21-128 and 15-151 problem sheet 2

Solutions to the following six exercises and optional bonus problem are to be submitted to Gradescope by

11 PM on Wednesday, the 10th of September, 2025.

Problem 1 (5 points)

Determine, with proof, which are tautologies, where A, B and C are non-empty subsets of $\mathbb{Z}$.

(a) $(A \cap B \cap C) \subset (A \cup B)$

(b) $(A \setminus B) \cap (B \setminus A) = \varnothing$

(c) $(A \cap B \neq \varnothing) \implies ((A \setminus B) \subset A)$

Note, we take both $\subsetneq$ and $\subset$ to mean *proper* subset, and $\subseteq$ to mean subset.

Problem 2 (5 points)

Prove that $\{x \in \mathbb{Z} : 6 \mid x\} = \{x \in \mathbb{Z} : 6 \mid (7x - 12)\}$.

Problem 3 (5 points)

Let $X \triangle Y = (X \setminus Y) \cup (Y \setminus X)$. Prove $C \setminus (A \triangle B) = (A \cap B \cap C) \cup (C \setminus (A \cup B))$.

Problem 4 (5 points)

For this problem, we will consider sets entirely composed of sets (for example, no numbers allowed). We call a set C a *Rod* set if and only if it has the property

$$((A \in B) \wedge (B \in C)) \implies (A \in C)$$

for all A, B. Show that if S is a *Rod* set, then $\mathcal{P}(S)$ is also a *Rod* set.

Problem 5 (5 points)

A set $S \subseteq \mathbb{R}$ is called *open* if

$$(\forall x \in S)(\exists \varepsilon \in \mathbb{R}^+)(\forall y \in \mathbb{R})(|x - y| < \varepsilon \implies y \in S).$$

Let $S \subseteq \mathbb{R}$ be an open set, with points $a, b \in S$, and $a < b$. Show that there exists some $c \in S$ such that $a < c < b$. Recall that $\mathbb{R}^+$ is the set of positive real numbers.

Problem 6 (5 points)

Let $\mathbb{N}^+$ denote the set of positive integers and consider the function $f : \mathbb{N}^+ \times \mathbb{N}^+ \to \mathbb{R}$ defined by

$$f(a, b) = \frac{(a + 1)(a + 2b)}{2}$$

(a) Show that the image of f is a subset of $\mathbb{N}^+$.

(b) Determine, with proof, exactly which positive integers are elements of the image of f. (**Hint:** Try plugging in values and look for a pattern.)

Bonus Problem (1 point)

A function $g : \mathbb{R} \to \mathbb{R}$ is *even* if $g(-x) = g(x)$ for all $x \in \mathbb{R}$, or *odd* if $h(-x) = -h(x)$ for all $x \in \mathbb{R}$.

Let $f : \mathbb{R} \to \mathbb{R}$.

(a) Prove that there exists a unique pair of functions $g, h : \mathbb{R} \to \mathbb{R}$ such that g is even, h is odd, and $f = g + h$. (**Hint:** Express both $f(x)$ and $f(-x)$ in terms of $g(x)$ and $h(x)$, and solve the resulting system of equations.)

(b) When f is a polynomial function, express g and h as in (a) in terms of the coefficients of f and show your work.