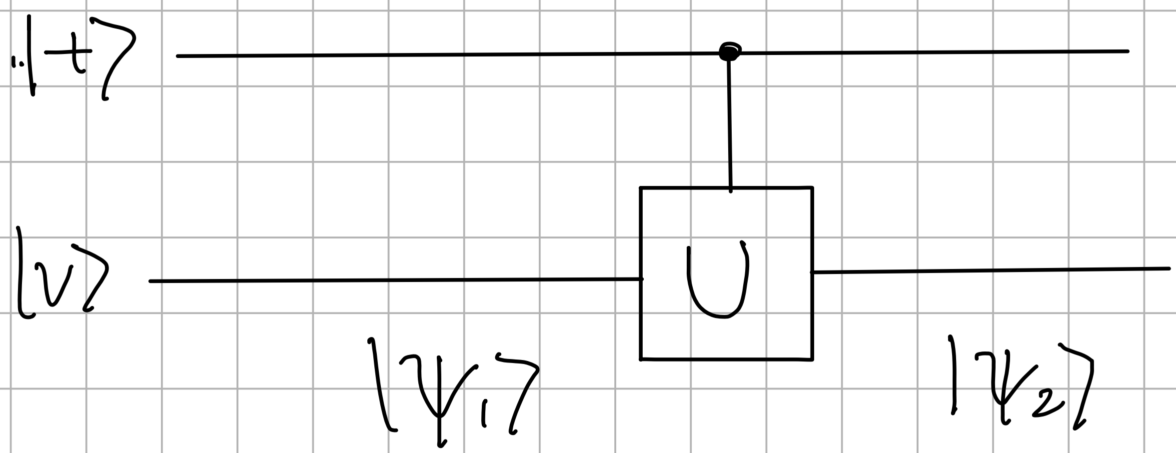
if we measure first qbit as 0, then state collapse to

$$|0\rangle \otimes \left(\frac{\sqrt{3}}{2}\,|0\rangle + \frac{1}{2}\,|1\rangle\right)$$

if we measure the first qbit as zero then

$$|1\rangle \otimes \left(\frac{1}{2}\,|0\rangle + \frac{\sqrt{3}}{2}\,|1\rangle\right)$$

2.6) Phase Kickback



Lets say $|v\rangle$ is an eigenvector of U

Then

$$U|v\rangle = e^{i\theta}|v\rangle$$

(as all eigen values can be represented as $e^{i\theta}$ in quantum computing)

$$|\psi_1\rangle = |+v\rangle$$

$$= \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) \otimes |v\rangle$$

$$= \frac{1}{\sqrt{2}}|0\rangle|v\rangle + \frac{1}{\sqrt{2}}|1\rangle|v\rangle$$

$$|\psi_2\rangle = \underbrace{C\text{-}U}_{\text{controlled gate } U}\left(\frac{1}{\sqrt{2}}\left(|0v\rangle + |1v\rangle\right)\right)$$

controlled gate U

variable

$$= \frac{1}{\sqrt{2}}\left(|0v\rangle + e^{i\theta}|1v\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\theta}|1\rangle\right)|v\rangle$$

This occurs if $|v\rangle$ is eigen vector of the gate U, by applying controlled-U gate with $|v\rangle$ as target, we can 'kick' the phase onto the control qubit

This is called phase kickback

3.1 Superdense Coding

Superdense coding
└ allows 2 bits of classic information
(00, 01, 10, 11)
using 1 qbit.

This uses entanglement

Lets say Alice is sending info of 2 bit to Bob

Alice and Bob agree on a entangled state $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$

Lets say they agree that Alice takes qbit 1 and bob takes qbit 2 if alice wants to send

00 $\rightarrow$ does nothing $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

01 $\rightarrow$ applies x gate on her qbit $|\psi\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$

10 $\rightarrow$ applies z gate on her qbit $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$

11 $\rightarrow$ applies x and z gate $|\psi\rangle = \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle)$

Bob side:

Bob will have one of them
He will do the following operations (zoom in)

$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \xrightarrow[\text{(1st qbit control, 2nd qbit target)}]{\text{CNOT}} \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) \xrightarrow[\text{left qbit}]{\text{H for}} H|+\rangle|0\rangle = |00\rangle$

similarly for others

Doubt

$|\psi\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle) \xrightarrow{\text{CNOT}} \frac{1}{\sqrt{2}}(|11\rangle + |01\rangle) \xrightarrow{H} H|+\rangle|1\rangle = |01\rangle$

$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \xrightarrow{\text{CNOT}} \frac{1}{\sqrt{2}}(|00\rangle - |10\rangle) \xrightarrow{H} H|-\rangle|0\rangle = |10\rangle$

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|10\rangle - |01\rangle\right) \xrightarrow{\text{CNOT}} \frac{1}{\sqrt{2}}\left(|11\rangle - |01\rangle\right) \xrightarrow{H} H|-\rangle|1\rangle = |11\rangle$$

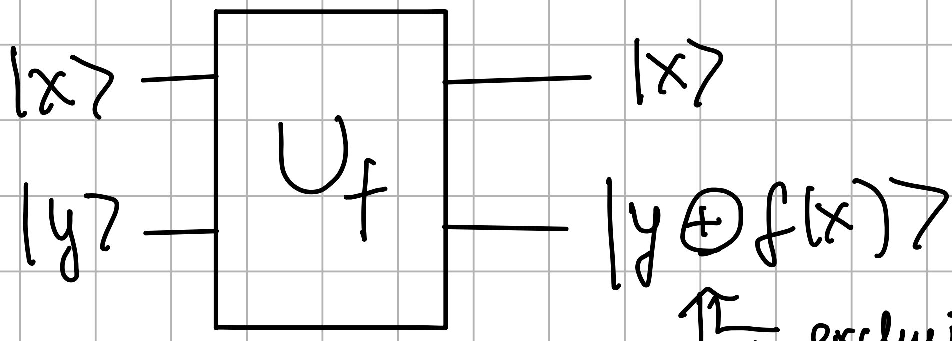
Revise: NOT, AND, OR

XOR

x	$f(x)$
0 0	0
0 1	1
1 0	1
1 1	0

3.2B) Functions on Quantum Computers

Standard Quantum function

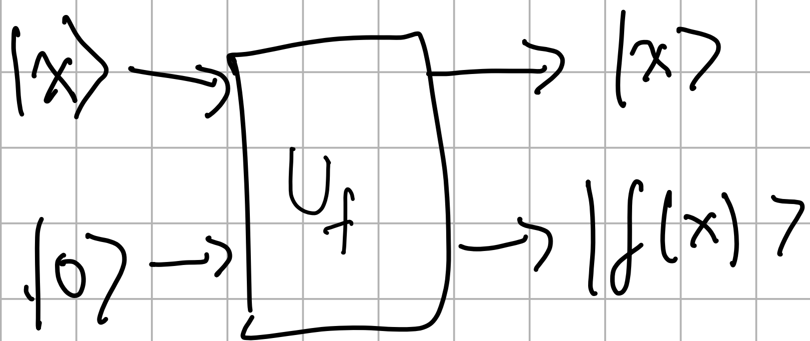


exclusive Or (x or)

set $y = |0\rangle$

note

we use x or for making the function one - one



So

$$U_f\,|x\rangle|0\rangle = |x\rangle\,|f(x)\rangle$$

Lets apply function U_f to $|x\rangle|-\rangle$

$$U_f|x\rangle|-\rangle$$

$$= U_f|x\rangle\left(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right)$$

$$= U_f \left(\frac{1}{\sqrt{2}} \left(|x\rangle|0\rangle - |x\rangle|1\rangle \right) \right)$$

$$= \frac{1}{\sqrt{2}} \left(U_f |x\rangle|0\rangle - U_f |x\rangle|1\rangle \right)$$

$$= \frac{1}{\sqrt{2}} \left(|x\rangle|f(x)\rangle - |x\rangle|\overline{f(x)}\rangle \right)$$

$\left[1 = \overline{f(x)} \right.$ because $0 = f(x)$ so $\left. \bar{0} = \overline{f(x)} = 1 \right]$

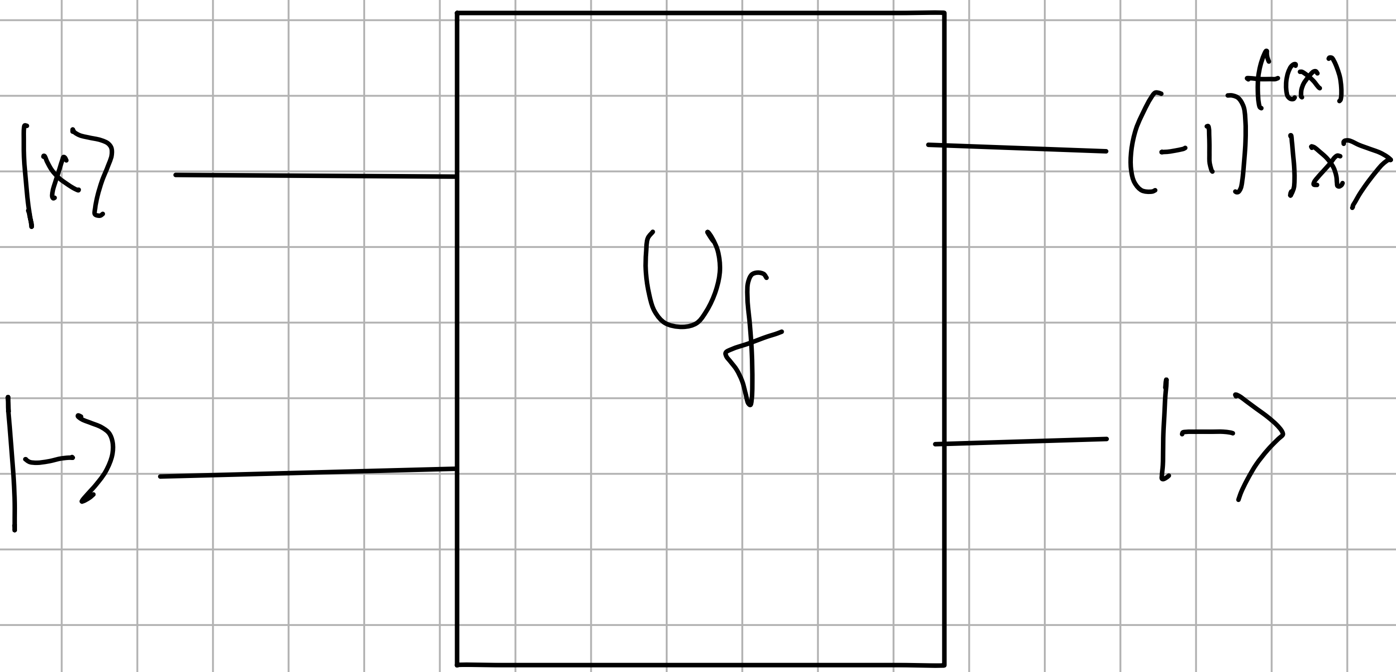
$$= \begin{cases} \frac{1}{\sqrt{2}} |x\rangle|0\rangle - |x\rangle|1\rangle & \text{if } f(x) = 0 \\ \frac{1}{\sqrt{2}} |x\rangle|1\rangle - |x\rangle|0\rangle & \text{if } f(x) = 1 \end{cases}$$

$$= \begin{cases} |x\rangle|-\rangle & \text{if } f(x) = 0 \\ -|x\rangle|-\rangle & \text{if } f(x) = 1 \end{cases}$$

we can combine these 2 to get

$$U_f |x\rangle|-\rangle = (-1)^{f(x)} |x\rangle|-\rangle$$

So now we can write as



- Like this, if output bit is in $|-\rangle$ state, it is called Phase Oracle
- Instead of the function output $f(x)$ being XOR'd with $|-\rangle$ register, a phase of $(-1)^{f(x)}$ was applied to the input to the function.

No cloning theorem

- we cannot copy qbits like classical bits.
- we need α and β values, simply measuring the qbit wont give any probablity

3.3) Deutsch's Algorithm

$$f : \{0,1\} \longrightarrow \{0,1\}$$

$\uparrow$ function $\uparrow$ input $\uparrow$ output

Constant functions

Constant One function

x	$f(x)$
0	1
1	1

Constant Zero function

x	$f(x)$
0	0
1	0

Balanced function

L Half 0s, Half 1

Identity

x	$f(x)$
0	0
1	1

Not

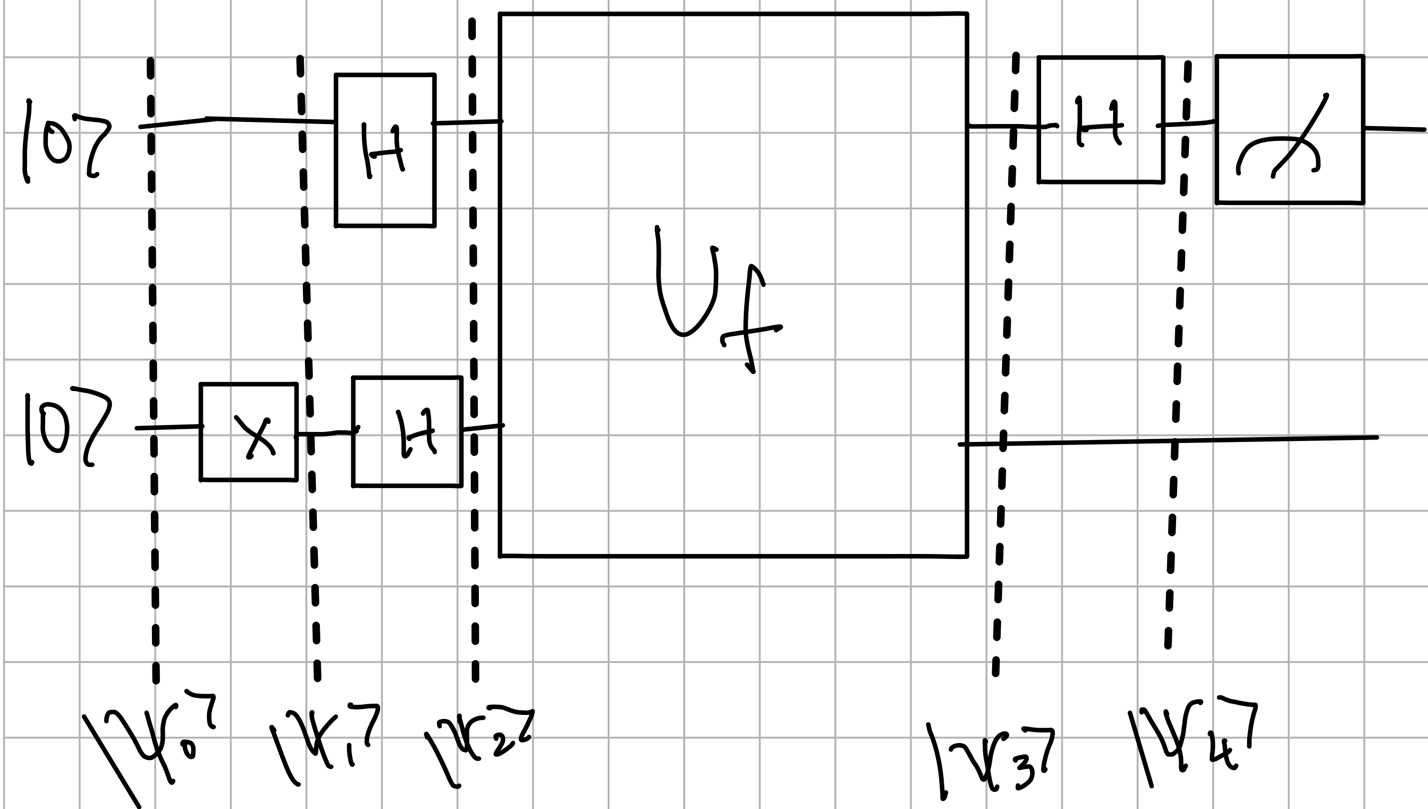
x	$f(x)$
0	1
1	0

On classical computer, to get if the function is balanced or constant, we need to have $f(0)$ and $f(1)$

So if $f(0) = f(1) \leftarrow$ constant

if $f(0) \neq f(1) \leftarrow$ balanced

But in Quantum Computers only need one query of the function to determine if it is constant or balanced!



$$|\psi_0\rangle = |0\rangle|0\rangle$$

$$|\psi_1\rangle = |0\rangle|1\rangle$$

$$|\psi_2\rangle = |+\rangle|-\rangle$$

$$|\psi_3\rangle = U_f |+\rangle|-\rangle$$

Lets expand $|+\rangle$ state only

$$= U_f\left(\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)|-\rangle\right)$$

$$= U_f\left(\frac{1}{\sqrt{2}}\left(|0\rangle|-\rangle + |1\rangle|-\rangle\right)\right)$$

$$= \frac{1}{\sqrt{2}}\left(\underbrace{U_f|0\rangle|-\rangle} + \underbrace{U_f|1\rangle|-\rangle}\right)$$

Phase Oracles

$$= \frac{1}{\sqrt{2}}\left((-1)^{f(0)}|0\rangle|-\rangle + (-1)^{f(1)}|1\rangle|-\rangle\right)$$

$|-\rangle$ state is not needed so we omit it from here on

$$|\psi_3\rangle = \frac{1}{\sqrt{2}}\left((-1)^{f(0)}|0\rangle + (-1)^{f(1)}|1\rangle\right)$$

So if (zoom in)

if $f(0)=f(1)$:

$|\psi_3\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle$

$|\psi_4\rangle = |0\rangle$

if $f(0) \neq f(1)$:

$|\psi_3\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = |-\rangle$

$|\psi_4\rangle = |1\rangle$

So,

- if we measure $|0\rangle$ then the function is constant
- if we measure $|1\rangle$ then the function is balanced.

3.4) Deutch-Jozsa Algorithm

This also identifys if a funtion is constant or balanced but the input is any no. of qbits (not just 1 qbit input)

$$f : \{0,1\}^n \longrightarrow \{0,1\} \qquad n \leftarrow \#\text{qbit input}$$

Constant Zero

x	$f(x)$
00...00	0
00...01	0
00...10	0
⋮	⋮
11...11	0

Constant 1

x	$f(x)$
anything	1
	1
	1
	1
	⋮
	1

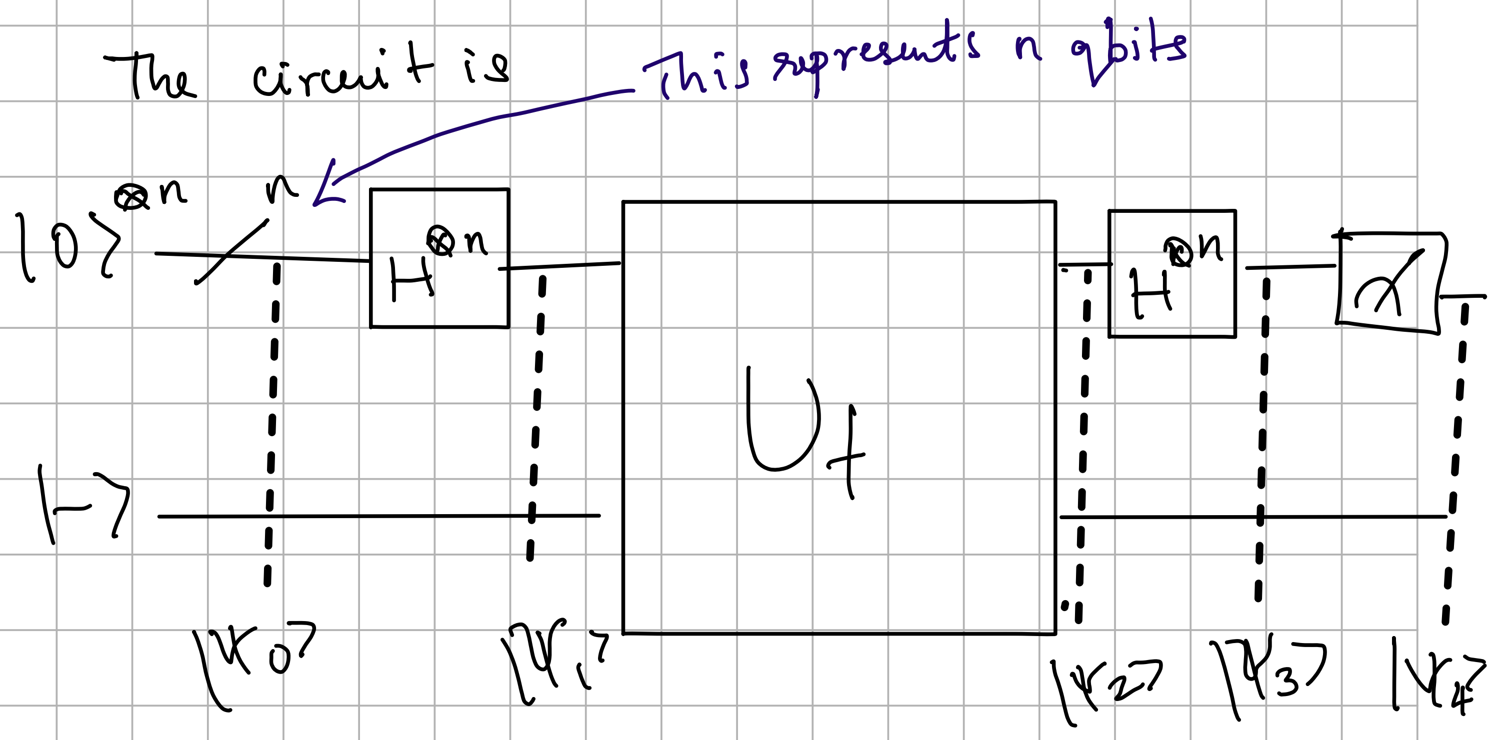
for classical computer, it will take

$2^{n-1}+1$ calculations to solve if we

will get constant or balanced function

(total # calculation is 2^n (as every value of 0 or 1 to n bits)
and for getting if it is balanced or not, we need
to calculate half+1 number of values
half values is 2^{n-1} and one more is $2^{n-1}+1$
So if
- $2^{n-1}+1$ values will give same output for $f(x)$
  then $f(x)$ is constant
- $2^{n-1}+1$ values give atleast 1 different value
  then $f(x)$ is balanced.)

The circuit is



$$|\psi_0\rangle = |0\rangle^{\otimes n} |-\rangle$$

$$|\psi_1\rangle = H^{\otimes n} |0\rangle^{\otimes n} |-\rangle$$

$$= \underbrace{H|0\rangle H|0\rangle \dots H|0\rangle}_{n \text{ times}} |-\rangle$$

$$= |+\rangle|+\rangle|+\rangle \dots |+\rangle |-|$$

note
$|+\rangle|+\rangle \dots |+\rangle$ n times is not $|+\rangle^{\otimes n}$

Lets take $n = 2$

$$|+\rangle|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$= \frac{1}{\sqrt{2^2}}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

$$= \frac{1}{\sqrt{2^2}} \sum_{x \in \{0,1\}^2} |x\rangle$$

So if $n=3$

$$|+\rangle|+\rangle|+\rangle = \frac{1}{\sqrt{2^3}} \sum_{x \in \{0,1\}^3} |x\rangle$$

So coming back to our equation

$$|\psi_1\rangle = \left(\frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle \right) |-\rangle$$

$$|\psi_2\rangle = U_f \left[\frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle |-\rangle \right]$$

$$= \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} \underbrace{U_f |x\rangle |-\rangle}_{\text{Phase Oracle form}}$$

$$= \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} |x\rangle |-\rangle$$

omit $|-\rangle$, not needed anymore

$$|\psi_2\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} |x\rangle|-\rangle$$

$$|\psi_3\rangle = H\left(\frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} |x\rangle\right.$$

$$= \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} H^{\otimes n} |x\rangle$$

note:

$$\boxed{H^{\otimes n}|x\rangle = \frac{1}{\sqrt{2^n}} \sum_{z \in \{0,1\}^n} (-1)^{x \cdot z} |z\rangle}$$

so

$$|\psi_3\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} \frac{1}{\sqrt{2^n}} \sum_{z \in \{0,1\}^n} (-1)^{x \cdot z} |z\rangle$$

$$= \sum_{x \in \{0,1\}^n} \frac{1}{2^n} \sum_{z \in \{0,1\}^n} (-1)^{f(x)} (-1)^{x \cdot z} |z\rangle$$

$$= \sum_{x \in \{0,1\}^n} \frac{1}{2^n} \sum_{z \in \{0,1\}^n} (-1)^{f(x) + x \cdot z} |z\rangle$$

now consider the input as $|00 \ldots 0\rangle$

then $\frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{f(x) + \underbrace{x \cdot 0 \ldots 0}}$

↳ This is zero

if f is constant :

$$\frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} = \frac{1}{2^n} \sum_{x \in \{0,1\}^n} 1 \quad \text{if } f(x) = 0 \text{ for all } x$$

$$= \frac{1}{2^n} 2^n$$

$$= 1$$

if $f(x) = 1$ then

$$\frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} = \frac{1}{2^n} \sum_{x \in \{0,1\}^n} -1$$

$$= \frac{1}{2^n} \cdot -2^n$$

$$= -1$$

So if the function is constant then amplitude of $|00\ldots0\rangle = \pm 1$

if f is balanced

$$\frac{1}{2^n} \sum_{x \in \{0,1\}} (-1)^{f(x)} = 0$$

Now we measure Qbit

$\Rightarrow$ if f is constant

Amplitude of $|00\ldots0\rangle$ is ± 1

Prob of $|00\ldots0\rangle$ is 1

$\Rightarrow$ if f is balanced

amplitude of $|00\ldots\rangle$ is 0

prob of $|00\ldots0\rangle$ is 0

Key takeaway

$$H^{\otimes n} |0\rangle^{\otimes n} = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle$$

$$H^{\otimes n} |x\rangle = \frac{1}{2^n} \sum_{z \in \{0,1\}^n} -1)^{x \cdot z} |z\rangle$$

3.5) Bernstein-Vazirani Algorithm

we have a function $f : \{0,1\}^n \to \{0,1\}$

that does the operation $f(x) = x \cdot s \pmod 2$

we are trying to solve for secret string s

This problem also uses the same diagram as Deutch Jozsa algo (the previous one)

The steps are same for this and Deutch Jozsa till the orange line is there (follow orange from prev pages)

$$|\psi_3\rangle = \frac{1}{2^n} \sum_{x \in \{0,1\}^n} \sum_{z \in \{0,1\}^n} (-1)^{f(x) + x \cdot z} |z\rangle$$

here $f(x) = x \cdot s \pmod 2$

$$|\psi_3\rangle = \frac{1}{2} \sum_{x \in \{0,1\}^n} \sum_{z \in \{0,1\}^n} (-1)^{x \cdot s + x \cdot z} |z\rangle$$

$$= \frac{1}{2} \sum_{x} \sum_{z} (-1)^{(s+z) \cdot x} |z\rangle$$

$+$ indicate bitwise XOR

Now we measure the qbit

amplitude of $|s\rangle$ state

$$= \frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{(s+s)\cdot x}$$

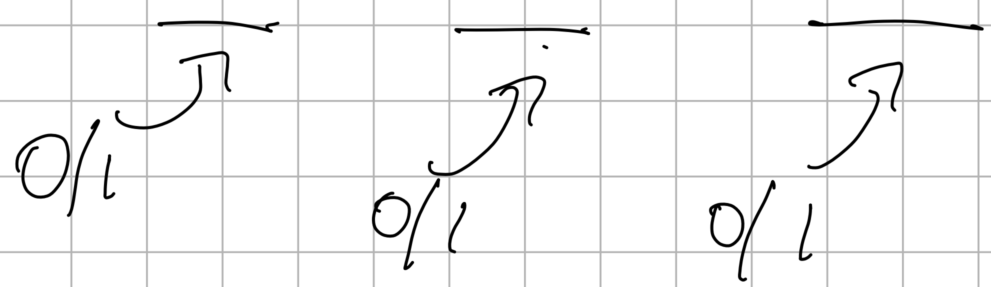
$$= \frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{00\ldots0 \cdot x}$$

$$= \frac{1}{2^n} \sum_{x \in \{0,1\}^n} -1^{0}$$

$$= \frac{2^n}{2^n} = 1$$

3.6) Quantum Fourier Transform

in normal computers, lets say 3 bits



They can take 0 or 1 for every value

But in qbits **after applying Quantum Fourier Transform**

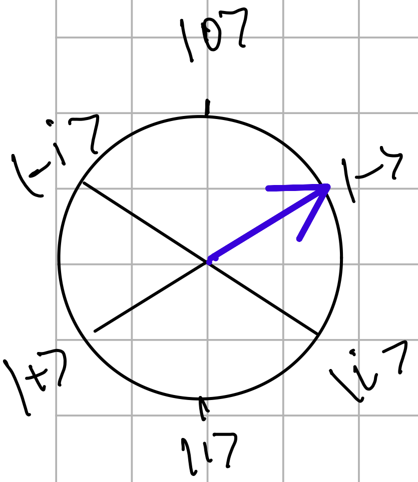
first qbit can take 2 states $|+\rangle$ or $|-\rangle$

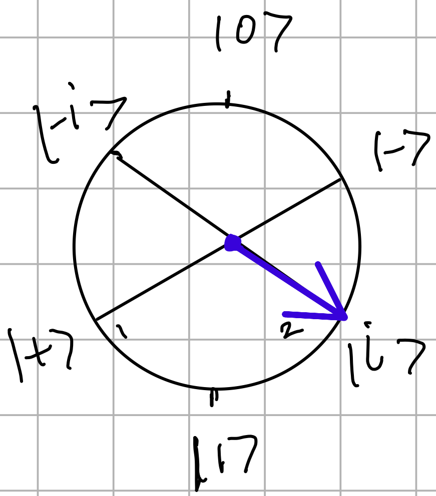
second qbit can take 4 states $|+\rangle |i\rangle |-\rangle |-i\rangle$

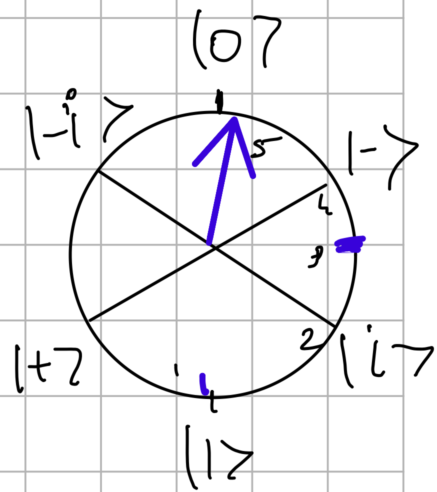
third qbit can take 8 states $|+\rangle$ $|i\rangle$ $|-\rangle$ $|-i\rangle$ and states at 45° between these (Try to imagine)

5 in binary

State for $|101\rangle$ after QFT is applied (quantum fourier transform)







Radian around z axis

$$2 \times \frac{5}{8} \times 2\pi$$

$$= \frac{\pi}{2}$$

Radians around z axis

$$= \frac{5}{8} \times 2\pi \text{ radians}$$

$$= \frac{5\pi}{4} \text{ Radian}$$

$$\cancel{2} \times \frac{\pi}{\cancel{2}}$$

$$\pi$$

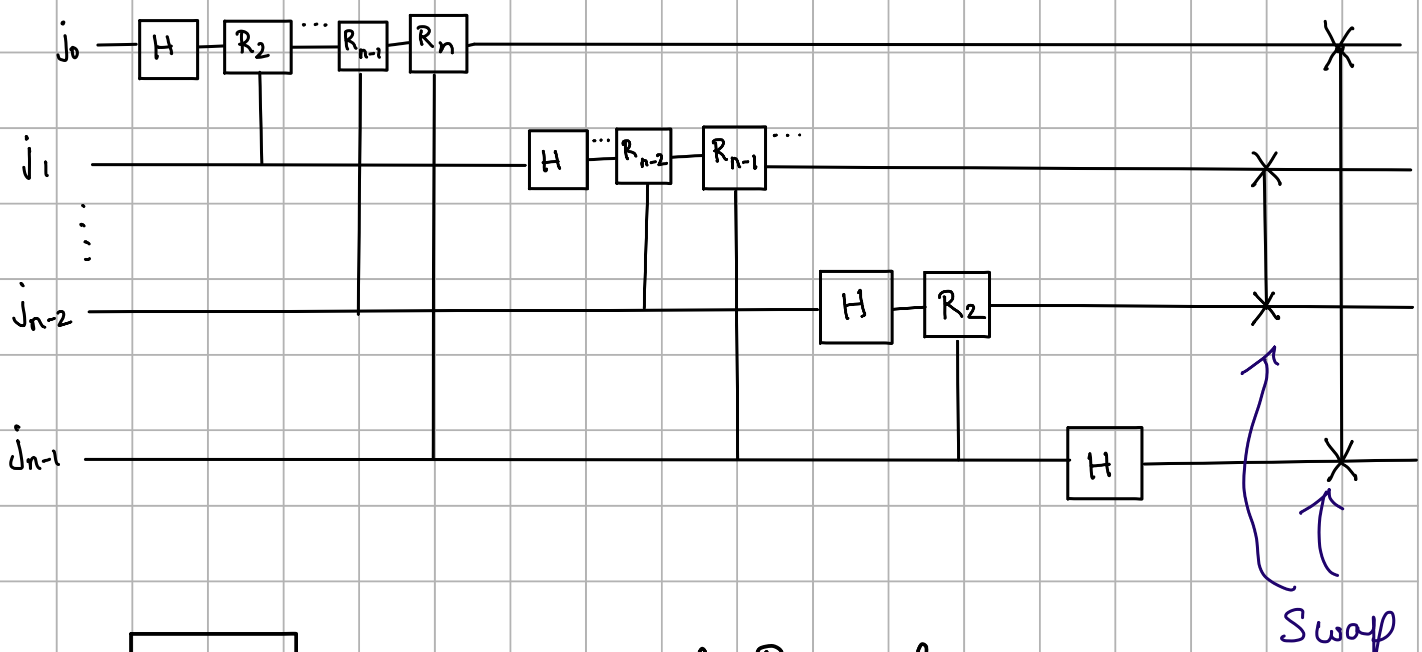
we find the right one and multiply by 2 to get the rest

So this means if we have $|101\rangle$ and apply QFT

$$QFT(|101\rangle) = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\pi}|1\rangle\right) \frac{1}{\sqrt{2}}\left(|0\rangle + e^{\pi i/2}|1\rangle\right)$$

$$\frac{1}{\sqrt{2}}\left(|0\rangle + e^{5i\pi/4}|1\rangle\right)$$

The circuit for QFT is as follows:

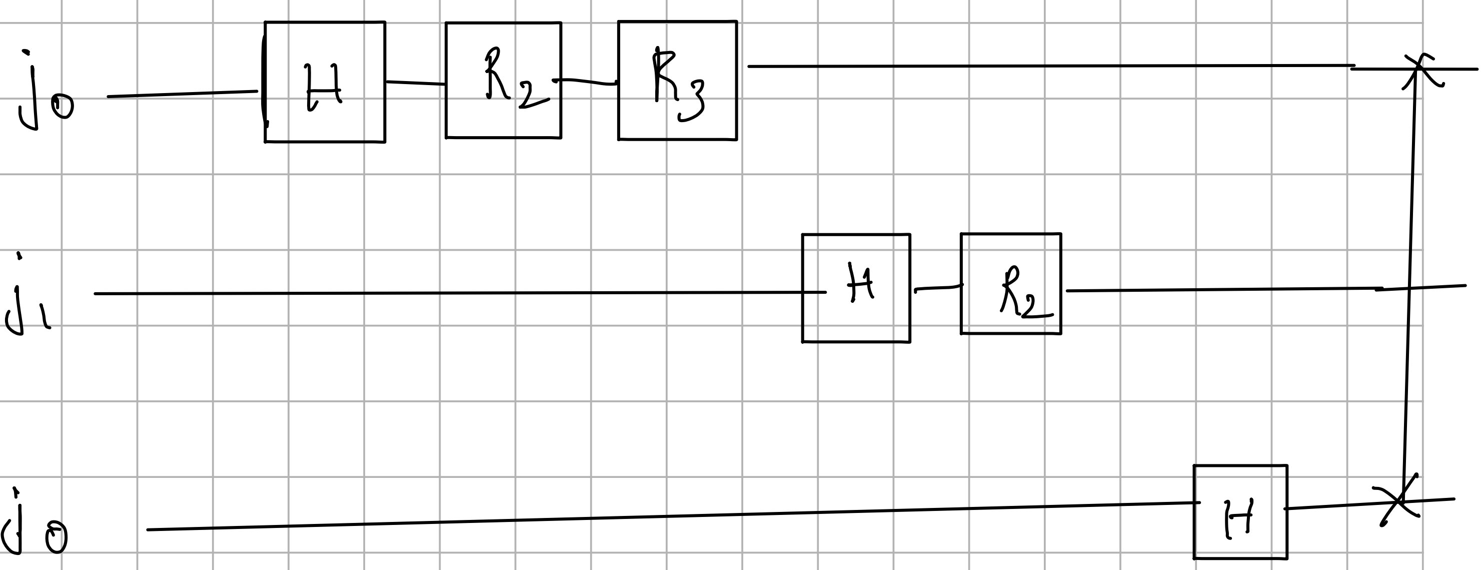


$\boxed{R_K}$ ← controlled R gate

$$R_K = \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{2\pi i}{2^k}} \end{bmatrix}$$

Lets take an example for 3 qbits

$$|j\rangle = |j_0 j_1 j_2\rangle$$



$$H|j_0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^{j_0}|1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i\left(\frac{j_0}{2}\right)}|1\rangle\right)$$

$$R_2 H|j_0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i \left(\frac{j_0}{2} + \frac{j_1}{4}\right)} |1\rangle\right)$$

$$R_3 R_2 H|j_0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i \left(\frac{j_0}{2} + \frac{j_1}{4} + \frac{j_2}{8}\right)} |1\rangle\right)$$

now going to $|j_1\rangle$

$$H|j_1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i \left(\frac{j_1}{2}\right)} |1\rangle\right)$$

$$R_2 H|j_1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i \left(\frac{j_1}{2} + \frac{j_2}{4}\right)} |1\rangle\right)$$

Lastly

$$H|j_2\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i\, j_2/2} |1\rangle\right)$$

Now we should swap j_0 and j_2 for final answer

$$\mathrm{QFT}|j\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i \frac{j_2}{2}}|1\rangle\right) \otimes$$

$$\frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i \left(\frac{j_1}{2}+\frac{j_2}{4}\right)}|1\rangle\right) \otimes$$

$$\frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i \left(\frac{j_0}{2}+\frac{j_1}{4}+\frac{j_2}{8}\right)}|1\rangle\right)$$

3.7) Quantum Phase Estimation

It allowes us to find eigen valuee of a eigen vector given the matrix

we have $U \leftarrow$ gate and state $|v\rangle \leftarrow$ eigenvector

$$U|v\rangle = e^{i\theta}|v\rangle$$

eigen value [which we can find out]