QUANT JOB INTERVIEW QUESTIONS

Contents

1. Basic Math

(1) One morning, in Springfield, somewhere in the US, it started snowing at a heavy but constant rate. Homer Simpson had just started his own snowplow business. His snowplow started out at 8:00 A.M. At 9:00 A.M. it had gone 2 miles. By 10:00 A.M. it had gone 3 miles. Assuming that the snowplow removes a constant volume of snow per hour, determine the time at which it started snowing.

(2) A hollow cone of semi-vertical angle $\arctan(\frac{1}{2})$ is fixed vertex-downwards with its axis vertical and is being filled with water at a constant rate k. A spider which can run at speed u is asleep at the end of a web, hanging at a height h_0 vertically above the vertex. Find, in terms of k and h_0, the minimum value of u that will enable the spider to escape a soaking, assuming that he starts to run upwards as soon as the water touches his feet.

(3) Let A be an $n \times n$ matrix, which is skew-symmetric, i.e. $A^T = -A$. If n is odd, prove that $Ker(A) \neq \vec{0}$, where $Ker(A)$ is the kernel or the null-space of A consisting of all vectors killed by A, i.e. all v such that $Av = 0$.

(4) If A is a skew-symmetric matrix, i.e. $A^T = -A$, then prove that $-A^2$ is symmetric and nonnegative definite matrix.

(5) Consider a full-rank matrix A of size m by n and a vector $b \in R^m$.
  (a) Assume that $m < n$. Find $x \in R^n$ with the smallest 2-norm that solves $Ax = b$.
  (b) Assume that $m > n$. Find $x \in R^n$ that minimizes $||Ax - b||^2$.

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(6) Let $\Phi(.)$ be the cumulative distribution function of a $N(0,1)$ random variable. Define a function $g(x)$ as

$$g(x) = \int_{-\ln x}^{x^2} \Phi(t+x)\, dt.$$

Evaluate $g'(1)$ in terms of $\Phi(.)$, where $g'(x)$ is the first derivative of $g(x)$.

(7) A cube of ice melts without changing shape at uniform rate $4cm^3/min$. Find the rate of change of the surface area of the cube when the volume is $125cm^3$.

(8) Solve the equation for $x \in \mathbb{R}$:

$$\sqrt{x + \sqrt{x + \sqrt{x + \sqrt{\ldots}}}} = x$$

(9) Find y where

$$y = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\vdots}}}}$$

2. Probability

(1) a and b are randomly chosen real numbers in the interval $[0,1]$, that is both a and b are standard uniform random variables. Find the probability that the quadratic equation $x^2 + ax + b = 0$ has real solutions.

(2) What is the expected number of (fair) coin flips to get two consecutive heads? Hint: Compute expectation by conditioning.

(3) You are trapped in a dark cave with three indistinguishable exits on the walls. One of the exits takes you 3 hours to travel and takes you outside. One of the other exits takes 1 hour to travel and the other takes 2 hours, but both drop you back in the original cave through the ceiling, which is unreachable from the floor of the cave. You have no way of marking which exits you have attempted. What is the expected time it takes for you to get outside?

(4) Consider a log-normal random variable X given by

$$X = e^{m+sZ},$$

where Z is a standard-normal random variable, m and s are real constants, and s is positive.

(a) Compute the probability density function of X.
(b) Compute mean, median, and mode of the distribution of X.

(5) Suppose X is a Gaussian variable of mean 0, and standard deviation σ, and define $N(u)$ as the CDF of a centered Gaussian variable of mean 0 and variance 1, i.e,

$$N(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-\frac{s^2}{2}} ds.$$

(a) Calculate the expectation of $N(X)$.
(b) Calculate the expectation of $N(X + a)$, where $a > 0$.

(6) By setting up and solving the appropriate convolution, show that the sum of two independent zero-mean unit-variance Gaussian distributions is Gaussian, with variance 2.

(7) If you repeatedly flip a coin whose probability of heads is p then what is the expected number of flips you need to do in order to get a head immediately followed by a tail?

(8) Suppose X is a random variable with $E[X^2] < \infty$. What is the constant c that minimizes $E[(X - c)^2]$?

(9) Let X be a continuous random variable such that $P(X \leq 0) = 0$. Let $f(\cdot)$ be its probability density function. Suppose that $f(x)/P(X > x)$ is equal to a constant λ for all $x > 0$. What is the distribution function of X?

(10) Let $f_X(\cdot)$ be the pdf of a continuous r.v. X. What is the pdf of e^X?

(11) You have 1000 coins, one of which is faulty: it has a head on both sides. You randomly draw a coin, and, without examining it, toss it 10 times. As it happens, you get 10 heads in a row. What's the probability that it's the faulty one?

(12) Let X be a random variable with density distribution function $f(X)$. Define $F(x) = Pr[X < x]$. What is $E[F(X)]$?

(13) Assume the arrival time, T, of a credit default event follows an exponential distribution such that

$$P(T < t) = 1 - e^{-ht}$$

for $t \geq 0$, and $P(T < t) = 0$ for $t < 0$. The quantity $h > 0$ is called the default rate. What is the expected value, $E[T]$, of the default event?

(14) Consider a game where we are allowed to throw a dice 3 times, and we win an amount equal to the number on the dice. We are allowed to stop when we want and obtain the value corresponding to the last throw, but if we

throw it 3 times, we must accept the last value. How much is this game worth?

(15) A bug crawls along the edges of a regular tetrahedron ABCD with edges length 1. It starts at A and at each vertex chooses its next edge at random (so it has a $\frac{1}{3}$ chance of going back along the edge it came on, and a $\frac{1}{3}$ chance of going along each of the other two). Find the probability that after it has crawled a distance 7 it is again at A.

(16) An ant and a blind spider are on opposite corners of a cube. The ant is stationary and the spider moves at random from one corner to another along the edges only. What is the expected number of turns before the spider reaches the ant?

(17) Two archers shoot at a target. The distance of each shot from the center of the target is uniformly distributed from 0 to 1, independently of the other shot. What is the PDF (probability density function) of the distance from the center for the winning shot?

(18) The height of men (in inches) in the US is approximately normally distributed with mean 68 inches and standard deviation of 5 inches.

   (a) A door frame manufacturer wants to build door frames such that only 2% of all men have to duck their head when walking through the door. How high does the door frame need to be?

   (b) The weight of men in the US has mean 168 pounds with standard deviation 25 pounds. The correlation between weight and height is 0.35. What conclusions can you make if the correlation were -0.5? What is your best guess of a man's height if you are told that he weighs 160 pounds?

(19) **The Inverse Transform Method for MC Simulation** Let U be a uniform random variable on the interval $(0, 1)$. For any continuous distribution function F define the random variable $X = F^{-1}(U)$. What is the distribution of X?

3. Markov Chains Applications

(1) What is the expected number of (fair) coin flips to get three consecutive heads? What about n consecutive heads?

4. Brownian Motion and Stochastic Calculus

(1) Let $W(t)$ be the standard Brownian motion, $W(0) = 0$. Find the probability $\mathbf{P}(W(t) \le 0, t = 0, 1, 2) =?$

(2) If W_t is the standard Brownian motion and $S_t = S_0 e^{\mu t + \sigma W_t}$ show that
 (a) $dS_t = (\mu + \frac{\sigma^2}{2}) S_t dt + \sigma S_t dW_t$.
 (b) $\mathbf{E}[S_t] - \mathbf{E}[S_0] = (\mu + \frac{\sigma^2}{2}) \int_0^t \mathbf{E}[S_\tau] d\tau$.
 (c) Conclude that $\mathbf{E}[S_t] = S_0 e^{(\mu + \frac{\sigma^2}{2})t}$.
 (d) Finally, if X is Gaussian with mean μ and standard deviation σ, then conclude that $\mathbf{E}[e^X] = e^{\mu + \frac{\sigma^2}{2}}$.

(3) For standard one-dimensional Brownian motion $W(t)$, calculate

$$\mathbf{E}\Big[\Big(\frac{1}{T}\int_0^T W_t\,dt\Big)^2\Big]$$

(4) Compute the probability

$$P\left(\int_0^1 W(t)dt > \frac{2}{\sqrt{3}}\right)$$

(5) Let W_t be a Brownian Motion starting at 0. Let

$$\beta_k(t) = \mathbf{E}\left[W_t^k\right]$$

Using Ito's formula show that

$$\beta_k(t) = \frac{1}{2}k(k-1)\int_0^t \beta_{k-2}(s)ds.$$

Using this show that $\mathbf{E}[W_t^4] = 3t^2$. What is $\mathbf{E}[W_t^6] =?$

(6) Consider the stock price process $dS(t) = \sigma dW(t)$ with initial level $S(0)$. Consider the average stock price process $A(T) = \frac{1}{T}\int_0^T S(t)dt$. What is the distribution of $A(T)$?

(7) Suppose $W(t)$ is the standard Brownian motion and $y(t)$ satisfies the SDE

$$\begin{aligned} dy(t) &= -cy(t)\,dt + \sigma\,dW(t), \\ y(0) &= x, \end{aligned}$$

where $C > 0$ and σ are constants. Find the mean and variance of $y(t)$.

(8) Consider the stock price pricess $dS(t) = \sigma dW(t)$. Assume that the initial level is $S(0)$, and there is a barrier at $H > S(0)$.

(a) What is the probability the barrier is breached between now and T? Express your answer in terms of the normal distribution.
(b) Let b denote the first time the process breaches the barrier at H. What is the density function of b?

(9) Let X be such that $dX = \mu X dt + \sigma X dW$, where $\sigma > 0$. Find $\alpha \neq 0$ such that X^α is a Martingale.

5. Option Pricing

(1) Consider a one-period model in which the stock has a value of 100 units today and either 120 or 80 units at the end of the period. Assume that the risk-free interest rate is zero. Compute the price of a call option expiring at the end of the period at the money, that is the strike is also 100. Compute this price in two different ways, once using risk-neutral valuation and again using a replicating portfolio.

(2) For a given stock, today the stock price is 12, and the forward price (with expiry at time T) is 12. Assume that at time T, the stock price can be either 10 or 20 (it cannot be any other value). Our stock analyst tells us that the real-world probability of the stock going to 20 is 60%. Assuming zero interest rates, what is the price of a call option with strike 15 on this stock?

(3) For this question, assume European options with expiry in one years time. Also, recall that “long” means you own the option, “short” means you’ve sold it.

Suppose a stock currently trades at 100. Can you say which of the portfolios is worth more? (a) long a call struck at 100, short two calls at 105 and long a call at 110, or, (b) long a call struck at 100, short a call struck at 110.

(4) Consider a one-year European call option on a 100 shares of stock whose spot price is \$94. Imagine we know that in one year, the stock price will be either \$92 or \$102, with 50% probability of each.
(a) What is the fair price of the option, if the strike is \$95? Ignore interest rates and dividends.
(b) What if the probability is 25% that the stock price will be \$92 and 75% that it is 102\$?

(5) If S_t follows a geometric Brownian motion with volatility σ and $S(0) = S_0$ and the prevailing interest rate is r , the value of an option with maturity T and payout $[S_T - K]^+$ at time 0 is $S_0 N(d_1) - Ke^{-rT} N(d_2)$. Here N denotes

the c.d.f of a normal distribution, $d_1 = \frac{ln\frac{S_0}{K}+(r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{(T)}}$ and $d_2 = d_1 - \sigma\sqrt{T}$. Give a probablilistic interpretation of $N(d_1)$.

(6) Consider an "infinite expiry" barrier option. There is an upper barrier at H and a lower barrier at 0. If the stock price touches the upper barrier you get a payoff of 1 and the trade terminates immediately. Likewise if the stock price touches the lower barrier, you get a payoff of 0 and the trade terminates. What is the price of the option? (Assume zero interest rates, and any reasonable stock price dynamics.)

(7) Consider a 1x2 swaption and a cap. The cap under consideration is a strip of two one year caplets, one expiring a year from now, the other expiring in two year's time. Comment on the Black vol of the swaption relative to the Black vol of the cap under the assumption that forward rates are correlated. Which is higher?

(8) What is the Put-Call parity for European options on non-dividend paying stock? Is it valid only in the Black-Scholes framework or is it model-independent? Derive the Put-Call Parity from first principles assuming constant and continuously compounded interest rate.

(9) Suppose S_1 and S_2 follow geometric Brownian motions which are instantaneously correlated with positive correlation ρ. Price an option that pays off $\max(S_T^1 - S_T^2, 0)$. If you are the owner of such an option, are you long or short correlation?

Recall that "long correlation" means you benefit from an increase in the value of the correlation, while short means you benefit from a decrease in the value.

(10) Given the price of a zero coupon bond maturing at T_1 (i.e., the price now of risklessly receiving \$1 at time T_1), denoted by B_1^T, and the price of a zero coupon bond maturing at T_2, denoted by B_2^T, what is the implied annual interest rate spanning the period T_1 to T_2? Show that this forward interest rate can be locked in now at no cost by a combination of positions on the above two zero coupon bonds.

(11) What is the fair swap-rate with payments on dates $T_0, T_1, ..., T_n$ if the corresponding zero coupon prices are $B_0^T, B_1^T, ..., B_n^T$? Assume funding at LIBOR, and the existence of a day-count function $\mathrm{DCF}(t_1, t_2)$, which computes the fraction of a year over which interest is accrued between a pair of dates t_1, $t_2 \geq t_1$.

(12) Suppose the risk-neutral price process of a stock follows $dS_t = \sigma dW_t$ and $S_0 = 0$. Price an option at $t = 0$ that pays out 1 at T iff the $S_T \geq K$ and S

touched level $M < 0 < K$ at least once before T. Hint: Use the Reflection Principle.