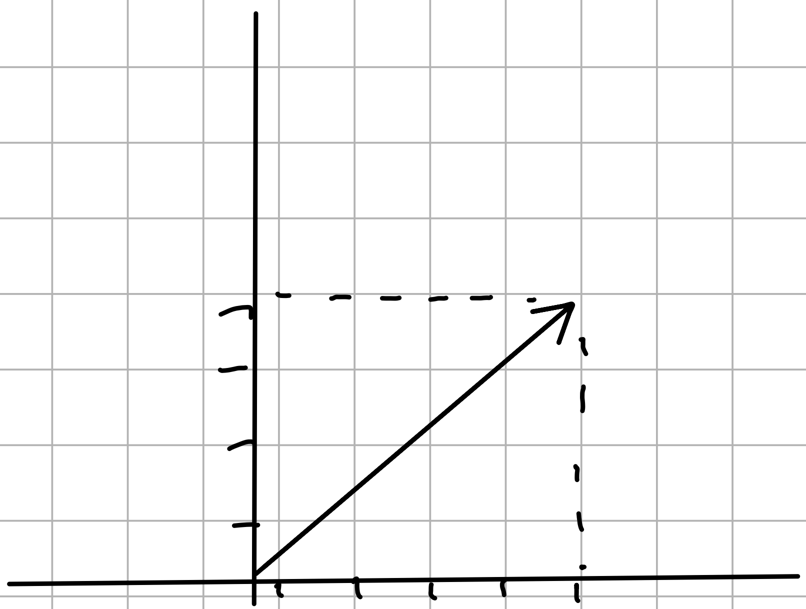
Week-1

Vectors

↑ magnitude and direction

Ex:

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$



$$u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}_{n\times 1}$$

n-component vector

for real life example

u may refer to a patient and

$u_1, u_2 \ldots u_n$ may be parameters of the patient

Field $\mathbb{F}$

$\langle \mathbb{F}, +, \times \rangle$ is a field

when

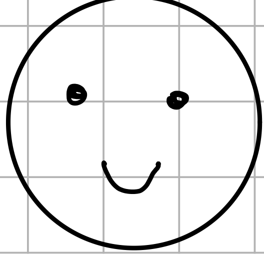
$\langle F, + \rangle$ is a group

and

$\langle F - \{0\}, \times \rangle$ is a group

additive identity

Prove $\mathbb{Z}_5$ is a Field



note

$\oplus_2$ ← xor gate

$\otimes_2$ ← and operator

Vector space V

Let F be any field. Let V be a non empty collection of objects called 'vectors'.

V is a vector space over F if rules for adding two vectors and scalar multiplication exists such that V is closed under vector addition and scalar multiplication

This means the result must come under the same set

(Example: $\oplus_5$ is closed under $\mathbb{Z}_5$)

So basically

i) for $u, v \in V$, $u+v \in V$

ii) for $\alpha \in F$, $u \in V$, $\alpha u \in V$

and few properties

- Commutativity: $u+v = v+u$
- associtivity: $(u+v)+w = u+(v+w)$
- Distibutivity: $\alpha(u+v) = \alpha u + \alpha v$
- additive identity exists: $0 \in V$ s.t. $0+u = u+0 = u \in V$
- additive inverse exists: for every $u \in V$ there exists $-u$ s.t. $u+(-u) = 0$
- multiplicative identity exists: $1u = v$ where $v \in V$

If all this hold good, V is a vector space

Examples of vector spaces

i) The field $\mathbb{F}$ itself is a vector space

ii) $\mathbb{R}^n$ for any finite n is a vector space over $\mathbb{R}$

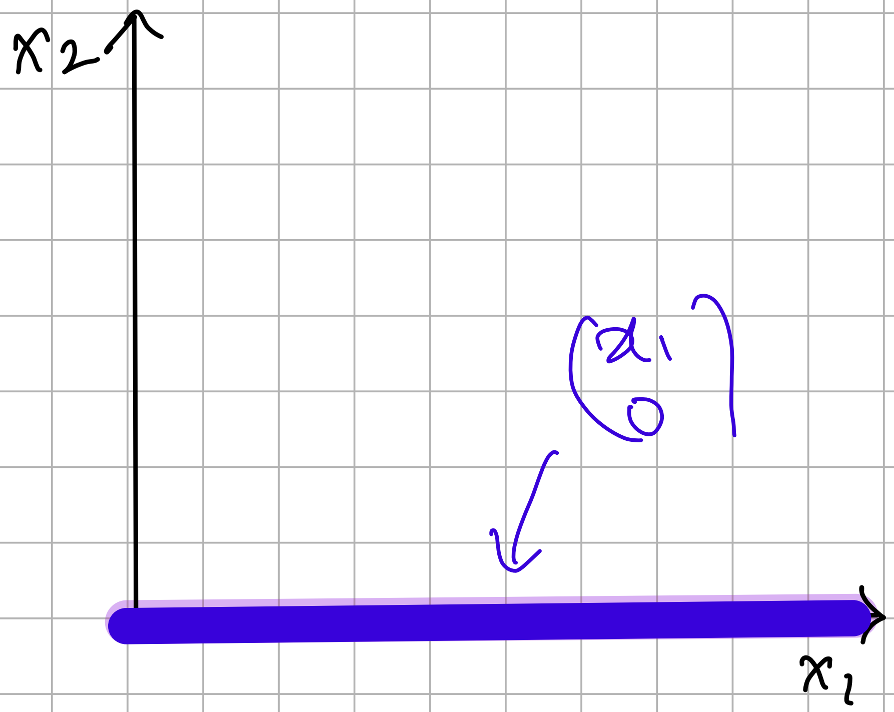
iii) Set of all polynomials of degree $\leq n$ and with coefficients real is a vector space over $\mathbb{R}$

iv) Set of all square matrices $(n \times n)$ over $\mathbb{R}$

v) Set of all real symmetric matrices over $\mathbb{R}$

vi) Set of all continuous functions of time t for t in $(-\infty, \infty)$ defined over $\mathbb{R}$

Consider $\mathbb{R}^2$



Subset of $\mathbb{R}^2$, $S_k = \left\{ \begin{pmatrix} x_1 \\ kx_1 \end{pmatrix} \; \begin{matrix} x_1 \in \mathbb{R} \\ k \in \mathbb{R} \end{matrix} \right\}$

$k=0$: $S_0 = \left\{ \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \; x_1 \in \mathbb{R} \right\}$

Let $u, v \in S_0$

$u = \begin{pmatrix} u_1 \\ 0 \end{pmatrix}$, $v = \begin{pmatrix} v_1 \\ 0 \end{pmatrix}$

$$u+v = \begin{pmatrix} u_1+v_1 \\ 0 \end{pmatrix} \in S_0$$

So S_0 is closed under vector add^n

$\alpha \in \mathbb{R}$

$$\alpha u = \alpha \begin{pmatrix} u_1 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha u_1 \\ 0 \end{pmatrix} \in S_0$$

So S_0 is closed under scalar Mulp^n

and

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in S_0$$

Hence

S_0 is a vector space over $\mathbb{R}$

Similary $S_1 = \begin{pmatrix} x_1 \\ x_1 \end{pmatrix}$ is VS (vector space)

$S_2 = \begin{pmatrix} x_1 \\ -x_1 \end{pmatrix}$ is VS

$\vdots$

So any line passing through the origin is a vector space over $\mathbb{R}$

Any Subset of a vector space V, which by itself is a vector space with operations as defined in V is called a vector Subspace of V

Remember:

i) Every set is a subset of itself

$\therefore$ any $\mathbb{R}^n$ is a trivial subspace of $\mathbb{R}^n$

ii) Consider the set $S_0 = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$

This is also a subspace of $\mathbb{R}^2$

Hence, Set containing only the zero vector is a vector subspace

iii) In any $\mathbb{R}^n$, a plane passing through the origin is a vector subspace.

Some Questions:

1) What happens if we add multiple vectors in a vectorspace

Meaning what is it say about combining vectors?

2) Suppose we want to transform an entire vectorspace, what is the effect

of the transformation on the vector space?

Combining vectors and Linear Independence

Linear combination of vectors

Suppose $u_1, u_2 \ldots u_K$ are k vectors of n components each

$$u_i = \begin{pmatrix} u_{i1} \\ u_{i2} \\ u_{i3} \\ \vdots \\ u_{in} \end{pmatrix} \quad \text{for } i = \{1, 2, \ldots k\}$$

Let $\alpha_1, \alpha_2, \ldots \alpha_K$ be scalars

u_i for $i = 1, \ldots k$. are real vectors and let $\alpha_1\ \alpha_2 \ldots \alpha_K$ be real as well

The vectors

$$v = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 \ldots \alpha_K u_K$$

is called linear combinations of the

k-vectors $u_1, u_2 \ldots u_K$

Example:

Suppose $u_1, u_2, u_3 \ldots u_K$ are standard unit vectors

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \qquad u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \ldots \ u_K = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

Kth component

Suppose $u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + v_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= v_1 u_1 + v_1 u_2$$

So vector v is linear combination of u

Suppose we have $u_1, u_2 \dots u_n$ all n component vectors

Let $\alpha_1, \alpha_2 \dots \alpha_n$ be scalars such that

$$\alpha_1 + \alpha_2 + \alpha_3 \dots \alpha_n = \frac{1}{n}$$

$$v = \alpha_1 v_1 + \alpha_2 u_2 + \alpha_3 v_3 + \dots \alpha_n v_n$$

$$= \frac{1}{n}(u_1 + v_2 + v_3 + \dots v_n)$$

$\Rightarrow$ Average of the vectors

If the coefficient or the scalars add up to 1, we call them as <u>affine</u> combination

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Suppose the coefficients in an affine combination are all non negative, we call this combination as

i) Convex Combination

ii) Weighted Average

Suppose we want to study the effect of a 'transformation' on a vector space

What strategy do we adopt to do this?

Consider $u_1, u_2, u_3 \dots u_n$ to be n component vectors

Look at that linear combination of the u vectors that result in n-comp zero vectors.

Vector A is independent

if x is not zero vector

in $Ax = 0$ [null space is non zero]

Keep in mind that if A contains

a column full of zeroes, then x is

always not zero vector

$$\begin{bmatrix} | & | & | & \cdots & \begin{smallmatrix} 1 \\ 0 \\ 1 \end{smallmatrix} \end{bmatrix} \quad x_1\, x_2 \cdots x_k]$$

$$x_1 \begin{bmatrix} | \end{bmatrix} + x_2 \begin{bmatrix} | \end{bmatrix} + \cdots \; x_k \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

so $x_1 = x_2 = \cdots = x_{k-1} = 0$

and x_k can be anything

Linearly independent

if null space of A is zero.
then A is Linearly independent

Some conclusions:

1) A linearly independent set cannot contain zero vector

2) A single vector is always linearly independent unless it is the zero vector

3) Any subset of a linearly independent set of vectors is always linearly independent

4) Any Superset of dependent vectors are linearly dependent

5) Two vectors are linearly indep if one is not a multiple of another

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Span of set of vectors

Let $u_1, u_2 \dots u_k$ be k vectors

Span := set of all possible Linear combinations of $u_1, u_2, u_3 \dots u_k$

Span is a vector space

Span of a set of linearly indep vectors

Let $v_1, v_2 \dots v_n$ be linearly indep set of vectors

Span: $\{ \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n , \alpha_i \in \mathbb{R} \}$

for $i = 1, 2, \dots n$

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$\Downarrow$

Smallest subspace that contains the set of linearly indep vectors

for example: $v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$\{ \alpha_1 v_1 + \alpha_2 v_2 , \ \alpha_1 \text{ and } \alpha_2 \in \mathbb{R} \}$$

$$\left\{ \alpha_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} , \ \alpha_1, \alpha_2 \in \mathbb{R} \right\}$$

i) $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in$ Spam of $\begin{pmatrix} 2 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\alpha_1 = \alpha_2 = 0$

ii) Spam $\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix} \& \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right)$ is closed under vector addition

iii) Spam $\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix} \& \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right)$ is closed under Scalar multiplication

Spam $\left(\begin{bmatrix}2\\1\end{bmatrix} \& \begin{pmatrix}1\\2\end{pmatrix}\right)$ is vector space

Basis: A set of n linearly indep n component vectors called Basis for the vector space that contains these n-linearly indep n-comp vectors

note

any vector in a vector space can be uniquely represented as basis of that vector space.