Video - 1

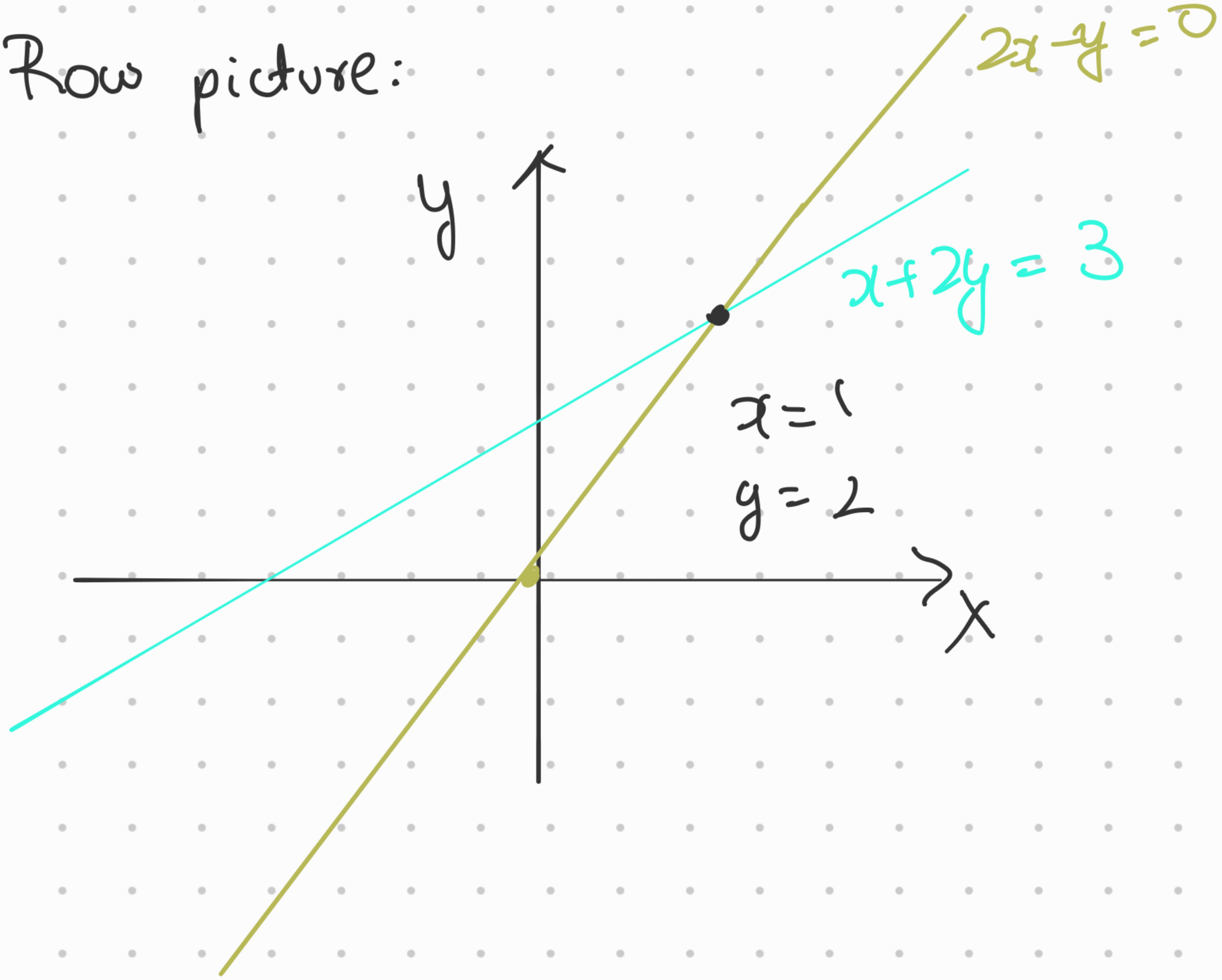
System of linear equations

$$2x - y = 0$$
$$x + 2y = 3$$

$$\Rightarrow \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$A \cdot X = b$$

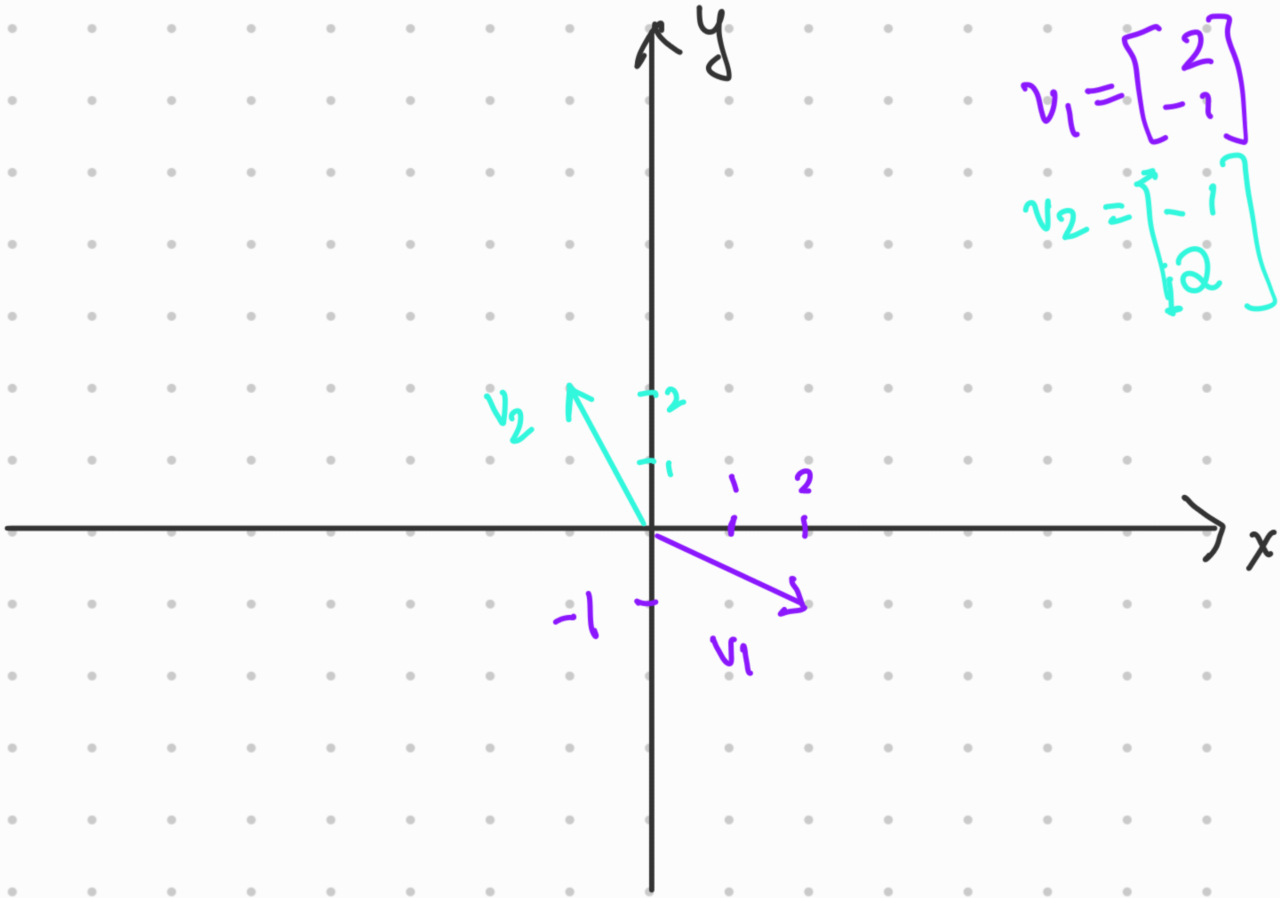
Row picture:



Column Picture

$$x\begin{bmatrix}2\\-1\end{bmatrix}+y\begin{bmatrix}-1\\2\end{bmatrix}=\begin{bmatrix}0\\3\end{bmatrix}$$

Linear combinations of columns

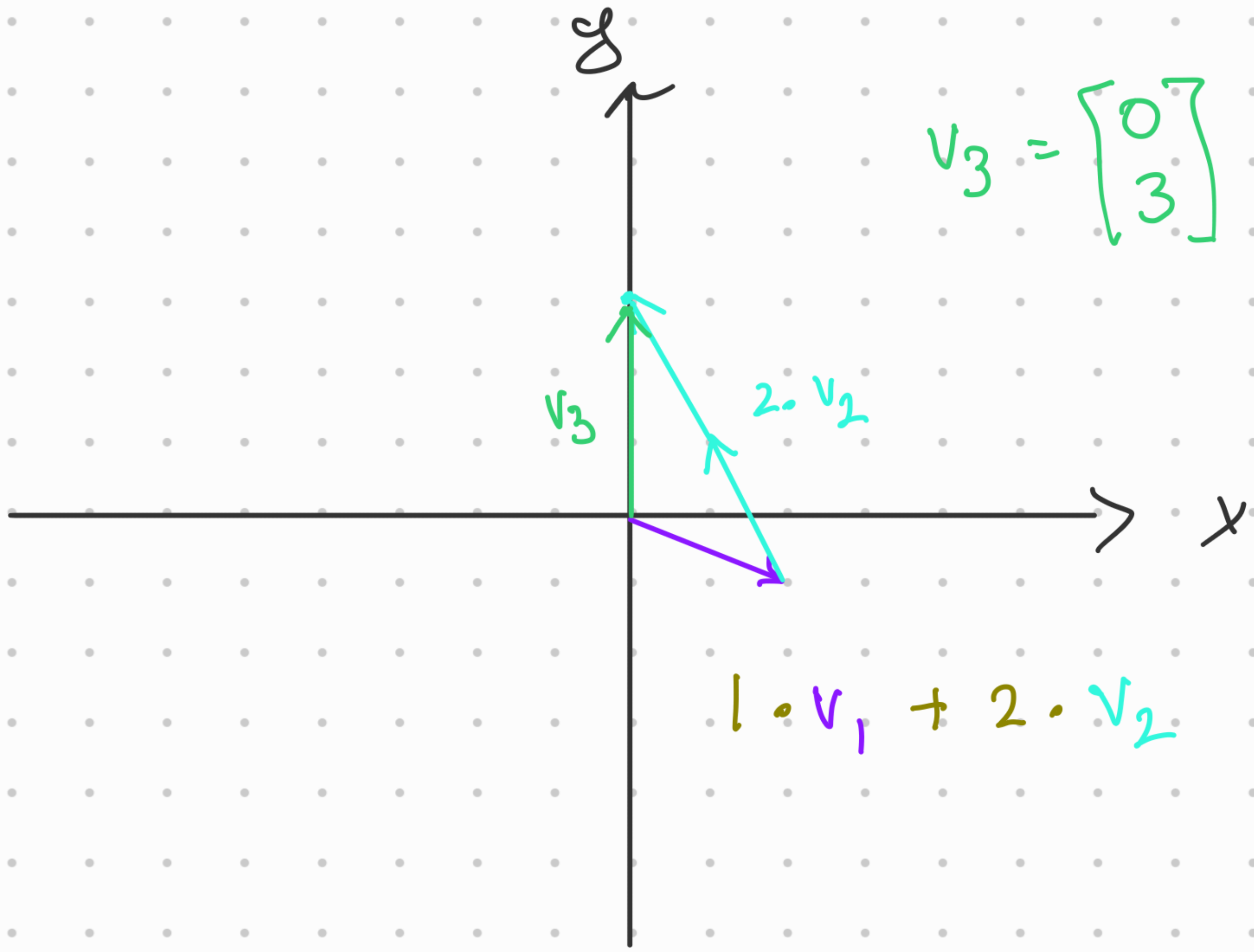


$v_1 = \begin{bmatrix}2\\-1\end{bmatrix}$

$v_2 = \begin{bmatrix}-1\\2\end{bmatrix}$

So we need x and y such that these vector addition will give $\begin{bmatrix}0\\3\end{bmatrix}$ vector

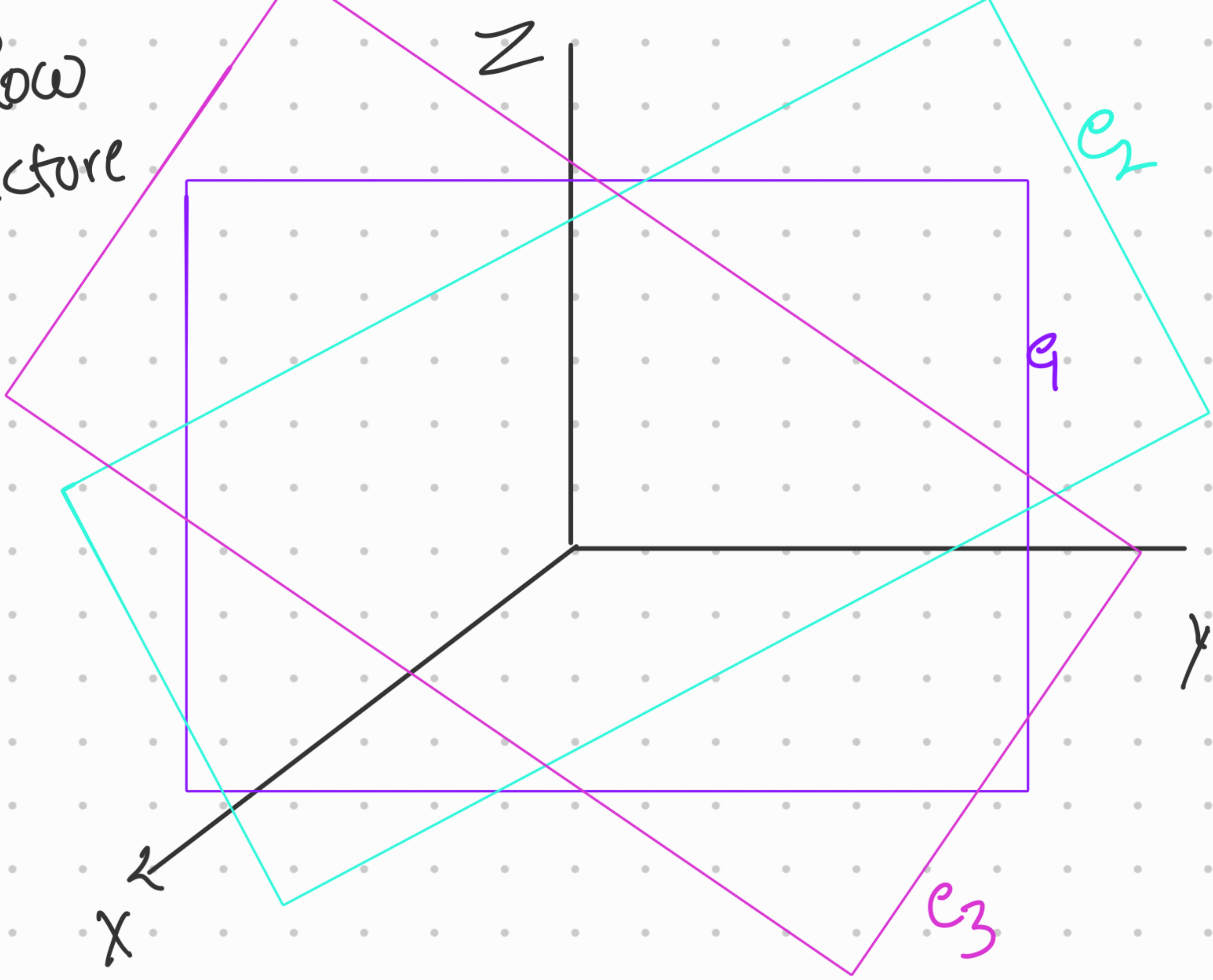
Let us visualize how $x = 1$ and $y = 2$ is the solution



Example for 3 variables

$$2x - y = 0 \text{ — } e_1$$
$$-x + 2y - z = -1 \text{ — } e_2$$
$$-3y + 4z = 4 \text{ — } e_3$$

Row picture

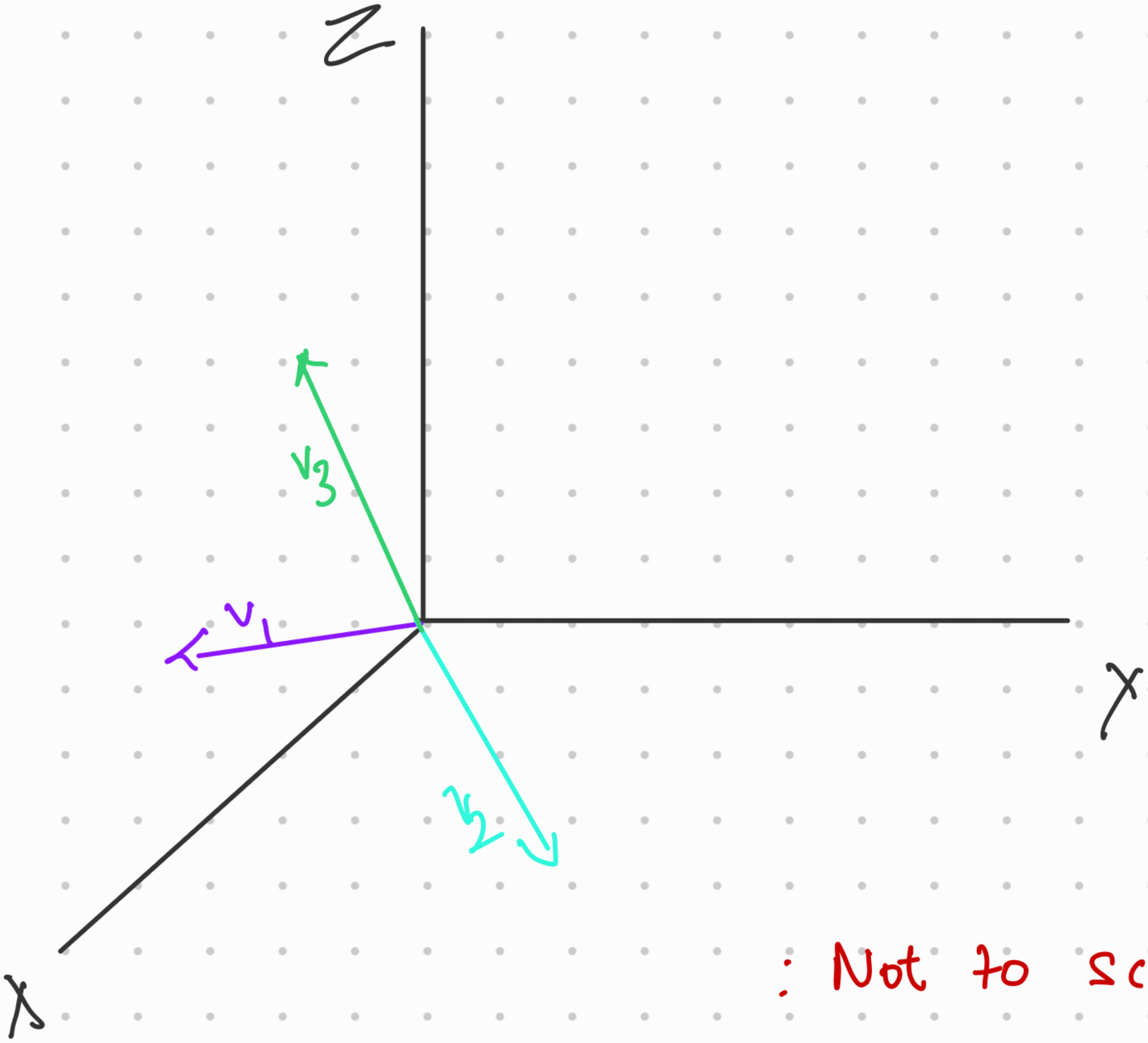


These planes will meet at a single point that will be the solution [if unique solution exists]

Column Picture :

$$x\begin{bmatrix}2\\-1\\0\end{bmatrix}+y\begin{bmatrix}-1\\2\\-3\end{bmatrix}+z\begin{bmatrix}0\\-1\\4\end{bmatrix}=\begin{bmatrix}0\\-1\\4\end{bmatrix}$$

v_1 $\quad v_2$ $\quad v_3$



: Not to scale

We have to get x, y, z such that we get the final vector

In the above $x = y = 0$ and $z = 1$ is the sol^n

Important question:

Can we get $Ax = b$ for every b

or

Do linear combinations of the coloums fill the entire 3D space?

Answer: Not for all matrices

Why?

↳ if the vectors point towards same direction, then every b is not possible [3D space]

Video-2 Elimination in Matrices

$x + 2y + z = 2$

$3x + 8y + z = 12$

$4y + z = 2$

pivot

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \quad R_2 = R_2 - 3R_1$$

Pivot

$$= \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} \quad R_3 = R_3 - 2R_2$$

$$= \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix} \Leftarrow$$ This is elimination

getting lower half as zeros

we should have augmented the RHS as well

$$\left[\begin{array}{ccc:c} 1 & 2 & 1 & 2 \\ 3 & 8 & 1 & 12 \\ 0 & 4 & 1 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc:c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 4 & 1 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc:c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 0 & 5 & -10 \end{array}\right]$$

⟸ we get this after elimination

now the equation are

$$x + 2y + z = 2$$
$$2y - 2z = 6$$
$$5z = -10$$

We back substitute and get the answer

$z = -2, \quad y = 1, \quad x = 2$

Matrix Multiplication

So in the previous example for elimination

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} \quad R_2 = R_2 - 3R_1$$

This can be written as matrix multiplication

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}$$

Lets call this E_{21}

to make (2, 1)th position zero

So next

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

E_{32}

$\Rightarrow U$

$$E_{32}(E_{21}A) = U$$

$$(E_{32}E_{21})A = U \quad \text{(Associative Law)}$$

So all elimination matrix can be combined to single matrix

To exchange Rows of matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

$\Rightarrow$ Permutation matrix

To exchange columns of matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

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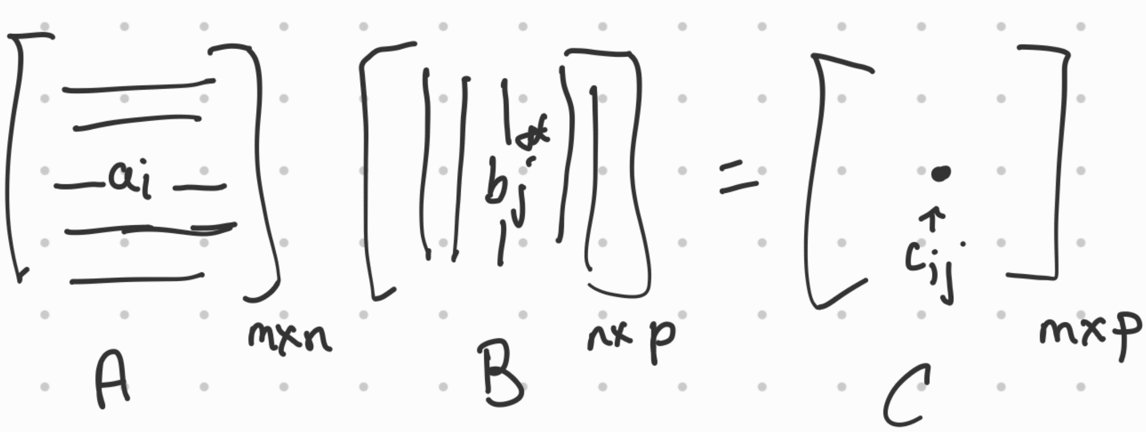
Matrix Multiplication → 4 ways

Way 1)



$$c_{ij} = \sum_{k=1}^{n} a_{ik} + b^{*}_{kj}$$ Generic Method

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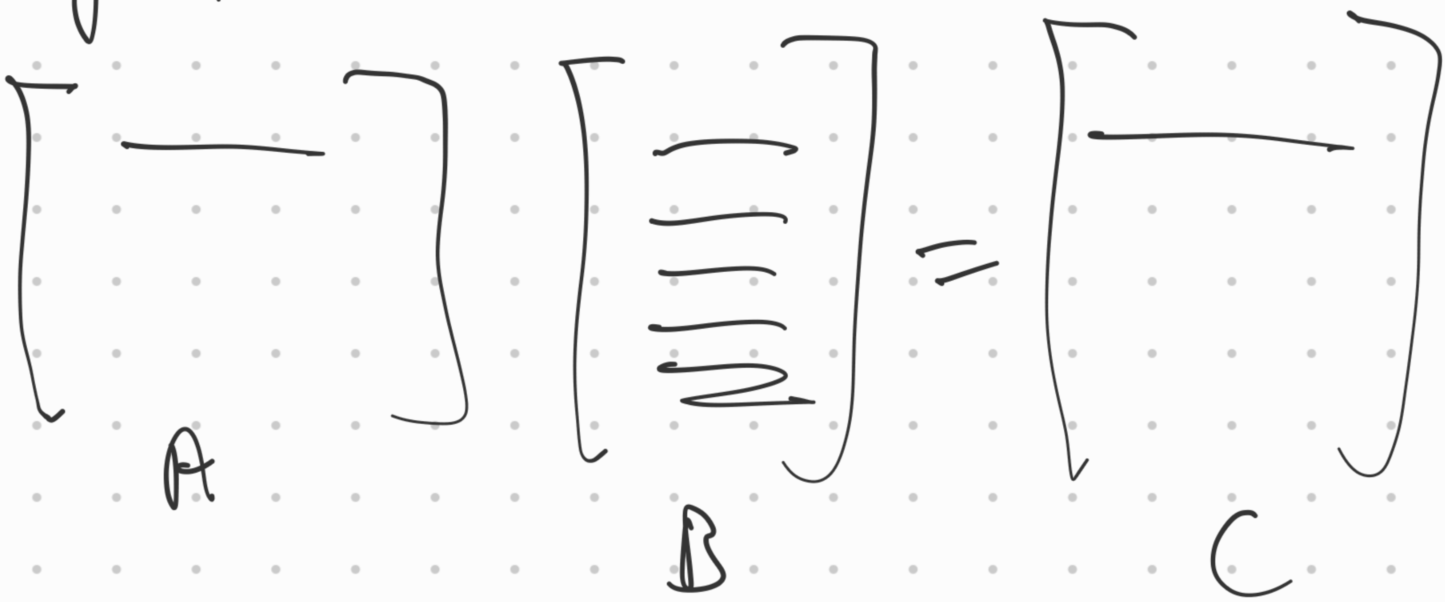
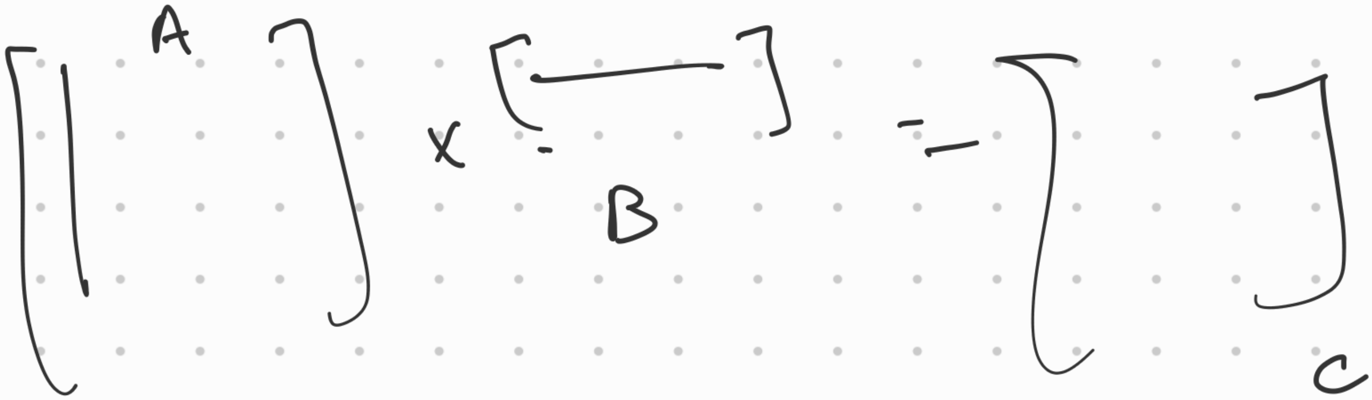
Way 2) Column wise

$$A \cdot B = \underbrace{[A b_1^* \;\; A b_2^* \;\; \cdots \;]}_{C}$$

way 3) Row wise



way 4) column(A) x Row(B) = C



Inverses

$$A^{-1} A = I$$

$\uparrow$ if exists

$AA^{-1} = I$ is also true

if inverse exists to A, then it is called invertable, non-singular

---

Example for singular

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \Longleftarrow \text{singular}$$

if $Ax = 0$, where $x \neq 0$ vector then it is singular

$$\underset{A}{\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}} \underset{x}{\begin{bmatrix} 3 \\ -1 \end{bmatrix}} \underset{=}{=} \underset{0}{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

So A is singular

Ex for non singular [where inverse exists]

$$\underset{A}{\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}} \underset{A^{-1}}{\begin{bmatrix} & \\ & \end{bmatrix}} = \underset{I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

$A \times$ column j of A^{-1} = column j of I

Gaussian Jorden [solve 2 eq^n at once]

$$\overset{A}{\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \leftarrow$$

2 equation to solve with same A

$$\underset{A}{\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \leftarrow$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right]$$

$\quad A \qquad I$

augmented matrix

$\Rightarrow$ we solve so A becomes I in augmented matrix to get A^{-1} at I place

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right]$$

$R_2 = R_2 - 2R_1$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

$R_1 = R_1 - 3R_2$

$$\left[\begin{array}{cc|cc} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

$$\underbrace{\qquad}_{I} \qquad \underbrace{\qquad}_{A^{-1}}$$

What actually is happening

$$E[A : I] = [I \;\_\_\_]$$

if E is elimination matrics combined together

$$E \cdot [A : I] = [I \;\_\_\_]$$

$E \cdot A = I$ so E must be A^{-1} to get I as result

$E \cdot I = E$, so we get A^{-1} after Elimination to get I in RHS

Video - 4

Let matrix A and B be invertable

$$(A \cdot B) \cdot \underbrace{(B^{-1} A^{-1})}_{\text{order is reverse}} = I$$

because

$$= A(B B^{-1}) A^{-1} \quad \text{(associative law)}$$

$$= (A \quad I) A^{-1}$$

$$(A A^{-1}) = I$$

---

if $A A^{-1} = I$

then $(A^{-1})^T A^T = I$

This is same as $(A^T)^{-1}$

Lets take $A = \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}$

$$\underset{E_{21}}{\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}} = \underset{U}{\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}}$$

So for

$$\underset{A}{\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}}$$

$$= \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}$$

for A (2×2) E_{21} all the elimination required

So for 2×2 $E = E_{21}$

for 3×3 A

$$E_{32} E_{31} E_{21} A = U$$

So

$$A = \underbrace{E_{21}^{-1} E_{31}^{-1} E_{32}^{-1}}_{} U$$

why this order refer to pink above

$$= L\,U$$

where L is product of inverse of E

Why are we calculating L instead of E?

Lets take example

$$\underset{E_{32}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}} \underset{E_{21}}{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \underset{E}{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 10 & -5 & 1 \end{bmatrix}}$$

Remember $EA = U$

for L

$$E_{21}^{-1} \quad E_{32}^{-1} \quad L$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}$$

where $A = LU$

So in L in position (3,1) we got rid of 10 from E, hence L is better

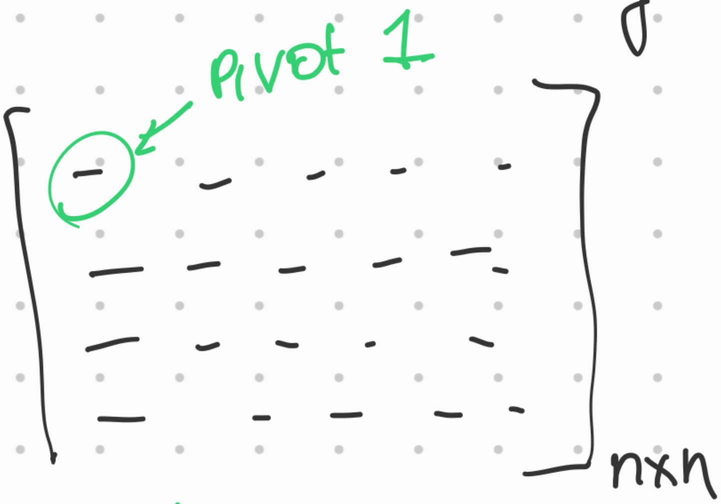
in $A = LU$

If no rows exchanges, then multipliers go directly into L

So the elimination matrix is always Lower triangular [therefore denoted as L]

How expensive is Elimination?

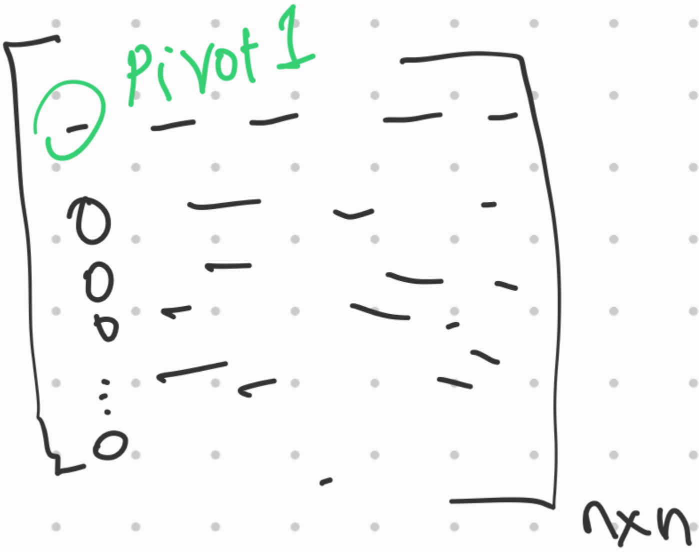
for $n \times n$ matrix, how much operation are we making?



note

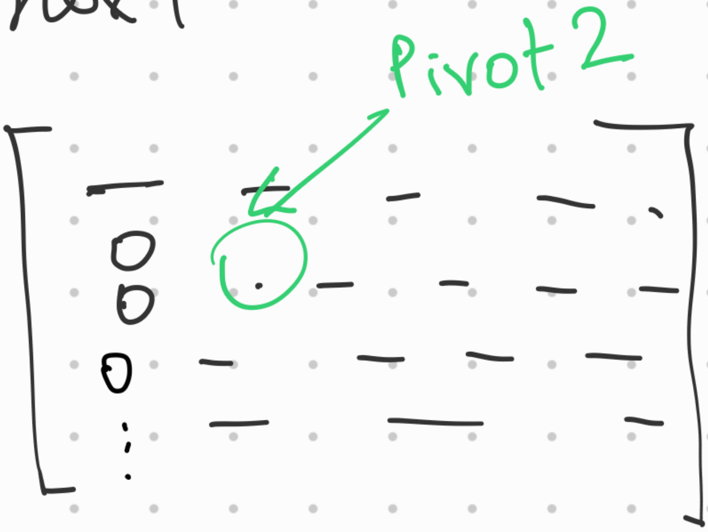
> one operation is one multiplication + one subtraction

$$R_b = \underbrace{R_b - \overbrace{x R_a}^{\text{multiplication}}}_{\text{Subtraction}}$$



here n^2 operations are made

next



here $(n-1)^2$ operations are made

and so on

So total count of operations are

$$= n^2 + (n-1)^2 + (n-2)^2 \ldots\ldots + 1^2$$

$$\simeq \frac{1}{3} n^3$$

This was for only A

but we need to augment b [RHS]

That cost is n^2

Permutations [Row exchange matrices]

note
we exchange when pivots are zero

all 3x3 permutations

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Leftarrow \text{does nothing}$$

$$P_{21} \Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Exchanges row2 with row 1

$$P_{13} \Rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$P_{23} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P_{\text{up cycle}} \Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$P_{\text{up cycle}} \cdot \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = \begin{bmatrix} R_2 \\ R_3 \\ R_1 \end{bmatrix}$$

$$P_{\text{down cycle}} \Rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P_{\text{down cycle}} \cdot \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = \begin{bmatrix} R_3 \\ R_1 \\ R_2 \end{bmatrix}$$

Inverses of P

$$\boxed{P^{-1} = P^{T}}$$

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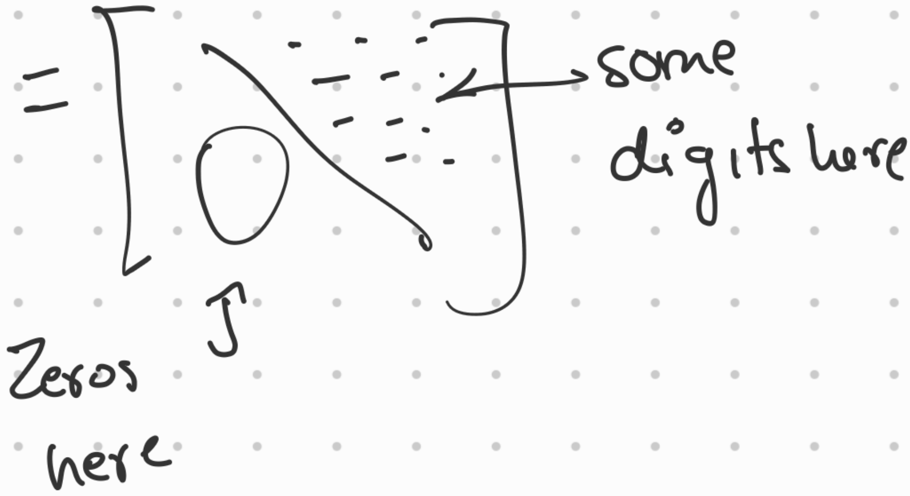
So till now we did

$A = LU$

where $L =$

and $U =$



But we never considered when pivots are Zero

So we need to include P

Hence $A = LU$ becomes

$$PA = LU \text{ for any invertable } A$$

P : permutations

$\Rightarrow$ identity matrix with reordered rows

for $n \times n$ matrix, $n!$ permutations are possible
[including identity]

and

$$\boxed{P\,P^T = I \quad \text{or} \quad P^{-1} = P^T}$$

Transposes:

$$\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix}$$

So

$$(A^T)_{ij} = A_{ji}$$

Symmetric matrix : $A^T = A$

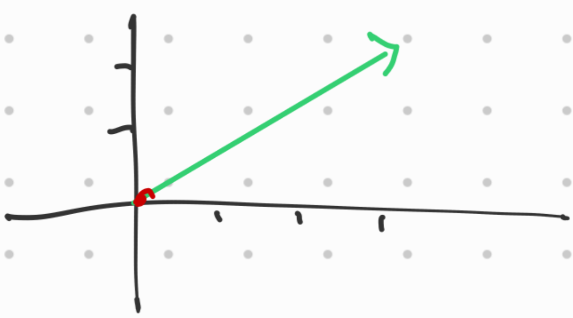
$R^T R$ is always symmetric

Why?

because $(R^T R)^T = R^T R^{TT} = R^T R$

Vector Spaces

Example: $\mathbb{R}^2 \Rightarrow \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \pi \\ e \end{bmatrix} \ldots$



and so on

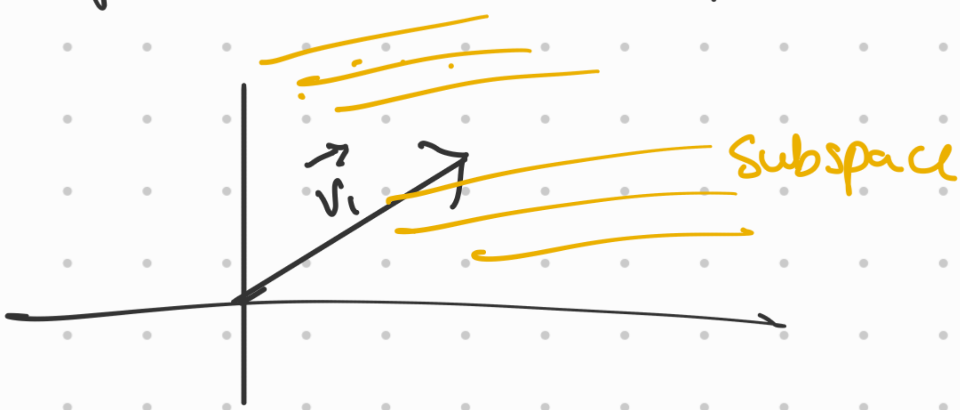
$R^2 \Rightarrow$ x-y plane

$R^3 \Rightarrow$ all vector with 3 components

$R^n \Rightarrow$ all vectors with n components

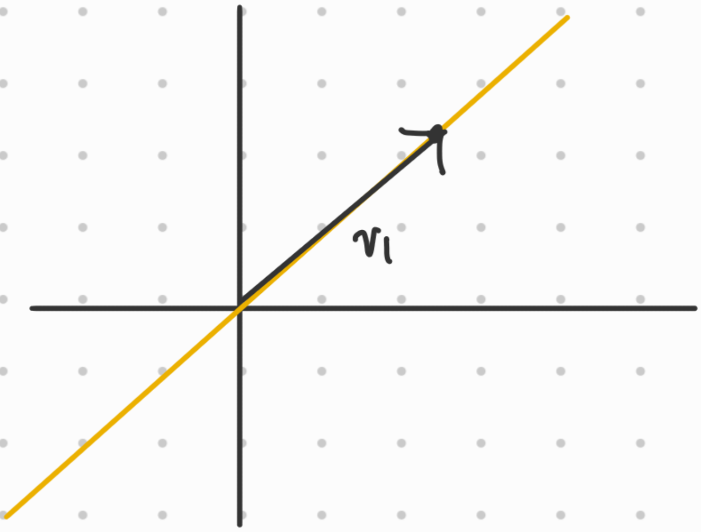
Vector spaces and subspaces

Ex: of not a subspace, Let subspace be quadrant 1



It cannot be a subspace as
$-1 \cdot v_1$ will result in vector outside subspace
so quadrant 1 is not a subspace in R^2

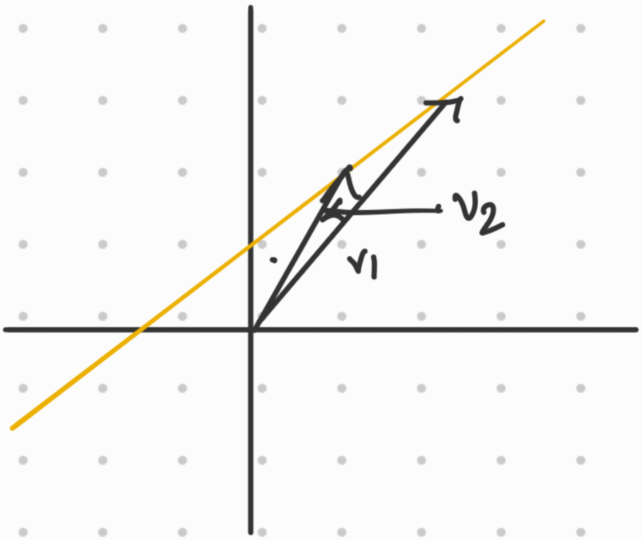
Ex of subspace



Subspace :
Line passing through origin

Let $v_1 = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$

this is a subspace as $x\, v_1$ for all x is on the subspace



Subspace :
Line not passing origin
[not a subspace]

adding v_1 and v_2 will be not in the subspace

hence Line not passing origin is not a subspace.

note

all subspaces in R^2 must pass through origin as $x v_1$ where $x = 0$, we must get zero

Subspaces of R^2

1) all of R^2
2) line passing through $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

3) only zero vector Z_0

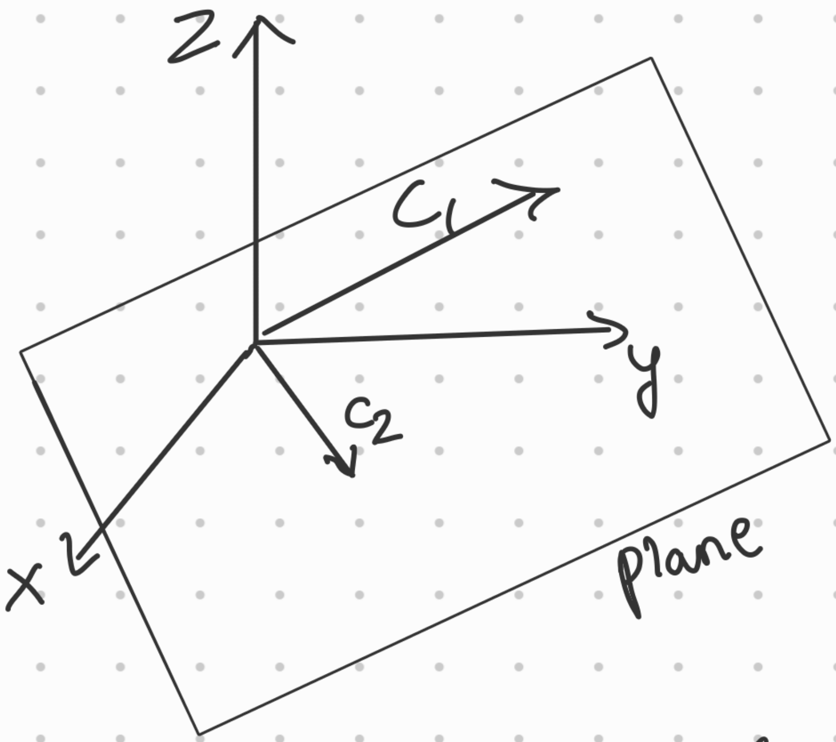
Subspace from Matrices

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}$$

columns in R^3

all their linear combinations form a subspace called column space $C(A)$

here for $A = [c_1 \ c_2]$



all combinations of c_1 and c_2 will give a plane passing through origin which is the column space of A [ie. $C(A)$]

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in R^3, subspaces are

1) entire R^3
2) plane through $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

3) line through $\begin{bmatrix}0\\0\\0\end{bmatrix}$

4) $\begin{bmatrix}0\\0\\0\end{bmatrix}$ only

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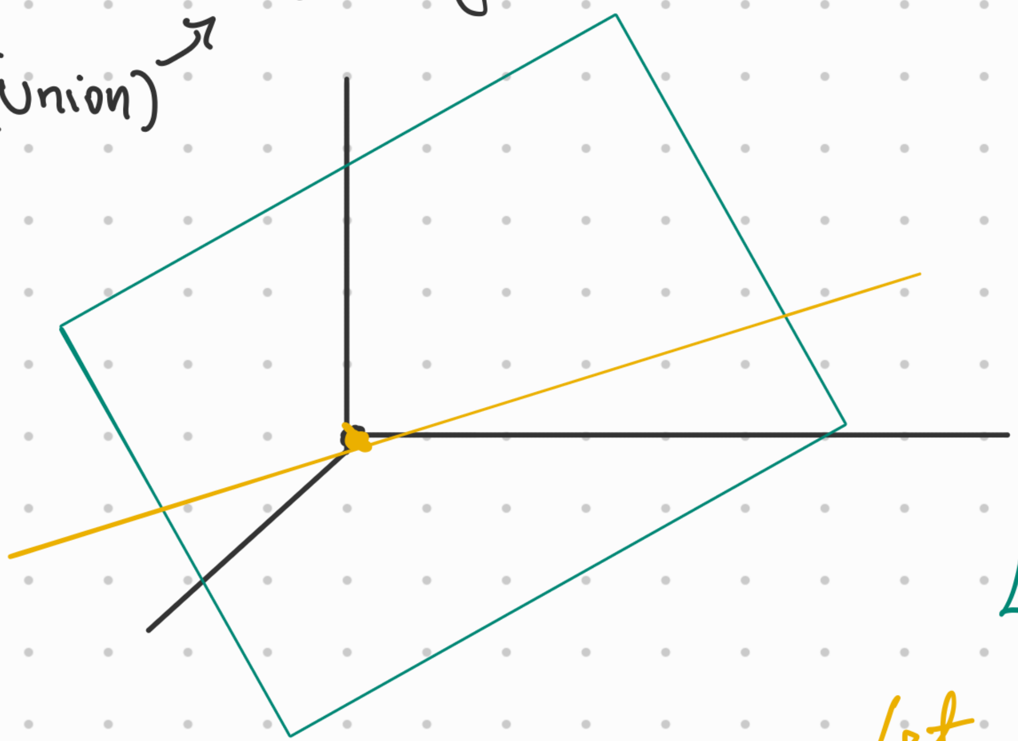
for 2 subspaces P and L :

Does

$P \cup L$ form a subspace?

(union)



Let P be plane

Let U be line

(both passing through origin)

No, $P \cup L$ does not form a subspace [visualize it, vectors go out of the plane and line when we add]

Does $P \cap L$ form a subspace?

It is a subspace

[visualize it, only $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is common for the above example of line and plane]

for 2 planes, intersection of 2 planes form a line passing through origin which is a subspace.

note

Vector space requirments

if v and w are 2 vectors in a space S and c and d are scalars

if $v+w \in S$ and $cv \in S$

then

all combinations of $cv+dw$ form the space S

Column spaces

$$C(A) \in R^4$$

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$$

$C(A)$ = all linear combinations of A

Does $Ax=b$ have solution for every b?

Same as saying, will Ax form all possible values for b, or can some b not be acessable by Ax

Answer: No, for some b it is possible

So,

which b's allows this system to be solved?

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

So when is b solvable for the above equation

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

then

$$x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

for $b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ then $x = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

and so on

note

So b allows the system of equations to be solved when b is in the column space of A

In $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$

$\quad\quad\quad\quad c_1 \quad c_2 \quad c_3$

C_1 and C_2 are adding new space to $C(A)$, but C_3 is not

This is because C_3 is dependent on C_1 and C_2

$$C_3 = C_1 + C_2$$

as

$$C(A) = \underbrace{C(c_1) \cup C(c_2)}_{\Downarrow} \cup C(c_3)$$

This already makes $C(c_3)$

hence union $C(c_3)$ is not required

note

So columns that add new spaces are called pivot columns.

pivot columns are important

Null spaces

Null space of A $=$ all solutions of $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

where $Ax = 0$

$C(A)$ is in $\mathbb{R}^4$

$N(A)$ is in $\mathbb{R}^3$ as $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{3\times 1}$

null space of A

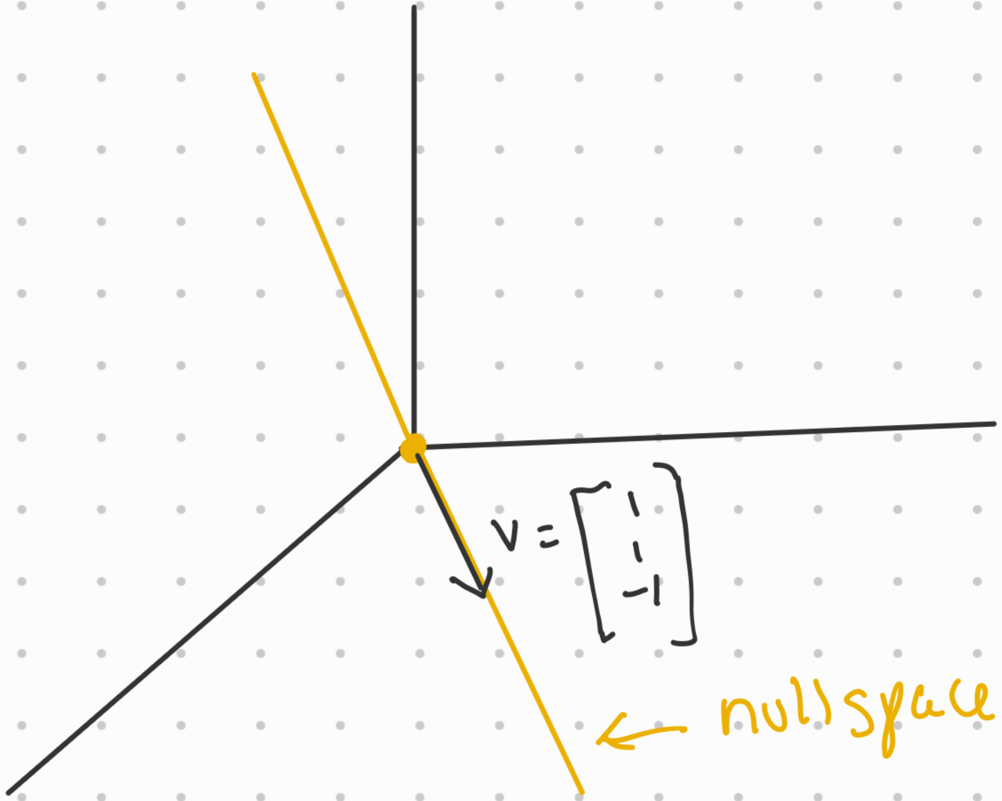
$Ax = 0$

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$X = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix} \cdots$$

So any $\begin{bmatrix} c \\ c \\ -c \end{bmatrix}$ or $c \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

So null space is a line in R^3 passing through origin



Checking null space is a space

if $Av = 0$ and $Aw = 0$, then $A(v+w) = 0$

Distributive Law works on matrices

$$A(v+w) = Av + Aw = 0$$

and

$$A(cv) = c(Av) = c \cdot 0 = 0$$

hence null space is a sub space in R^3

Does x form a subspace when

$Ax = b$ when $b \neq 0$

Answer : No

Example

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

So x does not pass origin, so the sum of vectors wont be in the same space

Hence x solution does not form Subspace when $b \neq 0$

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Computing null space $[Ax = 0]$

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

We can notice that C_2 is $2C_1$
so C_2 is not independent,
and Row-wise, $R_3 = R_1 + R_2$, so
R_3 is not independent.

We should get all these answers after doing elimination

Our goal is to solve $Ax = 0$

by solving using elimination, we are not changing the solutions of $Ax=0$ but we are changing $C(A)$ in the process

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

P_1

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix} \quad \begin{array}{l} R_2 = R_2 - 2R_1 \\ R_3 = R_3 - 3R_1 \end{array}$$

P_2

as P_2 is zero, go to the next one in rectangle matrix

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & \end{bmatrix} \quad R_3 = R_3 - R_1$$

$= U$ [all lower half is zero]

note:

So, the number of pivots is 2 which is called Rank of Matrix

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Columns with pivots (c_1 and c_3) are pivit columns

(c_2 and c_4) ← free variables

free columns meaning, we can freely decide on what value we want for x_1 and x_3, and accordingly

x_2 and x_4 will have a value

for example with A

$$A = \begin{bmatrix} \underline{1} & 2 & 2 & 2 \\ 0 & 0 & \underline{2} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

meaning

$$x_1 + 2x_2 + 2x_3 + 2x_4 = 0$$

$$2x_3 + 4x_4 = 0$$

Let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

So x_1 and x_3 pivot variables

x_2 and x_4 = free variables

Let $x_2 = 1$ and $x_4 = 0$

then we get

$$X = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$ Like this we can take other x_2 and x_4 values to get other vectors

$$X = c \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$ will form a line in null space

but this wont be the only solution

Let $x_2 = 0$ and $x_4 = 1$

then

$$X = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

So if r is the rank of the matrix of order $m \times n$

pivot variable = r

free variables = $n - r$

Reduced Row echelon form (rref)

the echelon form U has zeros on the lower part of the diagonal, but in rref we try to get zeros on the upper part of the diagonal as well

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathrm{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_1 = R_1 - R_2$$

$$R = \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2 = R_2/2$$

$\Uparrow$

rref(A)

<u>note</u>

rref(A) will give the simplest system of equations to solve A

So first we did

$$Ax = 0$$

then $Ux = 0$ and then $Rx = 0$

Notice in R

exclude this

$$R = \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

x_1 and x_3 have values I in them

So I part of x_1 and x_3 and free part which is x_2 and x_4

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 2 & -2 \\ 0 & 2 \end{bmatrix}$$

Regular rref form

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

Sometimes I and F are combined depending on the pivot column places

where I is identity columns and F is free columns

when $Rx = 0$, what are all possible x we can get

$$Rx = 0$$

$$\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{pivot} \\ x_{free} \end{bmatrix} = 0$$

$$x_{pivot} [I] + x_{Free} F = 0$$

$$x_{pivot} = -F x_{Free}$$

So $x = \begin{bmatrix} -F \\ I \end{bmatrix}$

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix}$$

Lets start elimination

pivot color

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix}$$

$R_2 = R_2 - 2R_1$
$R_3 = R_3 - 2R_1$
$R_4 = R_4 - 2R_1$

Row exchange R_2 and R_3

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 4 & 4 \end{bmatrix}$$

$R_4 = R_4 - 2R_2$

$$U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

echelon form

Rank = 2

2 pivot columns x_1 and x_2

1 free columns x_3

So for x_1 Let us set $x_3 = 1$ and solve for x_1 and x_2

$$x = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

so $x = c \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

So null space of A is $c\begin{bmatrix}-1\\-1\\1\end{bmatrix}$

---

further solving for rref(A)

$$R = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

I

F

So as we can see $R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$

and $x = \begin{bmatrix} -F \\ I \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

$-F$

single identity

Video - 8

Lets take the earlier system of equations

$$x_1 + 2x_2 + 2x_3 + 2x_4 = b_1$$

$$2x_1 + 4x_2 + 6x_3 + 8x_4 = b_2$$

$$3x_1 + 6x_2 + 8x_3 + 10x_4 = b_3$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

$$A \quad X = b$$

Lets start elimination with b augmented to A

P_1

$$\begin{bmatrix} 1 & 2 & 2 & 2 & : & b_1 \\ 2 & 4 & 6 & 8 & : & b_2 \\ 3 & 6 & 8 & 10 & : & b_3 \end{bmatrix} \Leftarrow \text{augmented matrix}$$

$$P_2 \quad \begin{bmatrix} 1 & 2 & 2 & 2 & : & b_1 \\ 0 & 0 & 2 & 4 & : & b_2 - 2b_1 \\ 0 & 0 & 2 & 4 & : & b_3 - 3b_1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 2 & 2 & : & b_1 \\ 0 & 0 & 2 & 4 & : & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & : & b_3 - b_1 - b_2 \end{bmatrix} = U$$

So if $0 = b_3 - b_2 - b_1$, then

$Ax = b$ solvable else not solvable

Solvability Condition on b

$Ax = b$ is solvable if b is in $C(A)$

or

If a combination of rows of A gives zero row, then same combinations of b must give 0.

Algorithm to find complete sol^n for $Ax = b$

1) x $_{\text{perticular}}$: set all free varibles to zero
Solve $Ax = b$ for pivot variables

$$x_1 + 2x_3 = 1 \qquad x_1 = 2 \qquad x_p = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}$$

$$2x_3 = 3 \qquad x_3 = 3/2$$

2) find x $_{\text{nullspace}}$

3) Complete sol^n $x = x_p + x_{null}$

why?

because $Ax_p = b$

$$\underline{Ax_n = 0}$$

$$A(x_p + x_n) = b$$

So we have completed x_n before for A

$$x_n = c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

So

$$X_{\text{Complete}} = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

---

for

$m \times n$ Matrix A with rank r

w.r.t $(r \leq m, r \leq n)$

when rank is full rank

that has 2 possiblities

if $r = n$ [full column rank]

pivots = n

free variable = None & 0

$N(A)$ = only zero vector
[no free variable]

$X = X_p$ [unique solution if it exists.
so either 0 or 1 sol^n]

Example:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{bmatrix} \quad r = 2$$

p_1 p_2

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

if $r = m$ [full row rank]

so every row has a pivot

so for what b can we solve it?

we can solve if for every b

visualize

$$\begin{bmatrix} a & - & - \\ 0 & b & - \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \overbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}^{\text{any } b}$$

so $c = b_3$

and we can solve it

So # pivots = m

free variables = n − m

Example

$$A = \begin{bmatrix} 1 & 2 & 6 & 5 \\ 3 & 1 & 1 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 0 & \_ & F & \_ \\ & 1 & 0 & \_ & \_ \end{bmatrix}$$

if $r = m = n$ for A

then $\mathrm{rref}(A) = I$

$N(A) =$ zero vector

only 1 solution.

To recap

if $r = m = n$

$R = I$, One solution only

if $r = n < m$

$R = \begin{bmatrix} I \\ 0 \end{bmatrix}$, 0 or 1 solution

if $r = m < n$

$R = \begin{bmatrix} I & F \end{bmatrix}$ ∞ solution

if $r < m, r < n$

$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$ 0 or ∞ Solution

Video - 9

Linear independence

Spanning a Space

Basis and dimension

Suppose A is a matrix of $m \times n$
with $m < n$
[meaning more # unknown variables than
equations]

Then conclusion is non Zero solutions in
$Ax = 0$
so this means there must be atleast
one free variables

Independence

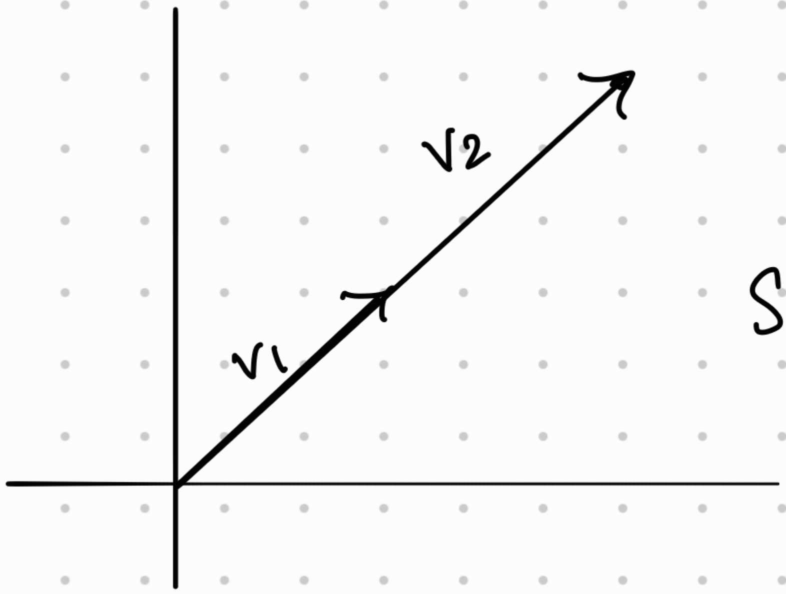
Vectors $x_1, x_2, x_3, \dots x_n$ are independent if no combinations give zero vector (except the zero combination)

Meaning

$$c_1x_1 + c_2x_2 + c_3x_3 \dots c_nx_n \neq 0$$

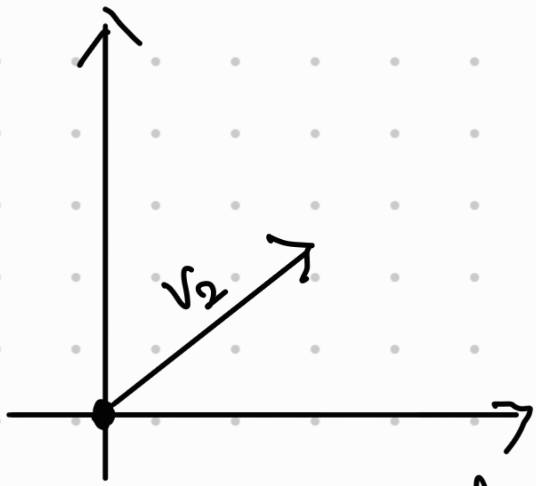
(except $c_i = 0$)

Examples:



$v_2 = 2v_1$

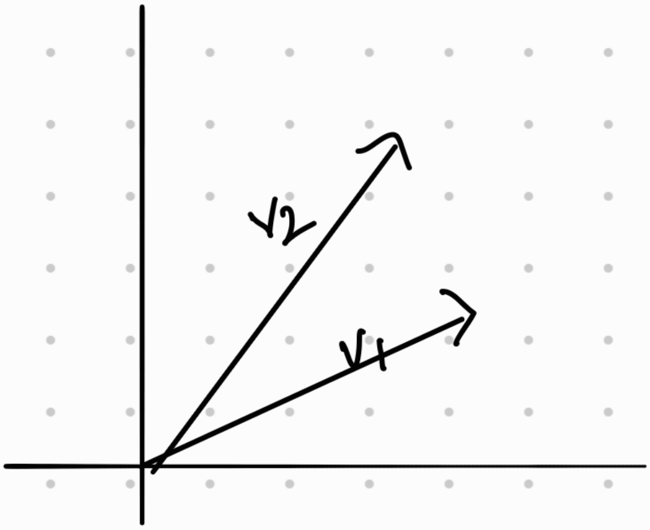
So v_1 and v_2 are dependent



v_1 and v_2 are dependent

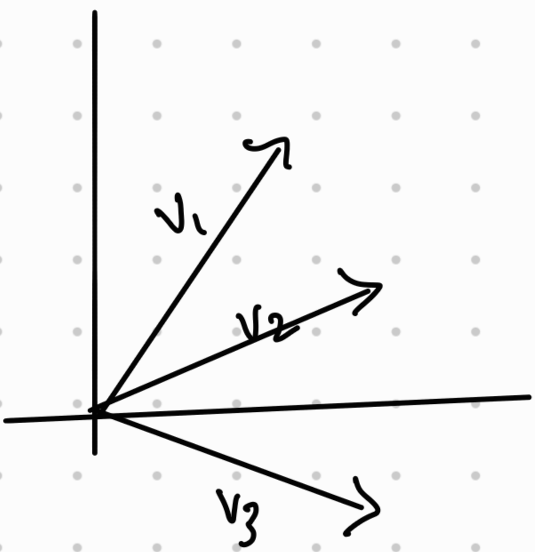
$v_1 =$ zerovector

$v_2 \Rightarrow$ non zero vector



v_1 and v_2 are independent

but



v_1, v_2 and v_3 are dependent

This is because we writing this in matrix form

$$A = \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix}$$

we have 3 variable and 2 equations
(as vectors are in 2D, they form 2 equations)

Lets assume

$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (it looks like this in the graph)

$v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$v_3 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

So

$$\underbrace{\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -1 \end{bmatrix}}_{A} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So to summarize

for vectors $v_1, v_2, v_3 \dots v_n$ in A

- vectors are independent if nullspace of A is zero vector ($rank = n$)
- vectors are dependent if null space of A is non zero vector ($rank < n$)

Vectors $V_1, V_2, V_3 \dots V_l$ spam a space means, the space consists of all combination of those spaces

---

Basis

Basis of a space is a sequence of vectors, $V_1, V_2 \dots V_b$ with

2 properties,

1) They are independent
2) They span the space

So these are the minimum $\underbrace{\#}_{\text{number of}}$ vectors required to spam that space

Example

Space is $\mathbb{R}^3$

One basis is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

are they independent? yes

cuz $c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0$

when $c_1 = c_2 = c_3 = 0$ only

Lets take 2 vectors that are independent

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$$

These two are independent, hence these can span a space [which would be a plane]

So if we choose vector that is not is the plane, we can span the entire R^3 using the 3 vectors

So Lets say

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 8 \end{bmatrix}$$

So these 3 vectors make another basis for R^3

note

Every basis for the space has same number of vectors

This number is called dimension of the space

So if R^3 then no. of vectors in basis also called dimension is 3

if R^n then no. of vectors or dimension is n

Example space is $C(A)$

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}$$

Are all the coloums independent ?

No, we can see that

$c_3 = c_1 + c_2$ and

$c_4 = c_1$

But to prove

if columns are idependent then

$N(A)$ should be 0

$$N(A) = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

Basis will be the independent coloums

that are c_1 and c_2 $\Rightarrow \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

Rank defination

Rank = # pivot columns

= dimension of $C(A)$

= 2 [in example]

Coming to null space

$N(A) = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ is one

the other is $\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

So $N(A) = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

does these 2 vectors span the entire null space?

<u>Yes!</u>

$\dim N(A) = \#$ free variables

$= n - r$

Video - 10

4 - fundamental subspaces

4 fundamental subspaces

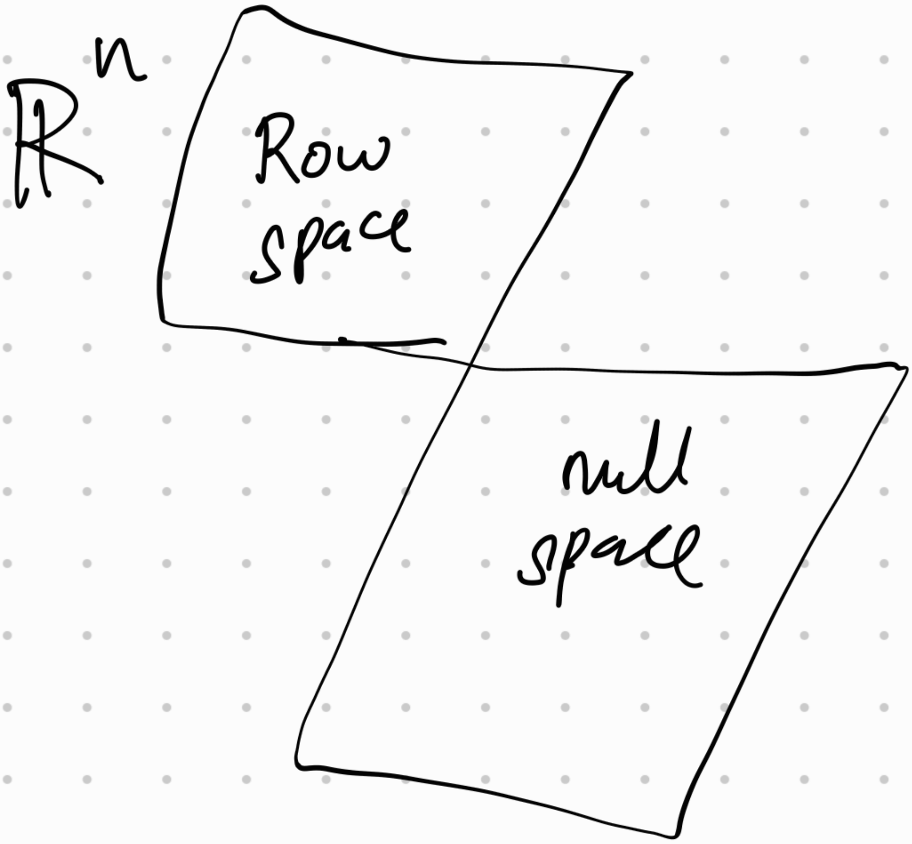
(1) column space of A $C(A)$ in R^m

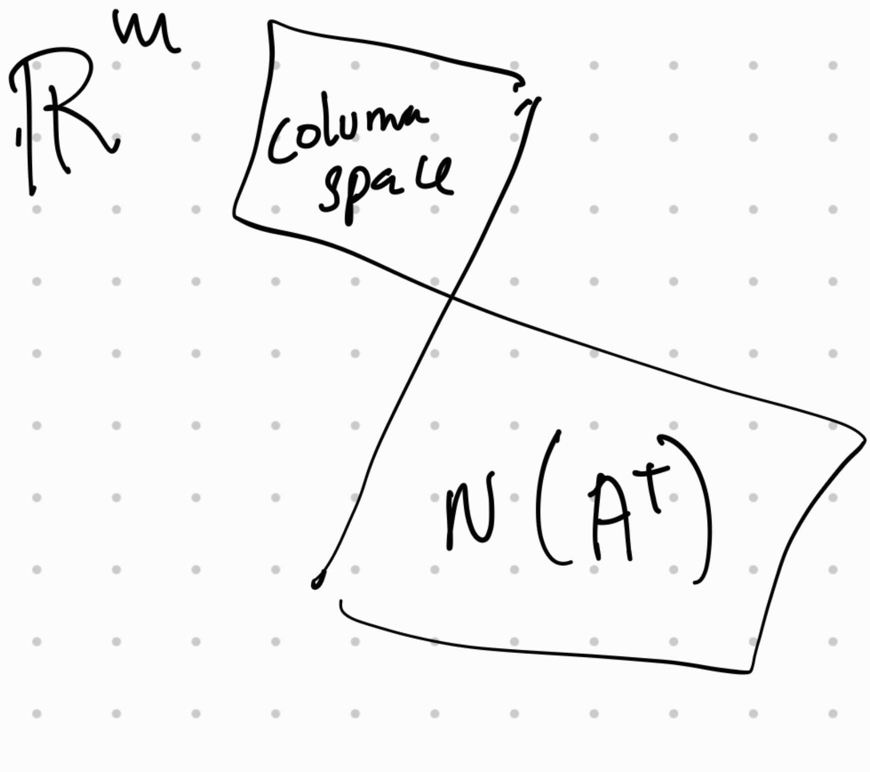
(2) null spaces of A $N(A)$ in R^n

(3) row spaces $\Rightarrow$ all combination of rows $= C(A^T)$ in R^n

(4) null space of $A^T \Rightarrow N(A^T)$ in R^m
Left

4 subspaces





for column space

basis = pivot columns

dimension = r [rank]

for row space:

dimension = r

for null space:

basis: speical solution

dimension: $n-r$

for null space of A^T

dimension = $m-r$

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

F

I

So basis of A for row space is first r rows in R for A
(rank)

so basis in the example is

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

4th space $N(A^T)$

$$A^T y = 0$$

$$\begin{bmatrix} & A & \end{bmatrix} \begin{bmatrix} y \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

Take Transpose on both side

$$(A^T y)^T = (0)^T$$

$$y^T A^{TT} = 0^T$$

$$y^T A = 0$$

$$\begin{bmatrix} y^T \end{bmatrix} \begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

rref matrix

$$E_{rref}[A : I] = [R : E_{rref}]$$

so

$$E_{rref} A = R$$

previously

$R = I$

$X[A : I] = [I : X]$

$XA = I$ so $X = A^{-1}$

So if we augment I to A

A : I and carry out to get R, we will get E

$$\left[\begin{array}{cccc|ccc} 1 & 2 & 3 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 1 & 0 & 0 & 1 \end{array}\right] \Rightarrow A : I$$

$$\left[\begin{array}{cccc|ccc} 1 & 0 & 1 & 1 & -1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{array}\right] \Rightarrow R : E_{rref}$$

So

$$E_{rref} A = R$$

$$\begin{bmatrix} -1 & 2 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Zero row [0 part of $Ax=0$]

this is the row with zero row

hence this is basis for getting our null space hence $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ is basis for $N(A^T)$

New vector space

Lets define a new vector

" All 3X3 matrices " be M

So we can do $A+B$ and cA

where A and B are 2 3X3 order matrices

Subspaces of M

upper triangular matrices

Symmetric matrices

diagonal matrices

So all previous ruls will apply for the new vector space

Like

for diagonal matrices

any 3 independent matrices would span all diagonal matrices

Ex:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 7 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So these 3 matrices can span the entire diagonal matrices

Video - 11

Basis of new vector spaces

Rank one matrices

Small world graphs

Basis for M => for all 3x3 s

they are

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdots\cdots \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so 9 matrices

so dimension $(M) = 9$

for symmetric matrices (S)

$$\text{dimension}(S) = 6$$

$$\begin{bmatrix} - & - & - \\ & - & - \\ & & - \end{bmatrix}$$

for upper triangular matrix U

$$\text{dimension}(U) = 6 \quad \begin{bmatrix} - & - & - \\ 0 & - & - \\ 0 & 0 & - \end{bmatrix}$$

$S \cap U =$ diagonal matrices D

$$\dim(S \cap U) = 3 \quad \begin{bmatrix} - & 0 & 0 \\ 0 & - & 0 \\ 0 & 0 & - \end{bmatrix}$$

$S \cup U \Rightarrow$ not a subspace

$S + U$ = any element of S + any element of U

sum of subspace

= all 3×3

dimension $(S+U) = 9$ [all 3×3]

---

so

$$\dim(S) + \dim(U) = \dim(S+U) + \dim(S \cap U)$$

$$6 + 6 = 9 + 3$$

---

new vector space

$$\frac{d^2y}{dx^2} + y = 0$$

Solution: $y = \cos x, \sin x$

What are all the complete solution for y ?

it is all the combination of

$$y = c_1 \cos x + c_2 \sin x$$

So basis are like $\cos x$ and $\sin x$

dimension of solⁿ = 2 [dont think too much]

Rank 1 matrix

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \end{bmatrix}$$

So dim $C(A)$ = dim $R(A) = 1$ = rank

So for rank 1 matrix, we can write

basis column $*$ basis row $= A$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \end{bmatrix}$$

So every rank 1 matrix are in the form

$A = u v^T$, where u and v are column vectors

---

for $A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$

$d(R(A)) = 1$, rank = 1

$d(C(A)) = 1$

$d(N(A)) = 3$

$d(N(A^T)) = 0$

Basis of $N(A)$

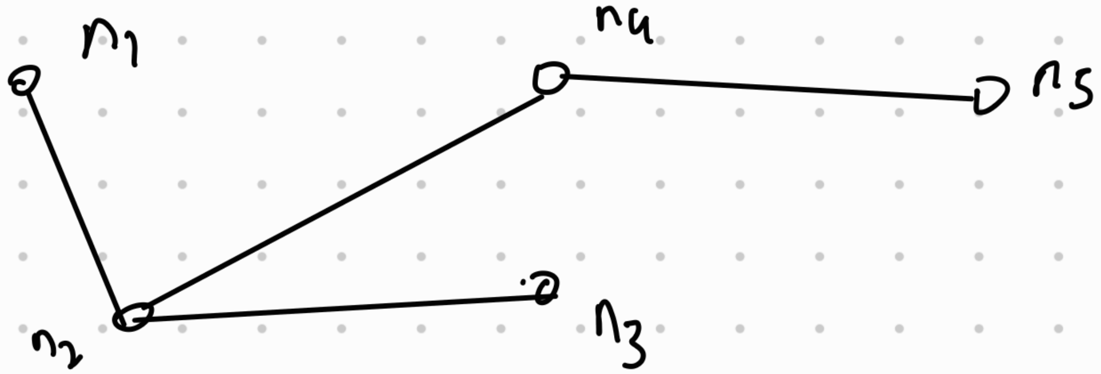
$$A = \begin{bmatrix} \underset{\substack{\uparrow \\ \text{pivot}}}{1} & \underbrace{1 \quad 1 \quad 1}_{\text{Free}} \end{bmatrix}$$

So

basis for $N(A) = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

Small world graphs

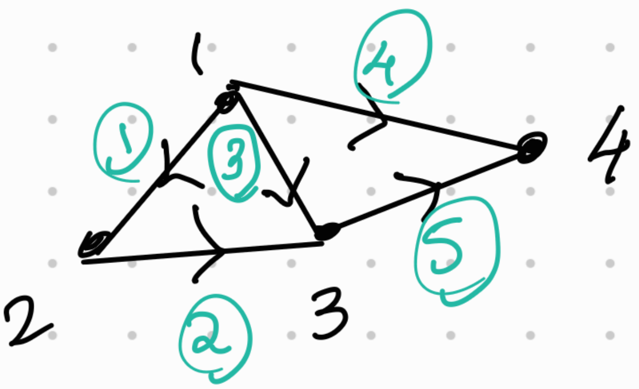
What are graphs? $\Rightarrow$ {nodes, edges}



Video - 12 [more of application]

- Graphs and Networks
- Incidence Matrices
- Kirchhoff's Law

Graphs $\Rightarrow$ nodes and edges



$n = 4$ (nodes)

$m = 5$ (edges)

Incidence matrix

$$
\begin{array}{c}
\begin{array}{cccc} \text{note} \; 1 & 2 & 3 & 4 \end{array} \\
A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}
\begin{array}{l} 1 \\ 2 \;\; \text{edges} \\ 3 \\ 4 \\ 5 \end{array}
\end{array}
$$

Notice that

node 1, 2 and 3 are a cycle

So adding edges 1 and 2

will give 3

What about null space?

$$\underbrace{\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ x_3 - x_1 \\ x_4 - x_1 \\ x_4 - x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Let $x = x_1, x_2, x_3, x_4 \ldots$ be potintials at nodes

$x_2 - x_1,\ x_3 - x_2 \ldots$

are potential differences

So a perticular solution is

$$x = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \text{ and basis is } c \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

rank of $A = 3$

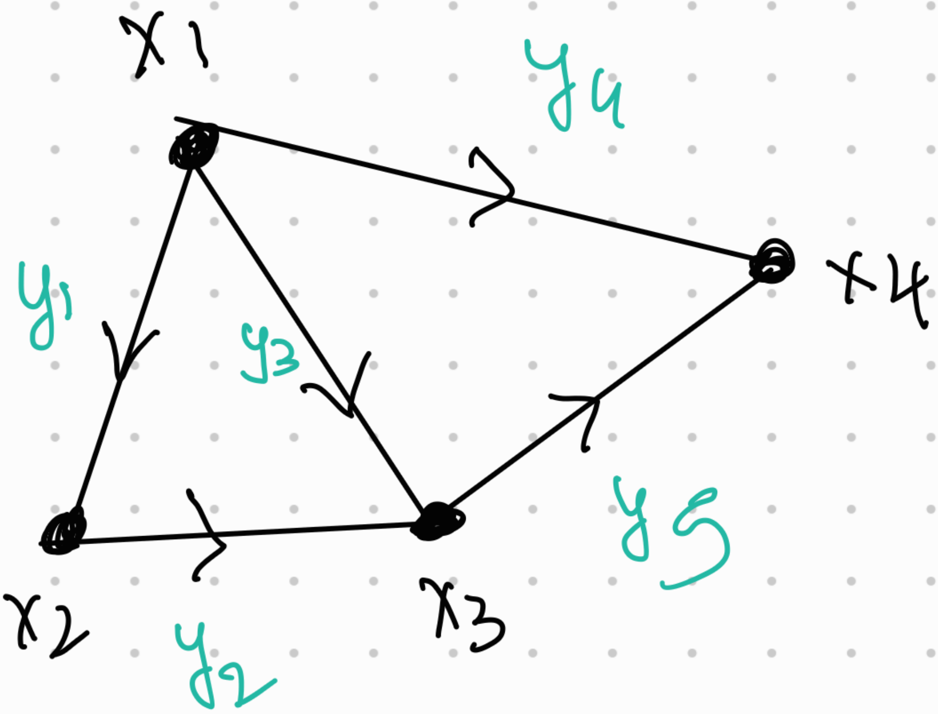
note
edges y_i can be visualized as currents and node values can be potentials

$$A^T y = 0$$

$$d(N(A^T)) = m - r$$
$$= 5 - 3$$
$$= 2$$

$A^T y = 0$ [Kirchoff's current Law]

$$\begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$



the first equation from $A^T y = 0$

$$-y_1 - y_3 - y_4 = 0$$ $\Leftarrow$ meaning all current is leaving from x_1 [net flow is zero]

$$y_1 - y_2 = 0$$ $\Leftarrow$ meaning $y_1 = y_2$ current leaving from x_1 to enter x_2 is same as current exiting x_2

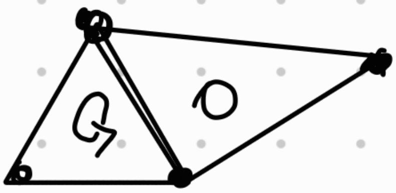
$y_2 + y_3 - y_5 = 0 \Leftarrow$ similarly

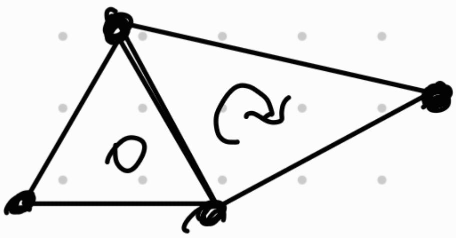
. . . .

Basis for $N(A^T)$

$$\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

$\downarrow$ $\downarrow$

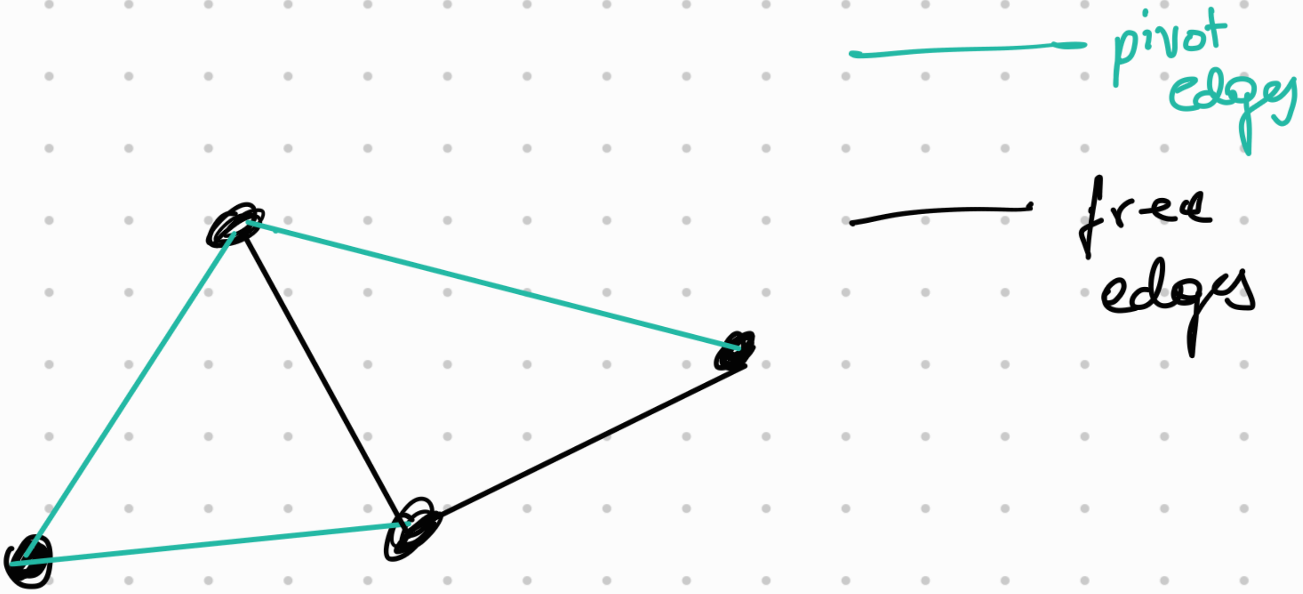




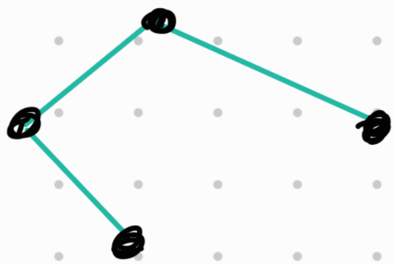
Pivot columns A^T are

c_1, c_2 and c_4

SO



graph with only pivot column edges never forms a loop & hence it is a tree



$\dim N(A^T) = m - r$

$\# \text{loops} = \# \text{edges} - (\# \text{nodes} - 1)$

$\text{rank} = n - 1$

#nodes − #edges + #loops = 1

↳ Euler's formula

Video - 13

Review

Rough work

$$B = \begin{bmatrix} u \\ 2 \cdot u \end{bmatrix}$$

Video-14

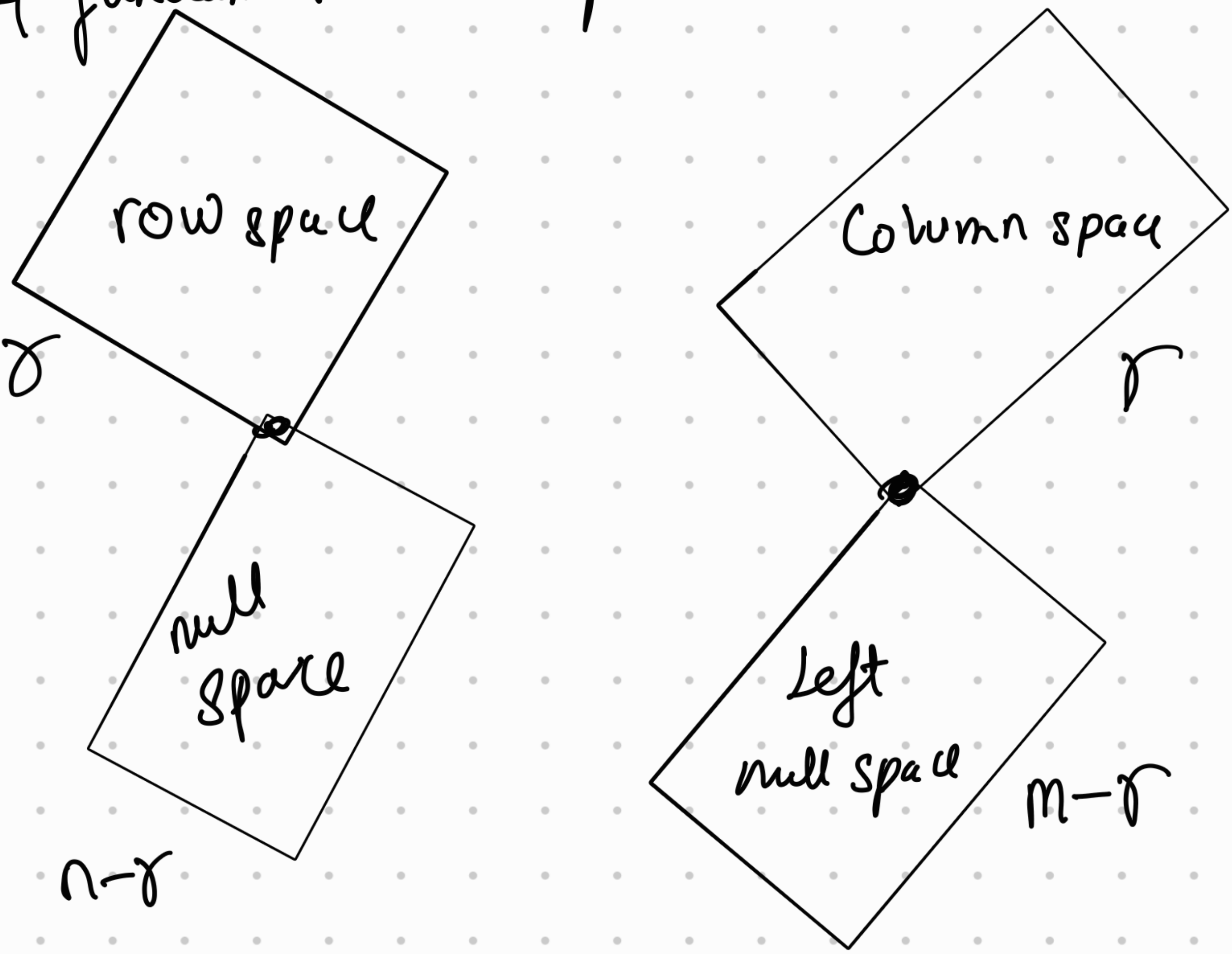
Orthogonal vectors and subspaces

null space $\perp$ row space

$$N(A^TA) = N(A)$$

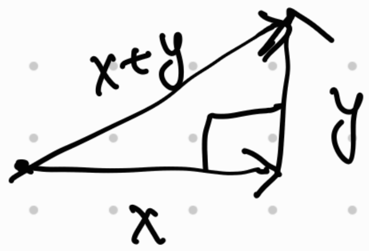
---

4 fundamental subspaces



Orthogonal vectors:

two vectors x and y are orthogonal if $x^T y = 0$



$$\|x\|^2 + \|y\|^2 = \|x+y\|^2 \text{ iff } x^T y = 0$$

Example

$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad y = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \quad x+y \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$$

$\|x\|^2 = 1+4+9$ $= 14$

$\|y\|^2 = 4+1+0$ $= 5$

$\|x+y\| = 9+1+9$ $= 19$

and $x^T y \Rightarrow \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

so $x^T y = 0$

---

if $x^T y = 0$

then $\|x\|^2 + \|y\|^2 = \|x+y\|^2$

also $x^T x + y^T y = (x+y)^T (x+y)$

simplify the above

$\cancel{x^T x} + \cancel{y^T y} = \cancel{x^T x} + x^T y + y^T x + \cancel{y^T y}$

as $x^T y = 0$, $y^T x$ will also be zero

$0 = 2x^T y$

or

$x^T y = 0$ [just checking]

[Dont overthink]

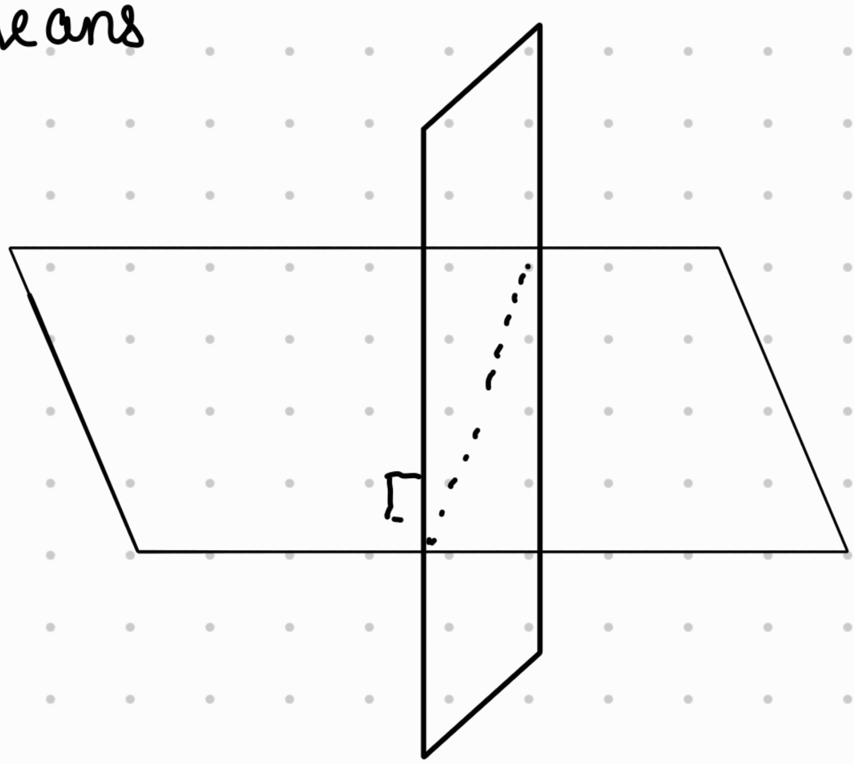
So,

if x is zero vector, then y could be anything, $x^T y$ is always 0

as $\begin{bmatrix} 0 & 0 & 0 & \cdots \end{bmatrix} \begin{bmatrix} - \\ - \\ - \\ \vdots \end{bmatrix} = 0$

Now about spaces and subspaces

Subspace S is orthogonal to subspace T means



if they are plane

row space is orthogonal to null space

Why?

Nullspace: x in $Ax = 0$

meaning
$$\begin{bmatrix} - r_1 - \\ - r_2 - \\ - r_3 - \\ \vdots \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \end{bmatrix}$$

$$r_1 \cdot x = 0$$

dot product

hence row dot product $x = 0$

so for row space, we mean all possible combination of row dot product x must be zero

So $(r_1)^T x = 0$

$(r_2)^T x = 0$

and so on.

Example

$$A = \begin{bmatrix} 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

$n = 3, \ r = 1$

$\dim N(A) = 2$

so subspace of x is orthogonal to row space of A

and their dimension add to the whole space

this is called orthogonal complements in R^n

Nullspace contains all vectors $\perp$ to row space

Next video, solving $Ax=b$ where there is no solution

where $m > n$ [just to make things easy]

This is required because sometimes some of the equations are not true as they can be recorded by physical instruments or recordings.

One solution is to discard the values [be it row or column] untill we get a nice solution

But this solution is not prefered as we are disregarding some information

for other sol^{n}s we need to learn about A^TA

$A^T A$

- → always square ($\underbrace{m \times n \;\; n \times m}$ = $m \times m$)
- → symmetric

So as $Ax = b$ wont have sol^n
we do the following:

$$A^T A \hat{x} = A^T b$$

Example

if $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix} \Rightarrow \text{rank} = 2$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 8 & 30 \end{bmatrix}$$

here

$N(A) =$ zero vector

hence $A^T A$ is invertable

but not always

note $\text{rank}(A^T A) = 2$

---

for example

$$A = \begin{bmatrix} 1 & 3 \\ 1 & 3 \\ 1 & 3 \end{bmatrix} \Rightarrow \text{rank} = 1$$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 3 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ 9 & 27 \end{bmatrix}$$

This is not invertable

[# pivot is one or rows r_1 and r_2

are not independent]

$\text{Rank}(A^T A) = \text{Rank}(A) = 1$

So,

$$\text{Rank}(A^T A) = \text{Rank}(A)$$

$$N(A^T A) = N(A)$$

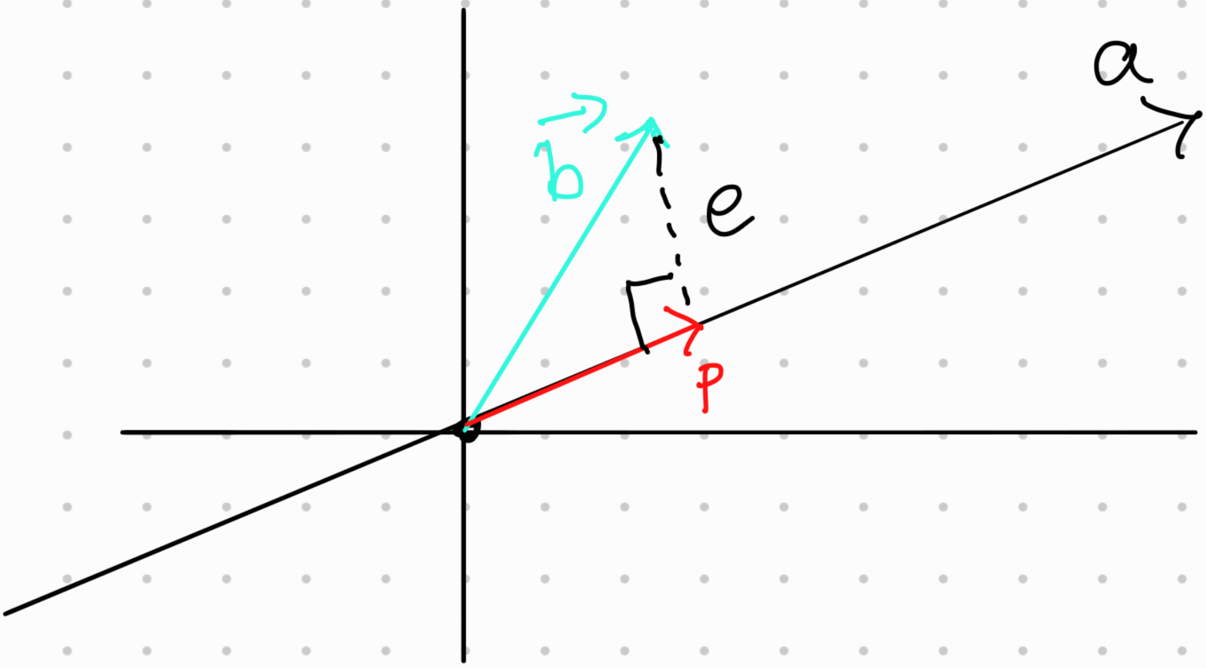
Video 15

Projections

Least Square

Projection matrix

Projection



as $\vec{p}$ is on $\vec{a}$, so p is some multiple of $\vec{a}$, so it can be written as $\vec{p} = x\vec{a}$

$e = b - p$ [as e is correction from b to p]

$e = b - xa$ [we need x for projection]

as $a \perp$ to e, then

$$a^T e = 0$$

$$a^T (b - xa) = 0$$

$$x\, a^T a = a^T b$$

$$\boxed{x = \frac{a^T b}{a^T a}, \quad p = a\,x}$$

p ← projection.

So $p = a\left(\frac{a^T b}{a^T a}\right)$

note

lets say we increase b by 2 ie $b = 2b$

then projection $p = a\left(\frac{a^T 2b}{a^T a}\right) \Rightarrow p \Rightarrow 2p$

So the projection increases 2 fold
thus it is intuitively correct.

if we increase a to $2a$

intuition says projection should not change

Lets see: $a = 2a$

$$p = \cancel{2}a \frac{(\cancel{2}a)^T b}{(\cancel{2}a)^T \cancel{2}a} = a \frac{a^T b}{a^T a}$$

So no effect on projection of b
satisfying our intuition

Projection is

$$\text{proj } p = P \; b$$

it is projection matrix

$$P = \frac{a\,a^T}{a^T a}$$

Lets take properties of P

- $C(P) =$ line through $\vec{a}$

  why?

  cuz projection must be a vector along $\vec{a}$ so

  $p = Pb$

  ↳ This must be along $\vec{a}$

- $\text{rank}(P) = 1$ $\left[P = \frac{a\,a^T}{a^T a} \Rightarrow \underset{\text{column}}{C}\ \underset{\text{row}}{R} \text{ form} \right]$

- Symmetric : $P = P^T$

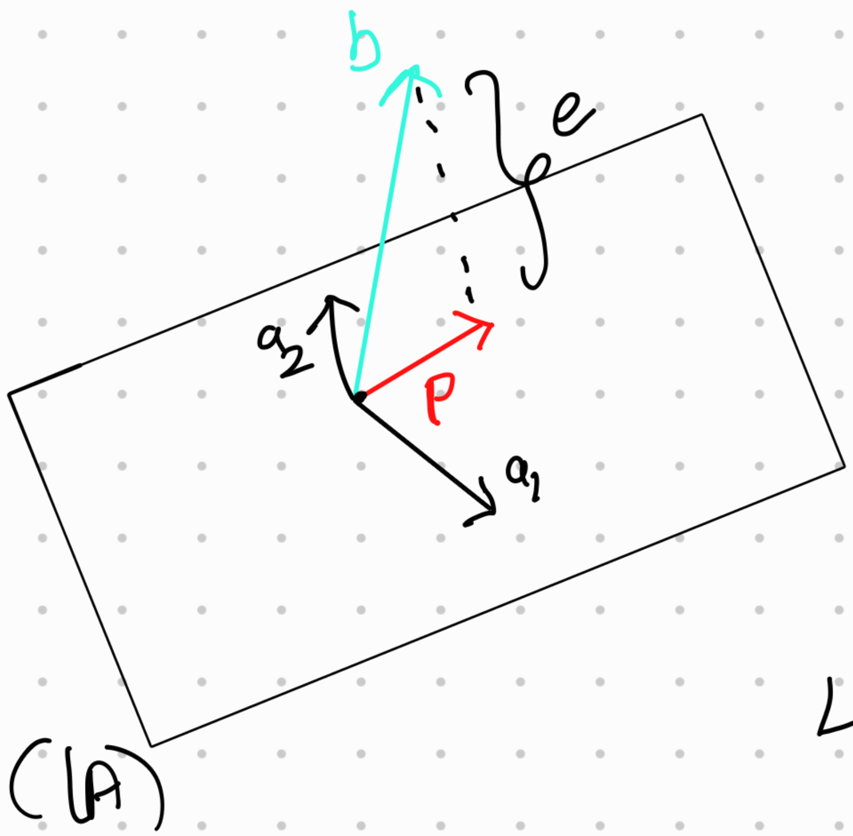
- $P^2 = P$ [projection of a line on $\vec{a}$ is itself]

Why are we projecting?

Because $Ax = b$ has no sol^{n}s

So we solve

$A\hat{x} = p$ where p is projection of b on $C(A)$



Let a_1 and a_2 be basis of $C(A)$

So

$$p = \hat{x}_1 a_1 + \hat{x}_2 a_2$$

$$p = A\hat{x}$$

Key $\quad b - A\hat{x} \perp$ plane A

$$A^T(b - A\hat{x}) = 0$$

$N(A^T)$ is in e

So $e \perp$ columnspace of A

So solving that equation

$$A^T A\hat{x} = A^T b$$

$$\hat{x} = (A^T A)^{-1} A^T b$$

$$p = A\hat{x} = \underbrace{A(A^T A)^{-1} A^T}_{} b$$

So projection matrix $P = A(A^T A)^{-1} A^T$

note

$$P = A(A^TA)^{-1}A^T \quad \text{step 1}$$

$$= A\,A^{-1}(A^T)^{-1}A^T \quad \text{step 2}$$

$$= I \text{ (identity)} \quad \text{step 3}$$

So why are we getting I

because of $(A^TA)^{-1} = A^{-1}(A^T)^{-1}$ only if

A is a square invertable matrix

So, when A is square and invertable

$C(A) \Rightarrow$ entire space [if R^3, then $C(A) = R^3$]

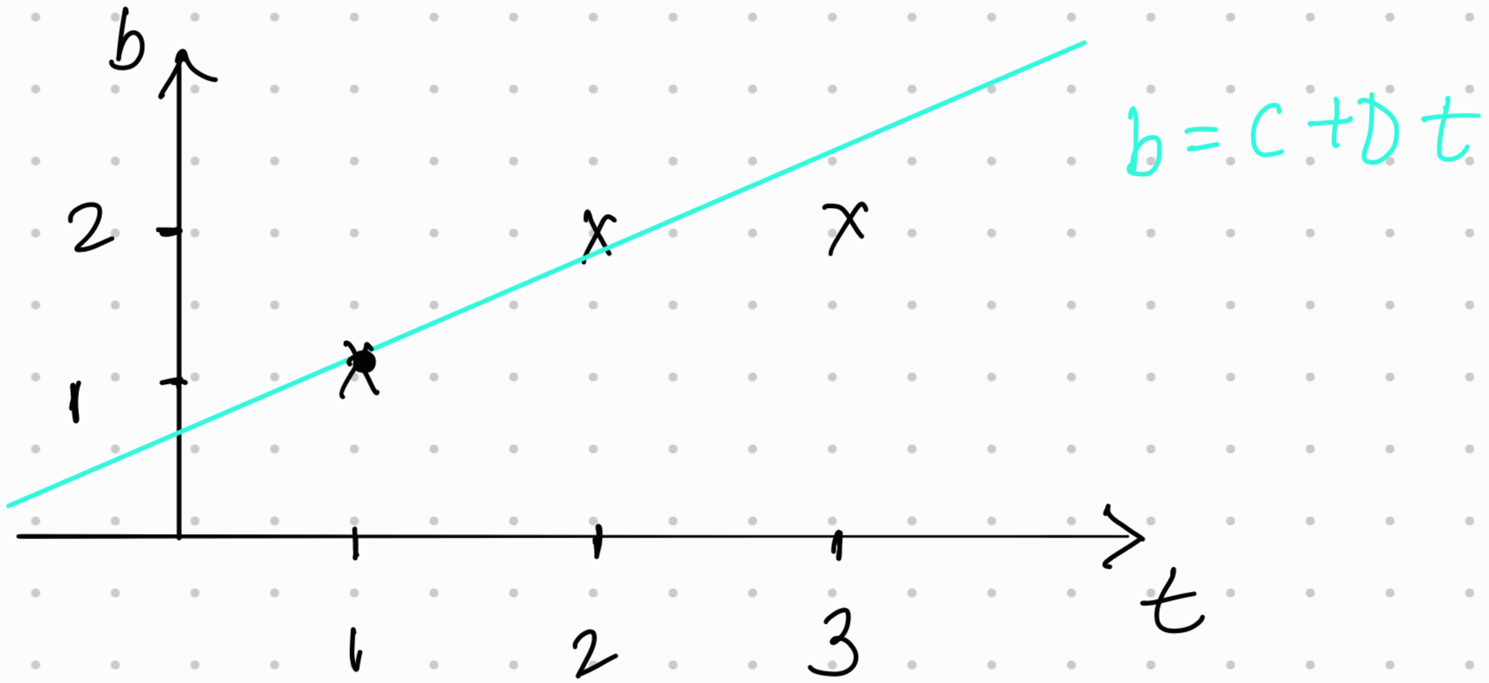
then the projection will the vector itself
as it will be included in the entire space

But generally
A will be rectangular [column matrix]

properties

$P = P^T$
$P^2 = P$
} both still works for higher dimension.

Applications of Projections



Fitting by a line $b = C + Dt$

$(1,1), (2,2), (3,2)$

So the line would surely go through $(1,1)$

$$\left.\begin{aligned} C + D &= 1 \\ C + 2D &= 2 \\ C + 3D &= 2 \end{aligned}\right\}$$ these are the equation it is trying to satisfy

in matrix form

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$A \quad x = b.$$ [not solvable here]

So we try and solve by using

$$A^T A \hat{x} = A^T b$$

Video - 16

Must Watch

Video-17

Orthogonal basis $q_1, q_2 \ldots q_n$

Orthogonal matrix Q

Gram-Schmidt $A \to Q$

Orthonomal vectors

if $$q_i^T q_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

then q_i and q_j are called orthonormal vectors

so lets take $Q = \begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{bmatrix}$

where every column in Q is orthonomal

to each other.

then

$$Q^T Q = ?$$

$$\begin{bmatrix} - & q_1 & - \\ - & q_2 & - \\ & \vdots & \\ - & q_n & - \end{bmatrix} \begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \end{bmatrix}$$

$$Q^T \qquad Q \quad = \quad I$$

---

if Q is square then $Q^T Q = I$ tells us $Q^{-1} = Q^T$

Example: $Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{Q^T} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = I$$

another example

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad Q^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}}_{Q^T} \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{Q} = \underbrace{\begin{bmatrix} \cos^2 + \sin^2 & 0 \\ 0 & \cos^2 + \sin^2 \end{bmatrix}}_{I}$$

$$Q = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$ $\Leftarrow$ another example

$$Q = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$Q = \frac{1}{3}\begin{bmatrix} 1 & -2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix}$$

note

Q generally dipicts a matrix with orthonormal columns

Project onto its column space

$$P = Q\underbrace{(Q^TQ)}^{-1}Q^T$$

↳ this part is I

$$= \underbrace{Q\,Q^T}$$

↳ this is I if Q is square matrix

Why we need Q?

$\Rightarrow$ $A^TA\hat{x} = A^Tb$ when we had no sol^n

but now $A = Q$

$$\underbrace{Q^TQ}_{I}\,\hat{x} = Q^Tb$$

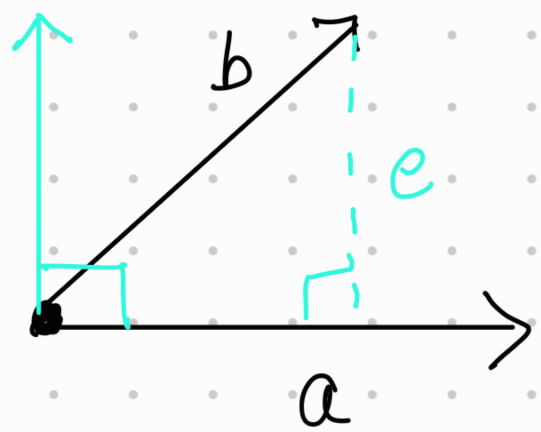
so $\hat{x} = Q^Tb$

or

$$\vec{x}_i = q_i^T b$$

Gram - Schmidt

vectors a and b [independent vectors]



previously we had e, now that will be the part of b we consider this time as it is $\perp^r$ to a

So if $A = a$

then $B = b - x A$ [x is some integer]

$$= b - \frac{A^T b}{A^T A} A$$

the orthonormal column would be

$$q_1 = \frac{A}{\|A\|}, \quad q_2 = \frac{B}{\|B\|}$$

for 3 vectors

a, b, c [independent]

all previous formula for a and b are valid

$$C = c - \frac{A^T c}{A^T A} A - \frac{B^T c}{B^T B} B$$

$\uparrow$ orthogonal vector for C

Example with numbers

$$a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$\text{So } A = a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$B = b - \frac{A^T b}{A^T A} A$$

$$= \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \frac{\cancel{3}}{\cancel{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}$$

So Gram-Schmidt is used for converting $A \longrightarrow Q$ $[A = QR]$

operation done on A

R is upper triangular

$$\underset{A}{\begin{bmatrix} | & | \\ a_1 & a_2 \\ | & | \end{bmatrix}} = \begin{bmatrix} | & | \\ q_1 & q_2 \\ | & | \end{bmatrix} \begin{bmatrix} a_1^T q_1 & \times \\ a_1^T q_2 & \times \end{bmatrix}$$

This will be zero

Video - 18

Determinants [det A or $|A|$]

Properties 1, 2, 3, 4 ... 10

$\pm$ signs

P-1 $\quad \det I = 1$

P-2 $\quad$ exchanging rows reverses the sign of determinants

So det Permutation is 1 or -1 depending on # row exchanges

$$\boxed{\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cd}$$

P-3

$$\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$\begin{vmatrix} a+y & b+z \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} y & z \\ c & d \end{vmatrix}$$

so determinent is linear iff it comes a single row

P-4 if 2 rows are equal, then det is zero

P-5 elimination steps will not the det value

P-6 Row of zeros $\rightarrow \det(A) = 0$

P-7

$$U = \begin{bmatrix} d_1 & * & * & * \\ 0 & d_2 & * & * \\ \vdots & 0 & \ddots & * \\ 0 & \cdots & 0 & d_n \end{bmatrix}$$

$$\det(U) = d_1 \times d_2 \times d_3 \cdots d_n$$

So determinent of any upper triangular matrix is product of its pivots

P-8 $\det A = 0$

if A is singular

[Because we get a row of zeros]

P-9 $\det A \cdot B = (\det A)(\det B)$

So conclusion:

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

as $A^{-1} A = I$

$$\det(A^{-1} A) = \det I$$

$$\det A^{-1} \det A = 1$$

$$\det A^{-1} = \frac{1}{\det A}$$

$\det A^2 = (\det A)^2$

$\det 2A = 2^m \det A$ [where $A_{m \times n}$]

P-10 $\det A^T = \det A$

So all properties for rows will work for columns as well

Video-19

Formula for det A ($n!$ terms)

Cofactor formula

Tridiagonal Matrices

P-1) det $I = 1$

P-2) sign reverses if row exchanges

P-3) det is linear in each row seperately

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix}$$

$$\begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}$$

These are zero because a row is zero

$$= \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix}$$

$$= ad - bc$$

So for a 3x3

What are the matrices that wont become zero?

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \Rightarrow \text{let this be the matrix}$$

The surviving matrices are

$$\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{vmatrix} +$$

$$+ a_{11} a_{22} a_{33} - a_{11} a_{23} a_{31}$$

minus is here because it is one row exchange from first matrix

$$\begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{vmatrix} +$$

$$\underline{-a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31}}$$

[2 exchanges

$$\begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{vmatrix}$$

$$\underline{+a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}}$$

So the full formula was

$$a_{11}a_{22}a_{33} - a_{11}a_{23}a_{31} - a_{12}a_{21}a_{33}$$

$$+ a_{12}a_{23}a_{31} + a_{13}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

So Generally the formula would be

$$\det A = \sum_{n!\text{ terms}} \pm a_{1\alpha} a_{2\beta} a_{3\gamma} \cdots a_{n\omega}$$

where

$(\alpha, \beta, \gamma, \ldots \omega) \Rightarrow$ Permutations of $(1, 2, 3, \ldots n)$

Taking 3x3 formula

$$\det A = a_{11}(a_{22}a_{33} - a_{23}a_{32})$$
$$+ a_{12}(-a_{21}a_{33} + a_{23}a_{31})$$
$$+ a_{13}(a_{21}a_{32} + a_{22}a_{31})$$

So cofactor of a_{11} is $(a_{22}a_{33} - a_{23}a_{32})$
and similarly for a_{13} and a_{13}

and also notice cofactor of a_{11} is det of smaller matrix

$$\begin{matrix} a_{11} & 0 & 0 \\ 0 & \multicolumn{2}{c}{\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}} \\ 0 & & \end{matrix}$$

So the cofactor of $a_{ij} = C_{ij}$

$\pm \det$ (n-1 matrix of row i and column j erased)

its $+$ if $i+j \Rightarrow$ even

$-$ if $i+j \Rightarrow$ odd

Cofactor formula

$$\boxed{\det A = a_{11} C_{11} + a_{12} C_{12} + \dots a_{1n} C_{1n}}$$

example

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = a \underbrace{d}_{\text{cofactor of } a} + b \underbrace{(-c)}_{\text{cofactor of } b}$$

Tridiagonal matrix

$$A_{4\times4} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$\det A_1 = 1$

$\det A_2 = 0$

$\det A_3 = -1$

$$|A_4| = 1 \cdot |A_3| - 1 |A_2|$$

So $|A_n| = |A_{n-1}| - |A_{n-2}|$

$|A_4| = |A_3| - |A_2|$

$= -1 - 0$

$= -1$

$A_5 = |A_4| - |A_3|$

$= 0$

similarly

$|A_6| = 1$

$|A_7| = 1$

$|A_8| = 0$

$|A_9| = -1$

$|A_{10}| = -1$

$|A_{11}| = 0$

$|A_{12}| = 1$

So there are repeating after 6

So we can use modulo for higher values.

$|A_{61}| = |A_1| = 1 \quad [61 \% 6 = 1]$

Video - 20

Formula for A^{-1}

Cramers Rule for $x = A^{-1}b$

$|\text{Det } A|$ = volume of the box

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

So inverse exists when det is not zero because det is in the denominator

$$A^{-1} = \frac{1}{|A|} \underbrace{C}^{T}$$

↳ cofactor matrix

So we should check

$$A\,C^T = |A|\,I$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & \cdots & C_{n1} \\ C_{12} & & C_{n2} \\ \vdots & & \vdots \\ C_{1n} & & C_{nn} \end{bmatrix}$$

Those both will give det A

$$= \begin{bmatrix} \det A & & & 0 \\ & \det A & & \\ 0 & & \det A & \\ & & & \ddots & \\ & & & & \det A \end{bmatrix}$$

$$= \frac{1}{\det A}\quad I$$

So now getting to $Ax = b$

$$x = A^{-1} b$$

$$x = \frac{1}{|A|} C^T b$$

Cramer's Rule :

for

$$x = A^{-1} b$$

$$= \frac{1}{\det A} C^T b$$

Notice that C^T will have cofactors and is being multiplied by matrix b hence that will give a determinant of a matrix

Let these matrices be denoted by B_j

$$X_1 = \frac{\det B_1}{\det A} \quad \text{a} \quad X_2 = \frac{\det B_2}{\det A}$$

Where

$$B_1 = \underbrace{\begin{bmatrix} | & | & | & & | \\ b & a_2 & a_3 & \cdots & a_n \\ | & | & | & & | \end{bmatrix}}$$

only first column is replaced by b and the rest is A

$$B_2 = \begin{bmatrix} | & | & | & | & & | \\ a_1 & b & a_3 & a_4 & \cdots & a_n \\ | & | & | & | & & | \end{bmatrix}$$

Same explaination as B_1

Why?

Because when we do multiplication

$$C^T b \Rightarrow C_{11} b_1 + C_{21} b_2 \cdots$$

b_1

$$\begin{bmatrix} | & | & | \\ b & a_2 & a_3 \\ | & | & | \end{bmatrix} = b_1 C_{11} \quad \text{and so on for other } B$$

this will be C_{11} as rest of the matrices is A

So B_j = A with column j reblaced by b

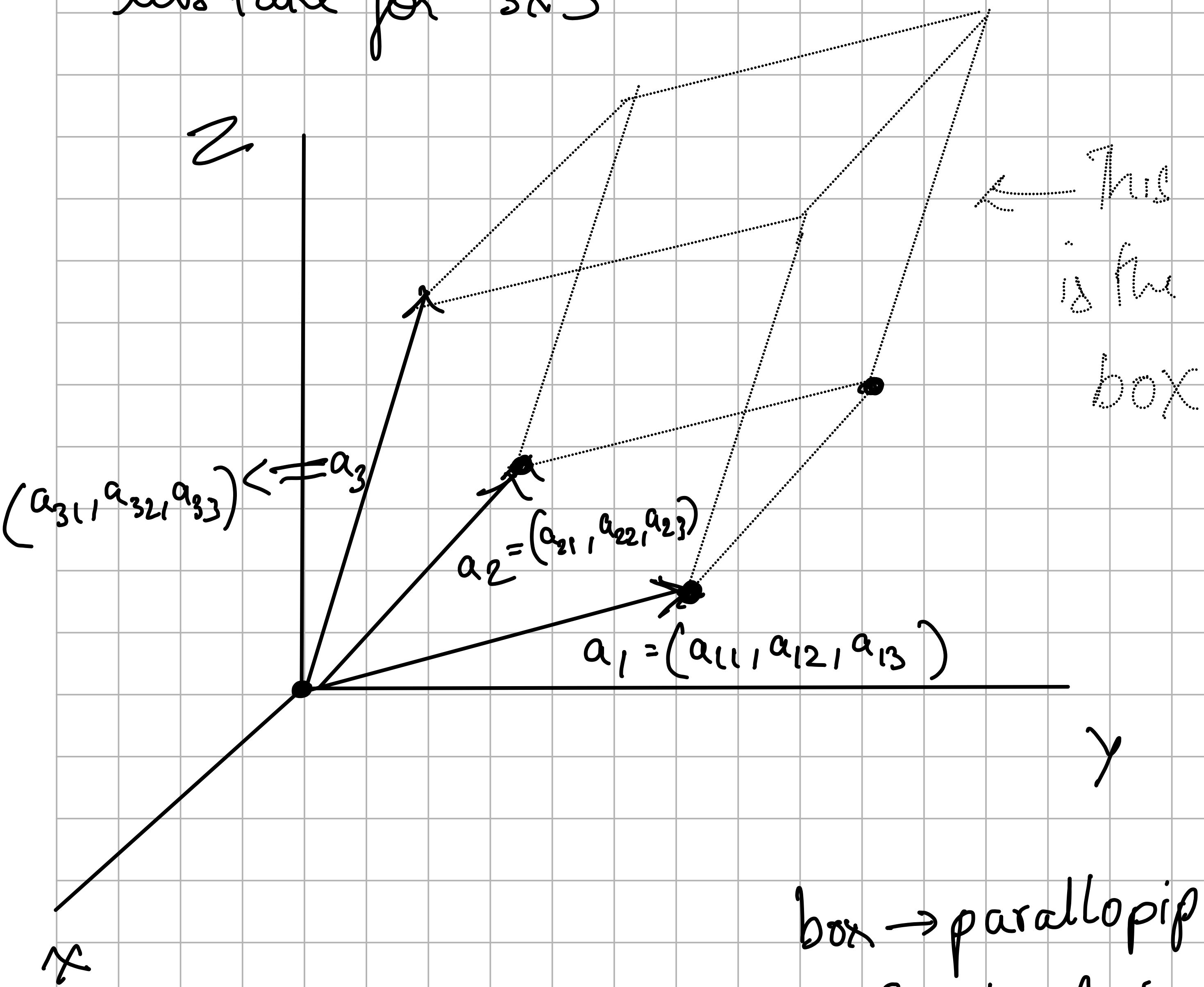
Cramer's rule

$$x_j = \frac{\det B_j}{\det A}$$

This rule is very lengthy to calculate inverse, elimination is much faster.
So Cramer's rule is not recommended to use for inverse calculation.

3) det A = volume of a box

lets take for 3×3



box → parallopiped
[each side is parallogram]

So speical case

if $A = I$ then the box is a unit cube alligned to the axis

if $A = Q$ then it is a cube but turned wrt axis

So if one of the vectors increases then the volume doubles,

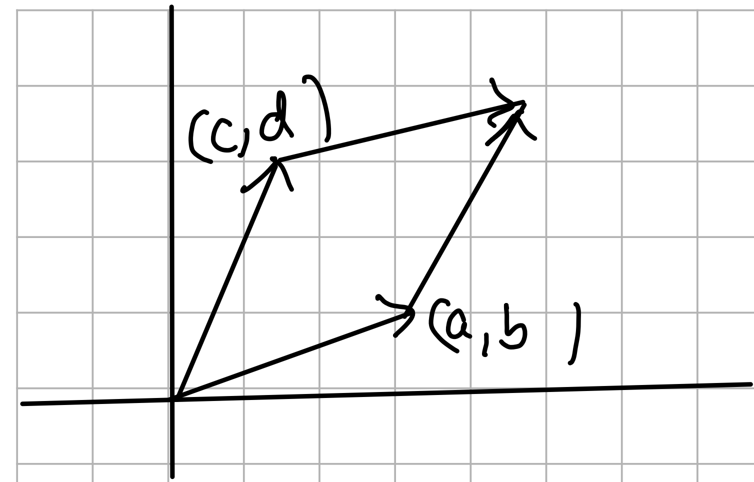
and we can multiply $2x$ with a single row

so it makes sense.

Proof not explained in videos properly because there was no time

but to conclude:

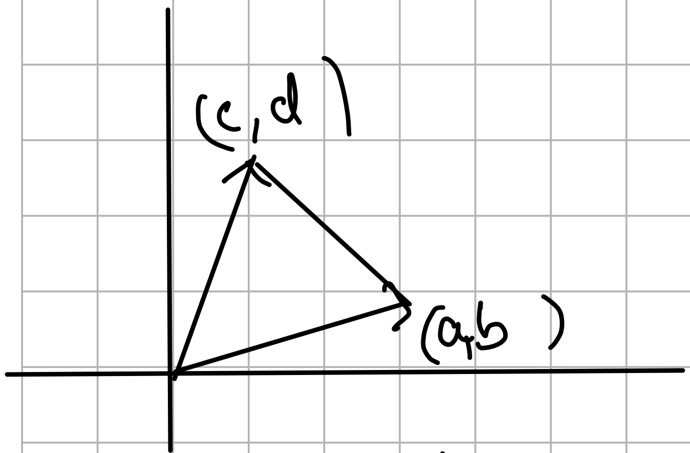
$\det A =$ volume of box



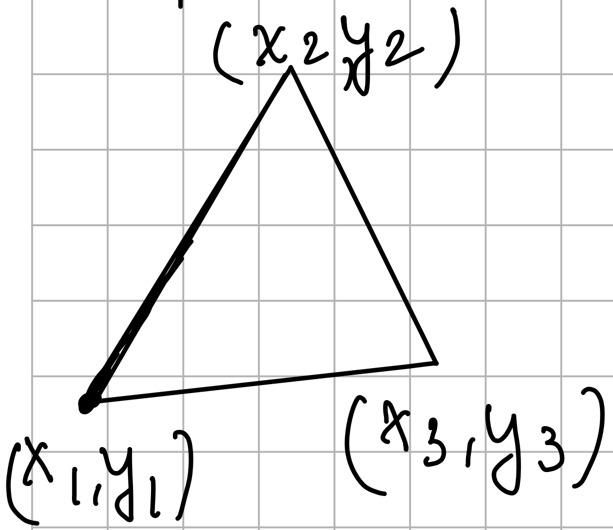
area of parallogram

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$= ad - bc$$



area of Δle is $\frac{1}{2}|A|$



area of Δle

$$= \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Video - 21

Eigen values & Eigen vectors

$$\det[A - \lambda I] = 0$$

$$\text{Trace} = \lambda_1 + \lambda_2 + \lambda_3 + \dots \lambda_n$$

---

from x we need Ax such that
Ax is parallel to x

These Ax are eigen vectors
So if they are in same direction
we can write it as

$$Ax = \lambda x$$

where λ will be the eigen value and
x will be the eigen vector

So if $Ax = 0$, then $\lambda = 0$, then 0 is eigenvalue

What are the eigenvalues for projection matrices

for x to be projected to plane of x

$$Px = x \quad \text{so } \lambda = 1$$

if we are projecting $\perp$ to plane of x, then

$$Px = 0 \quad \text{so } \lambda = 0$$

Lets see for permutation

$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, so for what x does row exchange do not affect the matrix

Answer is $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ q so $Ax = x$ so $\lambda = 1$

and for $x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$Ax = -x$ so $\lambda = -1$

fact

for $n \times n$ matrix, there are n eigenvalues and sum of eigen values (λ) is given by the following

$$\lambda_1 + \lambda_2 + \dots = \underbrace{a_{11} + a_{22} + a_{33} + \dots + a_{nn}}_{\text{sum of diagonal entries}}$$

How to solve $Ax = \lambda x$

$Ax = \lambda x$

$(A - \lambda I)x = 0$

So as x is eigenvector, then if $x = 0$, then there is no use as $x = 0$, λ does not matter in $Ax = \lambda x$ as both side $= 0$

So when $x \neq 0$, then $(A - \lambda I)$ must be singular because taking $(A - \lambda I)x = 0$ in the form $Ax = 0$ and where x is not zero [just now said why] then null space of $A - \lambda I$ [which is A in $Ax = 0$] will be singular

So as $A - \lambda I$ is singular then

$\det(A - \lambda I) = 0$

So now we can find λ from the above equation

Lets take an example

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix}$$

$$= (3-\lambda)^2 - 1$$

$$= \lambda^2 - 6\lambda + 8$$

$$= (\lambda - 4)(\lambda - 2)$$

note

6 will be the trace and 8 will be determinant value

$\lambda_1 = 4$

$\lambda_2 = 2$

So coming to x for each λ

$(A-4I)x_1=0$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} x_1 = 0$$

do elimination and get the basis for the null space

it is $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Same for the other $\lambda_2 = 2$

$(A-2I)x_2=0$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} x_2 = 0$$

$$x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

So notice the previous 2 examples

for $A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad \lambda_1^1 = 1 \quad \lambda_2^1 = -1$

$x_1^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad x_2^1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

and for

$A_2 = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ $\quad \lambda_1^2 = 4 \quad \lambda_2^2 = 2$

$x_1^2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad x_2^2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

So noticing the eigen values and the As

we notice that $\quad A_2 = (A_1 + 3I)$

and $\lambda_1^2 = \lambda_1^1 + 3$, $\lambda_2^2 = \lambda_2^1 + 3$

but x (eigenvectors) remain same

So if $Ax = \lambda x$

$$(A + nI)x = \lambda x + nx$$
$$= (\lambda + n)x$$

---

exampe: Lets take orthogonal matrics

$$Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

This matrix when multiplied rotates the other matrics by 90°

trace $= 0 = \lambda_1 + \lambda_2$ [Because diagonal elements are zero]

det $= 1 = \lambda_1 \lambda_2$

This does not seem possible because for 2 numbers to sum to zero, if one is positive, then other must be negative but their product could never be 1

This is not possible for any <u>real</u> no.

Lets solve the equation to see what we get

$$\det(Q - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0$$

$$\lambda_1 = i \quad \text{and} \quad \lambda_2 = -i$$

So orthonormal matrices gives us eigenvalues that are completely imaginary, whereas symmetric matrices give real eigenvalues

Lets take another example to get another possibility.

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$$ so A is upper triangular

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)^2 = 0$$

$$3 - \lambda = 0$$

$$\lambda = 3$$

so $\lambda_1 = 3$ and $\lambda_2 = 3$

But when coming to eigen vectors

$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad x_2 =$ no independent eigen vector

(we get this after solving for null space)

Video - 22

Diagonalizing a matrix $S^{-1}AS = \Lambda$

Powers of A / equation $u_{k+1} = Au_k$

Lets say A has n eigenvectors x_n

Lets stack all these eigenvectors as colums for matrix S

so $S = \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix}$

$\hookrightarrow$ eigen vector matrix

$$AS = A\begin{bmatrix} | & | & | & & | \\ x_1 & x_2 & x_3 & \cdots & x_n \\ | & | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \lambda_1 x_1 & \lambda_2 x_2 & \cdots & \lambda_n x_n \\ | & | & & | \end{bmatrix}$$

So $\begin{bmatrix} | & | & & | \\ \lambda_1 x_1 & \lambda_2 x_2 & \cdots & \lambda_n x_n \\ | & | & & | \end{bmatrix}$ can also be written as $\begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots \\ 0 & \lambda_2 & \cdots & \\ 0 & \cdot & \ddots & \lambda_n \end{bmatrix}$

(Because λ are values)

$$AS = S\Lambda$$

↳ matrix with only eigenvalues

Lets say S is invertable (meaning it has all its columns independent or nullspace = zero vector or $\det(S) \neq 0$)

for $AS = S\Lambda$

$$S^{-1}AS = \Lambda \quad \text{or} \quad A = S\Lambda S^{-1}$$

So now lets square A and find changes to eigenvalues and eigenvectors from the basic equation

$$Ax = \lambda x$$

multiply both side by A

$$A^2 x = A \lambda x = \lambda A x$$ [as λ is real no.] (most of the time)

$$A^2 x = \lambda (\lambda x) \quad [\text{as } Ax = \lambda x]$$

$$A^2 x = \lambda^2 x$$

so eigenvalue squares but eigen vectors remain the same

Now from the equation using Λ

$$A = S \Lambda S^{-1}$$

$$A^2 = S \Lambda \underbrace{S^{-1} S}_{I} \Lambda S^{-1}$$

$$= S \Lambda^2 S^{-1}$$

so eigen vectors (S) remains same but eigen values Λ squares for

$A^k = S \Lambda^k S^{-1}$

Theorem

for $A^k \to 0$ and $k \to \infty$

then all $|\lambda_i| < 1$

A is sure to have n independent vectors (and b is diagonalizabl)

if all the λ values are different (no repeat in λ's)

Equation $u_{k+1} = A u_k$

Starting with given vector u_0

$u_1 = A u_0$, $u_2 = A u_1$, --- so on

So: $\boxed{u_k = A^k u_0}$

To solve $u_{k+1} = A u_k$ from scratch

Lets do the following steps

$$u_0 = c_1 x_1 + c_2 x_2 + \cdots c_n x_n$$

[combination of parts of eigen vector]

$$A u_0 = c_1 \lambda_1 x_1 + c_2 \lambda_2 x_2 \cdots + c_n \lambda_n x_n$$

$$= \Lambda S c$$

Example

Fibonacci numbers: 0, 1, 1, 2, 3, 5, 8, 13, ...

100th term in the series is? $\rightarrow F_{100} = ?$

$$F_{k+2} = F_{k+1} + F_k$$

$F_{k+1} = F_{k+1}$ (This for making the single equation into system of equation with a dummy equation)

$$u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$$ This is the trick

So taking

$$u_{k+1} = A\, u_k$$

$$u_{k+1} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$$

$$u_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} u_k$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = \lambda^2 - \lambda - 1$$

$$\lambda = \frac{1 \pm \sqrt{1+4}}{2}$$

$$\lambda_1 = \frac{1}{2}(1+\sqrt{5})$$

$$\lambda_2 = \frac{1}{2}(1-\sqrt{5})$$

Video 23

Differential equations $\frac{du}{dt} = Au$

Exponential e^{At} of a matrix

Example

$$\frac{du_1}{dt} = -u_1 + 2u_2 \qquad u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\frac{du_2}{dt} = u_1 - 2u_2$$

$$A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \Leftarrow \text{matrix of the above equations}$$

Lets find eigen value and eigen vectors

$\lambda_1 = 0$ because A is singular so one $\lambda = 0$

and other $\lambda_2 = -3$ from trace $(-2 - 1 = -3 = \lambda_1 + \lambda_2)$

So coming to eigenvectors

$\lambda_1 = 0$ so $\begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\lambda_2 = -3$ $x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Solutions :

$$u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2$$

Checking : $\frac{du}{dt} = Au$ and plug in $e^{\lambda_1 t_1} x_1$

$$\lambda_1 \cancel{e^{\lambda_1 t_1}} x_1 = A \cancel{e^{\lambda_1 t_1}} x_1$$

so it checks out

going back to equation for our example

$$u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2$$

$$= c_1 \cdot 1 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

at $t=0$

$$c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Solu: $c_1 = \frac{1}{3}$, $c_2 = \frac{1}{3}$

So

$$u(t) = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{3} e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Steady state

$$u(\infty) = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \begin{bmatrix} \text{as } t \to \infty \text{ then} \\ e^{-3t} \to 0 \end{bmatrix}$$

Checking for the following with eigen values

1) Stability

if $u(t) \to 0$, then we need $e^{\lambda t} \to 0$

then real part of $\lambda < 0$

2) Steady state

when $\lambda_1 = 0$ and other eigen value are negative

3) Blow up if any real part $\lambda > 0$

Comment for 2x2 stability

a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is stable

when $\text{Re}\,\lambda_1 < 0$ and $\text{Re}\,\lambda_2 < 0$

and $\det A > 0$

$$\frac{du}{dt} = Au \quad , \quad \text{set } u = Sv$$

eigen vector matrix

$$\frac{dv}{dt} = S^{-1}ASv$$

$$= \Lambda v$$

So $v(t) = e^{\Lambda t} v(0)$

or

$$u(t) = Se^{\Lambda t}S^{-1}u(0) = e^{At}u(0)$$

Matrix exponential

$$e^{At} = I + At + \frac{(At)^2}{2} + \frac{(At)^3}{6} \ldots + \frac{(At)^n}{n!} + \ldots$$

<u>Must revise this</u>

<u>Part more throughly</u>

Video-24

Markov Matrices
Steady States

Fourier Series and Projections

Applications of eigenvalues;

Markov Matrices

$$A = \begin{bmatrix} .1 & 0.01 & .3 \\ .2 & .99 & .3 \\ .7 & 0 & .4 \end{bmatrix}$$

1) all entries must be ≥ 0

2) all columns add to 1

Study state in markov's matrix

is when $\lambda = 1$

So for our example

when $\lambda = 1$

$$A - 1 I = \begin{bmatrix} -.9 & .01 & .3 \\ .2 & -0.01 & .3 \\ .7 & 0 & -0.6 \end{bmatrix}$$

$A - I$ is singular (as rows seem to be dependent)

because $(1, 1, 1)$ is $n(A^T)$

and x_1 will be $n(A)$

note

eigen value of A = eigen value of A^T

And Markov's matrices always will have eigen value $\lambda = 1$ (the rest of them is not known)

Application of Markov matrix

$$\begin{bmatrix} u_{cal} \\ u_{mass} \end{bmatrix}_{k+1} = \begin{bmatrix} .9 & 0.2 \\ .1 & 0.8 \end{bmatrix} \begin{bmatrix} u_{cal} \\ u_{mass} \end{bmatrix}_{k}$$

and $\begin{bmatrix} u_{cal} \\ u_{mass} \end{bmatrix} = \begin{bmatrix} 0 \\ 1000 \end{bmatrix}$

So for $k=1$

$$\begin{bmatrix} u_{cal} \\ u_{mass} \end{bmatrix}_{1} = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} \begin{bmatrix} 0 \\ 1000 \end{bmatrix}$$

$$= \begin{bmatrix} 200 \\ 800 \end{bmatrix}$$

So we can easily find the # people over time with eigen values and eigen vectors.

$$\lambda_1 = 1 \quad \text{and} \quad x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\lambda_2 = .7 \quad \text{and} \quad x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$u_k = c_1 \, 1^k \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 (.7)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$u_0 = \begin{bmatrix} 0 \\ 1000 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\frac{0}{1000} = \frac{1000}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{2000}{3} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\text{So } c_1 = \frac{1000}{3} \text{ and } \frac{2000}{3}$$

projections with orthonormal basis

$q_1, q_2, \cdots q_n$

Any

$v = x_1 q_1 + x_2 q_2 \cdots + x_n q_n$

$q_1^T v = x_1 + x_2 \underbrace{q_2 q_1^T}_{0} \cdots x_n \underbrace{q_n q_1^T}_{0}$

as they are orthonormal

in matrix form

$Q x = v$

$x = Q^{-1} v$

$= Q^T v \quad [\text{as } Q^{-1} = Q^T]$

Fourier Series

$$f(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x$$
$$+ b_2 \sin 2x + \dots$$

for vectors to be orthonormal

$v^T w = 0$

$= v_1 w_1 + v_2 w_2 + v_3 w_3 \dots v_n w_n$

So for functions, lets say f and g to be orthonormal

$$f^T g = \int f(x)\, g(x)\, dx$$

limit depends on f and g

for the above series

$$f^T g = \int_0^{2\pi} f(x)\, g(x)\, dx$$

because the above function is perodic as $f(x) = f(x + 2\pi)$

So

$$= \int_0^{2\pi} \sin x \cos x \, dx$$

$$= \frac{1}{2}(\sin x)^2 \Big|_0^{2\pi} = 0$$

and the same would be true for other pairs of functions.

So to find a_1

$$a_1 \int_0^{2\pi} (\cos x)^2 dx = \int_0^{2\pi} f(x) \cos x \, dx$$

$$a_1 \pi = \int_0^{2\pi} f(x) \cos x \, dx$$

$$a_1 = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos x \, dx$$

Video - 25

Symmetric Matrices
Eigenvalues / Eigenvectors

Start: Positive Definite Matrices

Symmetric Matrices

So $A^T = A$

then

eigenvalue are <u>Real</u>

eigenvectors <u>are</u> orthonormal

$\Downarrow$

can be choosen

So usually

$$A = S \Lambda S^{-1}$$

but for symmetric A, S is orthonormal, So

$$A = Q \Lambda Q^{-1} = Q A Q^{T}$$

<u>Why real eigenvalues for</u>
<u>Symmetric real A</u>

$$Ax = \lambda x \quad \text{—— } 1$$

Lets say they [eigenvalue and eigenvector are complex]

Lets take conjugate

$$A\bar{x} = \bar{\lambda}\bar{x}$$

Lets take transpose on both side

$$\bar{x}^T A^T = \bar{x}^T \bar{\lambda}$$

as $A^T = A$, then

$$\bar{x}^T A = \bar{x}^T \bar{\lambda} \quad \text{—— } (2)$$

from (1)
Lets multiply $\bar{x}^T$ on both side from R

$$\bar{x}^T A x = \bar{x}^T \lambda x$$

from (2)
Lets multiply x on both side from left

$$\bar{x}^T A x = \bar{x}^T \bar{\lambda} x$$

So from the above result where LHS are equal

$$\bar{x}^T \lambda x = \bar{x}^T \bar{\lambda} x$$

So $\lambda = \bar{\lambda}$ so λ is real

This is true because $\bar{x}^T x$ is not zero and it must be real to cancel on both side.

Why?

Lets take $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$

So

$$\bar{x}^T x = [\bar{x}_1 \ \bar{x}_2 \ \bar{x}_3 \cdots \bar{x}_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Rough

$$(a+ib)(a-ib)$$

$$a^2 - \cancel{aib} + \cancel{aib} + b^2$$

$$a^2+b^2$$

So $\bar{x}_1 x_1 + \bar{x}_2 x_2 \cdots$ must be real

and we canceled that

Good matrices :

good meaning

1) Real λ
2) perpendicular x

Then matrix A is said to be good if

$$A = \bar{A}^T$$

but when A is real $A = A^T$ as $\bar{A} = A$
[no imaginary part]

---

for $A = A^T$

$A = Q \Lambda Q^T$ as discussed above

$$= \begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \lambda_n \end{bmatrix} \begin{bmatrix} q_1^T \\ q_2^T \\ \vdots \\ q_n^T \end{bmatrix}$$

$$= \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T + \dots + \lambda_n q_n q_n^T$$

$q_1 q_1^T \Rightarrow$ projection of something

So we can look as

All symmetric matrices are a combination of perpendicular projection matrices

---

fact

Signs of pivots are same as signs of eigenvalues in symmetric A

Positive definite symmetric Matrices:

all eigen values are <u>positive</u>

all pivots are <u>positive</u>

Example:

$$\begin{bmatrix} 5 & 2 \\ 2 & 3 \end{bmatrix}$$

pivots $5, \frac{11}{5}$

So for λ

$$\begin{bmatrix} 5-\lambda & 2 \\ 2 & 3-\lambda \end{bmatrix}$$

$$\lambda^2 - 8\lambda + 11 = 0$$

$$\lambda = 4 \pm \sqrt{5}$$

so both λ are positive