

# 21-128 and 15-151 problem sheet 1

Solutions to the following six exercises and optional bonus problem are to be submitted to  
Gradescope by

**11 PM on Wednesday, the 3rd of September, 2025.**

## Problem 1

Suppose that  $a$  and  $b$  are positive integers such that  $a$  divides  $b$ . Prove that  $a$  divides  $a^2 + b^2$ .

## Problem 2

Find, with proof, all solutions in real numbers to the equation:  $5 = 2\sqrt{x} + \sqrt{10 - x}$ .

## Problem 3

Let  $p(x, y)$  be the predicate ‘ $x + y$  is even’, where  $x$  and  $y$  range over the integers.

- (a) Prove that the following proposition is true:  $\forall x \exists y p(x, y)$ .
- (b) Prove that the following proposition is false:  $\exists y \forall x p(x, y)$ .

## Problem 4

- (a) Show that  $(p \Rightarrow q) \vee (p \Rightarrow r)$  and  $p \Rightarrow (q \vee r)$  are logically equivalent.
- (b) Show that  $(\forall x \in S P(x)) \vee (\forall x \in S Q(x))$  and  $\forall x \in S (P(x) \vee Q(x))$  are not logically equivalent, where  $P(x)$  and  $Q(x)$  are predicates and  $S$  is a set.  
*Hint:* What does it mean for two propositions to be logically equivalent? How might one prove that this does not hold?

## Problem 5

- (a) Show that the following proposition is false:  $(\forall a, x \in \mathbb{R})(\exists! y \in \mathbb{R})(x^4 y + ay + x = 0)$ .
- (b) Find, with proof, the set of real numbers  $a$  such that the following proposition is true:

$$(\forall x \in \mathbb{R})(\exists! y \in \mathbb{R})(x^4 y + ay + x = 0)$$

### **Problem 6**

Which of the following numbers are irrational for every choice of numbers  $r$ ,  $a$  and  $b$ , such that  $r$  is rational and  $a$  and  $b$  are irrational?

$$a + r \quad a + b \quad ar \quad a^b$$

Prove your claims, either by proving that the number must always be irrational or by providing a counterexample. If you claim that a number is irrational, then you should prove it or cite lecture/Clive.

### **Bonus Problem (1 points)**

Three brilliant, flawless logicians - Alana, Hasita, and Isha were blindfolded and each had a hat with a positive integer (possibly different for each) written on it placed on their heads.

Their blindfolds were then removed; they faced each other in a circle and each could see the hats the others were wearing, but not their own hat.

They were told that two of the numbers added up to the third. In order to be generously rewarded they needed to figure out what number was written on their hats.

Here is the conversation that took place:

Alana: I don't know what my number is.

Hasita: I don't know what my number is.

Isha: I don't know what my number is.

Alana: Now I know what my number is. It is 50.

(a) What are the other numbers?

(b) What combination(s) of numbers would allow Isha to solve the problem in round 1?