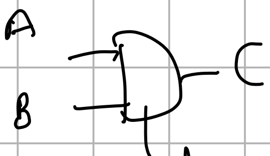
Week 1, Module-1

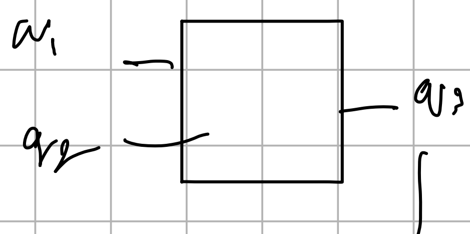
Quantum Computing Basics

	Classical computing	Quantum computing
bits	$b = \{0, 1\}$	$\lvert\psi\rangle = \alpha\lvert 0\rangle + \beta\lvert 1\rangle$
gates	and, Not, or etc	Polli-X, Y, Z, H etc



And gate

irreversable ✗



reversable ✓

[always reversable in quantum computing]

Notations

Dirac (Bra-ket) notation

ket : $|0\rangle$ $|1\rangle$

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$= \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \leftarrow \text{Hilbert space}$$

$$|\psi\rangle = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

↑ ket

bra $\langle \varphi | = [\, b_1^* \;\; b_2^* \;\; \dots \;\; b_n^* \,]$

bra-ket $\langle \varphi | \psi \rangle$

inner product $\rightarrow$ Like vector dot product

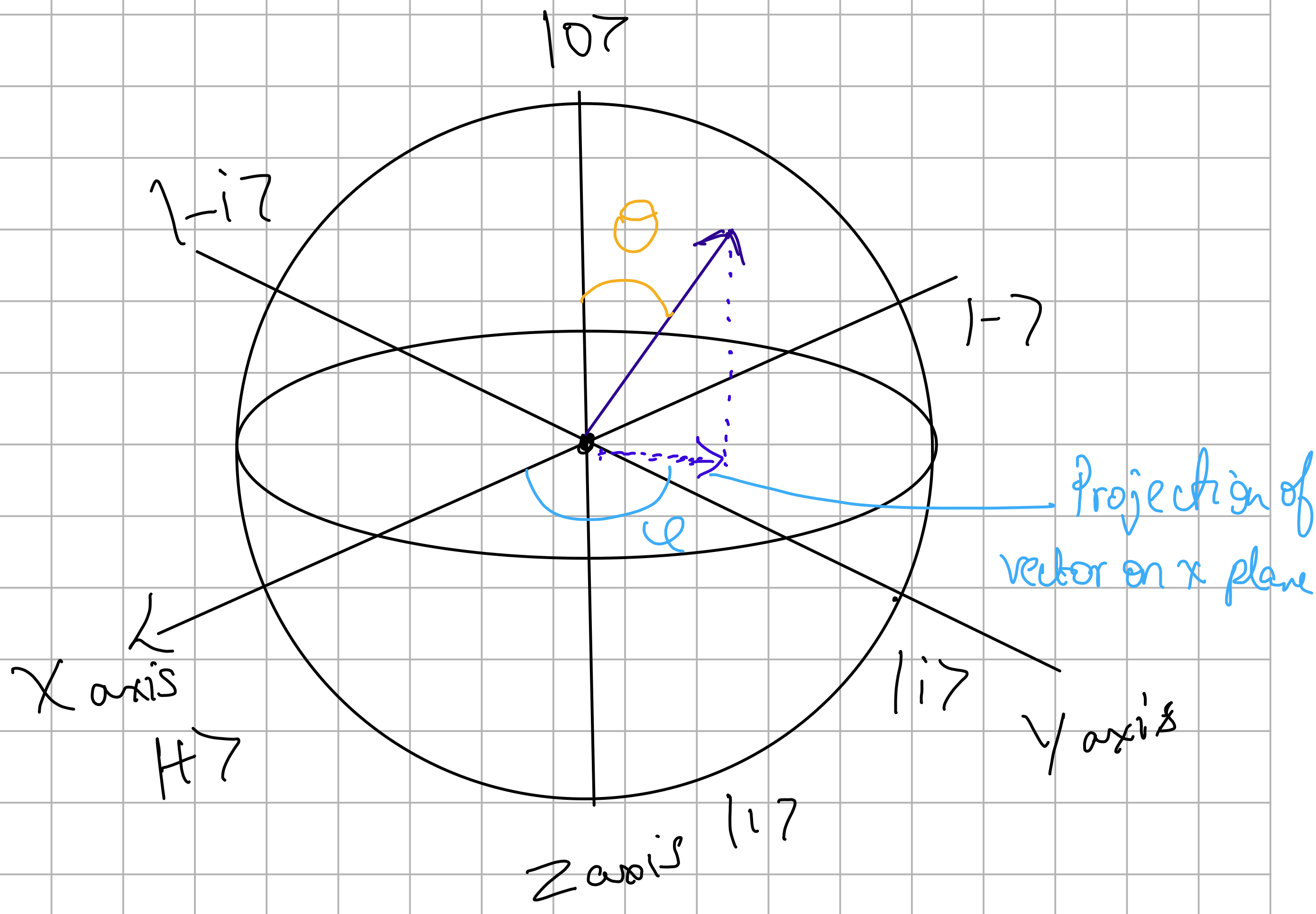
bra-ket $\rightarrow$ single value

ket bra $| \psi \rangle \langle \varphi |$

$\rightarrow$ output is $n \times n$ matrix

we can represent any qubit $|q\rangle$
in bloch sphere with 2 angels
θ and φ

where θ is angle of the vector with Z axis and φ is angle between vector's projection on x plane and x axis



$$|q\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$$

Module- 2

Recall:

any qubit

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Ket ψ Ket 0   Ket 1

and $|\alpha|^2 + |\beta|^2 = 1$

Aim: (1) physical description of qbit

$\hookrightarrow$ 2 level quantum Systems

(2) how do we represent physical properties of such system

(3) Time evolve?

Postulates of Quantum Mechanics

[* only finite dim quantum Systems

* Matrix formulation of quantum theory]

Postulate - 1:

The state of the quantum system is represented by a unit vector, in a complex inner product space (more generally, a Hilbert space)

State of a system: $|\psi\rangle \in \mathcal{H}$ (complex linear vector space)

(ket sai)

Recap

(a) Complex linear vector spaces

$$|v\rangle \in \mathbb{V}$$

$$\exists\, \lambda \in \mathbb{C} \text{ (complex id)} \ni \lambda|v\rangle \in \mathbb{V}$$

if $|v_1\rangle, |v_2\rangle \in \mathbb{V}$

then $\alpha|v_1\rangle + \beta|v_2\rangle \in \mathbb{V}$

(b) Notation bra-ket notation

$|\psi\rangle \leftarrow$ quantum state

ket si

$\langle\psi| \leftarrow$ complex conjugate Transpose (dagger of $|\psi\rangle$)

bra si

(c) Matrix representation

$|v\rangle \in$ d-dim complex linear vector space

then matrix multiplication ways

$|v\rangle = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}_{d\times 1}$ $v_1, v_2, v_3, \dots v_n \in \mathbb{C}$

Ket v

Column matrix

$$\langle v | = \left(\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}^* \right)^T$$

Bra v

(Row matrix)

$$= \begin{pmatrix} v_1^* & v_2^* & v_3^* & \cdots & v_n \end{pmatrix}_{1 \times d}$$

d) Inner product (single vaule - Scalar)

$\langle v | w \rangle$ => Inner product (matrix multiplication)

$$= \begin{pmatrix} v_1 & v_2 & \cdots & v_d \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_d \end{pmatrix}$$

= single v

e) Outer product (matrix $d \times d$)

Ket/bra

$$|v\rangle\langle w| = \begin{pmatrix} \\ \\ \end{pmatrix}_{d\times 1}\begin{pmatrix} & & \end{pmatrix}_{1\times d}$$

$$= \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}_{d\times d}$$

These give the linear operator on the vector in the vectorspace

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Postulate - 1

$|\psi\rangle \in H$

unit vectors $\rightarrow$ norm = 1 (length)

for any unit vector: $\boxed{\langle\psi|\psi\rangle = 1}$ (constraint is because they contain probability)

- over all phase (λ)

if we have $|v\rangle \in H$,

any $\lambda |v\rangle \in H$, physically $|\psi\rangle$ and

$\uparrow$

scalar

$\lambda|\psi\rangle$ corresponds to the same state

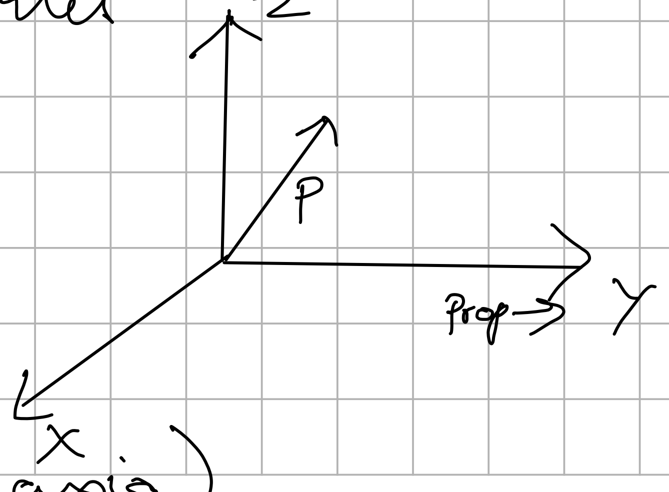
and λ is sometimes used for normilization

State space of Qubits

a) Physical example:

a1) Polarization state of light

plane of polarization is described by 2 $\perp^r$ states



$|H\rangle$ horizontal (Z axis)

or

$|v\rangle$ vertical (x axis)

(if propogation is on y axis)

$$|P\rangle = \alpha|H\rangle + \beta|V\rangle \in 2\text{dim}$$

Complex linear vector space

$\mathbb{C}^2$

a2) Spin $-\frac{1}{2}$ particles of e^- or p^+

$|\uparrow\rangle$ spin up $\quad |\downarrow\rangle$ spin down $\Longleftrightarrow |0\rangle, |1\rangle$

$|+\frac{1}{2}\rangle \quad |-\frac{1}{2}\rangle$

a3) Two level atom $\longrightarrow$ superconducting qubits

(artifical two level atoms)

other level unaccessable

$|e\rangle$ excited

$|g\rangle$ ground

Qbit state space :

2 orthogonal, normalized states
denoted by $|0\rangle$ and $|1\rangle$

$$\langle 0|1\rangle = \langle 1|0\rangle = 0$$

* $\{|0\rangle, |1\rangle\}$ are the orthonormal basis
for the qbit state space

* $\mathbb{C}^2 \equiv$ 2 dim complex linear vector space

* Any state of qbit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$
$(\langle\psi|\psi\rangle = 1)$

* We can paramatrize $|\psi\rangle$ in terms

$$\alpha = \cos\left(\frac{\theta}{2}\right) e^{i\phi/2}$$

$$\beta = \sin\left(\frac{\theta}{2}\right) e^{i\phi/2}$$

where

$\theta \in [0, \pi]$

$\phi_\beta, \phi_\alpha \in [0, 2\pi]$

$$|\psi\rangle = \cos\frac{\theta}{2}\, e^{i\phi_\alpha}|0\rangle + \sin\frac{\theta}{2}\, e^{i\phi_\beta}|1\rangle$$

$$= \underbrace{\cancel{e^{i\phi_\alpha}}}_{\substack{\text{global phase} \\ \text{(won't matter)}}} \left[\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}\, \underbrace{e^{i\phi}}_{\substack{\text{local phase} \\ \text{(matters!)}}}|1\rangle\right]$$

$$(\phi = \phi_\beta - \phi_\alpha)$$

So ψ can be represented by $\theta \in [0, \pi]$

and $\phi \in [0, 2\pi]$

It represents a surface of a sphere
(Bloch sphere)

Alternate Basis states of $\mathbb{C}^2$

$$|+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right);$$

$$|-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

check:

$$\left.\begin{aligned}\langle +|-\rangle &= \langle -|+\rangle = 0\\ \langle +|+\rangle &= \langle -|-\rangle = 1\end{aligned}\right\}$$ $\{|+\rangle, |-\rangle\}$ are orthonormal basis for $\mathbb{C}^2$

Another Basis

$$|i\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + i\,|1\rangle\right) \qquad |-i\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - i\,|1\rangle\right)$$

check $\{|+i\rangle, |-i\rangle\}$ are orthonormal basis for $\mathbb{C}^2$

* Basis transformations

Lets take $|0\rangle \longrightarrow |+\rangle,\ |1\rangle \longrightarrow |-\rangle$

$$H|0\rangle \rightarrow |+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$H|1\rangle \rightarrow |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$H = \frac{1}{\sqrt{2}}\left(|+\rangle\langle 0| + |-\rangle\langle 1| \right)$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle\langle 0| + |1\rangle\langle 0| + |0\rangle\langle 1| - |1\rangle\langle 1| \right)$$

for matrix representation:

$$|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad |-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Hadamard gate.

Postulate-2

Physical Quantities or Observables

Linear transformations (operators)

└ Hermatian Operations H

$$H = (H^{*})^{T} = H^{\dagger}$$ (dagger)

(adjoint)

Properties

└ Hermatian matrices have real eigen value

$$H|v\rangle = \lambda|v\rangle$$

└→ diagonalizable

Example of hermatian operators for 2 level systems

$$S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad S_x = S_x^{\dagger}$$

(physical dimensions of plank's constant)

Poshlate 2 : Physical quantilies (observables in quantum theory) are associated with Hermitian operators on the system Hilbert space

Eigenvalue of Hermatian operators are associated by the physical observable

Consider spin $\frac{1}{2}$ system

Let $|0\rangle = |+\frac{1}{2}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$|1\rangle = |-\frac{1}{2}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$S_z|0\rangle = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}|0\rangle$$

$$S_z|1\rangle = -\frac{\hbar}{2}|1\rangle$$

So S_z must be associated with the observable or quantity with spin along z direction

$+\frac{\hbar}{2}$ spin along z direction

$-\frac{\hbar}{2}$ spin if opposite to z direction

Similarly

$$\left.\begin{aligned} S_x|0\rangle &= \frac{\hbar}{2}|1\rangle \\ S_x|1\rangle &= \frac{\hbar}{2}|0\rangle \end{aligned}\right\}$$ not eigen value equation!

(change! not $H|v\rangle \neq \lambda|v\rangle$)

but for $|+\rangle$ and $|-\rangle$

$$\left.\begin{aligned} S_x|+\rangle &= \frac{\hbar}{2}|+\rangle \\ S_x|-\rangle &= -\frac{\hbar}{2}|-\rangle \end{aligned}\right\}$$ eigen values

So $+\frac{\hbar}{2}$ along x direction if $|+\rangle$

or $-\frac{\hbar}{2}$ along x direction if $|-\rangle$

Unitary operators (U)

$$U U^{\dagger} = U^{\dagger} U = I$$

↳ identity

Ex: $$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$H^{\dagger} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$H H^{\dagger} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So Hedamard gate is both hermatian and unitary

Postulate - 3 :

$|\psi(t_0)\rangle \leftarrow$ state of the system at time t_0

State at time t is obtained by the action of a unitary operator on the initial state

$$\boxed{\psi(t) = U(t, t_0)\,|\,|\psi(t_0)\rangle}$$

Hamiltonian (H)

H is a Hermitian operator associated with the energy of the quantum system

Schörindger eqn $= i\hbar \frac{\partial}{\partial t}\left(|\psi(t)\rangle\right) = H\,|\psi(t)\rangle$

This explains how quantum states evolve with time

Hamaitonian

$$\boxed{\text{Unitary operation preserve length (norm)}}$$

* quantum gates are essentially unitary operators / metrics !

Single qbit gates are 2x2 matrices

Ex: Hadamard $H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

Properties of $U(t, t_0)$:

i) $U(t_2, t_0) = U(t_2, t_1) U(t_1, t_0) \qquad t_0 \leq t_1 \leq t_2$

So in quantum gates, to do complex operations we stack up the quantum gate.

... etc $H \; G \; H$

$\leftarrow$

ii) $U^{\dagger}(t, t_0) = U(t_0, t)$

hence it means $U^{-1} = U^{\dagger}$

and also all quantum gates are reversable !!

Quantum Measurement

Postulate 4: Let A be an observable with eigenvalues $\{a_i\}$ and eigenvectors $\{|a_i\rangle\}$

$i \in \{1, 2, \ldots d\}$
↑ d dimensional eigen value

(4a) Measurement of observable A on state $|\psi\rangle$ gives outcome a_i with probibility

$$P(i) = |\langle\psi|a_i\rangle|^2$$

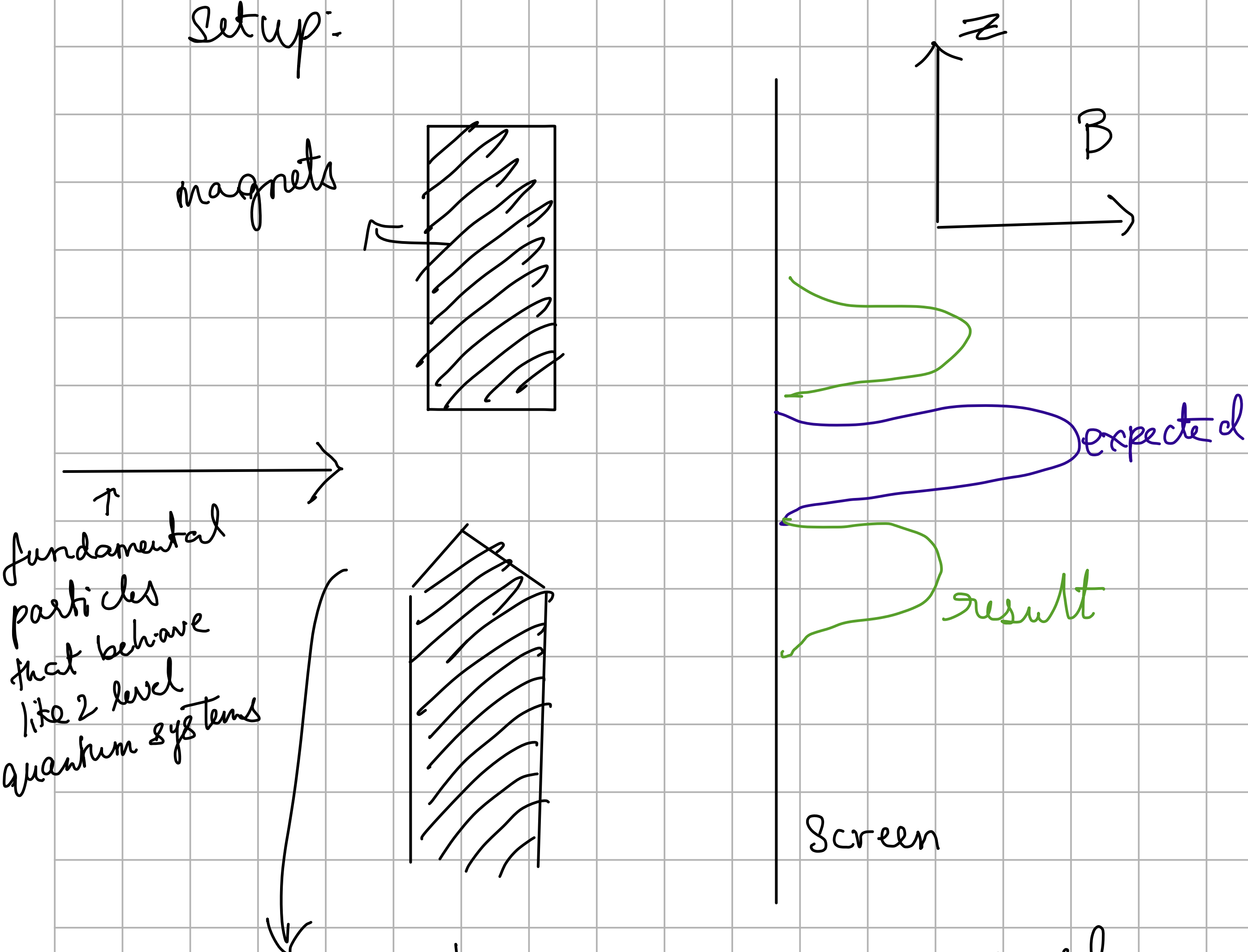
also called Born Rule

(4b) Post-measurement: if outcome a_i is obtained the state $|\psi\rangle$ "collapses" to the corresponding eigenstate $|a_i\rangle$

Physical model for quantum measurement
(Two level System)

Stern Gerlach experiment (1922)

Setup:



non uniform / inhomogenous magnetic field

- magnetic dipole ↗ undergoes deflection under a non uniform mag. field
- Spin angular momentum of spin $\frac{1}{2}$ particle ⟶ magnetic dipole moment $\vec{m}$

- clasically $\vec{m}$ could be oriented in any way w.r.t. z axis
- Let θ be angle made by $\vec{m}$ with z axis

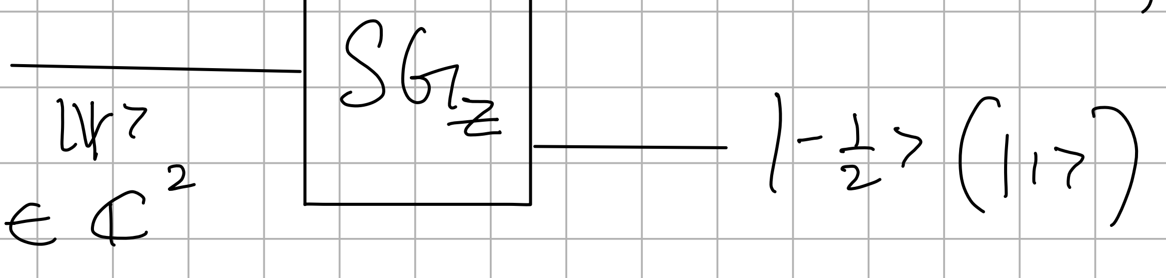
  Deflection is propotional to $\cos\theta$
- Actual outcome of the S-G experiment:

  two possible orientations of the spin angular momentum:

$$\left(\frac{\hbar}{2}\right), \left(-\frac{\hbar}{2}\right) \text{ or } |\uparrow\rangle, |\downarrow\rangle$$

$$\text{or } |0\rangle, |1\rangle$$

SG apparater



$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$

prob of spin value being $+\frac{1}{2} = \langle\psi|0\rangle = |\alpha|^2$

" " $-\frac{1}{2} = \langle\psi|1\rangle = |\beta|^2$

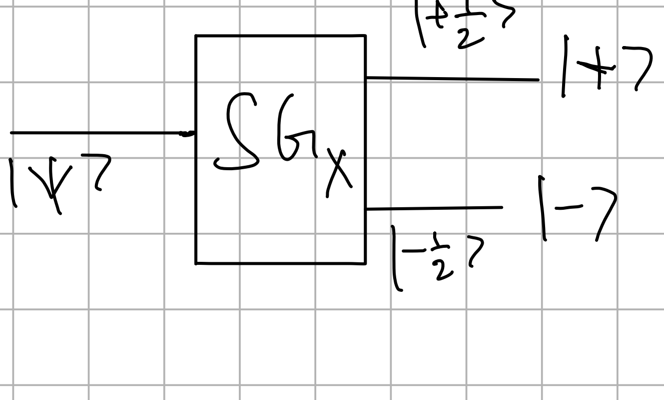
and $|\alpha|^2 + |\beta|^2 = 1$

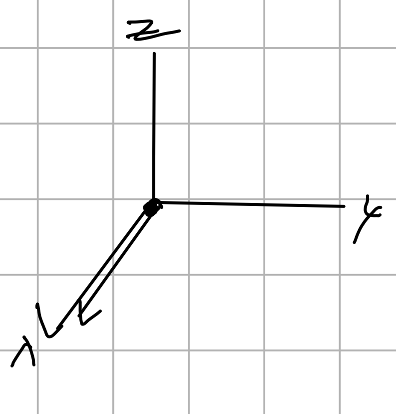
Example: if input is $|+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$

then prob$\left(+\frac{1}{2}\right) = \frac{1}{2}$

" " $-\frac{1}{2} = \frac{1}{2}$

* Spin along other axis?





$|+\rangle$, $|-\rangle$ were the results (Done before)
and they are the orthonormal basis for $\mathbb{C}^2$

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rightarrow \text{eigenstates} = \{|+\rangle, |-\rangle\}$$

$$|\psi\rangle = \gamma\,|+\rangle + \delta\,|-\rangle$$

prob of $+\frac{1}{2}$ spin $= \langle\psi|+\rangle = |\gamma|^2$

$+$

" $-\frac{1}{2}$ spin $= \langle\psi|-\rangle = |\delta|^2$

$\parallel$

1

* Spin along Y axis

$$S_y = \frac{\hbar}{2}\left(|i\rangle\langle i| - |-i\rangle\langle -i|\right)$$

$$= \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

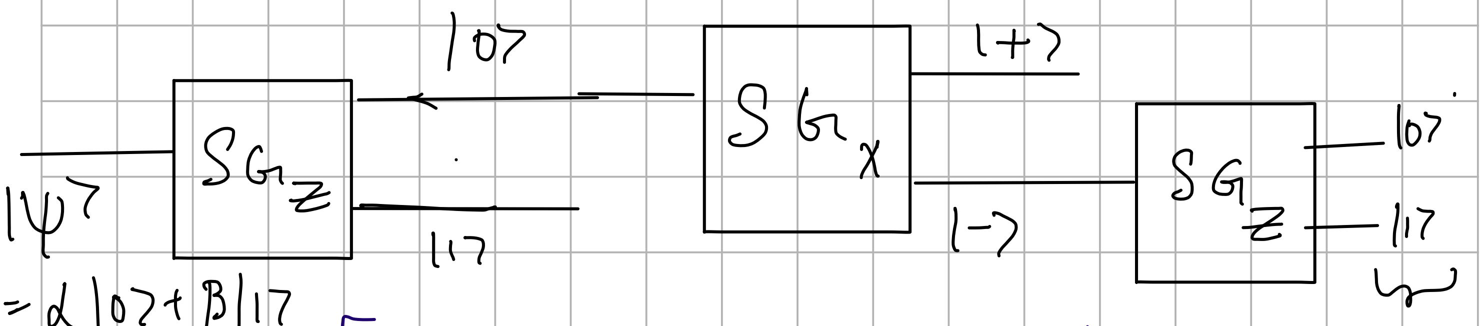
Uncertainty Principle

Sequential Stern Gerlach setup

note

$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$ $\quad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$

similarly $|0\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}}$ $\quad |1\rangle = \frac{|+\rangle - |-\rangle}{\sqrt{2}}$



both states are equally

(we expect only $|0\rangle$ because we initially measured and only passed on $|0\rangle$ state to the next SG but that is not happening)

So when we measure Z to be $|0\rangle$ and then measure $|-\rangle$ along X, we loose all info about Z somehow. This means that we can never measure along x and z at the same time.

* S_x, S_z are incompatible observable!

* In eigenstate of S_z, both $|0\rangle$, $|1\rangle$ outcome are equally likely

* In eigenstate of S_x; both $|+\rangle$ $|-\rangle$ outcome are equally likely

* It is impossible to have a state which hase a definite value for both S_x and S_z

$\hookrightarrow$ Uncertainty Principle

Quantum Gates

Single Qbit gate

$$|q\rangle \rightarrow \boxed{G} \rightarrow |q'\rangle$$

Every Gate is unitary in Quantum Computing