

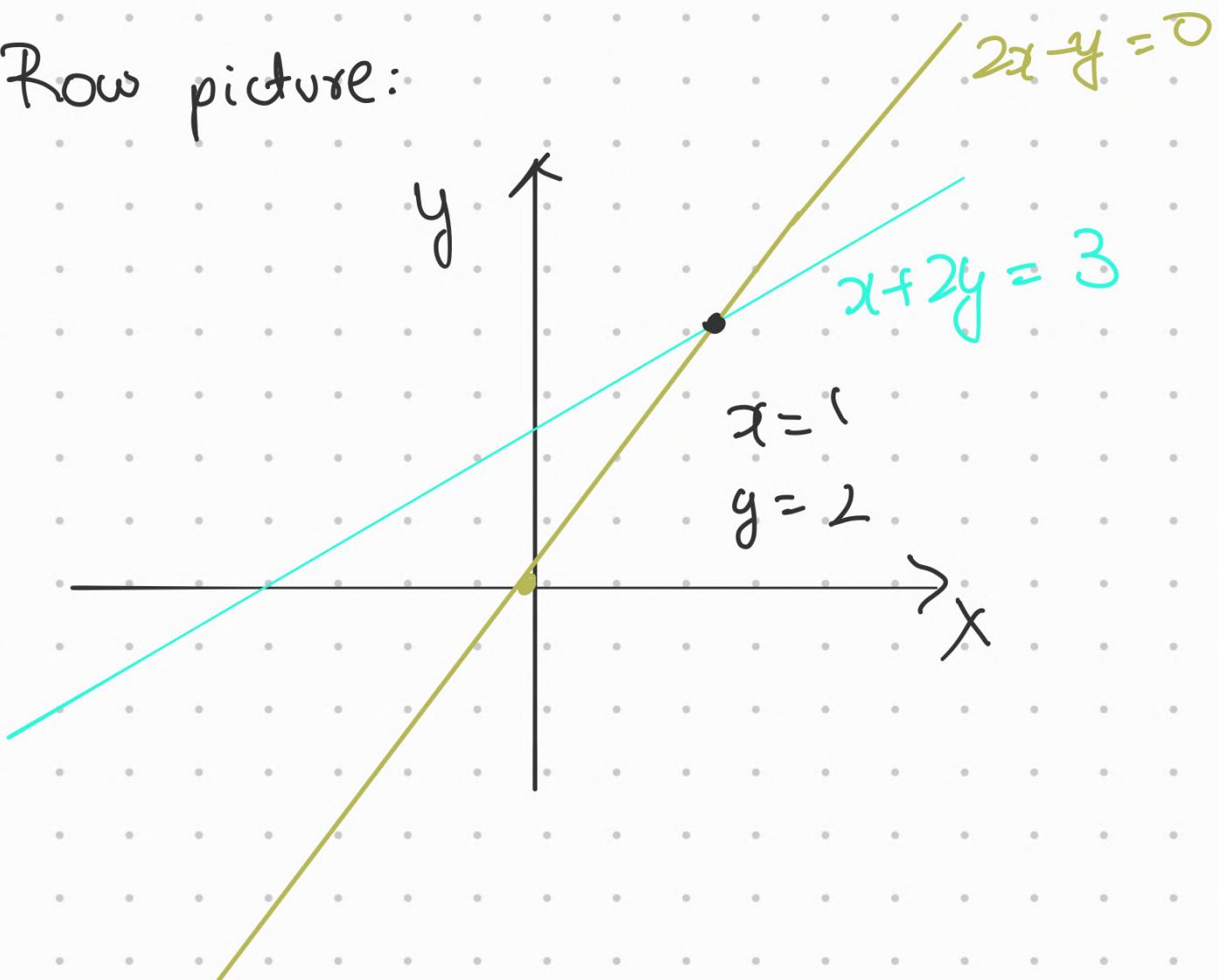
Video - 1

System of linear equations

$$\begin{aligned} 2x - y &= 0 \\ x + 2y &= 3 \end{aligned} \Rightarrow \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$A \cdot x = b$

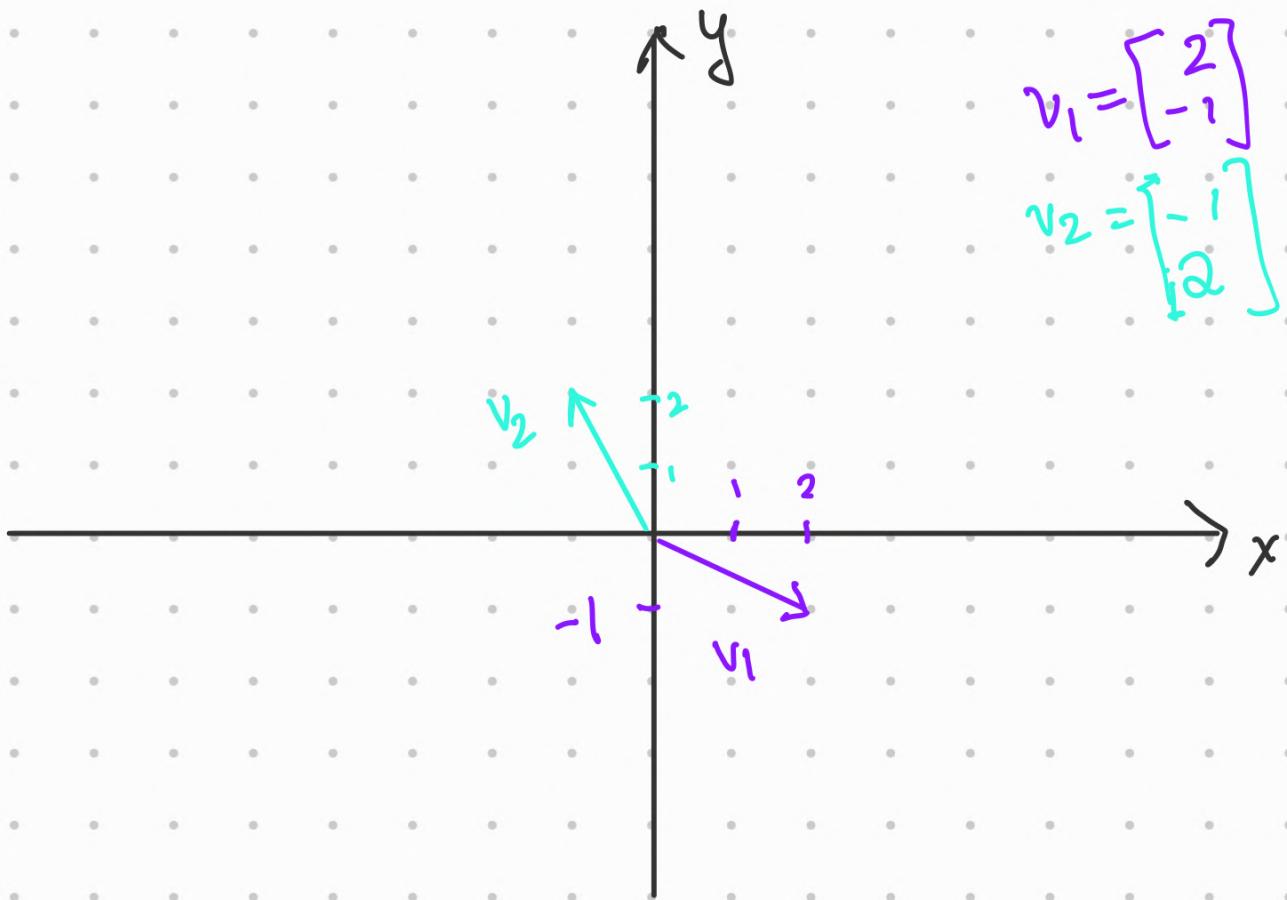
Row picture:



Column Picture

$$x \begin{bmatrix} 2 \\ -1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

Linear combinations of columns

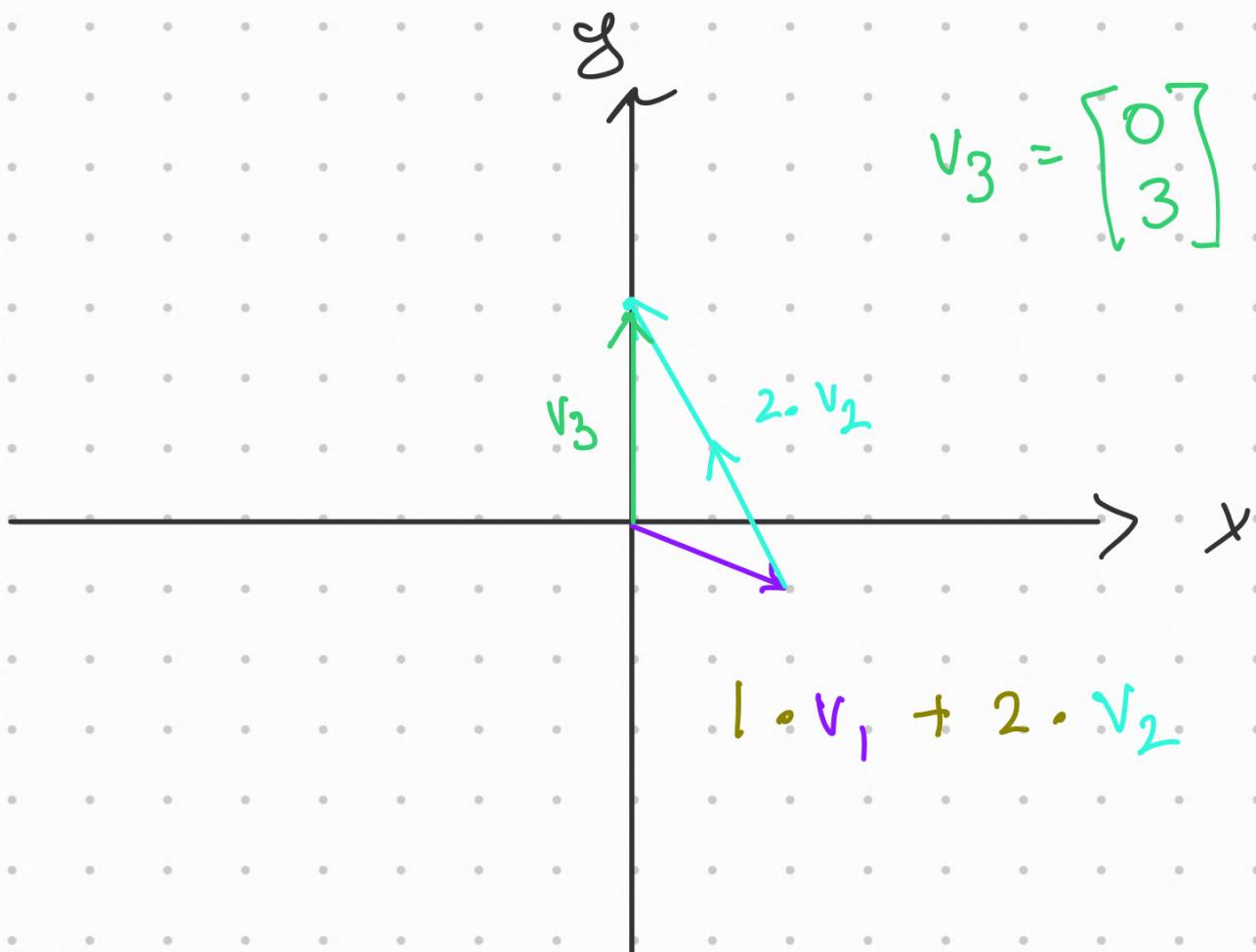


$$v_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$
$$v_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

So we need x and y such that

these vector addition will give $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$ vector

Let us visualize how $x=1$ and $y=2$ is
the solution



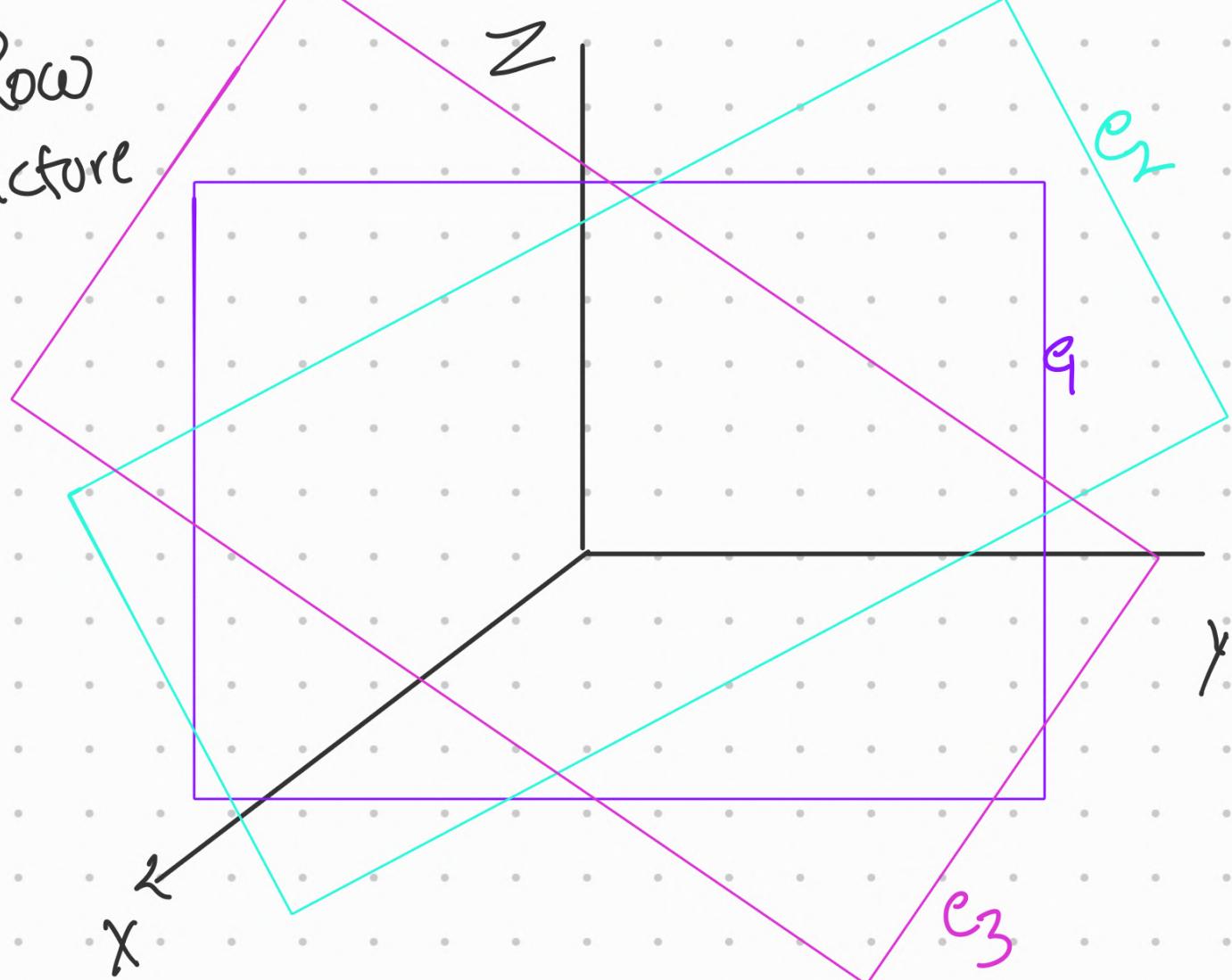
Example for 3 variables

$$2x - y = 0 - e_1$$

$$-x + 2y - z = -1 - e_2$$

$$-3y + 4z = 4 - e_3$$

Row
picture

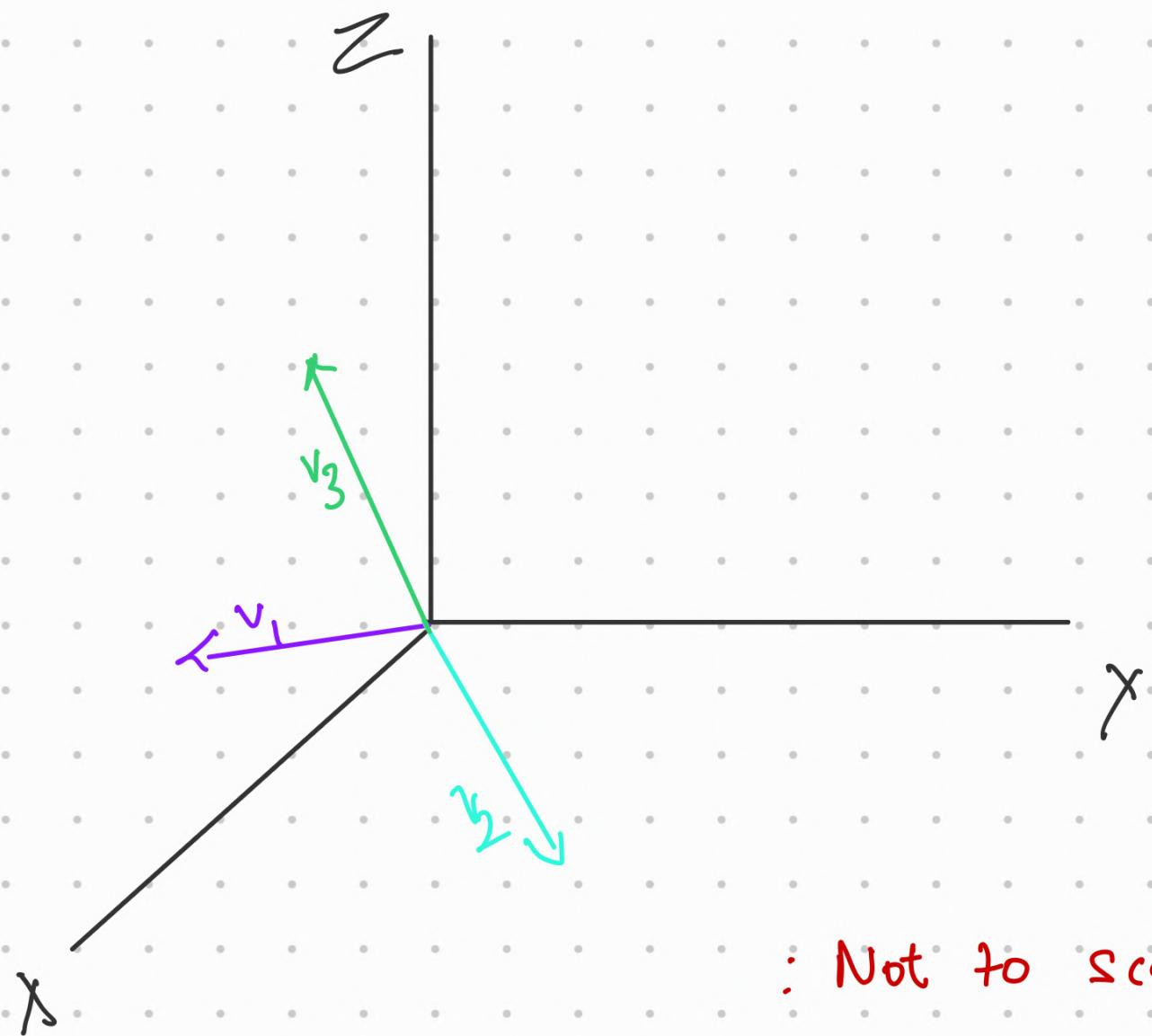


These planes will meet at a single point
that will be the solution [if unique
solution exists]

Column Picture:

$$x \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} + z \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

v_1 v_2 $\sqrt{3}$



We have to get x, y, z such that
we get the final vector

In the above $x = y = 0$ and $z = 1$ is the sol^u

Important question:

Can we get $Ax = b$ for every b
or

Do linear combinations of the columns
fill the entire 3D space?

Answer: Not for all matrices

Why?

→ if the vectors point towards
same direction, then every b
is not possible [3D space]

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Elimination in Matrices

$$x + 2y + z = 2$$

$$3x + 8y + z = 12$$

$$4y + z = 2$$

pivot

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}$$

$$R_2 = R_2 - 3R_1$$

Pivot

$$= \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}$$

$$R_3 = R_3 - 2R_2$$

$$= \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

← This is elimination

getting lower half as zeros

we should have augmented the RHS
as well

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 3 & 8 & 1 & 12 \\ 0 & 4 & 1 & 2 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 4 & 1 & 2 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 0 & 5 & -10 \end{array} \right]$$

we get
this after
elimination

Now the equation are

$$x + 2y + z = 2$$

$$2y - 2z = 6$$

$$5z = -10$$

We back substitute
and get the answer

$$z = -2, \underline{y = 1}, \underline{x = 2}$$

Matrix Multiplication

So in the previous example for elimination

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} \quad R_2 = R_2 - 3R_1$$

This can be written as matrix multiplication

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}$$

↳ Let's call

this E_{21}

↳ to make (2, 1)th position zero

So next

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

$\nearrow E_{32}$

$\nwarrow U$

$$E_{32} (E_{21} A) = U$$

$$(E_{32} E_{21}) A = U \quad (\text{Associative Law})$$

so all elimination matrix can be combined to single matrix

To exchange rows of matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

\Rightarrow Permutation matrix

To exchange columns of matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

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Matrix Multiplication \rightarrow 4 ways

Way 1)

$$A_{m \times n} \quad B_{n \times p} \quad C_{m \times p}$$

Diagram showing the multiplication of two matrices A and B to produce matrix C. Matrix A is an m by n matrix with rows labeled a_1, a_2, \dots, a_m . Matrix B is an n by p matrix with columns labeled $b_1^*, b_2^*, \dots, b_p^*$. Matrix C is an m by p matrix with entries c_{ij} .

$$C_{ij} = \sum_{k=1}^n a_{ik} + b_{kj}^* \quad \text{Generic Method}$$

Way 2)

column wise

A

B

C

$$\begin{bmatrix} & & & & \end{bmatrix} = \begin{bmatrix} & & & & \end{bmatrix}$$

Diagram showing the column-wise multiplication of matrices A and B to produce matrix C. Matrix A is a 2x2 matrix. Matrix B is a 2x3 matrix. Matrix C is a 2x3 matrix.

$$A = \begin{bmatrix} & \\ & \\ & \end{bmatrix}, \quad b^* = \begin{bmatrix} b_1^* \\ b_2^* \\ \vdots \end{bmatrix}$$

$$A \cdot B = [Ab_1^* \ Ab_2^* \ \dots]$$

$\underbrace{\hspace{10em}}$
 C

way 3) Row wise

$$\begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} & \\ & \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$

A B C

Way 4) column (A) \times Row (B) = C

$$\begin{bmatrix} & \\ & \end{bmatrix} \times \begin{bmatrix} & \\ & \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$

A B C

Inverses

$$A^{-1} A = I$$

I if exists

$$AA^{-1} = I \text{ is also true}$$

if inverse exists to A, then it is called
invertible, non-singular

Example for singular

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \Leftarrow \text{singular}$$

if $Ax = 0$, where $x \neq 0$ vector then it
is singular

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So A is singular

Ex for non singular [where inverse exists]

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} \quad \quad \\ A^{-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I$$

$A \times \text{column } j \text{ of } A^{-1} = \text{column } j \text{ of } I$

Gaussian Jorden [solve 2 eqⁿ at once]

$$A \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \leftarrow$$

2 equation

to solve with
same A

$$A \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \leftarrow$$

$$\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array} \Rightarrow \begin{array}{l} \text{we solve so } A \text{ becomes} \\ I \text{ in augmented matrix} \\ \text{to get } A^{-1} \text{ at } I \text{ place} \end{array}$$

augmented matrix

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array} \right]$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array} \right] \quad R_2 = R_2 - 2R_1$$

$$R_1 = R_1 - 3R_2$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & -3 \\ 0 & 1 & -2 & 1 \end{array} \right]$$

$\mathbb{I} \quad A^{-1}$

What actually is happening

$$E[A : I] = [\mathbb{I} -]$$

if E is elimination matrix combined together

$$E \cdot [A : I] = [\mathbb{I} -]$$

$E \cdot A = I$ so E must be A^{-1} to get I as result

$E \cdot I = E$, so we get A^{-1} after Elimination to get I in RHS

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Let matrix A and B be invertable

$$(A \cdot B) \cdot (B^{-1} A^{-1}) = I$$

order is reverse

because

$$= A (B B^{-1}) A^{-1} \quad (\text{associative law})$$

$$= (A \quad I) A^{-1}$$

$$(A A^{-1}) = I$$

$$\text{if } A A^{-1} = I$$

$$\text{then } (A^{-1})^T A^T = I$$

↑ This is same as $(A^T)^{-1}$

Let's take $A = \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$$

E_{21} A U

so for

$$\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$$

A L U

$$= \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$$

for $A (2 \times 2)$ E_{21} all the elimination
required

so for 2×2 $E = E_{21}$

for $3 \times 3 A$

$$E_{32} E_{31} E_{21} A = U$$

so

$$A = E_{21}^{-1} E_{31}^{-1} E_{32}^{-1} U$$

Why this order refer to pink above

$$= L U$$

where L is product of inverse of E

Why are we calculating L instead of E ?

Let's take example

$$\begin{matrix} E_{32} \\ \left[\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{matrix} \right] \end{matrix} \begin{matrix} E_{21} \\ \left[\begin{matrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \right] \end{matrix} = \begin{matrix} E \\ \left[\begin{matrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 10 & -5 & 1 \end{matrix} \right] \end{matrix}$$

Remember

$$EA = U$$

for L

$$E_{21}^{-1}$$

$$E_{32}^{-1}$$

L

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}$$

where

$$A = L U$$

So in L in position (3,1)

we got rid of 10 from E, hence L is better

$$\text{in } A = L U$$

If no rows exchanges, then multipliers go directly into L

So the elimination matrix is always lower triangular [therefore denoted

as LJ

How expensive is Elimination?

for $n \times n$ matrix, how much operation are we making?

A diagram of an $n \times n$ matrix. The top-left element is circled in green and labeled "Pivot 1". Dashed lines indicate the structure of the matrix. A bracket on the right indicates it is an $n \times n$ matrix.

note

One operation is
one multiplication
+ one subtraction

A diagram of an $n \times n$ matrix. The first row has its first element circled in green and labeled "Pivot 1". The second row has its first element circled in green. The third row has its first element circled in green. The fourth row has its first element circled in green. A bracket on the right indicates it is an $n \times n$ matrix.

$$R_b = R_b - x R_a$$

multiplication
Subtraction

here n^2 operations are made

next

A diagram of an $n \times n$ matrix. The second row has its second element circled in green and labeled "Pivot 2". The third row has its second element circled in green. The fourth row has its second element circled in green. A bracket on the right indicates it is an $n \times n$ matrix.

here $(n-1)^2$
operations are
made

and so on

So total count of operations
are

$$= n^2 + (n-1)^2 + (n-2)^2 + \dots + 1^2$$

$$\approx \frac{1}{3} n^3$$

This was for only A

but we need to augment b [RHS]

That cost is n^2

Permutations [Row exchange matrices]

note

we exchange when pivots are zero

all 3×3 permutations

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \leftarrow \text{does nothing}$$

$$P_{21} \Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Exchanges row 2
with row 1

$$P_{13} \Rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$P_{23} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$P_{\text{vp cycle}}$ \Rightarrow

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$P_{\text{vp cycle}} \cdot \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = \begin{bmatrix} R_2 \\ R_3 \\ R_1 \end{bmatrix}$$

$P_{\text{down cycle}} \Rightarrow$

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P_{\text{down cycle}} \cdot \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = \begin{bmatrix} R_3 \\ R_1 \\ R_2 \end{bmatrix}$$

Inverses of P

$$P^{-1} = P^T$$

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So till now we did

$$A = LU$$

where

$$L = \begin{bmatrix} 1 & & \\ -\frac{1}{2} & 1 & \\ -\frac{1}{3} & -\frac{1}{2} & 1 \end{bmatrix}$$

↑
zeros
↓
some
digit

and

$$U = \begin{bmatrix} 6 & -1 & 1 \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -\frac{1}{3} \end{bmatrix}$$

↑
some
digits here
↓
zeros here

But we never considered
when pivots are zero

So we need to include P

Hence $A = LU$ becomes

$PA = LU$ for any invertible A

P : permutations

↳ identity matrix with reordered rows

for $n \times n$ matrix, $n!$ permutations are possible
[including identity]

and

$$P P^T = I \quad \text{or} \quad P^{-1} = P^T$$

Transposes:

$$\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix}$$

so

$$(A^T)_{ij} = A_{ji}$$

Symmetric matrix : $A^T = A$

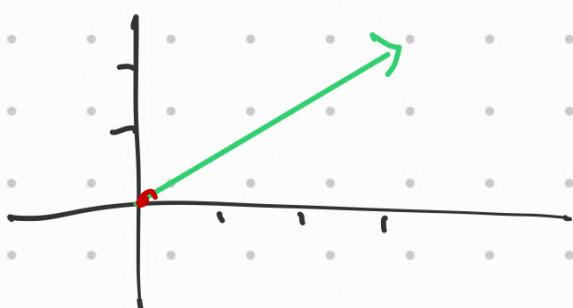
$R^T R$ is always symmetric

Why?

because $(R^T R)^T = R^T R^{TT} = R^T R$

Vector Spaces

Example: $\mathbb{R}^2 \Rightarrow \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \pi \\ e \end{bmatrix}, \dots$



and so on

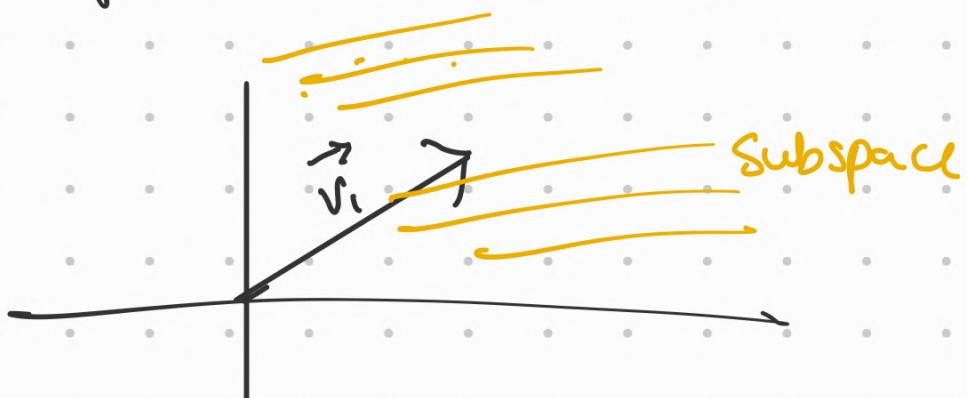
$\mathbb{R}^2 \Rightarrow$ x-y plane

$\mathbb{R}^3 \Rightarrow$ all vector with 3 components

$\mathbb{R}^n \Rightarrow$ all vectors with n components

Vector Spaces and subspaces

Ex. of not a subspace, let subspace be quadrant 1

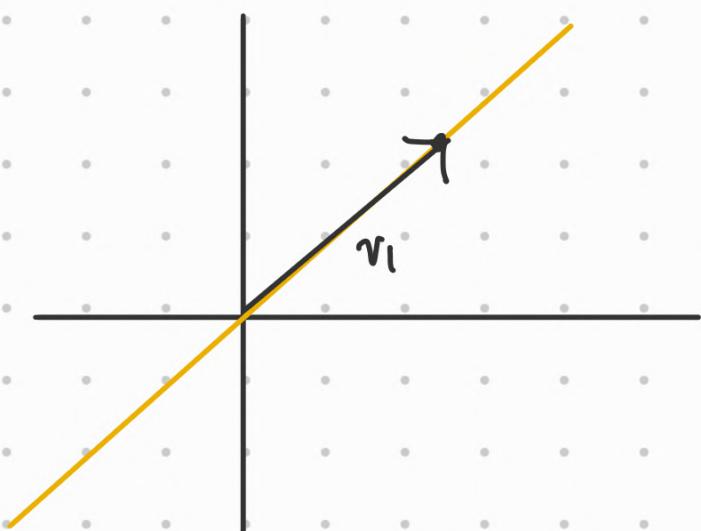


It cannot be a subspace as

- 1. v_1 will result in vector outside subspace

so quadrant 1 is not a subspace in \mathbb{R}^2

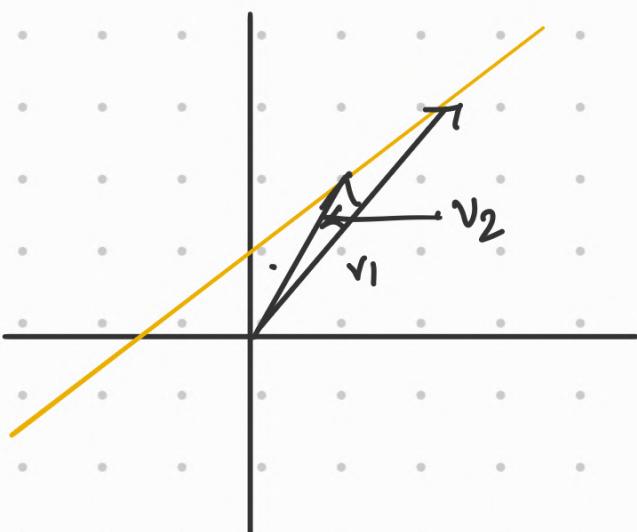
Ex of subspace



Subspace :
Line passing through
origin

let $v_1 = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$

this is a subspace as $x v_1$ for all x is on the subspace



Subspace :
Line not passing origin
[not a subspace]

adding v_1 and v_2 will be not in the subspace

hence Line not passing origin is
not a subspace.

note

all subspaces in \mathbb{R}^2 must pass through origin as xv_1 where $x=0$, we must get zero

Subspaces of \mathbb{R}^2

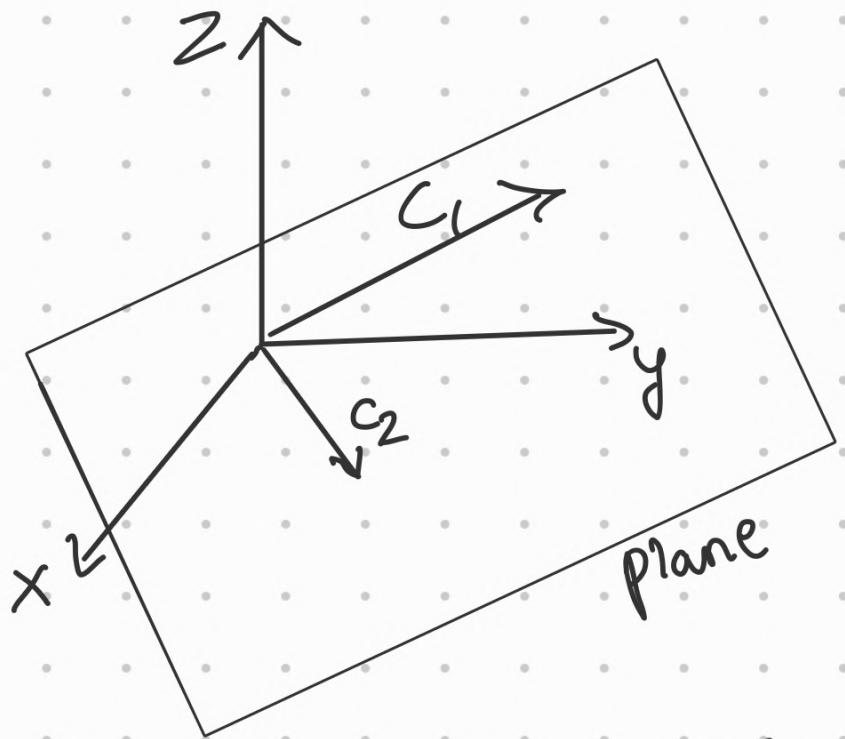
- 1) all of \mathbb{R}^2
- 2) line passing through $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$
- 3) only zero vector $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Subspace from Matrices

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix} \quad \text{columns in } \mathbb{R}^3$$

all their linear combinations form a subspace called column space $C(A)$

here for $A = [c_1 \ c_2]$



all combinations of c_1 and c_2 will give
a plane passing through origin

which is the column space of A [i.e. $C(A)$]

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in \mathbb{R}^3 , subspaces are

- 1) entire \mathbb{R}^3
- 2) plane through

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

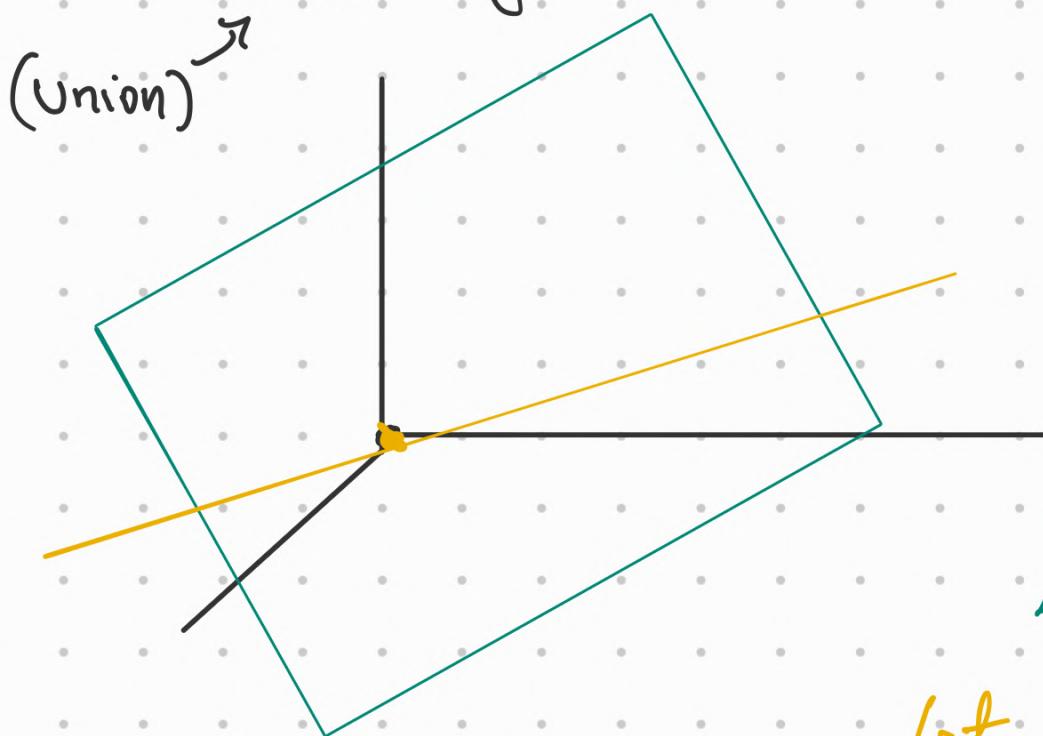
3) line through $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

A) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ only

for 2 subspaces P and L :

Does

P \cup L form a subspace?



Let P be plane

Let V be line
(both passing through origin)

No, $P \cup L$ does not form a Subspace [visualize it, vectors go out of the plane and line when we add]

Does $P \cap L$ form a subspace?

It is a subspace

[visualize it, only $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is common for the above example of line and plane]

for 2 planes, intersection of 2 planes form a line passing through origin which is a subspace.

note

Vector space requirements

if v and w are 2 vectors in a space S and
 c and d are scalars

if $v+w \in S$ and $cv \in S$
then

all combinations of $cv+dw$ form the space S

Column spaces

$$C(A) \in \mathbb{R}^4$$

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$$

$C(A)$ = all linear combinations of A

Does $Ax = b$ have solution for every b ?

Same as saying, will Ax form all possible values for b , or can some b not be accessible by Ax

Answer: No, for some b it is possible

So,

which b 's allows this system to be solved?

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix}$$

So when is b solvable for the above equation

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

then

$$x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

for $b = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$ then $x = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

and so on

note

So b allows the system of equations to be solved when b is in the column space of A

In $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$

C_1 and C_2 are adding new space

to $C(A)$, but C_3 is not

This is because C_3 is dependent on C_1 and C_2

$$C_3 = C_1 + C_2$$

as

$$C(A) = C(C_1) \cup C(C_2) \cup C(C_3)$$





This already makes $C(C_3)$
hence union $C(C_3)$ is not
required

note

So columns that add new spaces
are called pivot columns.

pivot columns are important

Null spaces

Null space = all solutions of $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$
of A where $Ax = 0$

$C(A)$ is in \mathbb{R}^4

$N(A)$ is in \mathbb{R}^3 as $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{3 \times 1}$

nullspace of A

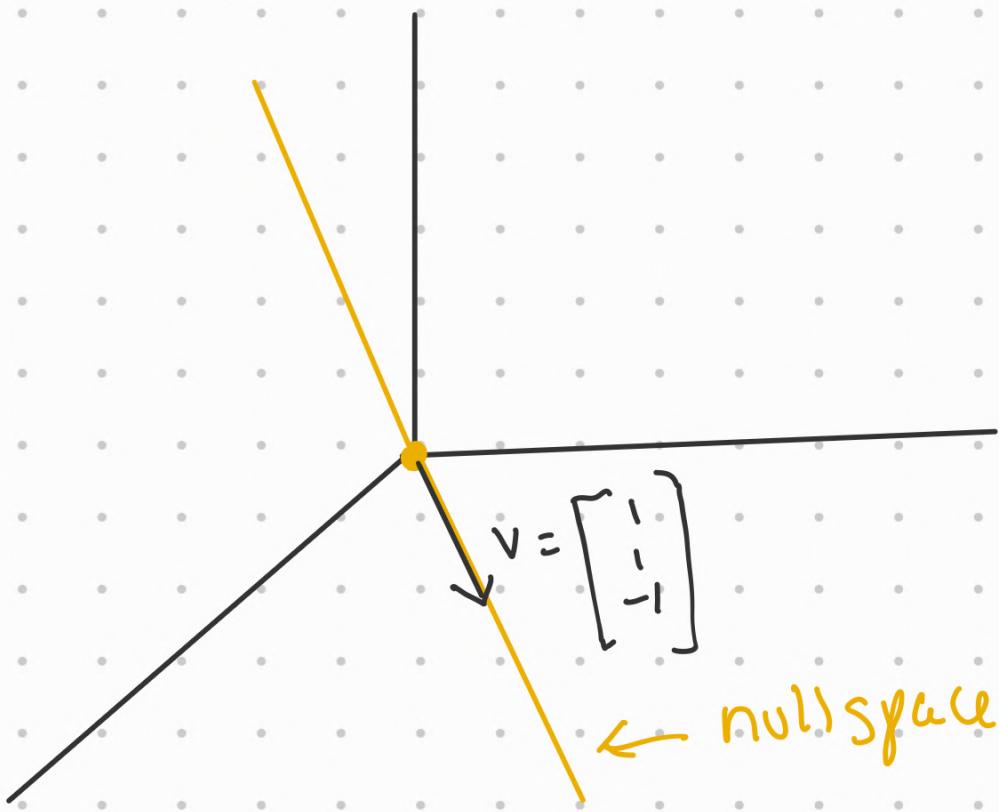
$$Ax = 0$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 6 \end{bmatrix}$$

$$x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix} \dots =$$

So any $\begin{bmatrix} c \\ c \\ -c \end{bmatrix}$ or $c \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

So null space is a line in \mathbb{R}^3
passing through origin



Checking null space is a space

if $Av=0$ and $Aw=0$, then $A(v+w)=0$

Distributive Law works on matrices

$$A(v+w) = Av + Aw = 0$$

and

$$A(cv) = c(Av) = c \cdot 0 = 0$$

hence null space is a subspace in \mathbb{R}^3

Does x form a subspace when

$$Ax = b \text{ when } b \neq 0$$

Answer : No

Example

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$$

So x does not pass origin, so the sum of vectors wont be in the same space

Hence X solution does not form Subspace
when $b \neq 0$

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Computing null space $[Ax = 0]$

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

We can notice that C_2 is $2C_1$
so C_2 is not independent,
and Row-wise, $R_3 = R_1 + R_2$, so
 R_3 is not independent.

We should get all these answers
after doing elimination

Our goal is to solve $Ax = 0$

by solving using elimination, we are not changing the solutions of $Ax=0$ but we are changing $C(A)$ in the process

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

P_1

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix} \quad R_2 = R_2 - 2R_1, \quad R_3 = R_3 - 3R_1$$

as P_2 is zero, go to the next one in rectangle matrix

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 = R_3 - R_1$$

$= U$ [all lower half is zero]

Note:

So, the number of pivots
is 2

which is called Rank of
Matrix

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

↑ ↑

Columns with ^(G and C₃) pivots are pivot columns
(C₂ and C₄) ← free variables
Free column meaning, we can freely
decide on what value we want
for x₁ and x₃, and accordingly

x_2 and x_4 will have a value
for example with A

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

meaning

$$x_1 + 2x_2 + 2x_3 + 2x_4 = 0$$

$$2x_3 + 4x_4 = 0$$

Let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

So x_1 and x_3 pivot variables

x_2 and x_4 = free variables

Let $x_2 = 1$ and $x_4 = 0$

then we get

$$x = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Like this we can take other x_2 and x_4 values to get other vectors

$$x = c \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

will form a line in null space

but this wont be the only solution

Let $x_2=0$ and $x_4=1$

then

$$x = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

So if r is the rank of the matrix of order $m \times n$

pivot variable = r

free variables = $n - r$

Reduced Row echelon form (rref)

the echelon form it has zeros on the lower part of the diagonal, but in rref we try to get zeros on the upper part of the diagonal as well

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$rref(\bar{A}) = \left[\begin{array}{cccc} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array} \right] R_1 = R_1 - R_2$$

$$\bar{R} = \left[\begin{array}{cccc} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right] R_2 = R_2 / 2$$

$$rref(A)$$

+ note

rref(\bar{A}) will give the simplest system of equations to solve A

So first we did

$$Ax = 0$$

then $Ux = 0$ and then $Rx = 0$

Notice in R

exclude this

$$R = \left[\begin{matrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{matrix} \right] \begin{matrix} -2 \\ 2 \\ 0 \end{matrix} \right]$$

x_1 and x_3 have values 1 in them

So 1 part free part
of x_1 and x_3 and which is x_2 and x_4

$$\left[\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \right]$$

$$\left[\begin{matrix} 2 & -2 \\ 0 & 2 \end{matrix} \right]$$

Regular rref form

Sometimes I and F
are combined depending on
the pivot column places

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

where I is identity columns
and F is free columns

When $Rx = 0$, what are all possible
 x we can get

$$Rx = 0$$

$$\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{\text{pivot}} \\ x_{\text{free}} \end{bmatrix} = 0$$

$$x_{\text{pivot}} [I] + x_{\text{Free}} F = 0$$

$$x_{\text{pivot}} = -F x_{\text{Free}}$$

So $x = \begin{bmatrix} -F \\ I \end{bmatrix}$

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix}$$

Let's start elimination

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix}$$

+ pivot colors

$$\begin{aligned} R_2 &= R_2 - 2R_1 \\ R_3 &= R_3 - 2R_1 \\ R_4 &= R_4 - 2R_1 \end{aligned}$$

Row exchange R_2 and R_3

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 4 & 4 \end{bmatrix}$$

$$R_4 = R_4 - 2R_2$$

$U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

echelon
form

Rank = 2

2 pivot columns x_1 and x_2

1 free columns x_3

So for x , let us set $x_3 = 1$ and solve for x_1 and x_2

$$x = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

so $x = C \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

So null space of A is $c \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

further solving for $\text{ref}(A)$

$$R = \left[\begin{array}{ccc|cc} 1 & 0 & 0 & -1 & F \\ 0 & 1 & 0 & -1 & \\ 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \end{array} \right]$$

So as we can see

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

$$\text{and } x = \begin{bmatrix} -F \\ I \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$-F$
single identity

Video - 8

Let's take the earlier system of equations

$$x_1 + 2x_2 + 2x_3 + 2x_4 = b_1$$

$$2x_1 + 4x_2 + 6x_3 + 8x_4 = b_2$$

$$3x_1 + 6x_2 + 8x_3 + 10x_4 = b_3$$

↓

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$
$$A \quad X = b$$

Let's start elimination with b augmented to A

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & : b_1 \\ 2 & 4 & 6 & 8 & : b_2 \\ 3 & 6 & 8 & 10 & : b_3 \end{array} \right]$$

Augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & : b_1 \\ 0 & 0 & 2 & 4 & : b_2 - 2b_1 \\ 0 & 0 & 2 & 4 & : b_3 - 3b_1 \end{array} \right]$$

P₂

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & : b_1 \\ 0 & 0 & 2 & 4 & : b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & : b_3 - b_1 - b_2 \end{array} \right] = U$$

So if $0 = b_3 - b_2 - b_1$, then

$Ax = b$ solvable else not solvable

Solvability Condition on b

$Ax = b$ is solvable if b is in $C(A)$
or

If a combination of rows of A gives
zero row, then same combinations of b
must give 0.

Algorithm to find complete solⁿ for $Ax = b$

- 1) $x_{\text{particular}}$: Set all free variables to zero
Solve $Ax = b$ for pivot variables

$$\begin{aligned}x_1 + 2x_3 &= 1 & x_1 &= 2 \\2x_3 &= 3 & x_3 &= \frac{3}{2}\end{aligned}$$

$$x_p = \begin{bmatrix} -2 \\ 0 \\ \frac{3}{2} \\ 0 \end{bmatrix}$$

- 2) find $x_{\text{nullspace}}$
- 3) Complete solⁿ $X = X_p + X_{\text{null}}$

Why?

because $Ax_p = b$

$$\underline{Ax_n = 0}$$

$$A(x_p + x_n) = b$$

So we have completed x_n before for A

$$x_n = c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

So

$$x_{\text{complete}} = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

for

$m \times n$ Matrix A with rank r
w.k.t ($r \leq m, r \leq n$)

When rank is full rank

that has 2 possibilities

if $r = n$ [full column rank]

pivots = n

free variable = None & 0

$N(A) =$ only zero vector
[no free variable]

$X = X_p$ [unique solution if it exists.
So either 0 or 1 solⁿ]

Example:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{bmatrix}$$

$$r = 2$$

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

if $r = m$ [full row rank]

so every row has a pivot

so for what b can we solve it?

we can solve if for every b

visualize

$$\left[\begin{array}{ccc|c} a & b & c & b_1 \\ 0 & 0 & 1 & b_2 \\ 0 & 0 & 0 & b_3 \end{array} \right] = \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \right]$$

any b

b_1
 b_2
 b_3

So $c = b_3$

and we can
solve it

So # pivots = m

free variables = n - m

Example

$$A \sim \begin{bmatrix} 1 & 2 & 6 & 5 \\ 3 & 1 & 1 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 0 & -\textcircled{F} & - \\ 1 & 0 & - & - \end{bmatrix}$$

if r = m = n for A

then $\text{rref}(A) = I$

$n(A) = \text{zero vector}$

only 1 solution

To recap

if $r = m = n$

$R = I$, One solution only

if $r = n < m$

$R = \begin{bmatrix} I \\ 0 \end{bmatrix}$, 0 or 1 solution

if $r = m < n$

$R = [I F]$ ∞ solution

if $r < m, r < n$

$R = \begin{bmatrix} I F \\ 0 0 \end{bmatrix}$ 0 or ∞ solution

Video - 9

Linear independence

Spanning a Space

Basis and dimension

Suppose A is a matrix of $m \times n$
with $m < n$

[meaning more # unknown variables than
equations]

Then conclusion is non zero solutions in
 $Ax = 0$

so this means there must be atleast
one free variables

Independence

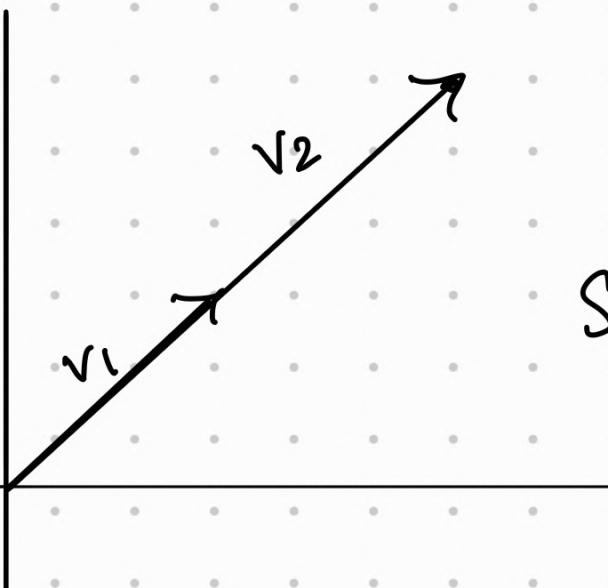
Vectors $x_1, x_2, x_3, \dots, x_n$ are independent if no combinations give zero vector (except the zero combination)

Meaning

$$c_1x_1 + c_2x_2 + c_3x_3 \dots c_nx_n \neq 0$$

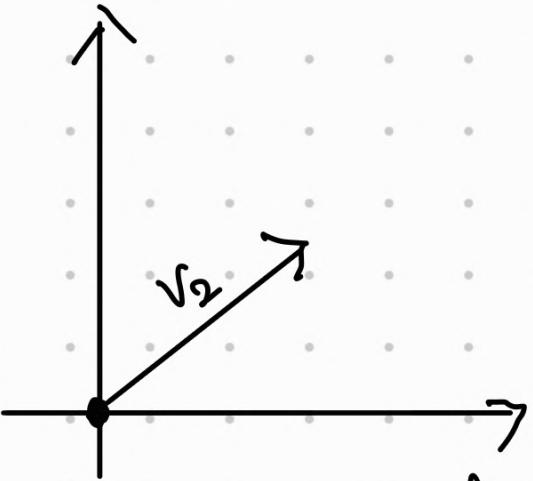
(except $c_i = 0$)

Examples:



$$v_2 = 2v_1$$

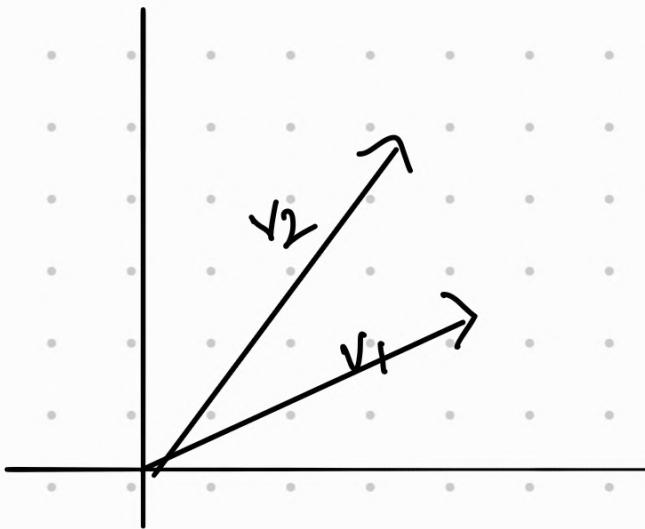
So v_1 and v_2 are dependent



v_1 = zero vector

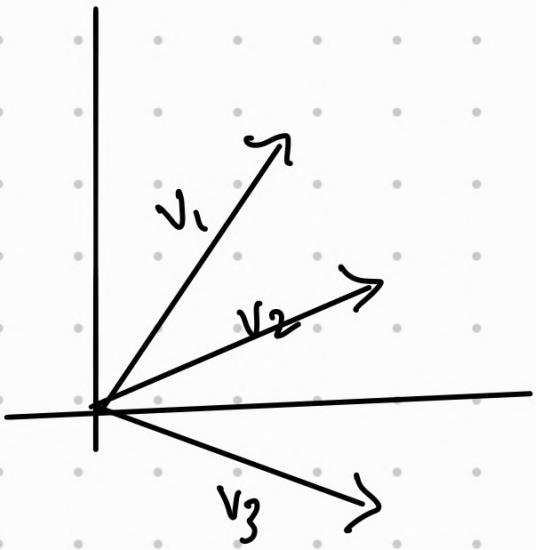
$v_2 \Rightarrow$ non zero vector

v_1 and v_2 are
dependent



v_1 and v_2 are
independent

but



v_1 , v_2 and v_3
are dependent

This is because we writing this in matrix form

$$A = \begin{bmatrix} 1 & 1 & 1 \\ v_1 & v_2 & v_3 \\ 1 & 1 & 1 \end{bmatrix}$$

We have 3 variable and 2 equations
[as vectors are in 2D, they form 2 equations]

Let's assume

$$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ (it looks like this in the graph)}$$

$$v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

so

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So to summarize

for vectors $v_1, v_2, v_3, \dots, v_n$ in A

- vectors are independent if null space of A is zero vector (rank = n)
 - vectors are dependent if null space of A is non zero vector (rank < n)
-

Vectors $v_1, v_2, v_3, \dots, v_\ell$ span a space

means, the space consists of all combination of those spaces

Basis

Basis of a space is a sequence of vectors, v_1, v_2, \dots, v_b with

2 properties,

- 1) They are independent
- 2) They span the Space

So these are the minimum $\underbrace{\# \text{ vectors required}}$
to span that space number of

Example

Space is \mathbb{R}^3

One basis is

$$\left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right], \left[\begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right], \left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right]$$

are they independent? yes

$$\text{cuz } c_1 \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right] + c_2 \left[\begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right] + c_3 \left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right] = 0$$

when $c_1 = c_2 = c_3 = 0$ only

Let's take 2 vectors that are independent

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$$

These two are independent, hence these can span a space [which would be a plane]

So if we choose vector that is not in the plane, we can span the entire \mathbb{R}^3 using the 3 vectors

So Let's say

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 8 \end{bmatrix}$$

So these 3 vectors make another basis for \mathbb{R}^3

note

Every basis for the space has same number of vectors

This number is called dimension of the space

So if \mathbb{R}^3 then no. of vectors in basis also called dimension is 3

if \mathbb{R}^n then no. of vectors or dimension is n

Example space is $C(A)$

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}$$

Are all the columns independent?

No, we can see that

$$C_3 = C_1 + C_2 \text{ and}$$

$$C_4 = C_1$$

But to prove

if columns are independent then

$N(A)$ should be 0

$$N(A) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Basis will be the independent columns

that are C_1 and $C_2 \Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

Rank definition

Rank = # pivot columns

= dimension of $C(A)$

= 2 [in example]

Coming to null space

$$N(A) = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \text{ is one}$$

The other is $\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$$\text{So } N(A) = \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

does these 2 vectors span the entire null space?

Yes!
=

$$\dim N(A) = \# \text{ free variables} = n - r$$

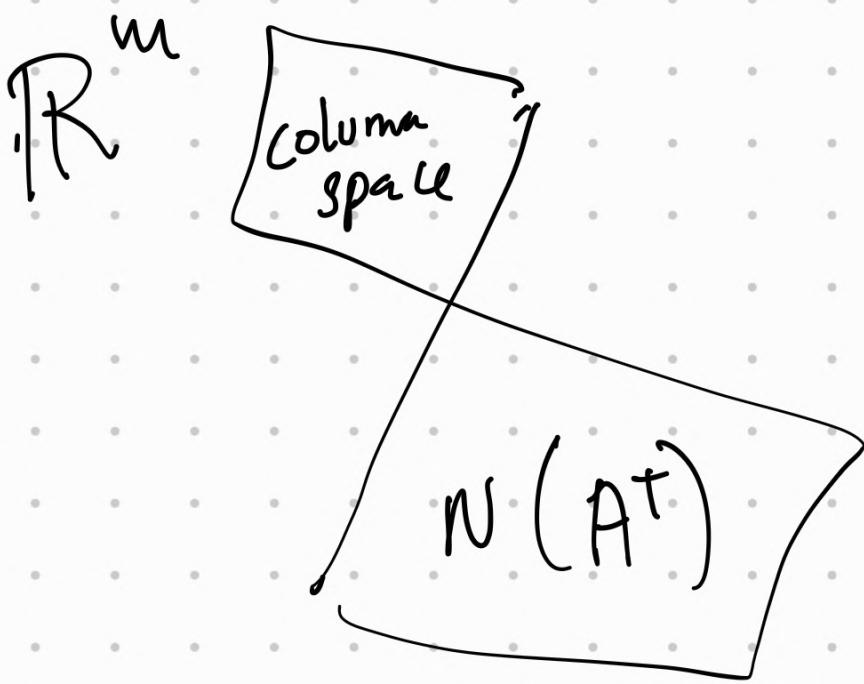
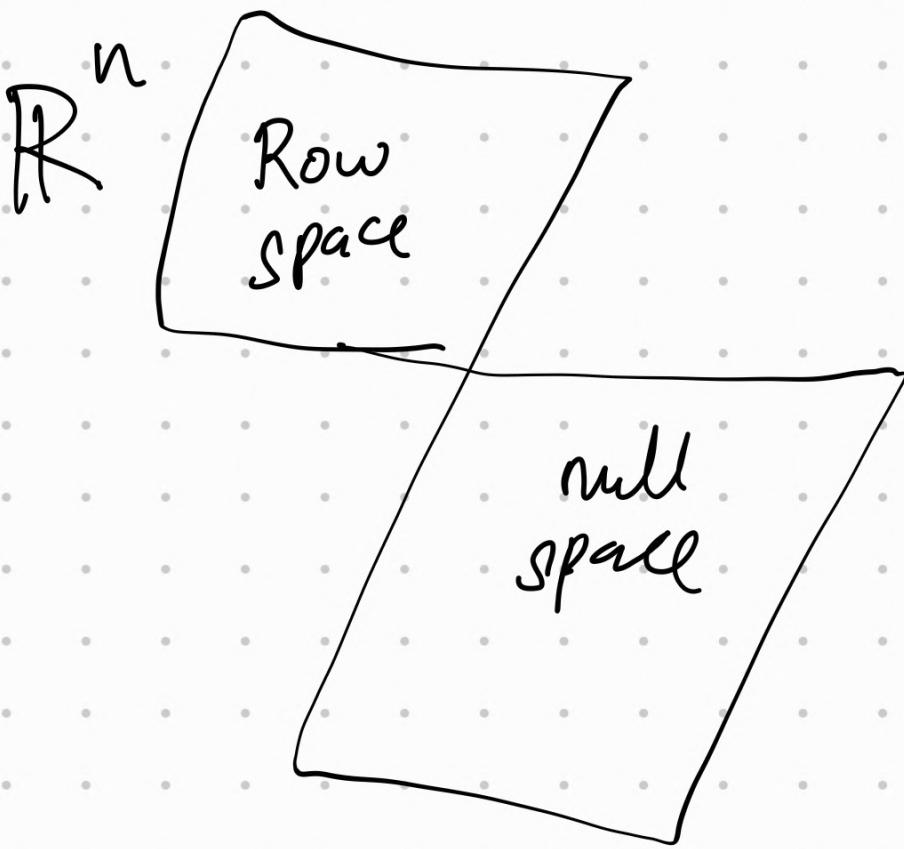
Video - 10

4-fundamental Subspaces

4 fundamental subspaces

- ① column space of A $C(A)$ in \mathbb{R}^m
- ② null spaces of A $N(A)$ in \mathbb{R}^n
- ③ row spaces \Rightarrow all combination of rows = $C(A^T)$ in \mathbb{R}^k
- ④ null space of $A^T \Rightarrow N(A^T)$ Left in \mathbb{R}^m

4 Subspaces



for column space

basis = pivot columns

dimension = r [rank]

for row space:

dimension = r

for null space:

basis: special solution

dimension: $n - r$

for null space of A^T

dimension = $m - r$

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

I *F*

So basis of A for row space is first

r rows in R for A
(rank)

So basis in the example is

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

4th space $N(A^T)$

$$A^T y = 0$$

$$[A] [y] = [0]$$

Take Transpose on both sides

$$(A^T y)^T = (0)^T$$

$$y^T A^{TT} = 0^T$$

$$y^T A = 0$$

$$[y^T] [A] = [0]$$

rref matrix

$$E_{rref} [A : I] = [R : E_{rref}]$$

so

$$E_{rref} A = R$$

Previously

$$R = I$$

$$X [A : I] = [I : X]$$

$$XA = I \text{ so } X = A^{-1}$$

So if we augment I to A

$A : I$ and carry out to
get R , we will get E

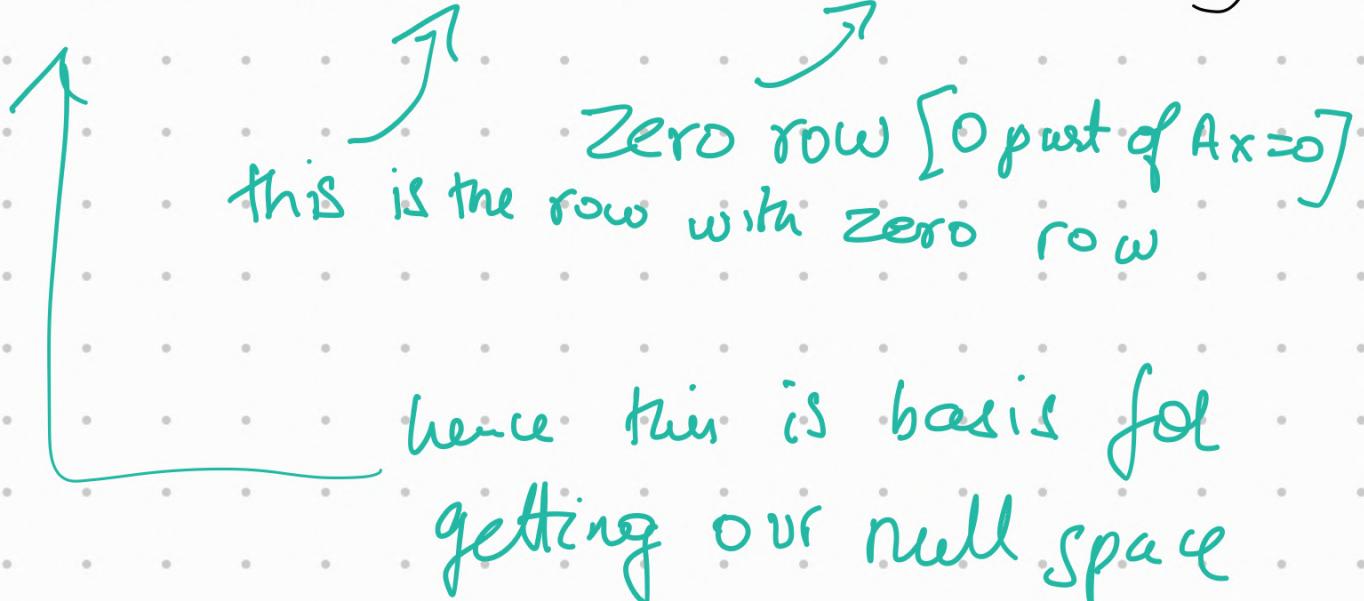
$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 2 & 3 & 1 \end{array} \right] \xrightarrow{\text{Row operations}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \Rightarrow A : I$$

$$\left[\begin{array}{cccc} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{\text{Row operations}} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{array} \right] \Rightarrow R : E_{\text{rrcf}}$$

So

$$E_{\text{rrcf}} A = R$$

$$\left[\begin{array}{ccc} 1 & 2 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{array} \right] \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{array} \right] = \left[\begin{array}{cccc} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$



hence this is basis for getting our null space

hence $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ is basis for $N(A^T)$

New vector space

Let's define a new vector

" All 3×3 matrices " be M

so we can do $A + B$ and cA

where A and B are 2 3×3 order matrices

Subspaces of M

upper triangular matrices

Symmetric matrices

diagonal matrices

So all previous rules will apply
for the new vector space

Like

for diagonal matrices

any 3 independent matrices would
span all diagonal matrices

Ex:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So these 3 matrices can span the
entire diagonal matrices

Video - 11

Basis of new vector spaces

Rank one matrices

Small world graphs

Basis for $M \Rightarrow$ for all 3×3 ,

They are

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So 9 matrices

So dimension (M) = 9

for symmetric matrices (S)

$$\text{dimension}(S) = 6$$

$$\begin{bmatrix} _ & _ & _ \\ _ & _ & _ \\ _ & _ & _ \end{bmatrix}$$

for upper triangular matrix U

$$\text{dimension}(U) = 6 \quad \begin{bmatrix} _ & _ & _ \\ 0 & _ & _ \\ 0 & 0 & _ \end{bmatrix}$$

$S \cap U = \text{diagonal matrices } D$

$$\dim(S \cap U) = 3$$

$$\begin{bmatrix} _ & 0 & 0 \\ 0 & _ & 0 \\ 0 & 0 & _ \end{bmatrix}$$

$S \cup U \Rightarrow \text{not a subspace}$

$S + U =$ any element of S + any element of U

Sum of
subspace

= all 3×3

dimension $(S + U) = 9$ [all 3×3]

so

$$\dim(S) + \dim(U) = \dim(S + U) + \dim(S \cap U)$$

$$6 + 6 = 9 + 3$$

new vector space

$$\frac{d^2y}{dx^2} + y = 0$$

Solution: $y = \cos x, \sin x$

What are all the complete solution for y ?

it is all the combination of

$$y = C_1 \cos x + C_2 \sin x$$

So basis are like $\cos x$ and $\sin x$

$$\text{dimension of sol}^n = 2 \quad [\text{don't think too much}]$$

Rank 1 matrix

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \end{bmatrix}$$

$$\text{So } \dim C(A) = \dim R(A) = 1 = \text{rank}$$

So for rank 1 matrix, we can write

$$\begin{array}{ccc} \text{basis} & * & \text{basis} \\ \text{column} & & \text{row} \end{array} = A$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \end{bmatrix}$$

So every rank 1 matrix are in the form

$$A = u v^T, \text{ where } u \text{ and } v \text{ are column vectors}$$

for $A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$

$$d(R(A)) = 1, \text{ rank} = 1$$

$$d(C(A)) = 1$$

$$d(N(A)) = 3$$

$$d(N(A^T)) = 0$$

Basis of $N(A)$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

↑ ⌊ ⌋
pivot Free

So

basis for $N(A) =$

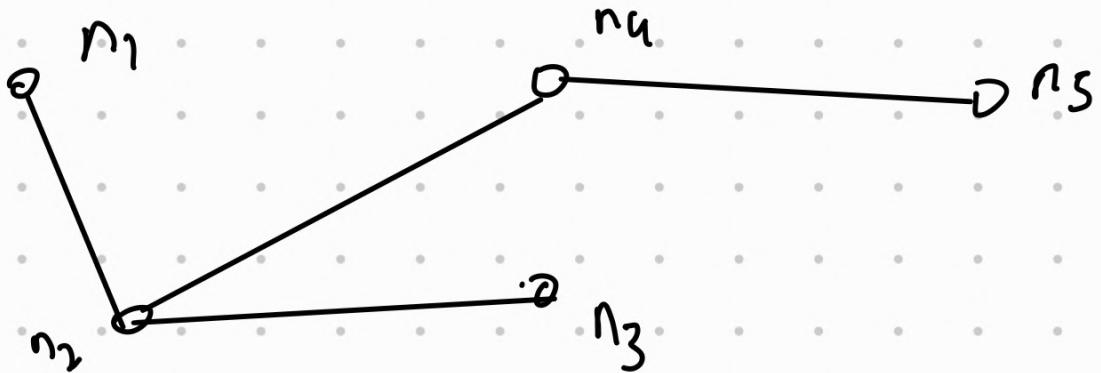
$$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

Small world graphs

What are graphs? $\Rightarrow \{ \text{nodes, edges} \}$



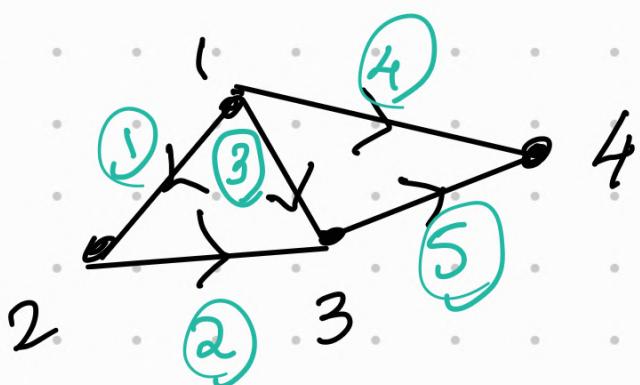
Video - 12 [more of application]

— Graphs and Networks

— Incidence Matrices

— Kirchhoff's Law

Graphs \Rightarrow nodes and edges



$n = 4$ (nodes)

$m = 5$ (edges)

note 1 2 3 4

Incidence
matrix

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

1
2 edges
3
4
5

Notice that

node 1, 2 and 3 are a cycle

so adding edges $\textcircled{1}$ and $\textcircled{2}$
will give $\textcircled{3}$

What about null space?

$$\begin{bmatrix} A & | & \mathbf{0} \\ \hline -1 & 1 & 0 & 0 & | & 0 \\ 0 & -1 & 1 & 0 & | & 0 \\ -1 & 0 & 1 & 0 & | & 0 \\ -1 & 0 & 0 & 1 & | & 0 \\ 0 & 0 & -1 & 1 & | & 0 \end{bmatrix}$$

$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

$$\begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ x_3 - x_1 \\ \vdots \\ x_n - x_1 \\ x_4 - x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Let $x = x_1, x_2, x_3, x_4, \dots$

be potentials at nodes

$x_2 - x_1, x_3 - x_2, \dots$

are potential differences

So a particular solution is

$$x = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{ and basis is } C \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

rank of $A = 3$

$$A^T y = 0$$

note

edges y_i can
be visualized as
currents and
node values can
be potentials

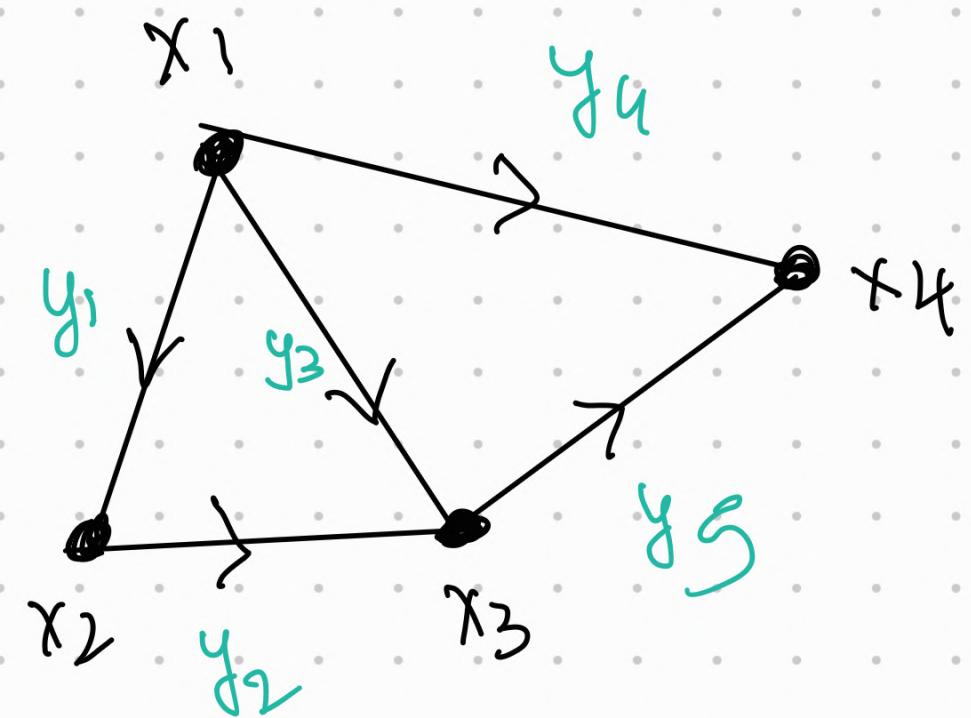
$$d(N(A^T)) = n - r$$

$$= 5 - 3$$

$$= 2$$

$A^T y = 0$ [Kirchoff's current
Law]

$$\begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$



the first equation from $A^T y = 0$

$$-y_1 - y_3 - y_4 = 0 \Leftarrow \text{meaning all current is leaving from } x_1$$

[net flow is zero]

$$y_1 - y_2 = 0 \Leftarrow \text{meaning } y_1 = y_2$$

current leaving from x_1 to enter x_2 is same as current exiting x_2

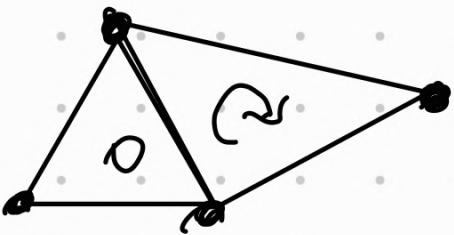
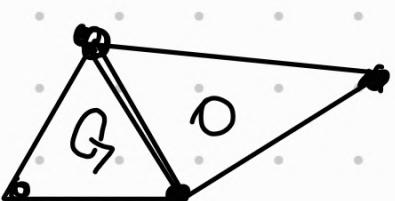
$$y_2 + y_3 - y_5 = 0 \leftarrow \text{similarly}$$

Basis for $N(A^T)$

$$\begin{bmatrix} 1 \\ -1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$



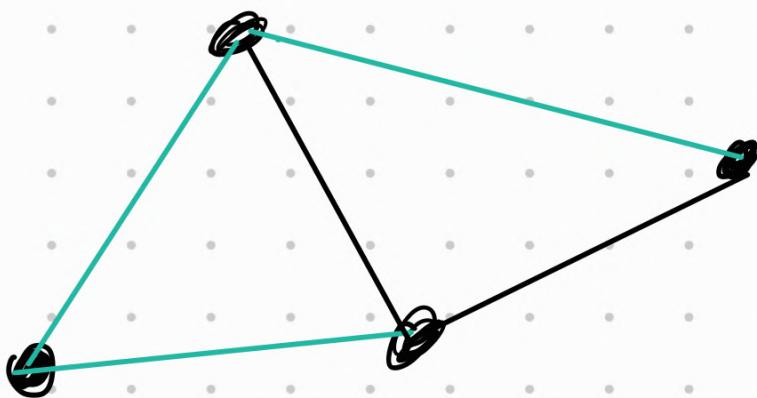
$$\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$



Pivot columns A^T are

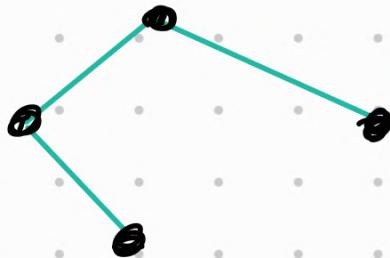
C_1, C_2 and C_4

SD



graph with only pivot column edges

never forms a loop & hence
it is a tree



$$\dim N(A^\top) = m - r$$

$$\# \text{loops} = \# \text{edges} - (\# \text{nodes} - 1)$$

$$\text{rank} = n - 1$$

$$\# \text{nodes} - \# \text{edges} + \# \text{loops} = 1$$

↳ Euler's formula

Video - 13

Review

Rough work

$$B = \begin{bmatrix} \mu \\ 2\cdot\mu \end{bmatrix}$$

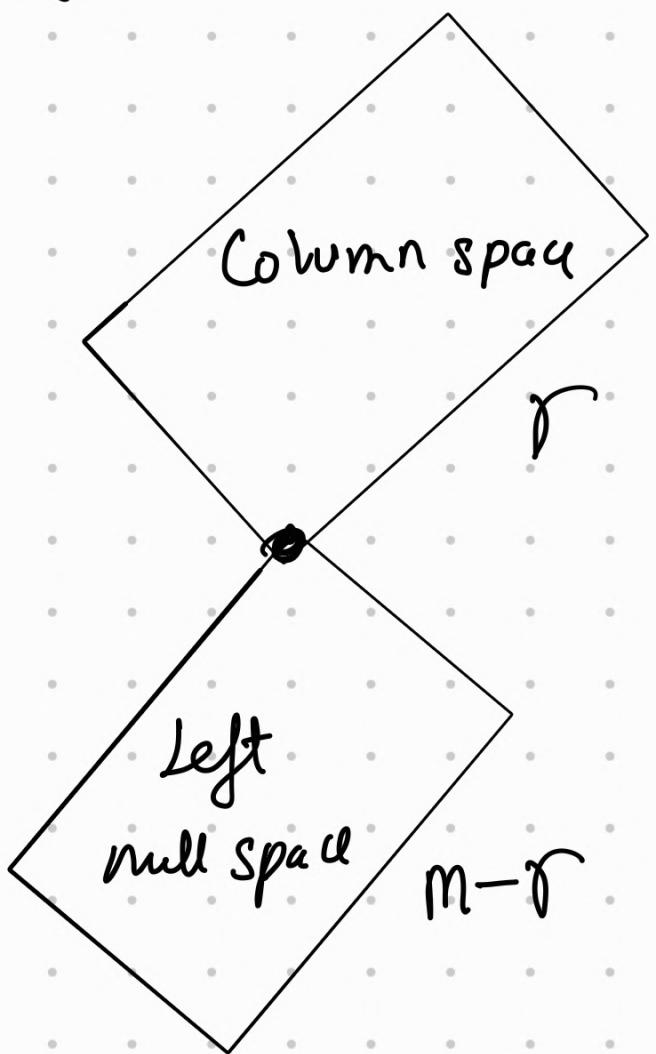
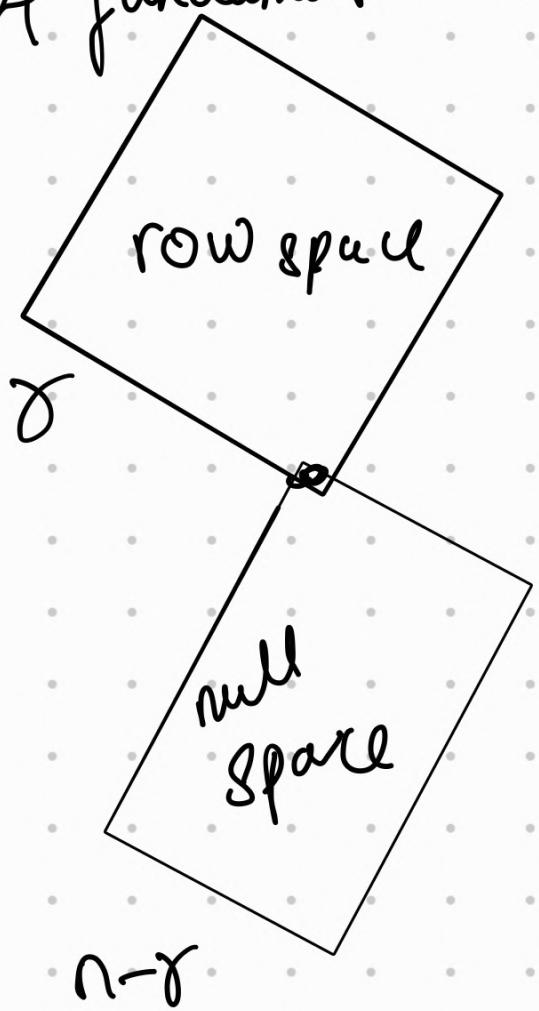
Video-14

Orthogonal vectors and Subspaces

null space \perp row space

$$N(A^T A) = N(A)$$

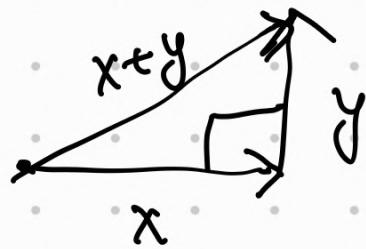
4 fundamental subspaces



Orthogonal vectors:

two vectors x and y are orthogonal

if $x^T y = 0$



$$\|x\|^2 + \|y\|^2 = \|x+y\|^2 \text{ iff } x^T y = 0$$

Example

$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, y = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, x+y = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$$

$$\begin{aligned} \|x\|^2 &= 1+4+9 & \|y\|^2 &= 4+1+0 & \|x+y\|^2 &= 9+1+9 \\ &= 14 & &= 5 & &= 19 \end{aligned}$$

$$\text{and } x^T y \Rightarrow \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{So } x^T y = 0$$

$$\text{if } x^T y = 0$$

$$\text{then } \|x\|^2 + \|y\|^2 = \|x+y\|^2$$

$$\text{also } x^T x + y^T y = (x+y)^T (x+y)$$

simplify the above

$$\cancel{x^T x} + \cancel{y^T y} = \cancel{x^T x} + x^T y + y^T x + \cancel{y^T y}$$

as $x^T y = 0$, $y^T x$ will also be zero

$$0 = 2x^T y$$

or

$$x^T y = 0 \text{ [just checking]}$$

[Don't overthink]

So,

if x is zero vector, then y could be anything, $x^T y$ is always 0

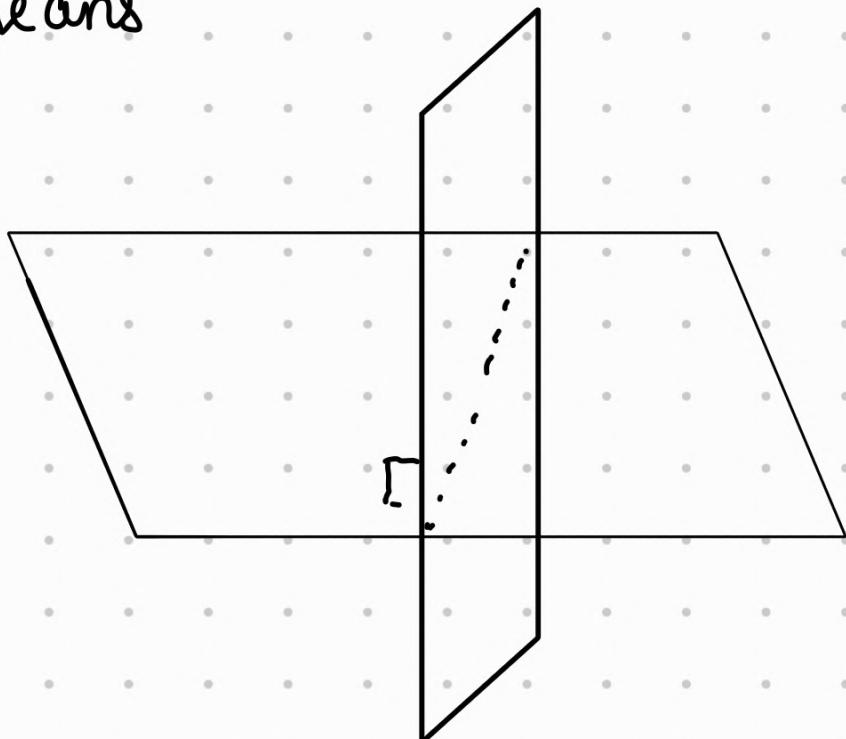
as

$$\begin{bmatrix} 0 & 0 & 0 & \dots \end{bmatrix} \begin{bmatrix} = \\ - \\ \vdots \end{bmatrix} = 0$$

Now about spaces and subspaces

Subspace S is orthogonal to subspace T

means



if they all
plane

row space is orthogonal to null space

Why?

Nullspace: X in $Ax = 0$

meaning

$$\begin{bmatrix} -r_1 \\ -r_2 \\ -r_3 \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \end{bmatrix}$$

$$r_1 \cdot x = 0$$



dot product

hence row dot product $x = 0$

so for row space, we mean all possible combination of row dot product x must be zero

$$\text{So } (r_1)^T x = 0$$

$$(r_2)^T x = 0$$

and so on:

Example

$$A = \begin{bmatrix} 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [0]$$

$$n=3, r=1$$

$$\dim N(A) = 2$$

so subspace of X is orthogonal to

row space of A

and their dimension add to the whole space

this is called orthogonal complements in \mathbb{R}^n

Nullspace contains all vectors \perp to
row space

Next video, solving $Ax = b$ where there is no solution

where $m > n$ [just to make things easy]

This is required because sometimes some of the equations are not true as they can be recorded by physical instruments or recordings.

One solution is to discard the values [be it row or column] until we get a nice solution

But this solution is not preferred as we are disregarding some information

for other sol's we need to learn about $A^T A$

$$\overbrace{A^T A}$$

→ always square ($m \times n \quad n \times m = m \times m$)
 → Symmetric

So as $Ax = b$ wont have sol^u
 we do the following:

$$A^T A \vec{x} = A^T b$$

Example

$$\text{if } A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix} \Rightarrow \text{rank} = 2$$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 8 & 30 \end{bmatrix}$$

here

$N(A) = \text{zero vector}$

hence $A^T A$ is invertable

but not always

note $\text{rank}(A^T A) = 2$

for example

$$A = \begin{bmatrix} 1 & 3 \\ 1 & 3 \\ 1 & 3 \end{bmatrix} \Rightarrow \text{rank} = 1$$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 3 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ 9 & 27 \end{bmatrix}$$

This is not invertable.

[# pivot is one of rows r_1 and r_2
are not independent]

$$\text{Rank}(A^T A) = \text{Rank}(A) = 1$$

So,

$$\text{Rank}(A^T A) = \text{Rank}(A)$$

$$N(A^T A) = N(A)$$

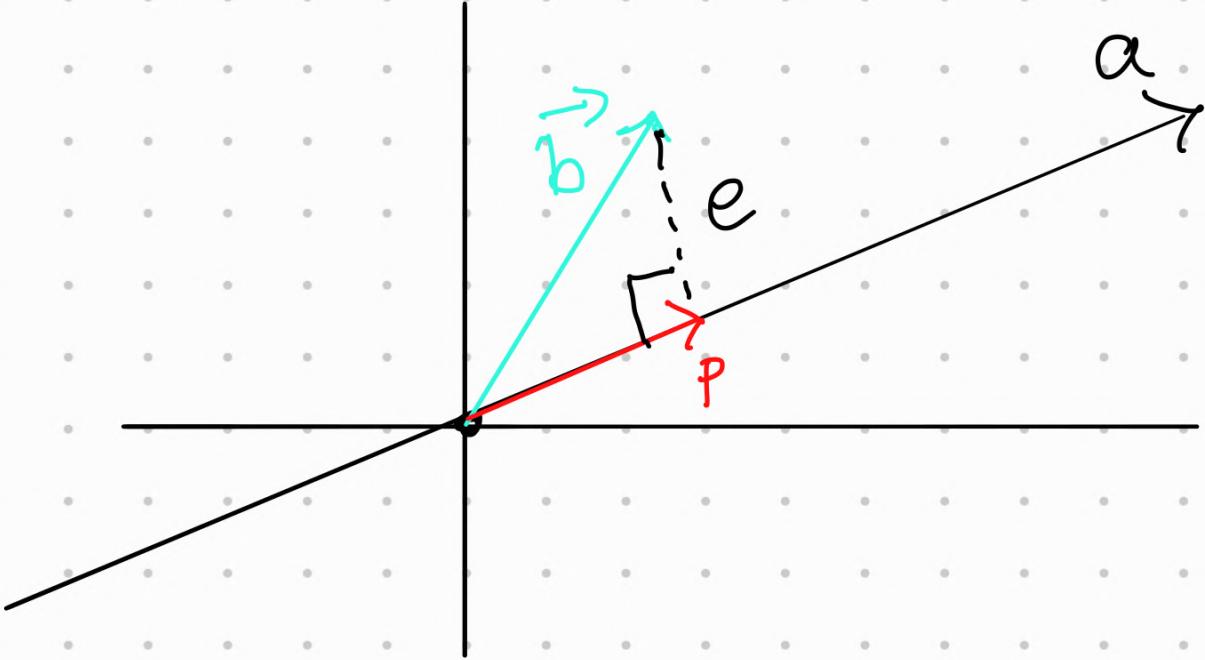
Video 15

Projections

Least Square

Projection matrix

Projection



as \vec{p} is on \vec{a} , so p is some multiple of \vec{a} , so it can be written as $\vec{p} = x \vec{a}$

$$e = b - p \quad [\text{as } e \text{ is correction from } b \text{ to } p]$$

$$e = b - x a \quad [\text{we need } x \text{ for projection}]$$

as $a \perp e$, then

$$a^T e = 0$$

$$a^T (b - x a) = 0$$

$$x a^T a = a^T b$$

$$x = \frac{a^T b}{a^T a} \quad \begin{matrix} \uparrow P = a \\ \uparrow \text{projection} \end{matrix}$$

$$\text{so } p = a \left(\frac{a^T b}{a^T a} \right)$$

note

lets say we increase b by 2 ie $b = 2b$

$$\text{then projection } p = a \left(\frac{a^T 2b}{a^T a} \right) \Rightarrow p = 2p$$

So the projection increases 2 fold
thus it is intuitively correct.

If we increase a to $2a$

intuition says projection should not change

Let's see: $a = 2a$

$$P = \cancel{2a} \frac{(2a)^T b}{(2a)^T 2a} = a \frac{a^T b}{a^T a}$$

So no effect on projection of b

Satisfying our intuition

Projection is

$$\text{proj } P = P b$$

↑ it is projection matrix

$$P = \frac{a a^T}{a^T a}$$

Let's take properties of \bar{P}

- $C(P) = \text{line through } \vec{a}$

Why?

cuz projection must be a vector along \vec{a} so

$$P = P b$$

\hookrightarrow This must be along \vec{a}

- $\text{rank}(P) = 1$ $\left[P = \frac{a a^T}{a^T a} \Rightarrow \begin{matrix} C & R \\ \text{column} & \text{row} \end{matrix} \right]$
form

- Symmetric: $P = P^T$

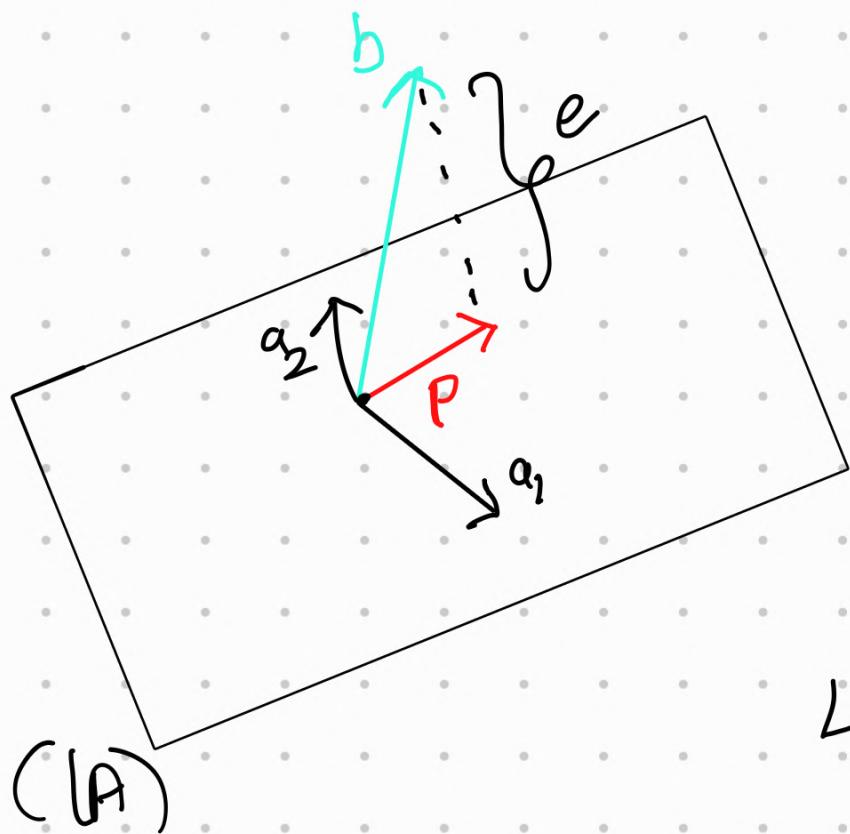
- $P^2 = P$ [projection of a line on \vec{a} in itself]

Why are we projecting?

Because $Ax = b$ has no solⁿs

So we solve

$\hat{Ax} = P$ where P is projection of b
on $C(A)$



Let a_1 and a_2
be basis of $C(A)$

So

$$P = \hat{x}_1 \alpha_1 + \hat{x}_2 \alpha_2$$

$$P = A \hat{x}$$

key $b - A\hat{x} \perp \text{plane } A$

$$\xrightarrow{\quad} A^T(b - A\hat{x}) = 0$$

$N(A^T)$ is in C

so $e \perp \text{column space of } A$

So solving that equation

$$A^T A \hat{x} = A^T b$$

$$\hat{x} = (A^T A)^{-1} A^T b$$

$$P = A \hat{x} = \underbrace{A(A^T A)^{-1}}_{\text{projection matrix}} \underbrace{A^T b}_{A^T}$$

So projection matrix $\hat{P} = A(A^T A)^{-1} A^T$

note

$$P = A(A^T A)^{-1} A^T \quad \text{step 1}$$

$$= A A^{-1} (A^T)^{-1} A^T \quad \text{step 2}$$

$$= I \text{ (identity)} \quad \text{step 3}$$

So why are we getting I

because of $(A^T A)^{-1} = A^{-1} (A^T)^{-1}$ only if
A is a square invertible matrix

So, when A is square and invertible

$C(A) \Rightarrow$ entire space [if R^3 , then $C(A) = R^3$]

then the projection will the vector itself
as it will be included in the entire space

But generally

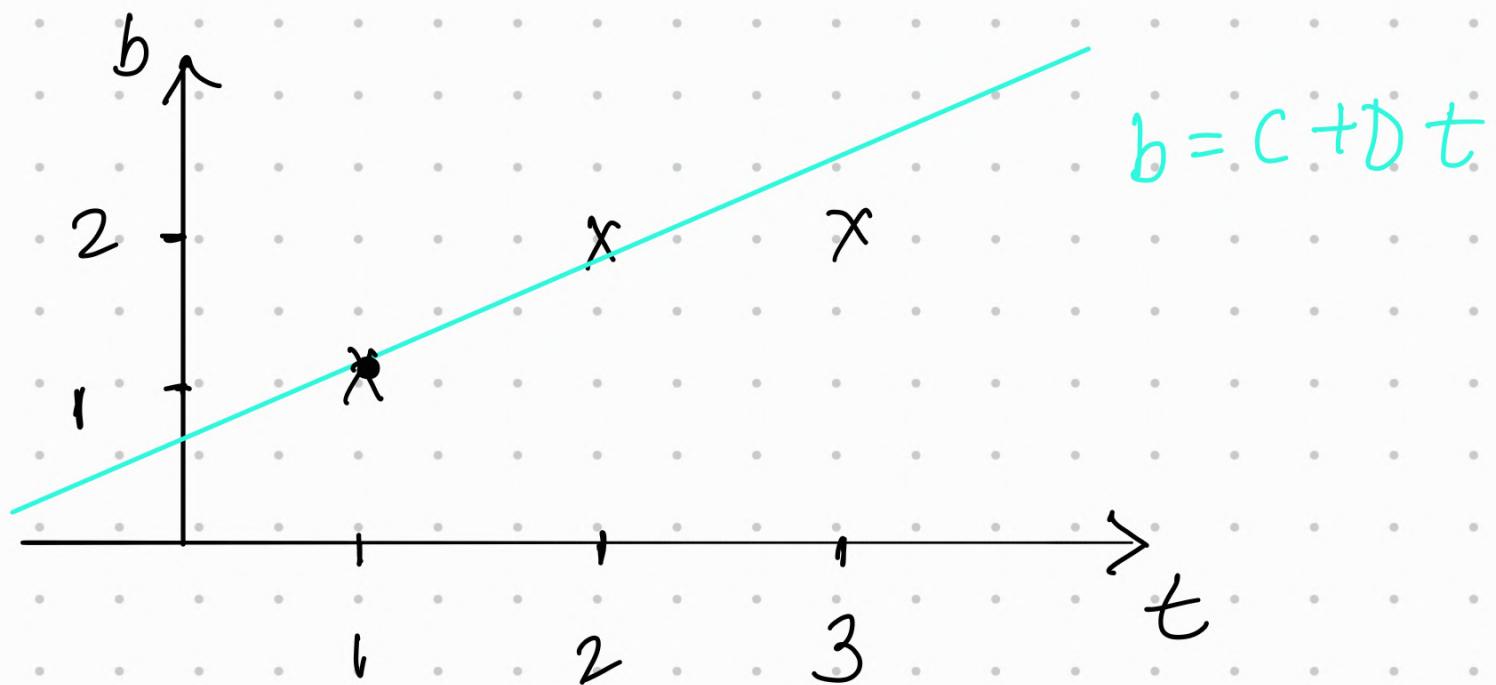
A will be rectangular [column matrix]

Properties

$$P = P^T \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{both still works for}$$

$$P^2 = P \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{higher dimension.}$$

Applications of Projections



Fitting by a line $b = c + Dt$

$(1, 1), (2, 2), (3, 2)$

So the line would surely go through $(1, 1)$

$$c + D = 1 \quad \left. \right\}$$

$$c + 2D = 2 \quad \left. \right\}$$

$$c + 3D = 2 \quad \left. \right\}$$

these are the equations it is trying to satisfy

in matrix form

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$A \cdot x = b \cdot [\text{not solvable here}]$$

So we try and solve by using

$$A^T A \hat{x} = A^T b$$

Video - 16



Video-17

Orthogonal basis q_1, q_2, \dots, q_n

Orthogonal matrix Q

Gram-Schmidt $A \rightarrow Q$

Orthonormal vectors

$$\text{if } q_i^T q_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

then q_i and q_j are called orthonormal vectors

$$\text{so let's take } Q = [q_1 \mid q_2 \mid \dots \mid q_n]$$

where every column in Q is orthonormal

to each other:

then

$$Q^T Q = ?$$

$$\begin{bmatrix} \vdots & q_1 & \vdots \\ -q_2 & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ -q_n & \vdots & \vdots \end{bmatrix} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ q_1 & q_2 & \cdots & q_n \\ 1 & 1 & \cdots & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 & \cdots \end{bmatrix}$$
$$Q^T Q = I$$

if Q is square then $Q^T Q = I$

tells us $Q^{-1} = Q^T$

Example : $Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Q^T

another example

$$Q = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

$$Q^T = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

$$\begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} \cos^2 \theta + \sin^2 \theta & 0 \\ 0 & \cos^2 \theta + \sin^2 \theta \end{bmatrix} = I$$

Q^T Q $= I$

$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \leftarrow \text{another example}$$

$$Q = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$Q = \frac{1}{3} \begin{bmatrix} 1 & -2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix}$$

note

Q generally depicts a matrix
with orthonormal columns

Project onto its column space

$$P = Q \underbrace{(Q^T Q)^{-1}}_{\text{this part is } I} Q^T$$
$$= Q Q^T$$

\hookrightarrow this is I if Q is square matrix

Why we need Q ?

$$\Rightarrow A^T A \hat{x} = A^T b \quad \text{when we had no } S^{-1}$$

but now $A = Q$

$$\underbrace{Q^T Q}_{I} \hat{x} = Q^T b$$

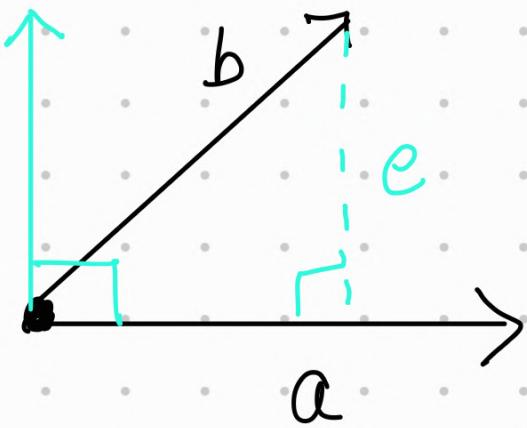
$$\text{so} \quad \hat{x} = Q^T b$$

or

$$\vec{x}_i = q_i^T b$$

Gram - Schmidt

vectors a and b [independent vectors]



Previously we had e , now that will be the part of b we consider this time as it is \perp^r to a

So if $A = a$

$$\text{then } B = b - x A \quad [x \text{ is some integer}]$$
$$= b - \frac{A^T b}{A^T A} A$$

the orthonormal column would be

$$q_1 = \frac{A}{\|A\|} \quad , \quad q_2 = \frac{B}{\|B\|}$$

for 3 vectors

a, b, c [independent]

all previous formulae for a and b
are valid

$$C = c - \frac{A^T C}{A^T A} A - \frac{B^T C}{B^T B} B$$

↑

orthogonal
vector for C

Example with numbers

$$a = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$\text{So } A = a = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$B = b - \frac{A^T b}{A^T A} A$$

$$= \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$$

$$Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}$$

So Gram-Schmidt is used for
converting $A \rightarrow Q$ $[A = QR]$
 \uparrow
Operation done
on A

R is upper triangular

$$A = \begin{bmatrix} 1 & 1 \\ a_1 & a_2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ a_{v1} & a_{v2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a_1^T a_1 & * \\ a_1^T a_2 & * \\ \downarrow h \end{bmatrix}$$

This will be zero

Video - 18

Determinants [det A or $|A|$]

Properties 1, 2, 3, 4 ... 10

\pm signs

P-1 $\det \mathbb{I} = 1$

P-2 exchanging rows reverses the sign
of determinants

So \det Permutation is 1 or -1
depending on
 $\#$ row exchanges

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cd$$

P-3

$$\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$\begin{vmatrix} a+y & b+z \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} y & z \\ c & d \end{vmatrix}$$

so determinant is linear iff
it comes a single row

P-4

if 2 rows are equal, then

det is zero

P-5

elimination steps will not
change the det value

P-6

Row of zeros $\rightarrow \det(A) = 0$

P-7

$$U = \begin{bmatrix} d_1 & * & * & * \\ 0 & d_2 & * & * \\ 0 & 0 & \ddots & * \\ 0 & \cdots & 0 & d_n \end{bmatrix}$$

$$\det(U) = d_1 \times d_2 \times d_3 \cdots d_n$$

So determinant of any upper triangular matrix is product of its pivots

P-8

$$\det A = 0$$

if A is singular

[Because we get a row of zeros]

P-9 : $\det A \cdot B = (\det A) (\det B)$

So conclusion:

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

as $A^{-1}A = I$

$$\det(A^{-1}A) = \det I$$

$$\det A^{-1} \det A = 1$$

$$\det A^{-1} = \frac{1}{\det A}$$

$$\det A^2 = (\det A)^2$$

$$\det 2A = 2^m \det A \quad [\text{where } A_{m \times n}]$$

P-10 $\det A^T = \det A$

So all properties for rows
will work for columns as well

Video - 19

Formula for $\det A$ ($n!$ terms)

Cofactor formula

Tridiagonal Matrices

P-1) $\det \underline{I} = 1$

P-2) sign reverses if row exchanges

P-3) \det is linear in each row separately

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix}$$

↑ ↙

$$\begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}$$

These are zero because a row is zero

$$= \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix}$$

$$= ad - bc$$

So for a 3×3

What are the matrices that won't become zero?

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \Rightarrow \text{let this be the matrix}$$

The surviving matrix are

$$\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{vmatrix} +$$

$$+ a_{11} a_{22} a_{33} - \begin{vmatrix} a_{11} & a_{23} & a_{31} \end{vmatrix}$$

minus is here
because it is one
row exchange
from first matrix

$$\left| \begin{array}{ccc} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{array} \right| + \left| \begin{array}{ccc} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{array} \right| +$$

$$- a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31}$$

2 exchanges

$$\left| \begin{array}{ccc} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{array} \right| + \left| \begin{array}{ccc} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{array} \right|$$

$$+ a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}$$

So the full formula was

$$a_{11} a_{22} a_{33} - a_{11} a_{23} a_{31} - a_{12} a_{21} a_{33}$$

$$+ a_{12} a_{23} a_{31} + a_{13} a_{21} a_{33} - a_{13} a_{22} a_{31}$$

So generally the formula would be

$$\det A = \sum_{\substack{\text{n! terms} \\ \bullet}} \pm a_{1\alpha} a_{2\beta} a_{3\gamma} \dots a_{n\omega}$$

Where

$(\alpha, \beta, \gamma, \dots, \omega) \Rightarrow$ Permutations of
 $(1, 2, 3, \dots, n)$

Taking 3×3 formula

$$\begin{aligned} \det A &= a_{11} (a_{22} a_{33} - a_{23} a_{32}) \\ &+ a_{12} (-a_{21} a_{33} + a_{23} a_{31}) \\ &+ a_{13} (a_{21} a_{32} + a_{22} a_{31}) \end{aligned}$$

So cofactor of a_{11} is $(a_{22} a_{33} - a_{23} a_{32})$
and similarly for a_{12} and a_{13}

and also notice cofactor of

a_{11} is det of smaller matrix

$$\begin{matrix} a_{11} & 0 & 0 \\ 0 & \boxed{a_{22} \quad a_{23}} \\ 0 & a_{32} & a_{33} \end{matrix}$$

So the cofactor of $a_{ij} = C_{ij}$

$\frac{+}{-} \det \left(\text{n-1 matrix of row } i \text{ and column } j \text{ erased} \right)$

its + if $i+j \Rightarrow \text{even}$

- if $i+j \Rightarrow \text{odd}$

Cofactor formula

$$\det A = a_{11} C_{11} + a_{12} C_{12} + \dots + a_{1n} C_{1n}$$

example

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = a \underbrace{d}_{\text{cofactor of } a} + b \underbrace{(-c)}_{\text{cofactor of } b}$$

Tri diagonal matrix

$$A_{4 \times 4} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \begin{aligned} \det A_1 &= 1 \\ \det A_2 &= 0 \\ \det A_3 &= -1 \end{aligned}$$

$$|A_4| = 1 \cdot |A_3| - 1 \cdot |A_2|$$

So

$$|A_n| = |A_{n-1}| - |A_{n-2}|$$

$$|A_4| = |A_3| - |A_2|$$

$$= -1 - 0$$

$$= -1$$

$$A_5 = |A_4| - |A_3|$$

$$= 0$$

Similarly

$$|A_6| = 1$$

$$|A_7| = 1$$

$$|A_8| = 0 \quad |A_{11}| = 0$$

$$|A_9| = -1 \quad |A_{12}| = 1$$

$$|A_{10}| = -1$$

So there are repeating after 6

So we can use modulo for higher values.

$$|A_{61}| = |A_1| = 1 \quad [61 \% 6 = 1]$$

Video - 20

Formula for A^{-1}

Cramers Rule for $x = A^{-1}b$

$|\text{Det } A| = \text{volume of the box}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

So inverse exists when det is not zero
because det is in the denominator

$$A^{-1} = \frac{1}{|A|} C^T$$

↳ cofactor matrix

So we should check

$$AC^T = |\mathbf{A}| \underline{\mathbf{I}}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} c_{11} & \cdots & c_{n1} \\ c_{12} & & \\ \vdots & & \\ c_{1n} & & c_{nn} \end{bmatrix} \quad \text{Those both will give } \det \mathbf{A}$$

$$= \begin{bmatrix} \det \mathbf{A} & & & \\ & \det \mathbf{A} & & \\ & & \det \mathbf{A} & \\ & & & \det \mathbf{A} \end{bmatrix}$$

$$= \frac{1}{\det \mathbf{A}} \mathbf{I}$$

So now getting to $Ax = b$

$$x = A^{-1} b$$

$$x = \frac{1}{|A|} C^T b$$

Cramer's Rule :

for

$$x = A^{-1} b$$

$$= \frac{1}{\det A} C^T b$$

Notice that C^T will have cofactors
and is being multiplied by matrix b

hence that will give a determinant
of a matrix

Let these matrices be denoted by B_j

$$X_1 = \frac{\det B_1}{\det A}$$

$$X_2 = \frac{\det B_2}{\det A}$$

Where

$$B_1 = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ b & a_2 & a_3 & \dots & a_n \\ 1 & 1 & 1 & \dots & 1 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ a_1 & b & a_3 & \dots & a_n \\ 1 & 1 & 1 & \dots & 1 \end{bmatrix}$$

only first column
is replaced by b
and the rest is A

Same explanation
as B_1

Why?

Because when we do multiplication

$$C^T b \Rightarrow C_{11} b_1 + C_{21} b_2 \dots$$

$$\begin{bmatrix} 1 & 1 & 1 \\ b & a_2 & a_3 \\ 1 & 1 & 1 \end{bmatrix} = b_1 C_{11} \text{ and so on for other } B$$

↑ this will be C_{11} as rest of the matrix is A

So B_j = A with column j replaced
by b

Cramer's rule

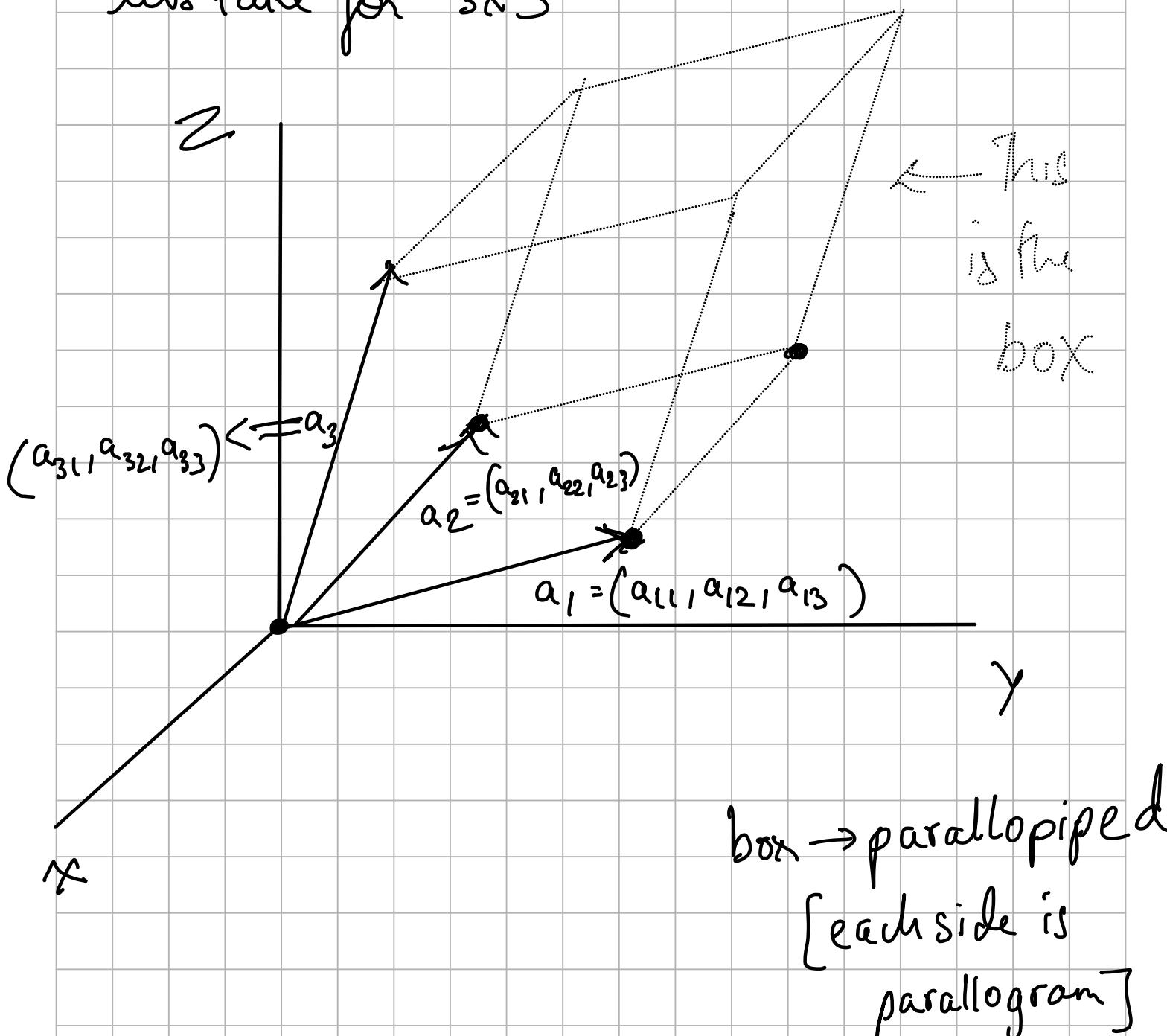
$$x_j = \frac{\det B_j}{\det A}$$

This rule is very lengthy to calculate inverse, elimination is much faster.

So Cramer's rule is not recommended to use for inverse calculation.

3) $\det A = \text{volume of a box}$

lets take for 3×3



box \rightarrow parallelepiped
[each side is parallelogram]

So Special case

if $A = I$ then the box is a unit cube aligned to the axis

if $A = Q$ then it is a cube but turned wrt axis

So if one of the vectors increases
Then the volume doubles,

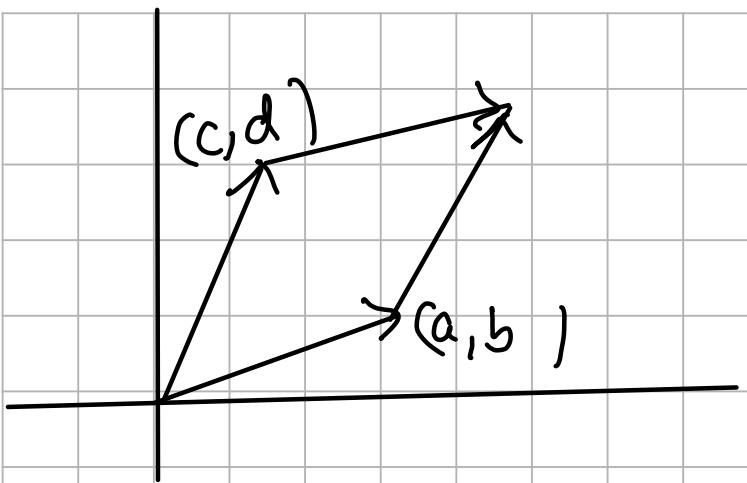
and we can multiply 2^{with}_x a single row

so it makes sense.

Proof not explained in videos properly
because there was no time

but to conclude :

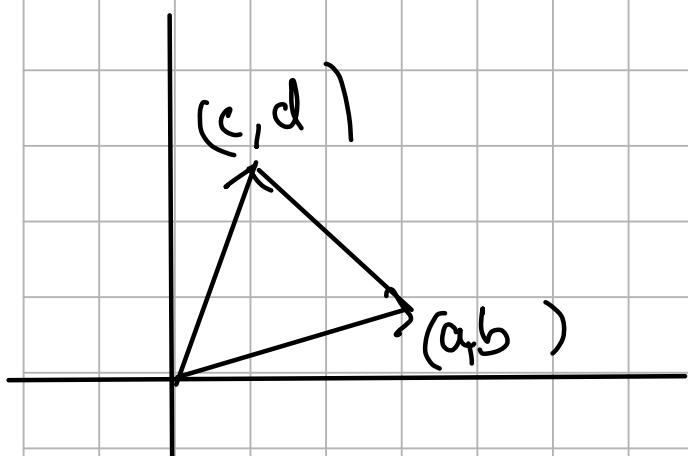
$\det A = \text{volume of box}$



area of parallelogram

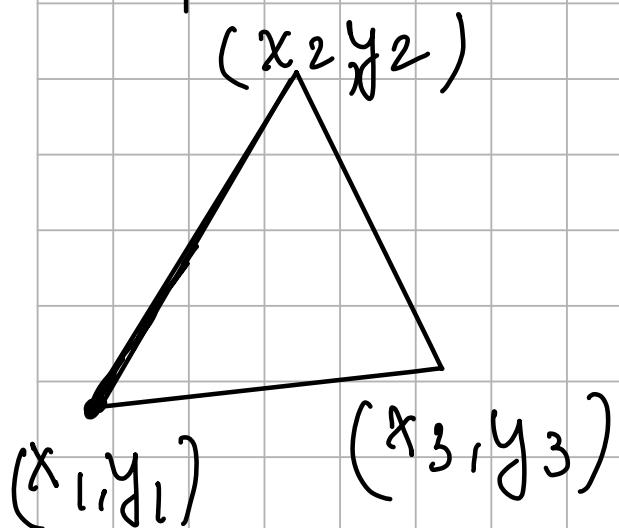
$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$= ad - bc$$



area of Δ is

$$\frac{1}{2} |A|$$



area of Δ

$$= \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Video - 21

Eigen values, Eigen vectors

$$\det [A - \lambda I] = 0$$

$$\text{Trace} = \lambda_1 + \lambda_2 + \lambda_3 + \dots + \lambda_n$$

from x we need Ax such that

Ax is parallel to x

These Ax are eigen vectors

So if they are in same direction

we can write it as

$$Ax = \lambda x$$

where λ will be the eigen value and

x will be the eigen vector

So if $Ax=0$, then $\lambda=0$, then 0 is eigenvalue

What are the eigenvalues for projection matrices

for x to be projected to plane of x

$$Px = x \text{ so } \lambda = 1$$

if we are projecting 1 to plane of x , then

$$Px = 0 \text{ so } \lambda = 0$$

lets see for permutation

$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, so for what x does
row exchange do not affect the matrix

Answer is $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\text{so } Ax = x \text{ so } \lambda = 1$$

and for $x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$Ax = -x \text{ so } \lambda = -1$$

fact

for $n \times n$ matrix, there are n eigenvalues

and sum of eigenvalues (λ) is given

by the following

$$\lambda_1 + \lambda_2 + \dots = a_{11} + a_{22} + a_{33} + \dots + a_{nn}$$

sum of diagonal entries

How to solve $Ax = \lambda x$

$$Ax = \lambda x$$

$$(A - \lambda I)x = 0$$

so as x is eigenvector, then if $x = 0$, then there is no use as $x = 0$, λ does not matter

in $Ax = \lambda x$ as both side = 0

So when $x \neq 0$, then $(A - \lambda I)$ must be singular because

taking $(A - \lambda I)x = 0$ in the form $Ax = 0$

and where x is not zero [just now said why]

then null space of $A - \lambda I$ [which is A in $Ax = 0$]

will be singular

So as $A - \lambda I$ is singular then

$$\det(A - \lambda I) = 0$$

So now we can find λ from the above equation

Let's take an example

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix}$$

$$= (3-\lambda)^2 - 1$$

$$= \lambda^2 - 6\lambda + 8$$

$$= (\lambda - 4)(\lambda - 2)$$

$$\lambda_1 = 4$$

$$\lambda_2 = 2$$

note
6 will be the trace and
8 will be determinant value

So coming to x for each λ

$$(A - 4I)x_1 = 0$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} x_1 = 0$$

do elimination and get the basis for the null space

it is $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Same for the other $\lambda_2 = 2$

$$(A - 2I)x_2 = 0$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} x_2 = 0$$

$$x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

So notice the previous 2 examples

for $A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$\lambda_1^1 = 1$$

$$x_1^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_2^1 = -1$$

$$x_2^1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and for

$$A_2^2 = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

$$\lambda_1^2 = 4$$

$$x_1^2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_2^2 = 2$$

$$x_2^2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

So noticing the eigenvalues and the A_2 s

we notice that

$$A_2 = (A_1 + 3I)$$

and $\lambda_1^2 = \lambda_1^1 + 3$, $\lambda_2^2 = \lambda_2^1 + 3$

but x (eigen vectors) remain same

$$\text{So if } Ax = \lambda x$$

$$(A + nI)x = \lambda x + nx \\ = (\lambda + n)x$$

example: Let's take orthogonal matrices

$$Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

This matrix
when multiplied
rotates the other
matrix by 90°

$$\text{trace} = 0 = \lambda_1 + \lambda_2 \quad [\text{Because diagonal elements are zero}]$$
$$\det = 1 = \lambda_1 \lambda_2$$

This does not seem possible because

for 2 numbers to sum to zero, if one
is positive, then other must be negative
but their product could never be 1

This is not possible for any real no.

Let's solve the equation to see what we get

$$\det(Q - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0$$

$$\lambda_1 = i \quad \text{and} \quad \lambda_2 = -i$$

So Orthonormal matrices gives us

eigen values that are completely
imaginary, whereas symmetric
matrices give real eigen values

lets take another example to get another possibility.

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \text{ so } A \text{ is upper triangular}$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)^2 = 0$$

$$3-\lambda = 0$$

$$\lambda = 3$$

$$\text{so } \lambda_1 = 3 \text{ and } \lambda_2 = 3$$

But when coming to eigen vectors

$$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

x_2 = no independent eigen vector

(we get this after solving for null space)

Video - 22

Diagonalizing a matrix $S^{-1}AS = \Lambda$

Powers of A / equation $U_{k+1} = AU_k$

Let's say A has n eigenvectors x_n

Let's stack all these eigenvectors as

columns for matrix S

$$\text{so } S = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & \dots x_n \\ 1 & 1 & 1 \end{bmatrix}$$

\hookrightarrow eigen vector matrix

$$AS = A \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 & \dots x_n \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 x_1 & \lambda_2 x_2 & \dots \lambda_n x_n \\ 1 & 1 & 1 \end{bmatrix}$$

$$\text{So } \begin{bmatrix} 1 & & & 1 \\ \lambda_1 x_1 & \lambda_2 x_2 & \dots \lambda_n x_n \\ 1 & 1 & & 1 \end{bmatrix} \text{ can also be}$$

written as $\begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ 1 & 1 & \dots & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 & \dots \\ 0 & \lambda_2 & 0 & \dots \\ 0 & 0 & \ddots & \dots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$

(Because λ are values)

$$AS = S \Lambda$$

(\Leftrightarrow matrix with only eigenvalues)

Let's say S is invertible (meaning it has all its columns independent or null space = zero vector or $\det(S) \neq 0$)

$$\text{for } AS = S \Lambda$$

$$S^{-1} AS = \Lambda \text{ or } A = S \Lambda S^{-1}$$

So now let's square A and find changes to eigenvalues and eigenvectors

from the basic equation

$$Ax = \lambda x$$

multiply both sides by A

$$A^2 x = A\lambda x = \lambda Ax \quad [\text{as } \lambda \text{ is real no.}]$$

(most of the time)

$$A^2 x = \lambda(\lambda x) \quad [\text{as } Ax = \lambda x]$$

$$A^2 x = \lambda^2 x$$

so eigenvalue squares but eigen
vectors remain the same

Now from the equation using Λ

$$A = S \Lambda S^{-1}$$

$$A^2 = S \Lambda S^{-1} \underbrace{S \Lambda S^{-1}}_{I}$$

$$= S \Lambda^2 S^{-1}$$

so eigen vectors (S) remains same
but eigen values Λ squares

for

$$A^k = S \Lambda^k S^{-1}$$

Theorem

for $A^k \rightarrow 0$ and $k \rightarrow \infty$

then all $|\lambda_i| < 1$

A is sure to have n independent vectors
(and b is diagonalizable)

if all the λ values are different
(no repeat in λ 's)

Equation $U_{k+1} = A U_k$

Starting with given vector U_0

$U_1 = A U_0$, $U_2 = A U_1$, ... so on

So : $U_k = A^k U_0$

To solve $U_{K+1} = A U_K$ from scratch

Let's do the following steps

$$U_0 = C_1 X_1 + C_2 X_2 + \dots + C_n X_n$$

[combination of parts of eigen vector]

$$A U_0 = C_1 \lambda_1 X_1 + C_2 \lambda_2 X_2 + \dots + C_n \lambda_n X_n$$

$$= \Lambda S C$$

Example

Fibonacci numbers: 0, 1, 1, 2, 3, 5, 8, 13, ...

100th term
in the series is? $\rightarrow F_{100} = ?$

$$F_{k+2} = F_{k+1} + F_k$$

$F_{k+1} = F_{k+1}$ (This for making the single equation into system of equation with a dummy equation)

$$u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$$

This is the trick

So taking

$$u_{k+1} = A u_k$$

$$u_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$$

A

$$u_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} u_k$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = \lambda^2 - \lambda - 1$$

$$\lambda = \frac{1 \pm \sqrt{1+4}}{2}$$

$$\lambda_1 = \frac{1}{2} (1 + \sqrt{5})$$

$$\lambda_2 = \frac{1}{2} (1 - \sqrt{5})$$

Video 23

Differential equations $\frac{du}{dt} = Au$

Exponential e^{At} of a matrix

Example

$$\frac{du_1}{dt} = -u_1 + 2u_2$$

$$u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\frac{du_2}{dt} = u_1 - 2u_2$$

$$A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \leftarrow \text{matrix of the above equations}$$

Let's find eigen value and eigen vectors

$\lambda_1 = 0$ because A is singular so one $\lambda = 0$

and other $\lambda_2 = -3$ from trace ($-2 - 1 = -3 = \lambda_1 + \lambda_2$)

So coming to eigen vectors

$$\lambda_1 = 0 \quad \text{so} \quad \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -3 \quad x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Solutions :

$$u(t) = C_1 e^{\lambda_1 t} x_1 + C_2 e^{\lambda_2 t} x_2$$

Checking: $\frac{du}{dt} = Au$ and putting in $e^{\lambda_1 t} x_1$

$$\lambda_1 e^{\lambda_1 t_1} x_1 = A e^{\lambda_1 t_1} x_1$$

so it checks out

going back to equation for our example

$$u(t) = C_1 e^{\lambda_1 t} x_1 + C_2 e^{\lambda_2 t} x_2$$

$$= C_1 \cdot 1 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} t + C_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

at $t=0$

$$C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Solu.

$$C_1 = \frac{1}{3}, C_2 = \frac{1}{3}$$

So

$$u(t) = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \frac{1}{3} e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Steady state

$$u(\infty) = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \left[\text{as } t \rightarrow \infty \text{ then } e^{-3t} \rightarrow 0 \right]$$

Checking for the following with

eigenvalues

1) Stability

if $v(t) \rightarrow 0$, then we need $e^{\lambda t} \rightarrow 0$
then real part of $\lambda < 0$

2) Steady state

when $\lambda_1 = 0$ and other eigenvalue are
negative

3) Blow up if any real part $\lambda > 0$

Comment for 2×2 stability

a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is stable

when $\operatorname{Re} \lambda_1 < 0$ and $\operatorname{Re} \lambda_2 < 0$

and $\det A > 0$

$$\frac{du}{dt} = Au \quad \text{set } u = Sv$$

\uparrow eigen vector
matrix

$$\frac{dv}{dt} = S^{-1}ASv$$

$$= \Lambda v$$

$$\text{So } v(t) = e^{\Lambda t} v(0)$$

or

$$u(t) = S e^{\Lambda t} S^{-1} u(0) = e^{At} u(0)$$

Matrix exponential

$$e^{At} = I + At + \frac{(At)^2}{2} + \frac{(At)^3}{6} \dots + \frac{(At)^n}{n!} + \dots$$

Must revise this

Part more thoroughly

Video-24

Markov Matrices Steady States

Fourier Series and Projections

Applications of eigenvalues;

Markov Matrices

$$A = \begin{bmatrix} .1 & .01 & .3 \\ .2 & .99 & .3 \\ .7 & 0 & .4 \end{bmatrix}$$

- 1) all entries must be ≥ 0
- 2) all columns add to 1

Study state in markov's matrix

is when $\lambda = 1$

So for one example

when $\lambda = 1$

$$A - 1 \cdot I = \begin{bmatrix} -0.9 & 0.01 & 0.3 \\ -0.2 & -0.01 & 0.3 \\ -0.7 & 0 & -0.6 \end{bmatrix}$$

$A - I$ is singular (as rows seem to be dependent)

because $(1, 1, 1)$ is $n(A^T)$

And x_1 will be $n(A)$

note

eigenvalue of A = eigen value of A^T

And Markov's matrices always will have eigen value $\lambda = 1$
(the rest of them is not known)

Application of Markov matrix

$$\begin{bmatrix} u_{cal} \\ u_{mass} \end{bmatrix}_{k \in I} = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \begin{bmatrix} u_{cal} \\ u_{mass} \end{bmatrix}_k$$

and $\begin{bmatrix} u_{cal} \\ u_{mass} \end{bmatrix} = \begin{bmatrix} 0 \\ 1000 \end{bmatrix}$

So for $k=1$

$$\begin{bmatrix} u_{cal} \\ u_{mass} \end{bmatrix}_1 = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \begin{bmatrix} 0 \\ 1000 \end{bmatrix}$$
$$= \begin{bmatrix} 200 \\ 800 \end{bmatrix}$$

So we can easily find the # people over time with eigen values and eigen vectors.

$$\lambda_1 = 1 \quad \text{and} \quad \chi_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -1 \quad \text{and} \quad \chi_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$u_k = c_1 1^k \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 (-1)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$u_0 = \begin{bmatrix} 0 \\ 000 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\frac{0}{000} = \frac{1000}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{2000}{3} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\text{So } c_1 = \frac{1000}{3} \text{ and } \frac{2000}{3}$$

projections with orthonormal basis

$$q_1, q_2, \dots, q_n$$

Any

$$v = x_1 q_1 + x_2 q_2 + \dots + x_n q_n$$

$$q_1^T v = x_1 + x_2 \underbrace{q_2 q_1^T}_0 + \dots + x_n \underbrace{q_n q_1^T}_0$$

as they are
orthonormal

in matrix form

$$Qx = v$$

$$x = Q^{-1} v$$

$$= Q^T v \quad \left[\text{as } Q^{-1} = Q^T \right]$$

Fourier Series

$$f(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x \\ + b_2 \sin 2x + \dots$$

for vectors to be orthonormal

$$v^T w = 0$$

$$= v_1 w_1 + v_2 w_2 + v_3 w_3 + \dots + v_n w_n$$

so for functions & lets say f and g
to be orthonormal

$$f^T g = \int f(x) g(x) dx$$

(limit depends on f and g)

for the above series

$$\int^T f g = \int_0^{2\pi} f(x) g(x) dx$$

because the above function is
periodic as $f(x) = f(x + 2\pi)$

So

$$\begin{aligned} &= \int_0^{2\pi} \sin x \cos x \, dx \\ &= \frac{1}{2} (\sin x)^2 \Big|_0^{2\pi} = 0 \end{aligned}$$

and the same would be true for other pairs of functions.

So to find a_1 ,

$$a_1 \int_0^{2\pi} (\cos x)^2 \, dx = \int_0^{2\pi} f(x) \cos x \, dx$$

$$a_1 \pi = \int_0^{2\pi} f(x) \cos x \, dx$$

$$a_1 = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos x \, dx$$

Video - 25

Symmetric Matrices

Eigen values / Eigen vectors

Start : Positive Definite Matrices

Symmetric Matrices

$$\text{So } A^T = A$$

then

eigen value are Real

eigen vectors are orthonormal

can be chosen

So usually

$$A = S \Lambda S^{-1}$$

but for symmetric A , S is orthonormal, so

$$A = Q \Lambda Q^{-1} = Q A Q^T$$

Why real eigenvalues for

Symmetric real A

$$A x = \lambda x \quad \text{--- } ①$$

Lets say they [eigenvalue and all complex]
eigenvector

Lets take conjugate

$$A \bar{x} = \bar{\lambda} \bar{x}$$

Lets take transpose on both side

$$\bar{x}^T A^T = \bar{x}^T \bar{\lambda}$$

as $A^T = A$, then

$$\bar{x}^T A = \bar{x}^T \bar{\lambda} \quad - \textcircled{2}$$

from ①

Let's multiply \bar{x}^T
on both side from R

$$\bar{x}^T \bar{x}^T A x = \bar{x}^T \bar{\lambda} x$$

from ②

Let's multiply x
on both side from left

$$\bar{x}^T A x = \bar{x}^T \bar{\lambda} x$$

So from the above result where LHS are equal

$$\bar{x}^T \bar{\lambda} x = \bar{x}^T \bar{\lambda} x$$

$$\text{so } \bar{\lambda} = \bar{\lambda} \text{ so } \bar{\lambda} \text{ is real}$$

This is true because $\bar{x}^T x$ is not zero

and it must be real to cancel
on both side.

Why?

Let's take $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$

so

$$\bar{x}^T x = [\bar{x}_1 \bar{x}_2 \bar{x}_3 \dots \bar{x}_n] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

Rough

$$(a+ib)(a-ib)$$

$$a^2 - \cancel{aib} + \cancel{aib} - b^2 + a^2 + b^2$$

$$a^2 + b^2$$

$$\text{So } \bar{x}_1 x_1 + \bar{x}_2 x_2 - \dots$$

must be real

and we canceled that

Good matrices :

good meaning

1) Real

2) perpendicular \times

Then matrix A is said to be good if

$$A = \bar{A}^T$$

but when A is real $A = A^T$ as $\bar{A} = A$

[no imaginary part]

for $A = A^T$

$$A = Q \Lambda Q^T \text{ as discussed above}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ q_1 & q_2 & \dots & q_n \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \begin{bmatrix} q_1^T \\ q_2^T \\ \vdots \\ q_n^T \end{bmatrix}$$

$$= \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T + \dots + \lambda_n q_n q_n^T$$

$q_1 q_1^T \Rightarrow$ projection of something

So we can look as

All symmetric matrices are a combination
of perpendicular projection matrices

fact

Signs of pivots are same as
signs of eigenvalues in symmetric A

Positive definite Symmetric Matrices:

all eigen values are positive

all pivots are positive

Example:

$$\begin{bmatrix} 5 & 2 \\ 2 & 3 \end{bmatrix}$$

pivots

$$5, \frac{11}{5}$$

So for λ

$$\begin{bmatrix} 5-\lambda & 2 \\ 2 & 3-\lambda \end{bmatrix}$$

$$\lambda^2 - 8\lambda + 11 = 0$$

$$\lambda = 4 \pm \sqrt{5}$$

so both λ are positive

